CHAPTER IV

Composite particles

1 *Introduction*

The covariant laws of electrodynamics of composite particles will be derived in this chapter on the basis of the equations that govern the fields and motion of charged point particles.

The equations for the fields generated by composite particles will be found by introducing covariantly defined multipole moments, through which the inner structure of the composite particles will be characterized. As a consequence the field equations will contain in their sources the effects of the motion of these multipoles. It will turn out that the electric and magnetic properties then appear in a more symmetric fashion than in the non-relativistic approximation.

The central problem in the derivation of equations of motion of composite particles in an electromagnetic field consists in defining a covariant centre of energy. It will be shown how such a definition can be obtained and how it leads to covariant equations of motion for particles endowed with multipole moments, at least in the case of weak fields.

2 *The field equations*

a. *The atomic series expansion*

Let us consider a system in which the charged point particles (electrons and nuclei) are grouped into stable entities (such as atoms, ions, molecules, free electrons), which will be called 'atoms' here. The particles will be labelled by double indices ki, where k numbers the atoms and i their constituent particles. Then the field equations (III.4–5) become:

$$\partial_\beta f^{\alpha\beta} = \sum_{k,i} e_{ki} \int u^\alpha_{ki}(s_{ki})\delta^{(4)}\{R_{ki}(s_{ki})-R\}\mathrm{d}s_{ki}. \qquad (1)$$

Here s_{ki} is the proper time along the world line of particle ki, $R^\alpha_{ki}(s_{ki})$ its four-position, $u^\alpha_{ki}(s_{ki})$ its four-velocity, and e_{ki} its charge. From the form of the integral it follows that it is not necessary to choose the proper time as the parameter along the world line. A different parametrization will indeed be introduced.

Let us now choose a (at the moment arbitrary) privileged world line R^α_k that describes the motion of atom k as a whole, with the proper time s_k. This parametrization can be carried over to the world lines of the constituent particles ki by defining the point $R^\alpha_{ki}(s_k)$ on the world line of ki with the help of the relation:

$$R_{ki}(s_k)\cdot n_k(s_k) = R_k(s_k)\cdot n_k(s_k), \qquad (2)$$

where $n^\alpha_k(s_k)$ is an arbitrary, fixed time-like unit vector depending on s_k. (Later on we shall make various choices for this unit vector.) Internal atomic parameters $r^\alpha_{ki}(s_k)$ are now introduced by

$$r^\alpha_{ki}(s_k) = R^\alpha_{ki}(s_k)-R^\alpha_k(s_k). \qquad (3)$$

Introducing this definition into (1) and expanding the delta function in powers of r^α_{ki}, one obtains

$$\partial_\beta f^{\alpha\beta} = \sum_{k,i} e_{ki} \int \left(u^\alpha_k(s_k)+\frac{\mathrm{d}r^\alpha_{ki}}{\mathrm{d}s_k}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(r_{ki}\cdot\partial)^n\delta^{(4)}\{R_k(s_k)-R\}\mathrm{d}s_k, \qquad (4)$$

where $u^\alpha_k(s_k) \equiv \mathrm{d}R^\alpha_k/\mathrm{d}s_k$ and where the fact has been used that $\partial/\partial R^\alpha_k$ acting on the delta function is the same as $-\partial_\alpha$ acting on it. The first term on the right-hand side with the term $n = 0$ of the series expansion is

$$c^{-1}j^\alpha(R) = \sum_k e_k \int u^\alpha_k(s_k)\delta^{(4)}\{R_k(s_k)-R\}\mathrm{d}s_k \qquad (5)$$

(with e_k the total charge $\sum_i e_{ki}$ of atom k). This four-vector has components $\alpha = 0$ and $\alpha = 1, 2, 3$, which are equal to the charge density and the current density (divided by c). Equation (4) can now be written as

$$\partial_\beta f^{\alpha\beta}(R) = c^{-1}j^\alpha(R)+\partial_\beta \sum_{k,i} e_{ki} \left\{\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\int u^\alpha_k r^\beta_{ki}(r_{ki}\cdot\partial)^{n-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int \frac{\mathrm{d}r^\alpha_{ki}}{\mathrm{d}s_k} r^\beta_{ki}(r_{ki}\cdot\partial)^{n-1}\right\}\delta^{(4)}(R_k-R)\mathrm{d}s_k. \qquad (6)$$

One can complete the second term at the right-hand side to the divergence of an antisymmetric tensor by subtracting a term of similar structure but with $r^\alpha_{ki}\mathrm{d}R^\beta_k/\mathrm{d}s_k$ instead of $(\mathrm{d}R^\alpha_k/\mathrm{d}s_k)r^\beta_{ki}$. If this extra term is partially integrated and then added to the last term, one obtains

$$\partial_\beta f^{\alpha\beta} = c^{-1} j^\alpha + \partial_\beta m^{\alpha\beta}. \tag{7}$$

This equation has already the form of Maxwell's inhomogeneous equations. It contains the antisymmetric tensor

$$m^{\alpha\beta} = \sum_{k,i} e_{ki} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int \left\{ r_{ki}^\alpha u_k^\beta - u_k^\alpha r_{ki}^\beta + \frac{n}{n+1}\left(r_{ki}^\alpha \frac{\mathrm{d}r_{ki}^\beta}{\mathrm{d}s_k} - \frac{\mathrm{d}r_{ki}^\alpha}{\mathrm{d}s_k} r_{ki}^\beta \right)\right\} (r_{ki}\cdot\partial)^{n-1}\delta^{(4)}(R_k - R)\mathrm{d}s_k, \tag{8}$$

which will be called the atomic polarization tensor.

The internal coordinates $r_{ki}^\alpha(s_k)$ enter this expression in certain combinations which we shall call the 'covariant electric and magnetic multipole moments', defined as

$$\mu_{(n)k}^{\alpha_1\ldots\alpha_n} = \frac{1}{n!}\sum_i e_{ki} r_{ki}^{\alpha_1} \ldots r_{ki}^{\alpha_n}, \qquad (n = 1, 2, \ldots),$$

$$\nu_{(n)k}^{\alpha_1\ldots\alpha_{n+1}} = c^{-1}\frac{n}{(n+1)!}\sum_i e_{ki} r_{ki}^{\alpha_1} \ldots r_{ki}^{\alpha_{n-1}} \left(r_{ki}^{\alpha_n} \frac{\mathrm{d}r_{ki}^{\alpha_{n+1}}}{\mathrm{d}s_k} - r_{ki}^{\alpha_{n+1}} \frac{\mathrm{d}r_{ki}^{\alpha_n}}{\mathrm{d}s_k}\right), \qquad (n = 1, 2, \ldots). \tag{9}$$

Their dependence on the time-like unit vector $n_k^\alpha(s_k)$ is indicated by the index (n). With the help of these quantities the polarization tensor becomes

$$m^{\alpha\beta} = \sum_k \sum_{n=1}^{\infty} (-1)^{n+1} \int (\mu_{(n)k}^{\alpha_1\ldots\alpha_{n-1}\alpha} u_k^\beta - \mu_{(n)k}^{\alpha_1\ldots\alpha_{n-1}\beta} u_k^\alpha + c\nu_{(n)k}^{\alpha_1\ldots\alpha_{n-1}\alpha\beta})\partial_{\alpha_1\ldots\alpha_{n-1}}\delta^{(4)}(R_k - R)\mathrm{d}s_k, \tag{10}$$

where we have written $\partial_{\alpha_1\ldots\alpha_n}$ for $\partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_n}$. The combination of electric and magnetic multipole moments which occurs here will be employed frequently, and will be denoted as

$$\mathrm{m}_{(n)k}^{\alpha_1\ldots\alpha_{n+1}} \equiv c^{-1}(\mu_{(n)k}^{\alpha_1\ldots\alpha_n} u_k^{\alpha_{n+1}} - \mu_{(n)k}^{\alpha_1\ldots\alpha_{n-1}\alpha_{n+1}} u_k^{\alpha_n}) + \nu_{(n)k}^{\alpha_1\ldots\alpha_{n+1}}. \tag{11}$$

In this way $m^{\alpha\beta}$ may be written in the compact form

$$m^{\alpha\beta} = c\sum_k \sum_{n=1}^{\infty} (-1)^{n+1} \int \mathrm{m}_{(n)k}^{\alpha_1\ldots\alpha_{n-1}\alpha\beta}\partial_{\alpha_1\ldots\alpha_{n-1}}\delta^{(4)}(R_k - R)\mathrm{d}s_k. \tag{12}$$

The inhomogeneous equation (7) with (5) and (12), together with the homogeneous one

$$\partial^\alpha f^{\beta\gamma} + \partial^\beta f^{\gamma\alpha} + \partial^\gamma f^{\alpha\beta} = 0 \tag{13}$$

form the atomic equations[1] for the fields generated by atoms with charges e_k and multipole moments $\mu_{(n)k}^{\alpha_1\ldots\alpha_n}$ and $\nu_{(n)k}^{\alpha_1\ldots\alpha_n}$ $(n = 1, 2, \ldots)$. The retarded solution of (7) and (13) may be written in terms of the Green function (III.44), derived in the preceding chapter. One finds (cf. (III.59)):

$$f^{\alpha\beta} = \sum_k \int \hat{f}_k^{\alpha\beta}\mathrm{d}s_k = \sum_k \int (\hat{f}_{k(e)}^{\alpha\beta} + \hat{f}_{k(m)}^{\alpha\beta})\mathrm{d}s_k, \tag{14}$$

where the contributions of the charges and multipoles are:

$$\hat{f}_{k(e)}^{\alpha\beta} \equiv -\frac{e_k}{2\pi}(u_k^\alpha\partial^\beta - u_k^\beta\partial^\alpha)\delta\{(R-R_k)^2\}\theta(R-R_k), \tag{15}$$

$$\hat{f}_{k(m)}^{\alpha\beta} \equiv -\sum_{n=1}^{\infty}\frac{(-1)^n c}{2\pi}\left\{\mathrm{m}_{(n)k}^{\alpha_1\ldots\alpha_n\alpha}\partial^\beta_{.\alpha_1\ldots\alpha_n} - c^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}s_k} - u_k\cdot\partial\right)\mu_k^{\alpha_1\ldots\alpha_{n-1}\alpha}\partial^\beta_{.\alpha_1\ldots\alpha_{n-1}}\right\}\delta\{(R-R_k)^2\}\theta(R-R_k) - (\alpha, \beta). \tag{16}$$

Here $\mathrm{d}/\mathrm{d}s_k$ acts only on the electric multipole moment μ_k. Furthermore $\theta(R-R_k)$ is the unit step function of $R^0 - R_k^0$. The symbol (α, β) indicates the preceding terms with α and β interchanged. If (16) is inserted into (14), the second term gives no contribution since the integral over s_k may be performed. The reason for retaining it is that (15) and (16) as they stand are together the multipole expanded form of the integrand of (III.59) (v. problem 4). This property will be useful later on.

It will be convenient to split the retarded field in the plus and minus fields (which are half the sum and half the difference of the retarded and advanced solutions respectively). The plus fields will satisfy an equation of the form (7) with (5) and (12), whereas the equation for the minus field will contain no sources. Since the advanced field has the same form as (14) but with $\theta(R_k - R)$ instead of $\theta(R - R_k)$, one finds then for the plus and minus fields

$$f_{\pm}^{\alpha\beta} = \sum_k \int \hat{f}_{\pm k}^{\alpha\beta}\mathrm{d}s_k = \sum_k \int (\hat{f}_{\pm k(e)}^{\alpha\beta} + \hat{f}_{\pm k(m)}^{\alpha\beta})\mathrm{d}s_k, \tag{17}$$

where the contributions of the charges and multipoles are for the plus field

$$\hat{f}_{+k(e)}^{\alpha\beta} \equiv -\frac{e_k}{4\pi}(u_k^\alpha\partial^\beta - u_k^\beta\partial^\alpha)\delta\{(R-R_k)^2\}, \tag{18}$$

[1] A. N. Kaufman, Ann. Physics **18**(1962)264; H. Bacry, Ann. Physique **8**(1963)197; S. R. de Groot and L. G. Suttorp, Physica **31**(1965)1713; L. G. Suttorp, On the covariant derivation of macroscopic electrodynamics from electron theory, thesis, Amsterdam (1968)

$$\hat{f}^{\alpha\beta}_{+k(m)} \equiv -\sum_{n=1}^{\infty} \frac{(-1)^n c}{4\pi} \left\{ \mathrm{m}^{\alpha_1 \ldots \alpha_n \alpha}_{(n)k} \partial^{\beta}_{.\alpha_1 \ldots \alpha_n} - c^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}s_k} - u_k \cdot \partial \right) \mu_k^{\alpha_1 \ldots \alpha_{n-1}\alpha} \partial^{\beta}_{.\alpha_1 \ldots \alpha_{n-1}} \right\} \delta\{(R-R_k)^2\} - (\alpha, \beta), \quad (19)$$

while the minus fields contain an extra factor $\varepsilon(R-R_k) \equiv \theta(R-R_k) - \theta(R_k - R)$ after the delta functions. The partial fields fulfil the equations:

$$\partial_\beta \hat{f}^{\alpha\beta}_{+k(e)} = e_k u_k^\alpha \delta^{(4)}(R_k - R) + \frac{e_k}{4\pi} \partial^\alpha u_k \cdot \partial \delta\{(R-R_k)^2\}, \quad (20)$$

$$\partial_\beta \hat{f}^{\alpha\beta}_{+k(m)} = \sum_{n=1}^{\infty} (-1)^n c \left\{ \mathrm{m}^{\alpha_1 \ldots \alpha_n \alpha}_{(n)k} \partial_{\alpha_1 \ldots \alpha_n} - c^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}s_k} - u_k \cdot \partial \right) \mu_k^{\alpha_1 \ldots \alpha_{n-1}\alpha} \partial_{\alpha_1 \ldots \alpha_{n-1}} \right\} \delta^{(4)}(R_k - R)$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4\pi} \left(\frac{\mathrm{d}}{\mathrm{d}s_k} - u_k \cdot \partial \right) \mu_k^{\alpha_1 \ldots \alpha_n} \partial^{\alpha}_{.\alpha_1 \ldots \alpha_n} \delta\{(R-R_k)^2\}, \quad (21)$$

$$\partial_\beta \hat{f}^{\alpha\beta}_{-k(e)} = \frac{e_k}{4\pi} \partial^\alpha u_k \cdot \partial \delta\{(R-R_k)^2\} \varepsilon(R-R_k), \quad (22)$$

$$\partial_\beta \hat{f}^{\alpha\beta}_{-k(m)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4\pi} \left(\frac{\mathrm{d}}{\mathrm{d}s_k} - u_k \cdot \partial \right) \mu_k^{\alpha_1 \ldots \alpha_n} \partial^{\alpha}_{.\alpha_1 \ldots \alpha_n} \delta\{(R-R_k)^2\} \varepsilon(R-R_k). \quad (23)$$

If one integrates these four equations over s_k only the first terms of the right-hand sides of (20) and (21) give contributions (v. (7) with (5) and (12)). The inhomogeneous atomic field equations may be written in an alternative form by introducing the atomic 'displacement tensor'

$$h^{\alpha\beta} \equiv f^{\alpha\beta} - m^{\alpha\beta}. \quad (24)$$

Then equation (7) becomes

$$\partial_\beta h^{\alpha\beta} = c^{-1} j^\alpha, \quad (25)$$

which has the same form as Maxwell's inhomogeneous equation.

