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THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

VI. THE DIFFERENCE BETWEEN THE ENERGY-MOMENTUM TENSORS IN THE PRESENCE AND IN THE ABSENCE OF EXTERNAL FIELDS*)

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Synopsis

The difference of the relativistic energy-momentum tensors of a substance with and without fields is derived for the case of thermodynamic equilibrium and linear constitutive relations. It may be looked upon as the field part of the total energy-momentum tensor. The remainder is then the corresponding material part. The negative divergence of the field tensor is the ponderomotive force corresponding to pressure and internal energy defined at zero fields. It constitutes a relativistic generalization of the Helmholtz force. A comparison with the tensors of Minkowski and Abraham shows that these cannot be justified from a microscopic point of view.

§ 1. *Introduction.* The relativistic second law of thermodynamics for a neutral polarized fluid was obtained[1]) in the form

$$T'Ds' = De' + p'Dv' - \tfrac{1}{2}F'_{\alpha\beta}D(v'M^{(1)\alpha\beta\prime}) + \tfrac{1}{2}v'M^{(2)\prime}_{\alpha\beta}DF^{\alpha\beta\prime}, \qquad (1)$$

where the time derivative $D = \mathrm{d}/\mathrm{d}t'$, the temperature T', the specific entropy s', the specific energy e', the pressure p', the specific volume $v' = (\varrho')^{-1}$, the electromagnetic field $F'_{\alpha\beta}$, and the polarizations $M^{(1)\prime}_{\alpha\beta}$ and $M^{(2)\prime}_{\alpha\beta}$ are measured in the permanent local rest frame, indicated by primes (T', p', $F'_{\alpha\beta}$ and $M^{(2)\prime}_{\alpha\beta}$ are equilibrium quantities). The polarization tensors $M^{(1)}_{\alpha\beta}$ and $M^{(2)}_{\alpha\beta}$ are defined in terms of the total polarization tensor $M_{\alpha\beta}$ and bulk velocity U_α by

$$M^{(1)}_{\alpha\beta} = -c^{-2}(U_\alpha U^\gamma M_{\gamma\beta} + U_\beta U^\gamma M_{\alpha\gamma}), \qquad (2)$$

$$M^{(2)}_{\alpha\beta} = \Delta^\gamma_\alpha \Delta^\varepsilon_\beta M_{\gamma\varepsilon}, \qquad (3)$$

where $\Delta_{\alpha\beta} = g_{\alpha\beta} + c^{-2}\,U_\alpha U_\beta$. Furthermore the parameters T', p' $F'_{\alpha\beta}$ and $M^{(2)\prime}_{\alpha\beta}$ are equilibrium quantities. The energy e' and the pressure p' form part of the material energy-momentum tensor $\overline{T}^{\alpha\beta}_{(m)}$ which in equilibrium and in

*) Articles I–V appeared in Physica **37** (1967) and **39** (1968).

the local rest frame reduces to [1] [2])

$$\overline{T}^{\alpha\beta\,\prime}_{(m)} = \begin{pmatrix} \overline{T}^{00\,\prime}_{(m)} & \overline{T}^{0i\,\prime}_{(m)} \\ \overline{T}^{i0\,\prime}_{(m)} & \overline{T}^{ij\,\prime}_{(m)} \end{pmatrix} = \begin{pmatrix} e'_v + \varrho' c^2 & 0 \\ 0 & p'\mathbf{U} \end{pmatrix}, \tag{4}$$

where e'_v is the energy density $\varrho' e'$ and $\mathbf{U}$ is the unit three-tensor. The material energy-momentum tensor occurs in the energy-momentum conservation laws

$$\partial_\beta (T^{\alpha\beta}_{(f)} + \overline{T}^{\alpha\beta}_{(m)}) = 0, \tag{5}$$

where $T^{\alpha\beta}_{(f)}$ is the field energy-momentum tensor, which in the local rest frame has the form

$$T^{\alpha\beta\,\prime}_{(f)} = \begin{pmatrix} \frac{1}{2}\boldsymbol{E}'^2 + \frac{1}{2}\boldsymbol{B}'^2 & \boldsymbol{E}' \wedge \boldsymbol{H}' \\ \boldsymbol{E}' \wedge \boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + (\frac{1}{2}\boldsymbol{E}'^2 + \frac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}')\mathbf{U} \end{pmatrix}. \tag{6}$$

The preceding formulae furnish the basis of the following discussion on the difference between the energy-momentum tensors for systems with and without external fields.