Owing to the antisymmetry of $h^{\alpha\beta}$ equation (25) is consistent with the law of conservation of charge

$$\partial_\alpha j^\alpha = 0, \quad (26)$$

which may be proved from expression (5) for the four-current.

b. *Multipole moments*

The polarization tensor (10) contains the covariant multipole moments (9) which depend on the atomic internal coordinates r_{ki}^α, measured in the observer's $(ct, \boldsymbol{R})$-frame. These covariant multipole moments have different values in different observer's Lorentz frames. They are therefore not constant properties which characterize the atoms. It is more convenient to characterize the internal electromagnetic structure of the atoms by means of parameters that are independent of the velocity of the atoms. This can be achieved with the help of parameters defined in a Lorentz frame in which the atom as a whole is at rest. Such an 'atomic frame' must have a (constant) velocity $\boldsymbol{v}$ equal to $(\mathrm{d}\boldsymbol{R}_k/\mathrm{d}t)_{t=t_0}$ (at a moment $t = t_0$) with respect to the reference frame $(ct, \boldsymbol{R})$ of the observer. Space-time coordinates of the reference frame $(ct, \boldsymbol{R})$ and of the atomic frame $(ct^{(0)}, \boldsymbol{R}^{(0)})$ are connected by a Lorentz transformation.

In the atomic frame the atom may be characterized by internal parameters $\boldsymbol{r}_{ki}^{(0)}$, which at the moment $t^{(0)} = t_0^{(0)}$ (corresponding to $t = t_0$ in the reference frame) are purely spatial vectors, i.e. $r_{ki}^{(0)0} = 0$. This corresponds to the choice $c^{-1}u_k^\alpha$ voor n_k^α as follows from (2) with (3). Since the atom suffers accelerations the atomic frame is just a momentary rest frame: only for $t = t_0$ does the atomic velocity $\mathrm{d}\boldsymbol{R}_k^{(0)}/\mathrm{d}t^{(0)}$ vanish. Hence one needs for a description in which the atomic parameters are independent of the velocity all the time a succession of momentary rest frames. This succession of Lorentz frames which is not a Lorentz frame itself will be called the permanent atomic rest frame (denoted by a prime). It coincides at time t_0 with the momentary atomic rest frame which has been denoted by (0).

The proper atomic multipole moments are certain useful combinations of the atomic internal parameters $\boldsymbol{r}'_{ki}$, defined in the permanent atomic frame[1]. The electric atomic 2^n-pole moment is defined as[2]

$$\boldsymbol{\mu}_k^{(n)} = \frac{1}{n!} \sum_i e_{ki} (\boldsymbol{r}'_{ki})^n, \qquad (n = 1, 2, \ldots) \quad (27)$$

and the magnetic atomic 2^n-pole moment as:

$$\boldsymbol{\nu}_k^{(n)} = \frac{n}{(n+1)!} \sum_i e_{ki} (\boldsymbol{r}'_{ki})^n \wedge \frac{\dot{\boldsymbol{r}}'_{ki}}{c}, \qquad (n = 1, 2, \ldots) \quad (28)$$

[1] A. N. Kaufman, op. cit. (dipole case), S. R. de Groot and J. Vlieger, Physica **31** (1965) 125 (quadrupole case); S. R. de Groot and L. G. Suttorp, op. cit.; L. G. Suttorp, op. cit.

[2] For atoms in uniform motion these atomic multipole moments coincide with the space-space and space-time components of the covariant multipole moments (11) – with the unit vector n^α chosen as u_k^α/c – in the rest frame. If the atom suffers accelerations the relation between the covariant and atomic multipole moments is slightly more complicated (v. appendix II).

(the powers indicate polyads of three-vectors). The dot indicates a time derivative defined in a special way as explained in appendix II.

The polarization tensor, which in (10) is given in terms of the covariant multipole moments, may be written now as a function of the atomic multipole moments. The connexion between the two kinds of multipole moments follows from a Lorentz transformation (v. appendix II). With the help of the relations (A6), (A41) and (A43) one obtains, performing the integral over s_k, for the components $m^{i0} = p^i$ of the polarization tensor:

$$\begin{aligned} \boldsymbol{p} = \sum_k \sum_{n=1}^{\infty} (-1)^{n+1} \Bigg[& \sum_{p=0}^{n-1} \binom{n-1}{p} \partial_0^p \{\boldsymbol{\Omega}_k(\gamma_k \boldsymbol{\beta}_k)^p (\nabla\cdot\boldsymbol{\Omega}_k^{-1})^{n-p-1} \vdots\, \boldsymbol{\mu}_k^{(n)} \delta(\boldsymbol{R}_k - \boldsymbol{R})\} \\ & - \sum_{p=0}^{n-1} \binom{n-1}{p} \partial_0^p \{(\gamma_k \boldsymbol{\beta}_k)^p (\nabla\cdot\boldsymbol{\Omega}_k^{-1})^{n-p-1} \vdots\, \boldsymbol{\nu}_k^{(n)} \wedge \boldsymbol{\beta}_k \delta(\boldsymbol{R}_k - \boldsymbol{R})\} \\ & + \sum_{p=0}^{n-2} \binom{n-2}{p} \frac{(n-1)(n+1)}{n} \partial_0^p \{(\gamma_k \boldsymbol{\beta}_k)^p (\nabla\cdot\boldsymbol{\Omega}_k^{-1})^{n-p-2} \gamma_k \partial_0(\gamma_k \boldsymbol{\beta}_k)\cdot\boldsymbol{\Omega}_k \\ & \qquad \vdots\, \boldsymbol{\nu}_k^{(n)} \wedge \boldsymbol{\beta}_k \delta(\boldsymbol{R}_k - \boldsymbol{R})\} \\ & - \sum_{p=0}^{n-2} \binom{n-2}{p} (n-1) \partial_0^p \{\boldsymbol{\Omega}_k(\gamma_k \boldsymbol{\beta}_k)^p (\nabla\cdot\boldsymbol{\Omega}_k^{-1})^{n-p-2} \gamma_k \partial_0(\gamma_k \boldsymbol{\beta}_k)\cdot\boldsymbol{\Omega}_k \\ & \qquad \vdots\, \boldsymbol{\mu}_k^{(n)} \delta(\boldsymbol{R}_k - \boldsymbol{R})\} \Bigg], \end{aligned} \tag{29}$$

where the triple dot stands for an n- or $(n-1)$-fold contraction. This expression contains as external variables the position vector $\boldsymbol{R}_k$ of the atoms (entering only in the delta function), the velocity $\mathrm{d}\boldsymbol{R}_k/\mathrm{d}t \equiv \boldsymbol{\beta}_k c$ and higher time derivatives. Furthermore we used the abbreviation $\gamma_k = (1-\boldsymbol{\beta}_k^2)^{-\frac{1}{2}}$ and the three-tensor ($\mathbf{U}$ is the unit three-tensor)

$$\boldsymbol{\Omega}_k = \mathbf{U} + (\gamma_k^{-1} - 1)\frac{\boldsymbol{\beta}_k \boldsymbol{\beta}_k}{\boldsymbol{\beta}_k^2}, \tag{30}$$

as well as its inverse

$$\boldsymbol{\Omega}_k^{-1} = \mathbf{U} + (\gamma_k - 1)\frac{\boldsymbol{\beta}_k \boldsymbol{\beta}_k}{\boldsymbol{\beta}_k^2}. \tag{31}$$

The three-tensor $\boldsymbol{\Omega}_k$ can be interpreted in terms of a Lorentz contraction, since for every three-vector $\boldsymbol{a} = \boldsymbol{a}_{//} + \boldsymbol{a}_{\perp}$ (split into a part parallel and a part perpendicular to the velocity $\boldsymbol{\beta}_k c$) one has according to definition (30)

$$\boldsymbol{\Omega}_k\cdot\boldsymbol{a} = \boldsymbol{a}_{\perp} + \sqrt{1-\boldsymbol{\beta}_k^2}\, \boldsymbol{a}_{//}. \tag{32}$$

This shows that the longitudinal component of the vector is subjected to a Lorentz contraction. In view of this property of $\boldsymbol{\Omega}_k$ we eliminate $\boldsymbol{\Omega}_k^{-1}$ from

(29) by means of the identity

$$\boldsymbol{\Omega}_k^{-1} = \boldsymbol{\Omega}_k + \gamma_k \boldsymbol{\beta}_k \boldsymbol{\beta}_k, \tag{33}$$

which is a consequence of (30) and (31). In the result for the polarization the atomic multipole moments then occur contracted with the tensor $\boldsymbol{\Omega}_k$. These quantities will be denoted with underlined symbols:

$$\underline{\boldsymbol{\mu}}_k^{(n)} \equiv \frac{1}{n!} \sum_i e_{ki} (\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki})^n, \qquad (n = 1, 2, \ldots), \tag{34}$$

$$\underline{\boldsymbol{\nu}}_k^{(n)} \equiv \frac{n}{(n+1)!} \sum_i e_{ki} (\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki})^{n-1} \boldsymbol{\Omega}_k \cdot \left(\boldsymbol{r}'_{ki} \wedge \frac{\dot{\boldsymbol{r}}'_{ki}}{c} \right), \qquad (n = 1, 2, \ldots). \tag{35}$$

According to (32) they represent atomic electromagnetic multipole moments of which the longitudinal components are submitted to a Lorentz contraction. With the help of (34) and (35) the electric polarization (29) becomes finally:

$$\begin{aligned} \boldsymbol{p} = & \sum_k \sum_{n=1}^{\infty} \sum_{p=0}^{n-1} \sum_{q=0}^{p} \frac{(-1)^{n-1}(n-1)!}{(n-1-p)!(p-q)!} \\ & \nabla^{n-1-p} \partial_0^{p-q-1} [(\gamma_k \boldsymbol{\beta}_k)^{p-q} \partial_0 \{\gamma_k^p (\gamma_k \boldsymbol{\beta}_k)^q \vdots\, (\underline{\boldsymbol{\mu}}_k^{(n)} - \underline{\boldsymbol{\nu}}_k^{(n)} \wedge \boldsymbol{\beta}_k)\}] D_q \delta(\boldsymbol{R}_k - \boldsymbol{R}) \\ & + \sum_k \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} \sum_{q=0}^{p-1} \frac{(-1)^{n-1}(n-1)!}{n(n-1-p)!(p-q)!} \\ & \nabla^{n-1-p} \partial_0^{p-q-1} [\{\partial_0 (\gamma_k \boldsymbol{\beta}_k)^{p-q}\} \{\gamma_k^p (\gamma_k \boldsymbol{\beta}_k)^q \vdots\, (\underline{\boldsymbol{\nu}}_k^{(n)} \wedge \boldsymbol{\beta}_k)\}] D_q \delta(\boldsymbol{R}_k - \boldsymbol{R}). \end{aligned} \tag{36}$$

The nabla operator differentiates $\boldsymbol{R}$, which occurs only in the delta function. Furthermore the product $\partial_0^{-1}\partial_0$, which occurs if $p = q$ in the first sum, is to be considered as unity. The triple dot stands for an $(n-1)$-fold contraction. The symbol D_q is defined by:

$$D_q \delta(\boldsymbol{R}_k - \boldsymbol{R}) \equiv \sum_{m=0}^{q} \frac{1}{m!(q-m)!} \partial_0^{q-m} \{(\boldsymbol{\beta}_k \cdot \nabla)^m \delta(\boldsymbol{R}_k - \boldsymbol{R})\}. \tag{37}$$

The magnetization $m^i = m^{jk}$ ($i, j, k = 1, 2, 3$ cycl.) follows by substitution of (A41) and (A42) into (10). Then one obtains an expression which is similar to (36) but with the replacements

$$\begin{aligned} \underline{\boldsymbol{\mu}}_k^{(n)} &\to \underline{\boldsymbol{\mu}}_k^{(n)} \wedge \boldsymbol{\beta}_k, \\ \underline{\boldsymbol{\nu}}_k^{(n)} \wedge \boldsymbol{\beta}_k &\to -\underline{\boldsymbol{\nu}}_k^{(n)}. \end{aligned} \tag{38}$$

This shows a symmetry between the components $\boldsymbol{p}$ and $\boldsymbol{m}$ of the polarization tensor $m^{\alpha\beta}$.

The polarization tensor obtained so far contains time derivatives of the delta function as is apparent from (37). They can be expressed in terms of spatial derivatives, because

$$\partial_0\,\delta\{\boldsymbol{R}_k(t)-\boldsymbol{R}\} = -\boldsymbol{\beta}_k\cdot\nabla\delta\{\boldsymbol{R}_k(t)-\boldsymbol{R}\}. \tag{39}$$

By means of the chain rule of differentiation one then obtains a sum of spatial derivatives of the delta function, where each term has a factor in front of it, which may contain $\boldsymbol{\beta}_k$ and time derivatives of $\boldsymbol{\beta}_k$. For the values $q = 0$ and 1 one finds directly from (37)

$$D_0 = 1, \qquad D_1 = 0. \tag{40}$$

For dipoles ($n = 1$) and quadrupoles ($n = 2$) one only needs these values.

c. *The field equations*

The atomic field equations (7) and (13) read in three-dimensional notation

$$\begin{aligned} \nabla\cdot\boldsymbol{e} &= \rho^{\mathrm{e}}-\nabla\cdot\boldsymbol{p},\\ -\partial_0\boldsymbol{e}+\nabla\wedge\boldsymbol{b} &= \boldsymbol{j}/c+\partial_0\boldsymbol{p}+\nabla\wedge\boldsymbol{m},\\ \nabla\cdot\boldsymbol{b} &= 0,\\ \partial_0\boldsymbol{b}+\nabla\wedge\boldsymbol{e} &= 0. \end{aligned} \tag{41}$$

Here $e^i = f^{0i}$, $b^i = f^{jk}$, $\rho^{\mathrm{e}} = j^0/c$, $j^i = j^i$, $p^i = -m^{0i}$ and $m^i = m^{jk}$ ($i, j, k = 1, 2, 3$ cycl.). These atomic field equations have the same form as Maxwell's equations. Their source terms include the atomic charge density

$$\rho^{\mathrm{e}} = \sum_k e_k\,\delta(\boldsymbol{R}_k-\boldsymbol{R}) \tag{42}$$

and the atomic current density

$$\boldsymbol{j} = \sum_k e_k\,\boldsymbol{\beta}_k\,c\delta(\boldsymbol{R}_k-\boldsymbol{R}), \tag{43}$$

as follows from (5). They satisfy the conservation law of charge (26) which reads in three-dimensional notation:

$$\frac{\partial\rho^{\mathrm{e}}}{\partial t}+\nabla\cdot\boldsymbol{j} = 0. \tag{44}$$

Furthermore the polarization vector is given in (36), while the magnetization follows with the replacement rules (38). They are sums of multipole contributions of various order n:

$$\boldsymbol{p} = \sum_{n=1}^{\infty}\boldsymbol{p}^{(n)}, \qquad \boldsymbol{m} = \sum_{n=1}^{\infty}\boldsymbol{m}^{(n)}, \tag{45}$$

of which the lowest orders will be given and discussed in the next subsection.

The sources $\rho^{\mathrm{e}}, \boldsymbol{j}, \boldsymbol{p}$ and $\boldsymbol{m}$ which are functions of $\boldsymbol{R}$ and t are all sums over the separate atoms k. They contain the internal atomic quantities e_k (which are scalars), $\boldsymbol{\mu}_k^{(n)}$ and $\boldsymbol{\nu}_k^{(n)}$ (which are defined in the atomic frames). They also depend on the external quantities of the atoms, which are defined in the observer's ($ct, \boldsymbol{R}$)-frame. The external quantities include the atomic positions, velocities and higher time derivatives of these. Time derivatives of the internal quantities $\boldsymbol{\mu}_k^{(n)}$ and $\boldsymbol{\nu}_k^{(n)}$ also occur. The atomic field equations are still equations for the microscopic fields $\boldsymbol{e}$ and $\boldsymbol{b}$, in which however the existence of stable groups of point particles is taken into account. They contain internal and external quantities, all referring to single atoms. Thus the atomic field equations can be said to be valid at the so-called *kinetic level* of the theory.

An alternative form of the atomic field equations follows by introducing the displacement vectors $d^i = h^{0i}$ and $h^i = h^{jk}$ ($i, j, k = 1, 2, 3$ cycl.) that form part of the displacement tensor (24). The latter definition reads for the components:

$$\boldsymbol{d} = \boldsymbol{e}+\boldsymbol{p}, \qquad \boldsymbol{h} = \boldsymbol{b}-\boldsymbol{m}, \tag{46}$$

so that one has for (41):

$$\begin{aligned} \nabla\cdot\boldsymbol{d} &= \rho^{\mathrm{e}},\\ -\partial_0\boldsymbol{d}+\nabla\wedge\boldsymbol{h} &= \boldsymbol{j}/c,\\ \nabla\cdot\boldsymbol{b} &= 0,\\ \partial_0\boldsymbol{b}+\nabla\wedge\boldsymbol{e} &= 0. \end{aligned} \tag{47}$$

These alternative atomic field equations have the form in which the macroscopic Maxwell equations are usually written.

d. *Explicit expressions for the polarization tensor*

In this subsection explicit expressions for a few special cases of the polarization tensor will be written down. The contributions from the electric and magnetic dipole moments ($n = 1$) that follow from (36–40) are:

$$\boldsymbol{p}^{(1)} = \sum_k(\underline{\boldsymbol{\mu}}_k^{(1)}-\underline{\boldsymbol{\nu}}_k^{(1)}\wedge\boldsymbol{\beta}_k)\delta(\boldsymbol{R}_k-\boldsymbol{R}), \tag{48}$$

$$\boldsymbol{m}^{(1)} = \sum_k(\underline{\boldsymbol{\nu}}_k^{(1)}+\underline{\boldsymbol{\mu}}_k^{(1)}\wedge\boldsymbol{\beta}_k)\delta(\boldsymbol{R}_k-\boldsymbol{R}). \tag{49}$$

Here the motion of the atom gives rise to two separate effects. In the first place the electric polarization (48) contains a term due to the magnetic dipole moment and similarly the magnetic polarization (49) a term due to the electric dipole moment. Furthermore the dipoles are subject to a Lorentz contraction, since $\underline{\mu}_k^{(1)}$ and $\underline{\nu}_k^{(1)}$ can be rewritten, if use is made of the property (32) of the $\boldsymbol{\Omega}_k$ tensor:

$$\underline{\mu}_k^{(1)} \equiv \boldsymbol{\Omega}_k\cdot\boldsymbol{\mu}_k^{(1)} = \boldsymbol{\mu}_{k,\perp}^{(1)}+\sqrt{1-\beta_k^2}\,\boldsymbol{\mu}_{k,//}^{(1)}, \tag{50}$$

$$\underline{\nu}_k^{(1)} \equiv \boldsymbol{\Omega}_k\cdot\boldsymbol{\nu}_k^{(1)} = \boldsymbol{\nu}_{k,\perp}^{(1)}+\sqrt{1-\beta_k^2}\,\boldsymbol{\nu}_{k,//}^{(1)}, \tag{51}$$

where $\boldsymbol{\mu}_k^{(1)}$ and $\boldsymbol{\nu}_k^{(1)}$ are split into a part parallel with and a part perpendicular to the velocity $c\boldsymbol{\beta}_k$.

The terms $\boldsymbol{p}^{(2)}$ and $\boldsymbol{m}^{(2)}$ with electric and magnetic quadrupoles ($n = 2$) read:

$$\boldsymbol{p}^{(2)} = \sum_k [-\nabla\cdot(\underline{\mu}_k^{(2)}-\underline{\nu}_k^{(2)}\wedge\boldsymbol{\beta}_k)-\gamma_k\boldsymbol{\beta}_k\cdot\partial_0\{\gamma_k(\underline{\mu}_k^{(2)}-\underline{\nu}_k^{(2)}\wedge\boldsymbol{\beta}_k)\}$$
$$-\tfrac{1}{2}\gamma_k\,\partial_0(\gamma_k\boldsymbol{\beta}_k)\cdot\underline{\nu}_k^{(2)}\wedge\boldsymbol{\beta}_k]\delta(\boldsymbol{R}_k-\boldsymbol{R}), \tag{52}$$

$$\boldsymbol{m}^{(2)} = \sum_k [-\nabla\cdot(\underline{\nu}_k^{(2)}+\underline{\mu}_k^{(2)}\wedge\boldsymbol{\beta}_k)-\gamma_k\boldsymbol{\beta}_k\cdot\partial_0\{\gamma_k(\underline{\nu}_k^{(2)}+\underline{\mu}_k^{(2)}\wedge\boldsymbol{\beta}_k)\}$$
$$+\tfrac{1}{2}\gamma_k\,\partial_0(\gamma_k\boldsymbol{\beta}_k)\cdot\underline{\nu}_k^{(2)}]\delta(\boldsymbol{R}_k-\boldsymbol{R}). \tag{53}$$

Due to the motion of the atoms the electric and magnetic quadrupoles are subject to a Lorentz contraction since we can write:

$$\underline{\mu}_k^{(2)} \equiv \boldsymbol{\Omega}_k\cdot\boldsymbol{\mu}_k^{(2)}\cdot\boldsymbol{\Omega}_k = \boldsymbol{\mu}_{k,\perp\perp}^{(2)}+\sqrt{1-\beta_k^2}(\boldsymbol{\mu}_{k,\perp//}^{(2)}+\boldsymbol{\mu}_{k,//\perp}^{(2)})+(1-\beta_k^2)\boldsymbol{\mu}_{k,////}^{(2)}, \tag{54}$$

and an analogous equation for $\underline{\nu}_k^{(2)}$. The leading terms of (52) and (53) contain the divergence of the electric and magnetic quadrupole densities. The second and third terms are relativistic corrections. They contain time derivatives of the atomic quadrupole moments which means that changing quadrupole moments contribute in a special way to the polarization tensor. This effect, which has a relativistic character, may be called the 'multipole fluxion effect'. Accelerated atoms carrying quadrupole moments also give rise to a special type of terms in the polarization tensor. This effect, which again does not exist in a non-relativistic theory may be called the 'acceleration effect'. The order of magnitude of these effects as compared to the main contributions in the polarization tensor will be discussed in the chapter on macroscopic theory.

Higher order multipole contributions may also be derived from (36–39). We shall not give these expressions explicitly.

The polarization tensor $(\boldsymbol{p}_k, \boldsymbol{m}_k)$ of a single atom k to all multipole orders takes a simple form in the momentary atomic rest frame $(ct^{(0)}, \boldsymbol{R}^{(0)})$. In fact $\boldsymbol{p}_k^{(0)}$ and $\boldsymbol{m}_k^{(0)}$ follow from (36) with (37) and (39) (without the summation

over k) and the corresponding expression obtained with (38), if one puts $\boldsymbol{\beta}_k^{(0)} = 0$. Then only $p = 0$ and $q = 0$ subsist in the leading term. The other term in (36) disappears altogether, whereas in $\boldsymbol{m}_k$ only its part with $q = 0$ is left over. Thus one gets

$$\boldsymbol{p}_k^{(0)} = \sum_{n=1}^{\infty}(-1)^{n-1}(\nabla^{(0)})^{n-1} \vdots\, \boldsymbol{\mu}_k^{(n)}\delta(\boldsymbol{R}_k^{(0)}-\boldsymbol{R}^{(0)}), \tag{55}$$

$$\boldsymbol{m}_k^{(0)} = \sum_{n=1}^{\infty}(-1)^{n-1}(\nabla^{(0)})^{n-1} \vdots\, \boldsymbol{\nu}_k^{(n)}\delta(\boldsymbol{R}_k^{(0)}-\boldsymbol{R}^{(0)})$$
$$-\sum_{n=2}^{\infty}\sum_{p=1}^{n-1}\frac{(-1)^{n-1}(n-1)!}{n(n-1-p)!}(\nabla^{(0)})^{n-1-p}(\partial_0^{(0)}\boldsymbol{\beta}_k^{(0)})^p \vdots\, \boldsymbol{\nu}_k^{(n)}\delta(\boldsymbol{R}_k^{(0)}-\boldsymbol{R}^{(0)}). \tag{56}$$

Apart from the leading terms, which are divergences of multipole densities, a term representing the acceleration effect (with quadrupoles and higher moments) appears in the magnetization vector for an atom in its momentary atomic rest frame.

e. *The non-relativistic and semi-relativistic limits*

In the so-called semi-relativistic approximation one retains terms of order c^{-1}, treating the atomic multipole moments as parameters characterizing the atom, without considering whether they contain a factor c^{-1}. Then, if terms of order c^{-2} and higher are neglected in (36) with (37–39) so that $\gamma_k \simeq 1$ and $\boldsymbol{\Omega}_k \simeq \mathbf{U}$, we find:

$$\boldsymbol{p}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty}(-1)^{n-1}\nabla^{n-1} \vdots\, (\boldsymbol{\mu}_k^{(n)}-\boldsymbol{\nu}_k^{(n)}\wedge\boldsymbol{\beta}_k)\delta(\boldsymbol{R}_k-\boldsymbol{R}), \tag{57}$$

$$\boldsymbol{m}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty}(-1)^{n-1}\nabla^{n-1} \vdots\, (\boldsymbol{\nu}_k^{(n)}+\boldsymbol{\mu}_k^{(n)}\wedge\boldsymbol{\beta}_k)\delta(\boldsymbol{R}_k-\boldsymbol{R}). \tag{58}$$

These formulae show a symmetry in the sense that $\boldsymbol{p}_k$ contains terms due to moving magnetic multipoles $\boldsymbol{\nu}_k^{(n)}$, just as $\boldsymbol{m}_k$ contains contributions of the same type from moving electric multipoles $\boldsymbol{\mu}_k^{(n)}$.

The non-relativistic limiting case is obtained if one takes into account the fact that the magnetic multipoles contain a factor c^{-1}. Then, up to terms of order c^{-1}, one is left with (cf. (I.34)):

$$\boldsymbol{p}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty}(-1)^{n-1}\nabla^{n-1} \vdots\, \boldsymbol{\mu}_k^{(n)}\delta(\boldsymbol{R}_k-\boldsymbol{R}), \tag{59}$$

$$\boldsymbol{m}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty}(-1)^{n-1}\nabla^{n-1} \vdots\, (\boldsymbol{\nu}_k^{(n)}+\boldsymbol{\mu}_k^{(n)}\wedge\boldsymbol{\beta}_k)\delta(\boldsymbol{R}_k-\boldsymbol{R}). \tag{60}$$

The symmetry of (57) and (58) is lost now, since $\boldsymbol{p}_k$ (59) contains no terms with moving magnetic multipoles.