§ 2. *The free energy for systems with linear constitutive relations.* In this paper we shall limit our discussions to isotropic media with linear constitutive relations, i.e. substances in which – in equilibrium – the polarizations are proportional to the electromagnetic fields

$$\boldsymbol{P}' = \kappa(v', T')\, \boldsymbol{E}', \tag{7}$$

$$\boldsymbol{M}' = \chi(v', T')\, \boldsymbol{B}', \tag{8}$$

or, in covariant notation,

$$M^{(1)\alpha\beta} = \kappa(v', T')\, c^{-2}(U^\alpha U_\gamma F^{\gamma\beta} + F^{\alpha\gamma} U_\gamma U^\beta), \tag{9}$$

$$M^{(2)\alpha\beta} = \chi(v', T')\, \Delta^\alpha_\gamma \Delta^\beta_\varepsilon F^{\gamma\varepsilon}. \tag{10}$$

The time derivative of the specific free energy

$$f' = e' - T's' \tag{11}$$

reads according to the entropy law (1)

$$Df' = -p'Dv' - s'DT' + \tfrac{1}{2}F'_{\alpha\beta} D(v' M^{(1)\alpha\beta\prime}) - \tfrac{1}{2}v' M^{(2)\prime}_{\alpha\beta}\, DF^{\alpha\beta\prime}. \tag{12}$$

This relation may be integrated at constant v' and T'. With the help of (2), (3), (9), (10) and the identity valid for arbitrary antisymmetric tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$:

$$A^{\alpha\beta\prime} DB'_{\alpha\beta} = A^{\alpha\beta} DB_{\alpha\beta} + 2c^{-2} U_\alpha (A^{\alpha\beta} B_{\beta\gamma} - B^{\alpha\beta} A_{\beta\gamma})\, DU^\gamma \tag{13}$$

one then gets the relation between the equilibrium values of the specific free energy f' *with* fields and f'_0 *without* fields (but at the same values of the specific volume v' and temperature T'):

$$f' = f'_0 - \tfrac{1}{4}v'\kappa^{-1}M^{(1)}_{\alpha\beta}M^{(1)\alpha\beta} - \tfrac{1}{4}v'\chi\Delta^{\alpha}_{\gamma}\Delta^{\beta}_{\varepsilon}F_{\alpha\beta}F^{\gamma\varepsilon}, \tag{14}$$

or, in three-dimensional notation,

$$f' = f'_0 + \tfrac{1}{2}v'\kappa^{-1}\boldsymbol{P}'^2 - \tfrac{1}{2}v'\chi\boldsymbol{B}'^2. \tag{15}$$

The pressure follows from the specific free energy by differentiation with respect to the specific volume at constant T', $v'M^{(1)\alpha\beta'}$ and $F^{\alpha\beta'}$. One obtains using (9) and (10)

$$p' = -\frac{\partial f'}{\partial v'} = p'_0 + \tfrac{1}{4}F_{\alpha\beta}M^{(1)\alpha\beta} + \tfrac{1}{4}F_{\alpha\beta}M^{(2)\alpha\beta} + \tfrac{1}{2}v'\frac{\partial\kappa}{\partial v'}c^{-2}F_{\alpha\beta}U^{\beta}U_{\gamma}F^{\alpha\gamma} + \tfrac{1}{4}v'\frac{\partial\chi}{\partial v'}F_{\alpha\beta}\Delta^{\alpha}_{\gamma}\Delta^{\beta}_{\varepsilon}F^{\gamma\varepsilon}, \tag{16}$$

where $p'_0 = -\partial f'_0/\partial v'$ is the pressure in the absence of fields. This formula may be written in three-dimensional notation as

$$p' = p'_0 + \tfrac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{P}' + \tfrac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{M}' + \tfrac{1}{2}\boldsymbol{E}'^2v'\frac{\partial\kappa}{\partial v'} + \tfrac{1}{2}\boldsymbol{B}'^2v'\frac{\partial\chi}{\partial v'}. \tag{17}$$