3 *The equations of motion of a composite particle in a field*

a. *Introduction*

To obtain equations of motion of composite particles we shall have to start from the corresponding equations for the constituent particles. In the preceding we have written the equations of motion for a set of point particles in the form (III.158):

$$\partial_\beta t^{\alpha\beta} = f^\alpha. \tag{61}$$

Here $t^{\alpha\beta}$ is the (symmetric) energy–momentum tensor, which is the sum of a material contribution and a field contribution. The latter contains the fields generated by the constituent particles. It has been given in (III.159) as

$$t^{\alpha\beta} = c\sum_i m_i \int u_i^\alpha u_i^\beta \delta^{(4)}(R_i - R)\mathrm{d}s_i + \sum_{i,j(i\neq j)} (f_i^{\alpha\gamma} f^\beta_{j.\gamma} - \tfrac{1}{4} f_i^{\gamma\varepsilon} f_{j\gamma\varepsilon} g^{\alpha\beta}) + \sum_i (f^{\alpha\gamma}_{+i} f^\beta_{-i.\gamma} + f^{\alpha\gamma}_{-i} f^\beta_{+i.\gamma} - \tfrac{1}{2} f^{\gamma\varepsilon}_{+i} f_{-i\gamma\varepsilon} g^{\alpha\beta}). \tag{62}$$

At the right-hand side appears the force density (III.160), which is the sum of the Lorentz forces acting on the individual particles:

$$f^\alpha = \sum_i e_i \int F^{\alpha\beta} u_{i\beta}\, \delta^{(4)}(R_i - R)\mathrm{d}s_i\,. \tag{63}$$

Starting from the energy–momentum balance given above we shall derive the equations of motion of a composite particle in an electromagnetic field. For that purpose we shall first have to define a covariant centre of energy. Subsequently the equations of motion will be obtained for charged composite particles which carry electromagnetic multipoles[1].

[1] The treatment will follow the derivation given in L. G. Suttorp and S. R. de Groot, N. Cim. **65A**(1970)245; v. also W. G. Dixon, N. Cim. **34**(1964)317; **38**(1965)1616. A treatment with a different definition of the centre of energy from the one used in the following was given in S.R. de Groot and L.G. Suttorp, Physica **37**(1967)284, 297; **39**(1968)84; L.G. Suttorp, On the covariant derivation of macroscopic electrodynamics from electron theory, thesis, Amsterdam (1968). Often equations have been postulated, i.e., either obtained from variational principles *ad hoc* or generalized from non-relativistic theory, e.g. J. Frenkel, Z. Phys. **37**(1926)243; V. Bargmann, L. Michel and V. L. Telegdi, Phys. Rev. Lett. **2**(1959) 435.

Part of this programme may also be performed with the use of the alternative energy–momentum balance discussed in the appendix of the preceding chapter. This will be done in an appendix to the present chapter.

b. *Definition of a covariant centre of energy*

In this subsection we shall be concerned with the general definition of a centre of energy for systems described by a symmetric energy–momentum tensor $t^{\alpha\beta}$, that fulfils a balance equation:

$$\partial_\beta t^{\alpha\beta} = f^\alpha, \tag{64}$$

with f^α a force density that has a finite support in space-like directions. The general scheme will subsequently be applied for the tensor $t^{\alpha\beta}$ of the preceding subsection.

Let us consider systems for which the total momentum p^α over a plane space-like surface Σ with normal n^α

$$p^\alpha = -c^{-1}\int_\Sigma t^{\alpha\beta} n_\beta\, \mathrm{d}^3\Sigma \tag{65}$$

is a finite time-like vector, with a positive time-component (then the total energy of the system is positive). The tensor $t^{\alpha\beta}$ will be supposed to diminish in space-like directions with increasing distances in such a way that this integral (and those which we shall need later) are (semi-)convergent. A covariant centre of energy may then be defined[1] by considering those plane surfaces Σ of which the normal n^α is parallel to p^α. In these surfaces one then determines the centre of energy

$$X^\alpha = \frac{\int_\Sigma R^\alpha n_\beta t^{\beta\gamma} n_\gamma\, \mathrm{d}^3\Sigma}{\int_\Sigma n_\varepsilon t^{\varepsilon\zeta} n_\zeta\, \mathrm{d}^3\Sigma}. \tag{66}$$

(In the rest frame of p^α this formula reads indeed $\boldsymbol{X} = \int \boldsymbol{R} t^{00}\mathrm{d}\boldsymbol{R}/\int t^{00}\mathrm{d}\boldsymbol{R}$.) These centres of energy then satisfy the relation

$$p_\alpha s^{\alpha\beta} = 0, \tag{67}$$

where the inner angular momentum is

$$s^{\alpha\beta} = -c^{-1}\int_\Sigma \{(R-X)^\alpha t^{\beta\gamma} - (R-X)^\beta t^{\alpha\gamma}\} n_\gamma\, \mathrm{d}^3\Sigma, \tag{68}$$

assumed to be a finite quantity for the system under consideration.

[1] T. Nakano, Progr. Theor. Phys. **15**(1956)333; W. Tulczyjew, Acta Phys. Polon. **18**(1959) 393; W. G. Dixon, N. Cim. **34**(1964)317.

We shall prove now that the set of centres of energy defined in this way forms one single world line (or several discrete ones). Consider such a point X^α determined in a plane surface Σ with normal parallel to p^α. One may ask oneself now if there exists a point $X^\alpha+\delta X^\alpha$ (with $p_\alpha\, \delta X^\alpha = 0$) in the infinitesimal neighbourhood of X^α, which is likewise a centre of energy, this time in a plane surface Σ' with normal parallel to the corresponding momentum $p^\alpha+\delta p^\alpha$. In the proper frame of p^α one has from (65), (67) and (68)

$$\int_\Sigma t^{i0}(R^0, \boldsymbol{R})\mathrm{d}\boldsymbol{R} = 0 \qquad (R^0 = X^0), \tag{69}$$

$$\int_\Sigma (\boldsymbol{R}-\boldsymbol{X})t^{00}(R^0, \boldsymbol{R})\mathrm{d}\boldsymbol{R} = 0. \tag{70}$$

The proper frame of $p^\alpha+\delta p^\alpha$ is connected to the proper frame of p^α by an infinitesimal pure Lorentz transformation:

$$R^{0\prime} = R^0+\boldsymbol{\varepsilon}\cdot\boldsymbol{R}, \qquad \boldsymbol{R}' = \boldsymbol{R}+\boldsymbol{\varepsilon}R^0 \tag{71}$$

for a certain value of $\boldsymbol{\varepsilon}$. The time-space point with coordinates $(R^0, \boldsymbol{X}+\delta\boldsymbol{X})$ has in the new frame the coordinates $(\hat{R}^{0\prime}, \boldsymbol{X}'+\delta\boldsymbol{X}')$ which are given by

$$\hat{R}^{0\prime} = R^0+\boldsymbol{\varepsilon}\cdot\boldsymbol{X}, \tag{72}$$

$$\boldsymbol{X}'+\delta\boldsymbol{X}' = \boldsymbol{X}+\delta\boldsymbol{X}+\boldsymbol{\varepsilon}R^0 \tag{73}$$

up to terms linear in $\boldsymbol{\varepsilon}$ and $\delta\boldsymbol{X}$. Furthermore the space coordinates of an arbitrary point in Σ' read in the new frame

$$\hat{\boldsymbol{R}}' = \hat{\boldsymbol{R}}+\boldsymbol{\varepsilon}\hat{R}^0, \tag{74}$$

where $\hat{\boldsymbol{R}}$ and $\hat{R}^0$ are connected by

$$\hat{R}^{0\prime} = \hat{R}^0+\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{R}}. \tag{75}$$

From (72), (74) and (75) it follows now that

$$\hat{R}^0 = R^0+\boldsymbol{\varepsilon}\cdot(\boldsymbol{X}-\hat{\boldsymbol{R}}'), \tag{76}$$

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}'-\boldsymbol{\varepsilon}R^0. \tag{77}$$

In the proper frame of $p^\alpha+\delta p^\alpha$ the space components of the total momentum vanish (cf. (69)):

$$\int_{\Sigma'} t^{i0\prime}(\hat{R}^{0\prime}, \hat{\boldsymbol{R}}')\mathrm{d}\hat{\boldsymbol{R}}' = 0. \tag{78}$$

Using the transformation properties of a tensor one gets

$$\int_{\Sigma'} \{t^{i0}(\hat{R}^0, \hat{\boldsymbol{R}})+\varepsilon^i t^{00}(\hat{R}^0, \hat{\boldsymbol{R}})+\varepsilon_j t^{ij}(\hat{R}^0, \hat{\boldsymbol{R}})\}\mathrm{d}\hat{\boldsymbol{R}}' = 0. \tag{79}$$

Introducing (76) and (77) we obtain ($\partial_0 \equiv \partial/\partial\boldsymbol{R}^0$):

$$\int \{t^{i0}(R^0, \hat{\boldsymbol{R}}')-R^0\boldsymbol{\varepsilon}\cdot(\partial/\partial\hat{\boldsymbol{R}}')t^{i0}(R^0, \hat{\boldsymbol{R}}')+\boldsymbol{\varepsilon}\cdot(\boldsymbol{X}-\hat{\boldsymbol{R}}')\partial_0\, t^{i0}(R^0, \hat{\boldsymbol{R}}') \\ +\varepsilon^i t^{00}(R^0, \hat{\boldsymbol{R}}')+\varepsilon_j t^{ij}(R^0, \hat{\boldsymbol{R}}')\}\mathrm{d}\hat{\boldsymbol{R}}' = 0. \tag{80}$$

With (69) it follows that the first term vanishes. From (72) and the fact that $\hat{R}^{0\prime}$ is constant in the integration it follows that R^0 is constant so that the second term gives no contribution either. Therefore we have:

$$\int \{\boldsymbol{\varepsilon}\cdot(\boldsymbol{X}-\hat{\boldsymbol{R}}')\partial_0\, t^{i0}(R^0, \hat{\boldsymbol{R}}')+\varepsilon^i t^{00}(R^0, \hat{\boldsymbol{R}}')+\varepsilon_j t^{ij}(R^0, \hat{\boldsymbol{R}}')\}\mathrm{d}\hat{\boldsymbol{R}}' = 0. \tag{81}$$

With the equation of motion (64) and a partial integration this becomes

$$\int \{\boldsymbol{\varepsilon}\cdot(\boldsymbol{X}-\boldsymbol{R})\boldsymbol{f}(R^0, \boldsymbol{R})+\boldsymbol{\varepsilon}t^{00}(R^0, \boldsymbol{R})\}\mathrm{d}\boldsymbol{R} = 0. \tag{82}$$

The first integral is in fact extended over a finite support since f^α has a finite support. Hence for sufficiently small force densities one has

$$\left|\boldsymbol{\varepsilon}\cdot\int(\boldsymbol{X}-\boldsymbol{R})\boldsymbol{f}(R^0, \boldsymbol{R})\mathrm{d}\boldsymbol{R}\right| < \left|\boldsymbol{\varepsilon}\int t^{00}\mathrm{d}\boldsymbol{R}\right|, \tag{83}$$

so that (82) cannot be satisfied. The conclusion is that no point exists in the infinitesimal neighbourhood of X^α, which is also a centre of energy; in other words the set of centres of energy determines a discrete number of world lines.

The condition (67) thus leads to a situation completely different from that following from the condition $u_\alpha s^{\alpha\beta} = 0$ (with $u^\alpha = \mathrm{d}X^\alpha/\mathrm{d}s$ where s is the proper time). As a matter of fact Møller proved[1] that the latter condition does not suffice to determine a world line. Moreover he showed that it cannot be supplemented by the requirement that p^α be parallel to u^α (in the

[1] C. Møller, Ann. Inst. H. Poincaré **11**(1949)251.

general case $f^{\alpha} \neq 0$)[1]. Nevertheless this condition is sometimes used[2], although it leads to peculiar solutions: even in the force-free case a special type of helical motions is possible. (In order to avoid this difficulty the form of the derived equations is sometimes[3] changed *ad hoc* by means of a so-called 'iteration process'.)

The general prescription for the construction of a centre of energy as given above will be applied now to the case of a composite particle (consisting of charged point particles) that moves in an external electromagnetic field. As we saw in the preceding subsection such a system may be described by means of the energy–momentum tensor (62). In order to be able to apply the construction of the energy centre given above, we must check whether all assumptions used there are justified. To begin with, the energy–momentum tensor $t^{\alpha\beta}$ (62) is indeed symmetric and the force density (63) has indeed a finite support in space-like directions. The next point to discuss is the convergence of the integrals, in particular of (65) and (68) for p^{α} and $s^{\alpha\beta}$. (The convergence of all other integrals occurring is determined by the latter two.) The integrals contain $t^{\alpha\beta}$, which is a quadratic function of the fields $f^{\alpha\beta}$ of which the behaviour for small and for large space-like distances follows from (III.110,111) and (III.96,98). The first two formulae show that the convergence at short distances presents no difficulties (note the presence of the condition $i \neq j$ in the second term). The latter formulae indicate that for large space-like distances the fields diminish inversely proportionally to those distances, so that the energy–momentum tensor diminishes only with the square of the distances. As a consequence the integral p^{α} (65) would diverge if no subsidiary conditions on the fields are imposed. This is a reflection of the fact that the total energy stored in the electromagnetic field would be infinitely great if the particles have been suffering accelerations from infinitely past to infinitely future times. Here we hit a well-known difficulty of classical theory. In such a theory a composite particle is not

[1] For the free composite particle centres of energy have been defined and discussed already by A. D. Fokker, Relativiteitstheorie (Noordhoff, Groningen 1929) 170; M. H. L. Pryce, Proc. Roy. Soc. **A195**(1949)62.

[2] H. Hönl and A. Papapetrou, Z. Phys. **112**(1939)512; **116**(1940)153; M. Mathisson, Proc. Cambr. Phil. Soc. **36**(1940)331; **38**(1942)40; H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. A **178**(1941)273; J. Weyssenhoff and A. Raabe, Acta Phys. Polon. **9**(1947)7, 19, 26, 34, 46; H. C. Corben, N. Cim. **20**(1961)529; Phys. Rev. **121**(1961)1833; A. Białas, Acta Phys. Polon. **22**(1962)499; P. Nyborg, N. Cim. **31**(1964)1209; **32**(1964)1131; W. G. Dixon, J. Math. Phys. **8**(1967)1591; J. Vlieger, Physica **37**(1967)165.