The specific entropy follows from the specific free energy by differentiation with respect to the temperature T' at constant v', $v'M^{(1)\alpha\beta'}$ and $F^{\alpha\beta'}$. One obtains using (9) and (10)

$$s' = -\frac{\partial f'}{\partial T'} = s'_0 + \tfrac{1}{2}c^{-2}v'\frac{\partial\kappa}{\partial T'}F_{\alpha\beta}U^{\beta}U_{\gamma}F^{\alpha\gamma} + \tfrac{1}{4}v'\frac{\partial\chi}{\partial T'}F_{\alpha\beta}\Delta^{\alpha}_{\gamma}\Delta^{\beta}_{\varepsilon}F^{\gamma\varepsilon}, \tag{18}$$

where $s'_0 = -\partial f'_0/\partial T'$ is the entropy in the absence of fields. The energy density $e'_v \equiv \varrho'e'$ follows from (11), (14) and (18):

$$e'_v = e'_{v0} + \tfrac{1}{4}F_{\alpha\beta}M^{(1)\alpha\beta} - \tfrac{1}{4}F_{\alpha\beta}M^{(2)\alpha\beta} + \tfrac{1}{2}c^{-2}T'\frac{\partial\kappa}{\partial T'}F_{\alpha\beta}U^{\beta}U_{\gamma}F^{\alpha\gamma} + \tfrac{1}{4}T'\frac{\partial\chi}{\partial T'}F_{\alpha\beta}\Delta^{\alpha}_{\gamma}\Delta^{\beta}_{\varepsilon}F^{\gamma\varepsilon}, \tag{19}$$

where $e'_{v0} \equiv \varrho'e'_0$ is the energy density at zero fields. Formula (19) reads alternatively

$$e'_v = e'_{v0} + \tfrac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{P}' - \tfrac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{M}' + \tfrac{1}{2}\boldsymbol{E}'^2T'\frac{\partial\kappa}{\partial T'} + \tfrac{1}{2}\boldsymbol{B}'^2T'\frac{\partial\chi}{\partial T'}. \tag{20}$$

In formulae (17) and (20) expressions have been obtained for the energy density $e'_v = \varrho'e'$ and the pressure p', occurring in the material energy-momentum tensor (4). These relations show the dependence of e'_v and p' on the fields.

§ 3. *The energy-momentum tensor in media with and without fields.* In the absence of electromagnetic fields the energy-momentum tensor for an isotropic fluid system in equilibrium (cf. formula (4)) reads in the local rest frame

$$\begin{pmatrix} e'_{v0} + \varrho' c^2 & 0 \\ 0 & p'_0 \mathrm{U} \end{pmatrix}. \tag{21}$$

In the presence of electromagnetic fields the energy-momentum tensor is equal to the sum of a material and a field part, which in equilibrium (in the rest frame) are given by (4) and (6). For the case of linear constitutive relations it follows from (17) and (20) that the difference of the tensor with field and the tensor without field is equal to

$$\begin{bmatrix} \frac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{D}' + \frac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{H}' + \frac{1}{2}\boldsymbol{E}'^2 T' \dfrac{\partial\kappa}{\partial T'} + \frac{1}{2}\boldsymbol{B}'^2 T' \dfrac{\partial\chi}{\partial T'} & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + \left(\frac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{D}' + \frac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{H}' + \frac{1}{2}\boldsymbol{E}'^2 v' \dfrac{\partial\kappa}{\partial v'} + \frac{1}{2}\boldsymbol{B}'^2 v' \dfrac{\partial\chi}{\partial v'}\right)\mathrm{U} \end{bmatrix} \tag{22}$$