[3] E. Plahte, Suppl. N. Cim. **4**(1966)291; J. Vlieger and S. Emid, Physica **41**(1969)368; S. Emid and J. Vlieger, Physica **52**(1971)329.

stable against radiation: as a consequence of the emitted radiation energy and momentum is lost unrestrictedly. This is a paradox if the total energy content in the initial state is finite. To obtain nevertheless classical equations of motion one imposes the subsidiary condition that in the remote past and future the particles are not accelerated (just as in the treatment of the foregoing chapter for a single particle). In this way the effects of radiation in the remote past and future are suppressed.

With the subsidiary condition it follows that for the discussion of the convergence of the integrals at large space-like distances only the velocity fields in (III.96) or (III.99) have to be taken into account. These retarded and advanced velocity fields due to particle i are

$$f^{\alpha\beta}_{\mathrm{r,a},i} = \pm \frac{e_i c^2}{4\pi(u_i \cdot r_i)^3} (r_i^{\alpha} u_i^{\beta} - r_i^{\beta} u_i^{\alpha})|_{\mathrm{r,a}}, \tag{84}$$

where $r_i^{\alpha} \equiv R^{\alpha} - R_i^{\alpha}$. If these fields are introduced into $t^{\alpha\beta}$ (62) it follows that the total momentum (65) over a space-like plane is indeed a finite four-vector (which is assumed to be time-like). However the integral (68) for the inner angular momentum is only conditionally convergent (as follows by counting the powers of R), so that a (Lorentz-invariant) prescription must be given for its evaluation. Since the integral is independent of the choice of the origin of coordinates (because only coordinate differences are involved) we may choose as origin a point lying in the plane Σ. The conditionally convergent integral splits into a convergent part with integrand $X^{\alpha}t^{\beta\gamma} - X^{\beta}t^{\alpha\gamma}$ and a semi-convergent part with integrand $R^{\alpha}t^{\beta\gamma} - R^{\beta}t^{\alpha\gamma}$. We shall confine our attention to the latter part. A prescription for the evaluation of this part is obtained by considering a three-sphere with radius ρ around a point in the plane Σ. We then evaluate the integral $\int(R^{\alpha}t^{\beta\gamma} - R^{\beta}t^{\alpha\gamma})\mathrm{d}^3\Sigma_{\gamma}$. We shall prove that this integral tends to a finite limiting value if the radius ρ tends to infinity. Furthermore we shall show that it is independent of the precise location of the centre of the sphere.

To start with the latter let us consider a sphere of radius ρ around the centre C_1^{α} and another sphere of the same radius around C_2^{α}. The difference between the integrals extended over the two spheres has an integrand which is of the order of ρ^{-3} (since for sufficiently large ρ only the velocity fields, which are proportional to ρ^{-2}, come into play) and is extended over a volume of the order of ρ^2. Hence this difference tends to zero if ρ grows indefinitely. Since the limiting value of the integral is now proved to be independent of the location of the sphere's centre, we shall choose this centre as the origin.

In order to prove the existence of the limit of the integral $\int(R^{\alpha}t^{\beta\gamma} - R^{\beta}t^{\alpha\gamma})$

$\times \mathrm{d}^3\Sigma_\gamma$ over the sphere for $\rho \to \infty$ we have to show that the contribution of a spherical shell, lying between two such spheres, tends to zero if the smallest radius tends to infinity. To that end we substitute $t^{\alpha\beta}$ (62) with $f_{\mathrm{r,\,a,}\,i}$ (84) into the integrand so that we get for the field dependent part:

$$\sum_{i,j(i\neq j)} e_i e_j c^4 \int R^\alpha \left\{ \frac{(r_i^\beta u_i^\varepsilon - r_i^\varepsilon u_i^\beta)(r_j^\gamma u_{j\varepsilon} - r_{j\varepsilon} u_j^\gamma)}{16\pi^2 (u_i\cdot r_i)^3 (u_j\cdot r_j)^3} - \frac{(r_i^\varepsilon u_i^\zeta - r_i^\zeta u_i^\varepsilon) r_{j\varepsilon} u_{j\zeta}\, g^{\beta\gamma}}{32\pi^2 (u_i\cdot r_i)^3 (u_j\cdot r_j)^3} \right\} \Bigg|_{\mathrm{r}} \mathrm{d}^3\Sigma_\gamma$$

$$+ \sum_i e_i^2 c^4 \int R^\alpha \left\{ \frac{(r_i^\beta u_i^\varepsilon - r_i^\varepsilon u_i^\beta)(r_i^\gamma u_{i\varepsilon} - r_{i\varepsilon} u_i^\gamma)}{16\pi^2 (u_i\cdot r_i)^6} \right.$$

$$\left. - \frac{(r_i^\varepsilon u_i^\zeta - r_i^\zeta u_i^\varepsilon) r_{i\varepsilon} u_{i\zeta}\, g^{\beta\gamma}}{32\pi^2 (u_i\cdot r_i)^6} \right\} \Bigg|_{-} \mathrm{d}^3\Sigma_\gamma - (\alpha, \beta), \qquad (85)$$

where we introduced the symbol $-$ at the bar to indicate half the difference of the retarded and the advanced contribution. The symbol (α, β) indicates the preceding expression with α and β interchanged. Since we have taken into account only velocity fields it is consistent to assume that the retarded position four-vector $R_i^\alpha|_{\mathrm{r}}$ that occurs here is parallel to the retarded four-velocity $u_i^\alpha|_{\mathrm{r}}$, at least for sufficiently large ρ, i.e. we write

$$R_i^\alpha|_{\mathrm{r}} = -c^{-2} u_i\cdot R_i\, u_i^\alpha|_{\mathrm{r}} + \xi_{i\mathrm{r}}^\alpha, \qquad (86)$$

where the factor in front of u_i^α is chosen such that the four-vector $\xi_{i\mathrm{r}}^\alpha$ is orthogonal to the velocity ($\xi_{i\mathrm{r}}\cdot u_i|_{\mathrm{r}} = 0$). If the radii of the spheres tend to infinity, both the left-hand side and the first term at the right-hand side blow up, while the last term remains finite. Similar remarks apply for the connexion between the advanced position $R_i^\alpha|_{\mathrm{a}}$ and the advanced velocity $u_i^\alpha|_{\mathrm{a}}$:

$$R_i^\alpha|_{\mathrm{a}} = -c^{-2} u_i\cdot R_i\, u_i^\alpha|_{\mathrm{a}} + \xi_{i\mathrm{a}}^\alpha. \qquad (87)$$

If we substitute (86) and (87) into (85) (using the definition $r_i^\alpha = R^\alpha - R_i^\alpha$), we find terms which are independent of ξ^α and terms that contain ξ^α. By counting the powers in R one notices that the latter terms give vanishing contributions if ρ tends to infinity. The remaining terms of the right-hand side of (85) read

$$\sum_{i,j(i\neq j)} e_i e_j c^4 \int \frac{R^\alpha\{u_i^\beta(R^2 u_j\cdot n - R\cdot u_j\, R\cdot n) - \tfrac{1}{2} n^\beta (R^2 u_i\cdot u_j - R\cdot u_i\, R\cdot u_j)\}}{16\pi^2 \{u_i\cdot(R-R_i)\}^3 \{u_j\cdot(R-R_j)\}^3} \Bigg|_{\mathrm{r}} \mathrm{d}^3\Sigma$$

$$+ \sum_i e_i^2 c^4 \int \frac{R^\alpha[u_i^\beta(R^2 u_i\cdot n - R\cdot u_i\, R\cdot n) - \tfrac{1}{2} n^\beta\{-R^2c^2 - (R\cdot u_i)^2\}]}{16\pi^2\{u_i\cdot(R-R_i)\}^6} \Bigg|_{-} \mathrm{d}^3\Sigma - (\alpha, \beta), \qquad (88)$$

where we employed the definition $\mathrm{d}^3\Sigma_\gamma = n_\gamma \mathrm{d}^3\Sigma$ (with n^α the normal to the surface Σ). Since R as well as the origin are situated on the surface Σ one has $R\cdot n = 0$. Furthermore we may use the light-cone equation $r_i^2 = 0$ to eliminate $u_i\cdot R_i$. Indeed if (86) or (87) is substituted into the light-cone equation written as $(R_i - R)^2 = 0$ one gets

$$u_i\cdot R_i|_{\mathrm{r,a}} = \{u_i\cdot R \pm \sqrt{(u_i\cdot R)^2 + R^2c^2}\}|_{\mathrm{r,a}}, \qquad (89)$$

where terms containing ξ have been suppressed at the right-hand side. Then the denominators of (88) get the form

$$\begin{aligned} &16\pi^2\{(u_i\cdot R)^2 + R^2c^2\}^{\frac{3}{2}}\{(u_j\cdot R)^2 + R^2c^2\}^{\frac{3}{2}}, \\ &16\pi^2\{(u_i\cdot R)^2 + R^2c^2\}^3. \end{aligned} \qquad (90)$$

If these denominators are used one finds immediately that the integral extended over the three-space between two spheres around the origin vanishes on grounds of symmetry (the integrand changes sign if R is replaced by $-R$). Thus we have proved now that the semi-convergent integral for the inner angular momentum tends to a definite limit if spheres of increasing radii are chosen as integration domain.

In the course of the proof on the uniqueness of the energy centre as given in the first part of this subsection we made use in an essential way of the assumption that the external force density f^α can be made arbitrarily small. In the present case where f^α is given by (63) in terms of the external fields, this assumption is certainly justified since these fields can be made arbitrarily small. This remark completes the discussion of the validity of the application of the general centre of energy construction to the special case of a system described by an energy–momentum tensor $t^{\alpha\beta}$ (62) and acted upon by a force density f^α (63). The asymptotic conditions employed are just what one has to expect in a classical theory in which radiative collapse of bound states would occur if the particles would be allowed to suffer accelerations in the remote past (and future).

c. *Charged dipole particles*

For a composite particle which satisfies the energy–momentum law (61) with the energy–momentum tensor $t^{\alpha\beta}$ (62) and the force density (63) the derivative of p^α with respect to the proper time s of the world line $X^\alpha(s)$ is given by [1]:

[1] One may prove from (69) and (70) that for sufficiently small fields the world line is time-like so that a proper time s along the world line may be introduced (see problem 2).

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = -\frac{c^{-1}}{\mathrm{d}s}\left\{\int_{\Sigma(s+\mathrm{d}s)} t^{\alpha\beta}(R)n_\beta(s+\mathrm{d}s)\mathrm{d}^3\Sigma - \int_{\Sigma(s)} t^{\alpha\beta}(R)n_\beta(s)\mathrm{d}^3\Sigma\right\}, \tag{91}$$

where the right-hand side is given by the difference of two integrals of the form (65) over the surfaces $\Sigma(s+\mathrm{d}s)$ and $\Sigma(s)$, divided by $\mathrm{d}s$. To be able to apply Gauss's theorem here we must discuss the contribution from the surface $\Sigma_\infty(s, \mathrm{d}s)$ at infinity which closes the volume between the surfaces $\Sigma(s+\mathrm{d}s)$ and $\Sigma(s)$. Since in the remote past and future the particles suffer no accelerations, only the velocity fields have to be inserted in the integral

$$\int_{\Sigma_\infty(s,\mathrm{d}s)} t^{\alpha\beta}(R)n_{\infty,\beta}(R)\mathrm{d}^3\Sigma \tag{92}$$

(with $n^\alpha_\infty(R)$ the outward pointing normal on the surface $\Sigma_\infty(s, \mathrm{d}s)$). By employing the expressions (84) and counting the powers in R one finds that this integral tends to zero, if the surface tends to infinity. Therefore one may apply Gauss's theorem to (91), with the result

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = \frac{c^{-1}}{\mathrm{d}s}\int_{\Sigma(s)}^{\Sigma(s+\mathrm{d}s)} f^\alpha \mathrm{d}^4V, \tag{93}$$

where the integral is extended over the volume bounded by the surfaces $\Sigma(s)$ and $\Sigma(s+\mathrm{d}s)$. The volume element d^4V may be written as

$$\mathrm{d}^4V = J(R, s)\mathrm{d}s\,\mathrm{d}^3\Sigma, \tag{94}$$

where the Jacobian $J(R, s)$ is:

$$J(R, s) = -\frac{u_\alpha p^\alpha}{\sqrt{-p^2}}\left\{1 - \frac{\dot{p}_\beta(R-X)^\beta}{u_\gamma p^\gamma}\right\} \tag{95}$$

(v. problem 3), with X^α, $u^\alpha \equiv \mathrm{d}X^\alpha/\mathrm{d}s$ and p^α functions of the proper time s. The equation of motion (93) becomes with (94)

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = \mathfrak{f}^\alpha, \tag{96}$$

where $\mathfrak{f}^\alpha$ is the total four-force expressed in terms of the force density $f^\alpha(R)$:

$$\mathfrak{f}^\alpha(s) \equiv c^{-1}\int_{\Sigma(s)} f^\alpha(R)J(R, s)\mathrm{d}^3\Sigma. \tag{97}$$