in the rest frame. (Here $\boldsymbol{D}' \equiv \boldsymbol{E}' + \boldsymbol{P}'$ and $\boldsymbol{H}' \equiv \boldsymbol{B}' - \boldsymbol{M}'$). The expressions (21) and (22) together form the total energy-momentum tensor in the presence of fields in the local rest frame of the equilibrium system under consideration. Hence the part (22) contains the complete effect of the switching-on of the fields. In view of this property expression (22) may be considered as a field energy-momentum tensor $T^{\alpha\beta\,\prime}_{[f]}$ and (21) as its corresponding material energy-momentum tensor $T^{\alpha\beta\,\prime}_{[m]}$ (the dash indicates the rest frame). These tensors read in arbitrary frames

$$\begin{aligned} T^{\alpha\beta}_{[f]} = {} & F^{\alpha\gamma}H^{\beta}_{\cdot\gamma} - \tfrac{1}{4}F_{\gamma\varepsilon}H^{\gamma\varepsilon}g^{\alpha\beta} + c^{-2}U^{\beta}(F^{\alpha\gamma}M_{\gamma\varepsilon} - M^{\alpha\gamma}F_{\gamma\varepsilon})\,U^{\varepsilon} \\ & + \tfrac{1}{2}\Delta^{\alpha\beta}\left(c^{-2}v'\frac{\partial\kappa}{\partial v'}F_{\gamma\varepsilon}U^{\varepsilon}U_{\zeta}F^{\gamma\zeta} + \tfrac{1}{2}v'\frac{\partial\chi}{\partial v'}F_{\gamma\varepsilon}\Delta^{\gamma}_{\zeta}\Delta^{\varepsilon}_{\eta}F^{\zeta\eta}\right) \\ & + \tfrac{1}{2}c^{-2}U^{\alpha}U^{\beta}\left(c^{-2}T'\frac{\partial\kappa}{\partial T'}F_{\gamma\varepsilon}U^{\varepsilon}U_{\zeta}F^{\gamma\zeta} + \tfrac{1}{2}T'\frac{\partial\chi}{\partial T'}F_{\gamma\varepsilon}\Delta^{\gamma}_{\zeta}\Delta^{\varepsilon}_{\eta}F^{\zeta\eta}\right), \end{aligned} \tag{23}$$

$$T^{\alpha\beta}_{[m]} = c^{-2}U^{\alpha}U^{\beta}(e'_{v0} + \varrho' c^2) + \Delta^{\alpha\beta}p'_0. \tag{24}$$

These tensors were derived for isotropic fluid media in equilibrium using linear constitutive relations. They will be further specified in the next section.

§ 4. *Induced dipole and permanent dipole substances.* The field energy-momentum tensor (23) contains derivatives of the electric and magnetic susceptibilities κ and χ. These may be expressed in κ and χ themselves if the dipole character of the medium is further specified. First we consider *induced dipole substances* obeying Clausius-Mossotti laws of the type

$$\frac{\kappa}{\kappa+3} \sim \frac{1}{v'}, \quad \frac{\chi}{3-2\chi} \sim \frac{1}{v'}, \tag{25}$$

while κ and χ are independent of the temperature T'. With these laws expression (22), which is the rest frame form of (23), becomes

$$T^{\alpha\beta\,\prime}_{[f]} = \begin{bmatrix} \tfrac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{D}' + \tfrac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{H}' & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & \begin{array}{c} -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' \\ + (\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' - \tfrac{1}{6}\boldsymbol{P}'^2 + \tfrac{1}{3}\boldsymbol{M}'^2)\mathrm{U} \end{array} \end{bmatrix}. \tag{26}$$

It may be noted that whereas the combination $\tfrac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{D}' + \tfrac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{H}'$ subsisted in the energy density, it disappeared from the diagonal elements of the field pressure tensor*).

As a second example we treat *permanent dipole substances* with susceptibilities obeying Clausius-Mossotti laws in their density dependence and Langevin-Debye laws in their temperature dependence:

$$\frac{\kappa}{\kappa+3} \sim \frac{1}{v'T'}, \quad \frac{\chi}{3-2\chi} \sim \frac{1}{v'T'} \tag{27}$$

With this behaviour the tensor (22) becomes

$$T^{\alpha\beta\,\prime}_{[f]} = \begin{bmatrix} \begin{array}{l} \tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' \\ -\tfrac{1}{6}\boldsymbol{P}'^2 + \tfrac{1}{3}\boldsymbol{M}'^2 \end{array} & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & \begin{array}{c} -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' \\ + (\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' - \tfrac{1}{6}\boldsymbol{P}'^2 + \tfrac{1}{3}\boldsymbol{M}'^2)\,\mathrm{U} \end{array} \end{bmatrix}. \tag{28}$$

In this expression no polarization energy of the type $\tfrac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{P}' - \tfrac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{M}'$ occurs. Instead $-\tfrac{1}{6}\boldsymbol{P}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' + \tfrac{1}{3}\boldsymbol{M}'^2$ appears together with $\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2$ in the energy density. The expression for the energy density has the same form as part of the diagonal elements of the field pressure tensor.

For *diluted media,* where terms quadratic in the susceptibilities may be neglected, the tensor (26) for induced dipole substances reduces to**):

$$T^{\alpha\beta\,\prime}_{[f]} = \begin{pmatrix} \tfrac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{D}' + \tfrac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{H}' & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + (\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}')\mathrm{U} \end{pmatrix}, \tag{29}$$

while the tensor (28) for permanent dipole substances gets the form

$$T^{\alpha\beta\,\prime}_{[f]} = \begin{pmatrix} \tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' & \boldsymbol{E}'\wedge\boldsymbol{H}' \\ \boldsymbol{E}'\wedge\boldsymbol{H}' & -\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + (\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}')\mathrm{U} \end{pmatrix}. \tag{30}$$

*) The combination $\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}' - \tfrac{1}{6}\boldsymbol{P}'^2 + \tfrac{1}{3}\boldsymbol{M}'^2$ may be written alternatively as $\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{H}'^2 - \tfrac{1}{6}\boldsymbol{P}'^2 - \tfrac{1}{6}\boldsymbol{M}'^2$.

**) In the approximation used here the combination $\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{B}'^2 - \boldsymbol{B}'\cdot\boldsymbol{M}'$ is equal to $\tfrac{1}{2}\boldsymbol{E}'^2 + \tfrac{1}{2}\boldsymbol{H}'^2$.

The tensor (29), derived for diluted induced dipole substances, shows some similarity to the tensors proposed by Minkowski[3]) and Abraham[4]), whose expressions contain the same energy density and energy flow. However, Minkowski's momentum density $c^{-1}\boldsymbol{D}' \wedge \boldsymbol{B}'$ is not found, and neither is the field pressure tensor $-\boldsymbol{E}'\boldsymbol{D}' - \boldsymbol{H}'\boldsymbol{B}' + (\frac{1}{2}\boldsymbol{E}'\cdot\boldsymbol{D}' + \frac{1}{2}\boldsymbol{B}'\cdot\boldsymbol{H}')\,\mathsf{U}$ proposed by both these authors. Thus neither Minkowski's nor Abraham's tensor may be justified from microscopic theory.