For the derivative of the inner angular momentum (68) with respect to

the proper time s one finds

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = -\frac{c^{-1}}{\mathrm{d}s}\left\{\int_{\Sigma(s+\mathrm{d}s)} \{R^\alpha - X^\alpha(s+\mathrm{d}s)\}t^{\beta\gamma}(R)n_\gamma(s+\mathrm{d}s)\mathrm{d}^3\Sigma - \int_{\Sigma(s)} \{R^\alpha - X^\alpha(s)\}t^{\beta\gamma}(R)n_\gamma(s)\mathrm{d}^3\Sigma\right\} - (\alpha, \beta) \tag{98}$$

again with $n^\alpha = p^\alpha/\sqrt{-p^2}$, depending on $s+\mathrm{d}s$ in the first integral and on s in the second. The integrals over $\Sigma(s)$ and $\Sigma(s+\mathrm{d}s)$ are to be read as the limits of integrals over three-spheres with increasing radii in the planes $\Sigma(s)$ and $\Sigma(s+\mathrm{d}s)$. In order to apply Gauss's theorem we close the region between the two surfaces $\Sigma(s)$ and $\Sigma(s+\mathrm{d}s)$ by a four-sphere of large radius, of which the centre is the origin of coordinates. (The intersections of this four-sphere with the surfaces $\Sigma(s)$ and $\Sigma(s+\mathrm{d}s)$ consist of two three-spheres.) With the use of Gauss's theorem one may write then

$$\begin{aligned}\int_{\Sigma(s)}^{\Sigma(s+\mathrm{d}s)} &\partial_\gamma[\{R^\alpha - X^\alpha(s)\}t^{\beta\gamma} - \{R^\beta - X^\beta(s)\}t^{\alpha\gamma}]\mathrm{d}^4V \\ &= -\int_{\Sigma(s+\mathrm{d}s)} \{R^\alpha - X^\alpha(s)\}t^{\beta\gamma}(R)n_\gamma(s+\mathrm{d}s)\mathrm{d}^3\Sigma \\ &\quad + \int_{\Sigma(s)} \{R^\alpha - X^\alpha(s)\}t^{\beta\gamma}(R)n_\gamma(s)\mathrm{d}^3\Sigma \\ &\quad + \int_{\Sigma(s,\mathrm{d}s)} \{R^\alpha - X^\alpha(s)\}t^{\beta\gamma}(R)n_{\infty,\gamma}(R)\mathrm{d}^3\Sigma - (\alpha, \beta),\end{aligned} \tag{99}$$

where the normal n^α_∞ is equal to $R^\alpha/\sqrt{R^2}$. The first two terms at the right-hand side may be written as

$$c\,\mathrm{d}s\left(\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} + u^\alpha p^\beta - u^\beta p^\alpha\right), \tag{100}$$

as follows by comparison with (98) and (65). The last term, which extends over the large four-sphere, vanishes with increasing radius, as may be seen in the following way. The velocity fields which are to be used in $t^{\alpha\beta}$ (62) decrease with the square of the inverse radius. As a consequence the part with $X^\alpha(s)$ goes to zero as counting of the powers in R shows. As to the term with R^α, it has the same form as (88) with n^α replaced by $n^\alpha_\infty = R^\alpha/\sqrt{R^2}$. Thus it vanishes. In this way we have found the change of inner angular momentum

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = \frac{c^{-1}}{\mathrm{d}s}\int_{\Sigma(s)}^{\Sigma(s+\mathrm{d}s)} \partial_\gamma[\{R^\alpha - X^\alpha(s)\}t^{\beta\gamma} - \{R^\beta - X^\beta(s)\}t^{\alpha\gamma}]\mathrm{d}^4V - (u^\alpha p^\beta - u^\beta p^\alpha). \tag{101}$$

If one uses the symmetry of the energy–momentum tensor $t^{\alpha\beta}$ (62), the energy–momentum law (61) and the expression (94) for the volume element one finds the inner angular momentum law

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = \mathfrak{d}^{\alpha\beta} - (u^\alpha p^\beta - u^\beta p^\alpha) \tag{102}$$

with the total torque

$$\mathfrak{d}^{\alpha\beta}(s) = c^{-1} \int_{\Sigma(s)} [\{R^\alpha - X^\alpha(s)\} f^\beta(R) - \{R^\beta - X^\beta(s)\} f^\alpha(R)] J(R, s) \mathrm{d}^3\Sigma, \tag{103}$$

containing the force density $f^\alpha(R)$.

With the explicit form (63) for the force density $f^\alpha(R)$ the total force (97) and the total torque (103) are completely specified:

$$\mathfrak{f}^\alpha(s) = c^{-1} \iint_{\Sigma(s)} F^{\alpha\beta}(R) \sum_i e_i \frac{\mathrm{d}R_{i\beta}}{\mathrm{d}s'} \delta^{(4)}\{R_i(s') - R\} J(R, s) \mathrm{d}^3\Sigma \, \mathrm{d}s', \tag{104}$$

$$\mathfrak{d}^{\alpha\beta}(s) = c^{-1} \iint_{\Sigma(s)} [\{R^\alpha - X^\alpha(s)\} F^{\beta\gamma}(R) - \{R^\beta - X^\beta(s)\} F^{\alpha\gamma}(R)] \sum_i e_i \frac{\mathrm{d}R_{i\gamma}}{\mathrm{d}s'} \delta^{(4)}\{R_i(s') - R\} J(R, s) \mathrm{d}^3\Sigma \, \mathrm{d}s'. \tag{105}$$

(For convenience's sake we choose a different parametrization of the world lines of the constituent particles, namely the parameter s', which may be induced with the help of the surfaces $\Sigma(s')$ starting from the parametrization of the world line of the centre of energy.)

Because of the occurrence of the four-dimensional delta functions only the intersection points of $\Sigma(s)$ with the world lines of the constituent particles contribute to the integrals over $\Sigma(s)$. Therefore we may perform the integrations in (104) and (105). One obtains then

$$\mathfrak{f}^\alpha(s) = c^{-1} \sum_i e_i F^{\alpha\beta}\{R_i(s)\} \frac{\mathrm{d}R_{i\beta}}{\mathrm{d}s}, \tag{106}$$

$$\mathfrak{d}^{\alpha\beta}(s) = c^{-1} \sum_i e_i [\{R_i^\alpha(s) - X^\alpha(s)\} F^{\beta\gamma}\{R_i(s)\} - \{R_i^\beta(s) - X^\beta(s)\} F^{\alpha\gamma}\{R_i(s)\}] \frac{\mathrm{d}R_{i\gamma}}{\mathrm{d}s}. \tag{107}$$

The external fields $F^{\alpha\beta}$ in these expressions depend on the positions R_i^α of the constituent particles. We may expand them in Taylor series around the centre of energy $X^\alpha(s)$ of the composite particle. To that purpose let us in-

troduce the relative positions

$$r_i^\alpha(s) \equiv R_i^\alpha(s) - X^\alpha(s) \tag{108}$$

of the constituent particles inside the composite particle. They fulfil the orthogonality relation (this follows from the construction of the centre of energy, described in the preceding subsection):

$$r_{i\alpha}(s) p^\alpha(s) = 0. \tag{109}$$

In the expanded expressions for (106) and (107) the internal coordinates $r_i^\alpha(s)$ may be grouped in such a way that only the covariant multipole moments (9) occur, with $p^\alpha/\sqrt{-p^2}$ as the time-like unit vector n^α. (In the following we shall omit the index $(n) = (p/\sqrt{-p^2})$ of the multipole moments.) If we limit ourselves (in this subsection) to the contributions of the covariant electric and magnetic dipoles ($n = 1$):

$$\begin{aligned} \mu^\alpha &= \sum_i e_i r_i^\alpha, \\ \nu^{\alpha\beta} &= \tfrac{1}{2} c^{-1} \sum_i e_i \left(r_i^\alpha \frac{\mathrm{d}r_i^\beta}{\mathrm{d}s} - r_i^\beta \frac{\mathrm{d}r_i^\alpha}{\mathrm{d}s} \right), \end{aligned} \tag{110}$$

i.e. to slowly varying external fields, we find from (106) and (107)

$$\begin{aligned} \mathfrak{f}^\alpha(s) = {} & c^{-1} e F^{\alpha\beta}\{X(s)\} u_\beta(s) \\ & + \tfrac{1}{2} \partial^\alpha F^{\beta\gamma}\{X(s)\} [c^{-1}\{\mu_\beta(s) u_\gamma(s) - \mu_\gamma(s) u_\beta(s)\} + \nu_{\beta\gamma}(s)] \\ & + c^{-1} \frac{\mathrm{d}}{\mathrm{d}s} [F^{\alpha\beta}\{X(s)\} \mu_\beta(s)], \end{aligned} \tag{111}$$

$$\begin{aligned} \mathfrak{d}^{\alpha\beta}(s) = {} & F^{\alpha\gamma}\{X(s)\}\{-c^{-1} u_\gamma(s) \mu^\beta(s) + \nu_\gamma^{\ \beta}(s)\} \\ & - F^{\beta\gamma}\{X(s)\}\{-c^{-1} u_\gamma(s) \mu^\alpha(s) + \nu_\gamma^{\ \alpha}(s)\}, \end{aligned} \tag{112}$$

where $e = \sum_i e_i$ is the charge of the composite particle and where the homogeneous field equations have been used in the second term at the right-hand side of (111). It will be convenient to introduce the electromagnetic dipole moment tensor ((11) with two indices $\alpha_1, \alpha_2 = \alpha, \beta$):

$$\mathfrak{m}^{\alpha\beta}(s) = c^{-1}\{\mu^\alpha(s) u^\beta(s) - \mu^\beta(s) u^\alpha(s)\} + \nu^{\alpha\beta}(s). \tag{113}$$

The covariant electric dipole moment may be expressed in terms of this tensor. If one uses the orthogonality relation (109) and the fact that the covariant electric quadrupole moment $\frac{1}{2}\sum_i e_i r_i^\alpha r_i^\beta$ is neglected in this subsection, one gets:

$$\mu^\alpha(s) = c \mathfrak{m}^{\alpha\beta}(s) p_\beta(s) / u_\gamma(s) p^\gamma(s). \tag{114}$$

If the definition (113) and the relation (114) are used in the expressions (111) and (112) for the force and the torque, we obtain:

$$\mathfrak{f}^{\alpha} = c^{-1}eF^{\alpha\beta}(X)u_{\beta}+\tfrac{1}{2}\{\partial^{\alpha}F^{\beta\gamma}(X)\}\mathfrak{m}_{\beta\gamma}+\frac{\mathrm{d}}{\mathrm{d}s}\left\{F^{\alpha\beta}(X)\mathfrak{m}_{\beta\gamma}\frac{p^{\gamma}}{u_{\varepsilon}p^{\varepsilon}}\right\}, \quad (115)$$

$$\mathfrak{d}^{\alpha\beta} = F^{\alpha\gamma}(X)\mathfrak{m}_{\gamma}^{\cdot\beta}-F^{\beta\gamma}(X)\mathfrak{m}_{\gamma}^{\cdot\alpha}+\{u^{\alpha}F^{\beta\gamma}(X)-u^{\beta}F^{\alpha\gamma}(X)\}\mathfrak{m}_{\gamma\varepsilon}\frac{p^{\varepsilon}}{u_{\zeta}p^{\zeta}}. \quad (116)$$

These expressions are to be inserted in (96) and (102). Together with the supplementary condition (67) one has obtained then the equation of motion and of inner angular momentum of a composite particle with charge and dipole moments in an external field.

In order to discuss them we first consider the *field-free* case. Then the equations reduce to

$$\frac{\mathrm{d}p^{\alpha}}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = p^{\alpha}u^{\beta}-p^{\beta}u^{\alpha}. \quad (117)$$

By differentiating the condition (67) one finds from these equations

$$p_{\alpha}p^{\alpha}u^{\beta}-p_{\alpha}u^{\alpha}p^{\beta} = 0. \quad (118)$$

Hence the four-vectors p^{α} and u^{α} are parallel, so that

$$p^{\alpha} = mu^{\alpha}, \quad (119)$$

where m is defined as:

$$m = -c^{-2}p_{\alpha}u^{\alpha}. \quad (120)$$

Now (117) reduces to

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = 0, \quad (121)$$

since $\mathrm{d}m/\mathrm{d}s$ vanishes as follows from (119) and the first equation of (117).

In the case *with fields* differentiation of (67) and substitution of (96) and (102) leads to

$$p^{\gamma}u_{\beta}p^{\beta} = u^{\alpha}p_{\beta}p^{\beta}-s^{\alpha\beta}\mathfrak{f}_{\beta}-\mathfrak{d}^{\alpha\beta}p_{\beta}. \quad (122)$$

Hence now p^{α} is not parallel to u^{α}. If this relation is multiplied by u_{α} one obtains the equality

$$p_{\alpha}p^{\alpha} = -c^{-2}(u_{\alpha}p^{\alpha})^{2}-c^{-2}u_{\alpha}s^{\alpha\beta}\mathfrak{f}_{\beta}-c^{-2}u_{\alpha}\mathfrak{d}^{\alpha\beta}p_{\beta}. \quad (123)$$

According to (119) and the condition (67) all terms on the right-hand side but the first are at least of second order in the fields, since the leading terms

of $\mathfrak{f}^{\alpha}$ and $\mathfrak{d}^{\alpha\beta}$ are linear in the fields. Hence if one wants to confine oneself to terms linear in the fields the equality (122) may be written as

$$p^{\alpha} = mu^{\alpha}+\frac{1}{mc^{2}}s^{\alpha\beta}\mathfrak{f}_{\beta}+c^{-2}\mathfrak{d}^{\alpha\beta}u_{\beta}, \quad (124)$$

so that now the total momentum p^{α} is expressed in terms of u^{α}, m, which is again defined as in (120), $s^{\alpha\beta}$, $\mathfrak{f}^{\alpha}$ and $\mathfrak{d}^{\alpha\beta}$. With this equality the equations of motion and of spin (96) and (102) become[1]:

$$\frac{\mathrm{d}(mu^{\alpha})}{\mathrm{d}s} = \mathfrak{f}^{\alpha}-c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}(s^{\alpha\beta}\mathfrak{f}_{\beta}/m+\mathfrak{d}^{\alpha\beta}u_{\beta}), \quad (125)$$

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = \Delta^{\alpha}_{\gamma}\Delta^{\beta}_{\varepsilon}\mathfrak{d}^{\gamma\varepsilon}+c^{-2}(s^{\alpha\gamma}u^{\beta}-s^{\beta\gamma}u^{\alpha})\mathfrak{f}_{\gamma}/m. \quad (126)$$

with the tensor:

$$\Delta^{\alpha\beta} \equiv \Delta^{\alpha\beta}(u) \equiv g^{\alpha\beta}+c^{-2}u^{\alpha}u^{\beta}. \quad (127)$$

The force and torque have been given by (115), (116) or, if again only terms linear in the fields are retained, by:

$$\mathfrak{f}^{\alpha} = c^{-1}eF^{\alpha\beta}u_{\beta}+\tfrac{1}{2}(\partial^{\alpha}F^{\beta\gamma})\mathfrak{m}_{\beta\gamma}-c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}(F^{\alpha\beta}\mathfrak{m}_{\beta\gamma}u^{\gamma}), \quad (128)$$

$$\mathfrak{d}^{\alpha\beta} = F^{\alpha\gamma}\mathfrak{m}_{\gamma}^{\cdot\varepsilon}\Delta^{\beta}_{\varepsilon}-F^{\beta\gamma}\mathfrak{m}_{\gamma}^{\cdot\varepsilon}\Delta^{\alpha}_{\varepsilon}. \quad (129)$$

Introducing these expressions into (125) and (126) we obtain the equations

[1] From (126) it follows that $s^{\alpha\beta}\mathfrak{f}_{\beta}/m = -(\mathrm{d}s^{\alpha\beta}/\mathrm{d}s)u_{\beta}$ up to terms linear in the fields. If one inserts this equality into (125) and (126) one gets (with the explicit expressions (128) and (129) for $\mathfrak{f}^{\alpha}$ and $\mathfrak{d}^{\alpha\beta}$) equations which have been discussed earlier[2] in connexion with the condition $u_{\alpha}s^{\alpha\beta}=0$. Owing to the use of this different subsidiary condition it is then not possible to go back to (125) and (126); this fact is connected with the appearance of unwanted helical solutions, even in the field-free case. (In ref. 3, equations of the type just described were derived on the basis of an explicit construction of a central point and with the use of the Darwin approximation for the intra-atomic fields. Helical motions of macroscopic dimensions are then excluded.)