§ 5. *The ponderomotive forces.* The energy-momentum conservation laws may be written as

$$\partial_\beta(T^{\alpha\beta}_{[f]} + T^{\alpha\beta}_{[m]}) = 0, \tag{31}$$

where $T^{\alpha\beta}_{[f]}$ is the field energy-momentum tensor (23) and $T^{\alpha\beta}_{[m]}$ the material energy-momentum tensor, which was defined as the energy-momentum tensor in the absence of fields. The conservation laws can alternatively be written as

$$\partial_\beta T^{\alpha\beta}_{[m]} = \mathscr{F}^\alpha, \tag{32}$$

where a force density $\mathscr{F}^\alpha$ is introduced which is given by

$$\mathscr{F}^\alpha \equiv -\partial_\beta T^{\alpha\beta}_{[f]}. \tag{33}$$

It corresponds thus with a material pressure and internal energy defined at zero fields.

With the help of the field tensor $T^{\alpha\beta}_{[f]}$ an expression may be obtained for the force density $\mathscr{F}^\alpha$. Let us consider explicitly the special case of a medium at rest with negligible acceleration. The expression for $\mathscr{F}^\alpha$ may then be found from the rest frame formula (22) for $T^{\alpha\beta}_{[f]}$ with the help of the Maxwell equations for a neutral current-free medium. The time component ($\alpha = 0$) turns out to be

$$\mathscr{F}^0 = \tfrac{1}{2}\boldsymbol{E}^2\partial_0\kappa - \partial_0\left(\tfrac{1}{2}T'\frac{\partial\kappa}{\partial T'}\,\boldsymbol{E}^2\right) + \tfrac{1}{2}\boldsymbol{B}^2\partial_0\chi - \partial_0\left(\tfrac{1}{2}T'\frac{\partial\chi}{\partial T'}\,\boldsymbol{B}^2\right). \tag{34}$$

The space components ($\alpha = 1, 2, 3$) form the macroscopic force density

$$\mathscr{F} = -\tfrac{1}{2}\boldsymbol{E}^2\nabla\kappa - \nabla\left(\tfrac{1}{2}v'\frac{\partial\kappa}{\partial v'}\,\boldsymbol{E}^2\right) - \tfrac{1}{2}\boldsymbol{B}^2\nabla\chi$$
$$-\nabla\left(\tfrac{1}{2}v'\frac{\partial\chi}{\partial v'}\,\boldsymbol{B}^2\right) + \partial_0\{(\kappa+\chi)\,\boldsymbol{E}\wedge\boldsymbol{B}\}. \tag{35}$$

Here the first two terms form together Helmholtz's ponderomotive force*). They have been found before from thermodynamical considerations[5]), and

*) Sometimes a magnetic susceptibility $\tilde{\chi}$ is defined by the relation $\boldsymbol{M} = \tilde{\chi}\boldsymbol{H}$. Then the third and fourth term of (34) become $\frac{1}{2}\boldsymbol{H}^2\partial_0\tilde{\chi} - \frac{1}{2}\partial_0[T'(\partial\tilde{\chi}/\partial T')\,\boldsymbol{H}^2]$ and the third and fourth term of (35) become $-\frac{1}{2}\boldsymbol{H}^2\nabla\tilde{\chi} - \nabla[\frac{1}{2}v'(\partial\tilde{\chi}/\partial v')\,\boldsymbol{H}^2]$.

in a statistical treatment for static electric dipoles[6]) as the ponderomotive force corresponding to a material pressure defined at zero fields. The third and fourth terms are analogous magnetic contributions. Finally a term appears, which contributes only if the fields vary in time; it may be written alternatively as $\partial_0(\boldsymbol{P} \wedge \boldsymbol{B} - \boldsymbol{M} \wedge \boldsymbol{E})$. The part with the magnetization is found in the present relativistic treatment of the system; in a non relativistic theory only the term with the electric polarization can be derived.

Conclusion. In this series of articles the relativistic energy-momentum tensor in polarized media was derived from microscopic theory by means of covariant averaging. The tensor could be looked upon as the sum of a material and a field part in various ways. Depending on whether the material properties, such as pressure and internal energy, were defined for states with or without fields, different expressions were obtained for the field energy-momentum tensor.

These results will be discussed in connexion with the literature on this subject in a final paper of this series.

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