[2] C. Møller, op. cit.; cf. papers mentioned in footnote 2 on page 184.

[3] S. R. de Groot and L. G. Suttorp, Physica **37**(1967)284, 297; **39**(1968)84; L. G. Suttorp, On the covariant derivation of macroscopic electrodynamics from electron theory, thesis Amsterdam (1968).

of motion and of spin:

$$\frac{\mathrm{d}(mu^\alpha)}{\mathrm{d}s} = c^{-1}eF^{\alpha\beta}u_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\mathfrak{m}_{\beta\gamma} + c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}\Bigg[\Delta^\alpha_\beta \mathfrak{m}^{\beta\gamma}F_{\gamma\varepsilon}u^\varepsilon - F^{\alpha\beta}\mathfrak{m}_{\beta\gamma}u^\gamma$$
$$- \frac{s^{\alpha\beta}}{m}\left\{c^{-1}eF_{\beta\gamma}u^\gamma + \tfrac{1}{2}(\partial_\beta F_{\gamma\varepsilon})\mathfrak{m}^{\gamma\varepsilon} - c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}(F_{\beta\gamma}\mathfrak{m}^{\gamma\varepsilon}u_\varepsilon)\right\}\Bigg], \tag{130}$$

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = \Delta^\alpha_\gamma \Delta^\beta_\varepsilon (F^{\gamma\zeta}\mathfrak{m}_\zeta^{\cdot\varepsilon} - \mathfrak{m}^{\gamma\zeta}F_\zeta^{\cdot\varepsilon}) + c^{-2}\frac{1}{m}(s^{\alpha\gamma}u^\beta - s^{\beta\gamma}u^\alpha)$$
$$\left\{c^{-1}eF_{\gamma\varepsilon}u^\varepsilon + \tfrac{1}{2}(\partial_\gamma F_{\varepsilon\zeta})\mathfrak{m}^{\varepsilon\zeta} - c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}(F_{\gamma\varepsilon}\mathfrak{m}^{\varepsilon\zeta}u_\zeta)\right\}. \tag{131}$$

The tensor $\mathfrak{m}^{\alpha\beta}(s)$ (113) contains the covariant electric and magnetic dipole moments μ^α and $\nu^{\alpha\beta}$, which are defined with respect to the time-like unit vector $n^\alpha = p^\alpha/\sqrt{-p^2}$. Since we neglected quadratic field terms and since $\mathfrak{m}^{\alpha\beta}$ is always multiplied by the field, we may replace them by the covariant dipole moments, defined with respect to $c^{-1}u^\alpha$. (They will be denoted by the same symbols.) The latter multipole moments have been studied in detail in section 2.

The space parts of (130) and (131) will be written in three-dimensional notation. The four-velocity u^α is $(\gamma c, \gamma \boldsymbol{v})$ with $\gamma = (1-\boldsymbol{\beta}^2)^{-\frac{1}{2}}$ and $\boldsymbol{\beta} = \boldsymbol{v}/c$. Furthermore the space-space components of $s^{\alpha\beta}$ will be denoted by the vector $\boldsymbol{s}$ with components $s^i = \frac{1}{2}\varepsilon^{ijk}s_{jk}$. As far as they occur at the right-hand side of (130) and (131), the space-time components s^{i0} of $s^{\alpha\beta}$ may be written as $(\boldsymbol{\beta}\wedge\boldsymbol{s})^i$ by the use of (67) with (124) together with the fact that we neglected quadratic field terms throughout. The field $F^{\alpha\beta}$ has the components $F^{ij} = \varepsilon^{ijk}B_k$ and $F^{0i} = E^i$. The covariant multipole moments occurring in $\mathfrak{m}^{\alpha\beta}$ may be expressed in terms of the atomic multipole moments, which are independent of the atomic velocities. One finds from (A41–43)

$$\mu^0 = \gamma\boldsymbol{\beta}\cdot\boldsymbol{\mu}^{(1)}, \qquad \mu^i = (\boldsymbol{\Omega}^{-1}\cdot\boldsymbol{\mu}^{(1)})^i, \qquad (i = 1, 2, 3),$$
$$\nu^{ij} = \gamma(\boldsymbol{\Omega}\cdot\boldsymbol{\nu}^{(1)})^k, \quad (i, j, k = 1, 2, 3 \text{ cycl.}), \qquad \nu^{i0} = -\gamma(\boldsymbol{\nu}^{(1)}\wedge\boldsymbol{\beta})^i, \tag{132}$$
$$(i = 1, 2, 3),$$

if only the atomic dipole moments are retained. The $\boldsymbol{\Omega}$-tensor is defined as

$$\boldsymbol{\Omega} = \mathbf{U} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta\beta}. \tag{133}$$

If these formulae are substituted into the definition (113) of $\mathfrak{m}^{\alpha\beta}$ one obtains

with (A6)[1]

$$\mathfrak{m}^{i0} \equiv \mathfrak{p}^i = \gamma(\boldsymbol{\Omega}\cdot\boldsymbol{\mu}^{(1)} + \boldsymbol{\beta}\wedge\boldsymbol{\nu}^{(1)})^i, \qquad (i = 1, 2, 3),$$
$$\mathfrak{m}^{ij} \equiv \mathfrak{m}^k = \gamma(\boldsymbol{\Omega}\cdot\boldsymbol{\nu}^{(1)} - \boldsymbol{\beta}\wedge\boldsymbol{\mu}^{(1)})^k, \qquad (i, j, k = 1, 2, 3 \text{ cycl.}). \tag{134}$$

We may now write the equations (130) and (131) in three-dimensional notation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\boldsymbol{v}) = e(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) + \gamma^{-1}\{(\nabla\boldsymbol{E})\cdot\boldsymbol{\mathfrak{p}} + (\nabla\boldsymbol{B})\cdot\boldsymbol{\mathfrak{m}}\}$$
$$+ c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}[\gamma\{\boldsymbol{\mathfrak{p}}\wedge\boldsymbol{B} - \boldsymbol{\mathfrak{m}}\wedge\boldsymbol{E}) - \boldsymbol{\beta}\wedge(\boldsymbol{\mathfrak{p}}\wedge\boldsymbol{E} + \boldsymbol{\mathfrak{m}}\wedge\boldsymbol{B})\}]$$
$$+ c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\{\gamma^3\boldsymbol{\beta}(\boldsymbol{\mathfrak{p}} - \boldsymbol{\beta}\wedge\boldsymbol{\mathfrak{m}})\cdot\boldsymbol{\Omega}^2\cdot(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B})\} + c^{-1}\frac{\mathrm{d}\boldsymbol{\Phi}_s}{\mathrm{d}t}, \tag{135}$$

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \gamma\boldsymbol{\Omega}^2\cdot(\boldsymbol{\mathfrak{p}}\wedge\boldsymbol{E} + \boldsymbol{\mathfrak{m}}\wedge\boldsymbol{B}) + \gamma\boldsymbol{\beta}\wedge(\boldsymbol{\mathfrak{p}}\wedge\boldsymbol{B} - \boldsymbol{\mathfrak{m}}\wedge\boldsymbol{E}) + \boldsymbol{\beta}\wedge\boldsymbol{\Phi}_s, \tag{136}$$

with the inner angular momentum terms $\boldsymbol{\Phi}_s$ given by:

$$\boldsymbol{\Phi}_s \equiv \frac{e}{mc}\gamma\{\boldsymbol{s}\wedge(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) - \boldsymbol{s}\wedge\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{E}\}$$
$$+ \frac{1}{mc}[\boldsymbol{s}\wedge\{(\nabla\boldsymbol{E})\cdot\boldsymbol{\mathfrak{p}} + (\nabla\boldsymbol{B})\cdot\boldsymbol{\mathfrak{m}}\} + \boldsymbol{s}\wedge\boldsymbol{\beta}\{(\partial_0\boldsymbol{E})\cdot\boldsymbol{\mathfrak{p}} + (\partial_0\boldsymbol{B})\cdot\boldsymbol{\mathfrak{m}}\}]$$
$$+ \frac{1}{mc^2}\gamma\boldsymbol{s}\wedge\frac{\mathrm{d}}{\mathrm{d}t}\{\gamma(\boldsymbol{\mathfrak{p}} - \boldsymbol{\beta}\wedge\boldsymbol{\mathfrak{m}})\wedge\boldsymbol{B} + \gamma\boldsymbol{E}\boldsymbol{\beta}\cdot\boldsymbol{\mathfrak{p}}\}$$
$$- \frac{1}{mc^2}\gamma\boldsymbol{s}\wedge\boldsymbol{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\{\gamma(\boldsymbol{\mathfrak{p}} - \boldsymbol{\beta}\wedge\boldsymbol{\mathfrak{m}})\cdot\boldsymbol{E}\}. \tag{137}$$

(For convenience the expressions (134) for $\boldsymbol{\mathfrak{p}}$ and $\boldsymbol{\mathfrak{m}}$ in terms of the atomic dipole moments have not been inserted in these formulae.)

The equation of motion (135) contains at its right-hand side the Lorentz force on a charge e, the 'Kelvin forces' on the electric and magnetic dipole moments $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\nu}^{(1)}$, and three terms which are time derivatives of quantities of which the leading terms are

$$c^{-1}\left\{\boldsymbol{\mu}^{(1)}\wedge\boldsymbol{B} - \left(\boldsymbol{\nu}^{(1)} - \frac{e}{mc}\boldsymbol{s}\right)\wedge\boldsymbol{E}\right\}, \tag{138}$$

[1] The vector $\mathfrak{p}$ which is defined here should not be confused with the space part of p^α (65).

as follows with (134). The time derivative of the first term is the electrodynamic effect found already in the non-relativistic theory (v. (I.155)). The time derivative of the other terms (which are sometimes called the 'hidden momentum' of a magnetic dipole particle with inner angular momentum) is an analogous magnetodynamic effect, which contains the vector product of the electric field and a combination of the magnetic moment and the inner angular momentum. Such a magnetodynamic effect, which occurs already in Frenkel's work, was discussed extensively[1] in recent years on the basis of various *ad hoc* arguments, such as the assumed equality of action and reaction for the forces exerted by a charge and a magnetic dipole moment on each other. One finds then only the magnetic dipole term, not the inner angular momentum term of (138). (The latter is for an atom or a molecule small as compared to the former since the mass m of the composite particle as a whole is much greater than the electronic masses which contribute to $\boldsymbol{s}$). Often the question of the equivalence of a magnetic dipole consisting of charged particles (or a current loop) and one consisting of two 'magnetic charges' plays a role in these discussions[2].

Since the terms given in (138) are of order c^{-2} the mechanism which leads to this contribution to the force can be studied already in a theory which gives all terms up to order c^{-2}. Such a treatment is presented in the fifth appendix of this chapter[3].

The equation of inner angular momentum (136) shows at the right-hand side the change of $\boldsymbol{s}$ due to a torque of which the leading term is $\boldsymbol{\mu}^{(1)}\wedge\boldsymbol{E}+\boldsymbol{\nu}^{(1)}\wedge\boldsymbol{B}$, as follows from (134).

When one wants to compare the results (135–137) with the non-relativistic and the so-called 'semi-relativistic' ones (see appendix V), it should in the first place be borne in mind that the expression (137) is of order c^{-1}, so that it contributes neither in the non-relativistic nor in the semi-relativistic approximation of the equations (135) and (136). Furthermore one should also remember that in the non-relativistic and semi-relativistic limit the magnetic moment $\boldsymbol{\nu}^{(1)}$ is considered to be of order c^{-1} and c^0 respectively. Then if one limits oneself to terms of order c^{-1} one finds indeed the equa-

[1] P. Penfield jr. and H. A. Haus, The electrodynamics of moving media (M.I.T. Press, Cambridge, Mass. 1967) p. 215; Phys. Lett. **26A**(1968)412; Physica **42**(1969)447; O. Costa de Beauregard, Compt. Rend. **263B**(1966)1007, **264B**(1967)565, 731, **266B**(1968) 364, 1181; Phys. Lett. **24A**(1967)177, **25A**(1967)95, **26A**(1967)48; Cah. Physique **206**(1967) 373; N. Cim. **63B**(1969)611; W. Shockley and R. P. James, Phys. Rev. Lett. **18**(1967)876.

[2] v. B. D. H. Tellegen, Am. J. Phys. **30**(1962)650. From the present results it follows that the force on a magnetic dipole consisting of charged particles is exactly the same as that which is assumed to be valid for a 'magnetic charge' dipole.

[3] S. Coleman and J. H. Van Vleck, Phys. Rev. **171**(1968)1370 followed a similar approach.

tions (I.55) and (I.79) of the non-relativistic theory and (A118) and (A136) of the semi-relativistic theory (cf. also problems 6–8).

d. *Charged particles with magnetic dipole moment proportional to their inner angular momentum*

Let us consider the special case of a charged composite particle without electric dipole moment and with a magnetic dipole moment proportional to the inner angular momentum; then

$$\mathfrak{m}^{\alpha\beta} = \kappa s^{\alpha\beta}. \tag{139}$$

The equations of motion and of spin have been given by (96), (102), (124) with (67):

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = \mathfrak{f}^\alpha, \tag{140}$$

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = \mathfrak{d}^{\alpha\beta} - (u^\alpha p^\beta - u^\beta p^\alpha), \tag{141}$$

$$p^\alpha = mu^\alpha + \frac{1}{mc^2} s^{\alpha\beta}\mathfrak{f}_\beta + c^{-2}\mathfrak{d}^{\alpha\beta}u_\beta, \tag{142}$$

$$p_\alpha s^{\alpha\beta} = 0. \tag{143}$$

The force and torque that follow from (128) and (129) with (139) are in this case:

$$\mathfrak{f}^\alpha = c^{-1}eF^{\alpha\beta}u_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\mathfrak{m}_{\beta\gamma}, \tag{144}$$

$$\mathfrak{d}^{\alpha\beta} = F^{\alpha\gamma}\mathfrak{m}_\gamma^{\cdot\beta} - F^{\beta\gamma}\mathfrak{m}_\gamma^{\cdot\alpha}, \tag{145}$$

where only terms linear in the field have been retained. If these expressions are inserted into (142) we obtain:

$$p^\alpha = mu^\alpha - c^{-2}\mathfrak{m}^{\alpha\beta}F_{\beta\gamma}u^\gamma + \frac{e}{mc^3}s^{\alpha\beta}F_{\beta\gamma}u^\gamma + \frac{1}{2mc^2}s^{\alpha\beta}(\partial_\beta F_{\gamma\varepsilon})\mathfrak{m}^{\gamma\varepsilon}. \tag{146}$$

From these equations one may prove that the square of the inner angular momentum $s_{\alpha\beta}s^{\alpha\beta}$ and the quantity

$$m^* = m + \tfrac{1}{2}c^{-2}F_{\alpha\beta}\mathfrak{m}^{\alpha\beta} \tag{147}$$

are conserved. In fact, multiplying (141) by $s_{\alpha\beta}$ and using (143) and (145), one gets

$$\frac{\mathrm{d}}{\mathrm{d}s}(s_{\alpha\beta}s^{\alpha\beta}) = 4s_{\alpha\beta}F^{\alpha\gamma}\mathfrak{m}_\gamma^{\cdot\beta}, \tag{148}$$

which vanishes, as follows if (139) is introduced. Furthermore the time derivative of m (120) becomes with (140), (144) and (146)

$$\frac{\mathrm{d}m}{\mathrm{d}s} = -\tfrac{1}{2}c^{-2}\frac{\mathrm{d}F^{\alpha\beta}}{\mathrm{d}s}\mathfrak{m}_{\alpha\beta}+c^{-4}\frac{\mathrm{d}u_\alpha}{\mathrm{d}s}\left\{\mathfrak{m}^{\alpha\beta}F_{\beta\gamma}u^\gamma-\frac{e}{mc}s^{\alpha\beta}F_{\beta\gamma}u^\gamma-\frac{1}{2m}s^{\alpha\beta}(\partial_\beta F_{\gamma\varepsilon})\mathfrak{m}^{\gamma\varepsilon}\right\}. \quad (149)$$

Since $\mathrm{d}u^\alpha/\mathrm{d}s$ vanishes in the field-free case, this equality becomes up to first order in the fields

$$\frac{\mathrm{d}m}{\mathrm{d}s}+\tfrac{1}{2}c^{-2}\frac{\mathrm{d}F^{\alpha\beta}}{\mathrm{d}s}\mathfrak{m}_{\alpha\beta} = 0. \quad (150)$$

Finally, if (141) is multiplied by $F_{\alpha\beta}$ one finds with (139), (145) and (146)

$$F_{\alpha\beta}\frac{\mathrm{d}\mathfrak{m}^{\alpha\beta}}{\mathrm{d}s} = 0, \quad (151)$$

if again only linear field terms are retained. From (150) and (151) it follows indeed that m^* (147) is conserved.

Since m^* is conserved we shall use it instead of m in (146). With (147) this expression becomes up to terms linear in the fields:

$$p^\alpha = m^*u^\alpha-\tfrac{1}{2}c^{-2}F_{\beta\gamma}\mathfrak{m}^{\beta\gamma}u^\alpha-c^{-2}\mathfrak{m}^{\alpha\beta}F_{\beta\gamma}u^\gamma+\frac{e}{m^*c^3}s^{\alpha\beta}F_{\beta\gamma}u^\gamma+\frac{1}{2m^*c^2}s^{\alpha\beta}(\partial_\beta F_{\gamma\varepsilon})\mathfrak{m}^{\gamma\varepsilon}. \quad (152)$$

In this expression appears the inner angular momentum $s^{\alpha\beta}$ together with the factor e/m^*c containing the total charge of the composite particle. We shall call this combination the 'normal magnetic moment'

$$\mathfrak{m}^{\alpha\beta}_{(\mathrm{n})} = \frac{e}{m^*c}s^{\alpha\beta}. \quad (153)$$

The total magnetic moment (139) is the sum of this normal part and an 'anomalous magnetic moment' $\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})}$:

$$\mathfrak{m}^{\alpha\beta} = \mathfrak{m}^{\alpha\beta}_{(\mathrm{n})}+\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})}, \quad (154)$$

$$\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})} \equiv \mathfrak{m}^{\alpha\beta}-\frac{e}{m^*c}s^{\alpha\beta} = \left(\kappa-\frac{e}{m^*c}\right)s^{\alpha\beta}. \quad (155)$$

For convenience one could introduce (for a charged composite particle) the 'gyromagnetic factor' g by means of

$$\kappa \equiv \frac{ge}{2m^*c}. \quad (156)$$

Then we may write the anomalous magnetic moment alternatively in the form

$$\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})} = \frac{(g-2)e}{2m^*c}s^{\alpha\beta}. \quad (157)$$

With (152) and (155) the equations of motion and spin (140) and (141) with (144) and (145) become:

$$m^*\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} = c^{-1}eF^{\alpha\beta}u_\beta+\tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\mathfrak{m}_{\beta\gamma}+\tfrac{1}{2}c^{-2}\frac{\mathrm{d}F^{\beta\gamma}}{\mathrm{d}s}\mathfrak{m}_{\beta\gamma}u^\alpha+c^{-2}\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})}\frac{\mathrm{d}F_{\beta\gamma}}{\mathrm{d}s}u^\gamma-\frac{1}{2m^*c^2}s^{\alpha\beta}\left(\partial_\beta\frac{\mathrm{d}F_{\gamma\varepsilon}}{\mathrm{d}s}\right)\mathfrak{m}^{\gamma\varepsilon}, \quad (158)$$

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = F^{\alpha\gamma}\mathfrak{m}_\gamma^{\ \beta}-F^{\beta\gamma}\mathfrak{m}_\gamma^{\ \alpha}+c^{-2}(u^\alpha\mathfrak{m}^{\beta\gamma}_{(\mathrm{a})}-u^\beta\mathfrak{m}^{\alpha\gamma}_{(\mathrm{a})})F_{\gamma\varepsilon}u^\varepsilon+\frac{1}{2m^*c^2}\{s^{\alpha\gamma}(\partial_\gamma F_{\varepsilon\zeta})\mathfrak{m}^{\varepsilon\zeta}u^\beta-s^{\beta\gamma}(\partial_\gamma F_{\varepsilon\zeta})\mathfrak{m}^{\varepsilon\zeta}u^\alpha\}. \quad (159)$$

In the last three terms at the right-hand side of equation (158) only the fields $F^{\alpha\beta}$ have been differentiated with respect to s, not the polarization tensors $\mathfrak{m}^{\alpha\beta}$, $\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})}$, the inner angular momentum tensor $s^{\alpha\beta}$ and the four-velocity u^α. The reason for this is that differentiation of the latter would have given rise to terms quadratic in the fields. (To see this use must be made of the proportionality of the polarization tensors $\mathfrak{m}^{\alpha\beta}$, $\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})}$ and the inner angular momentum tensor $s^{\alpha\beta}$ together with the equations (121) for the field-free case.)

In the right-hand sides of (158) and (159) the leading terms with the polarization tensors contain the first derivatives of the field and the field itself respectively. Hence if the fields are sufficiently homogeneous the last terms in (158) and (159) may be discarded, so that we then have the simpler equations:

$$m^*\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} = c^{-1}eF^{\alpha\beta}u_\beta+\tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\mathfrak{m}_{\beta\gamma}+\tfrac{1}{2}c^{-2}\frac{\mathrm{d}F^{\beta\gamma}}{\mathrm{d}s}\mathfrak{m}_{\beta\gamma}u^\alpha+c^{-2}\mathfrak{m}^{\alpha\beta}_{(\mathrm{a})}\frac{\mathrm{d}F_{\beta\gamma}}{\mathrm{d}s}u^\gamma, \quad (160)$$

$$\frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = F^{\alpha\gamma}\mathfrak{m}_\gamma^{\ \beta}-F^{\beta\gamma}\mathfrak{m}_\gamma^{\ \alpha}+c^{-2}(u^\alpha\mathfrak{m}^{\beta\gamma}_{(\mathrm{a})}-u^\beta\mathfrak{m}^{\alpha\gamma}_{(\mathrm{a})})F_{\gamma\varepsilon}u^\varepsilon. \quad (161)$$

These are the equations of motion and of spin for a composite particle with charge and magnetic dipole moment (proportional to the inner angular momentum) in sufficiently homogeneous external fields. The covariant magnetic dipole moment occurring in $\mathfrak{m}^{\alpha\beta}$ was defined with respect to the normal unit vector $p^\alpha/\sqrt{-p^2}$. As in the preceding subsection one might use as well the covariant magnetic dipole moment defined with respect to $c^{-1}u^\alpha$, because quadratic field terms have been neglected.

As in the preceding subsection we may write the covariant equations (160) and (161) in three-dimensional notation. The four-velocity u^α has components $(\gamma c, \gamma c\boldsymbol{\beta})$ with $\boldsymbol{\beta} = \boldsymbol{v}/c$, while the field components are $F^{0i} = E^i$, $F^{ij} = B^k$ $(i, j, k = 1, 2, 3 \text{ cycl.})$. Since the electric dipole moment vanishes here, one may write the components $\mathfrak{m}^{i0} = \mathfrak{p}^i$ and $\mathfrak{m}^{ij} = \mathfrak{m}^k$ $(i, j, k = 1, 2, 3$ cycl.) of the magnetic moment tensor according to (134) as: $\boldsymbol{\mathfrak{p}} = \boldsymbol{\beta} \wedge \boldsymbol{\mathfrak{m}}$ and $\boldsymbol{\mathfrak{m}} = \gamma\boldsymbol{\Omega}\cdot\boldsymbol{v}^{(1)}$. From the proportionality of the anomalous and total polarization tensor (v. (139), (156) and (157)) it follows that the space-space component $\mathfrak{m}^i_{(a)} = \mathfrak{m}^{jk}_{(a)}$ $(i, j, k = 1, 2, 3$ cycl.) is $\mathfrak{m}^i_{(a)} = \{(g-2)/g\}\mathfrak{m}^i$, while the space-time component $\mathfrak{m}^{i0}_{(a)}$ is equal to $(\boldsymbol{\beta} \wedge \boldsymbol{\mathfrak{m}}_{(a)})^i$. In this way the equations of motion and of spin (160) and (161) get the form:

$$m^* \frac{\mathrm{d}(\gamma \boldsymbol{v})}{\mathrm{d}t} = e(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) + \gamma^{-1}(\nabla\boldsymbol{E})\cdot(\boldsymbol{\beta}\wedge\boldsymbol{\mathfrak{m}}) + \gamma^{-1}(\nabla\boldsymbol{B})\cdot\boldsymbol{\mathfrak{m}}$$
$$+c^{-1}\gamma\frac{\mathrm{d}}{\mathrm{d}t}\{\boldsymbol{\beta}\boldsymbol{\mathfrak{m}}\cdot(\boldsymbol{B}-\boldsymbol{\beta}\wedge\boldsymbol{E}) - \boldsymbol{\mathfrak{m}}_{(a)}\wedge(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) + \boldsymbol{\mathfrak{m}}_{(a)}\wedge\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{E}\}, \qquad (162)$$

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \gamma^{-1}\boldsymbol{\mathfrak{m}}\wedge\boldsymbol{B} + \gamma^{-1}(\boldsymbol{\beta}\wedge\boldsymbol{\mathfrak{m}})\wedge\boldsymbol{E} + \gamma\boldsymbol{\beta}\cdot\boldsymbol{\mathfrak{m}}_{(a)}(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) - \gamma^{-1}\boldsymbol{\mathfrak{m}}_{(a)}\boldsymbol{\beta}\cdot\boldsymbol{E}$$
$$-\gamma\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{\mathfrak{m}}_{(a)}\boldsymbol{\beta}\cdot\boldsymbol{E}. \qquad (163)$$

(Time and space differentiations in the right-hand sides operate only on the fields, just as before.)

If the composite particle is momentarily at rest it is described by equations (162) and (163) with $\boldsymbol{\beta} = 0$. Then these equations reduce to:

$$m^* \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = e\boldsymbol{E} + (\nabla\boldsymbol{B})\cdot\boldsymbol{\mathfrak{m}} - c^{-1}\boldsymbol{\mathfrak{m}}_{(a)}\wedge\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}, \qquad (164)$$

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{\mathfrak{m}}\wedge\boldsymbol{B}. \qquad (165)$$

Equation (164) or (162) contains the 'magnetodynamic effect', i.e. $c^{-1}\boldsymbol{\mathfrak{m}}_{(a)} \wedge \mathrm{d}\boldsymbol{E}/\mathrm{d}t$ or alternatively $c^{-1}(\mathrm{d}/\mathrm{d}t)(\boldsymbol{\mathfrak{m}}_{(a)} \wedge \boldsymbol{E})$ (the difference between these two expressions is of second order in the fields). Thus this time derivative of the so-called 'hidden momentum' $c^{-1}\boldsymbol{\mathfrak{m}}_{(a)} \wedge \boldsymbol{E}$ contains only the *anomalous* magnetic moment.

e. *Composite particles in an arbitrarily varying electromagnetic field*

In subsection *c* we derived the equations of motion (125) and (126) with the expressions (106) and (107) for the total force and torque. By making a Taylor expansion of the fields and confining ourselves to terms with the charge and electromagnetic dipole moments we found the expressions (115) and (116) (or (128–129)) for the total force and torque. If the fields in which the composite particle moves change rapidly, the limitation to dipole terms in the Taylor expansion is no longer justified. We must then consider the complete Taylor expansion of the total force (106) around the centre of energy X^α

$$\mathfrak{f}^\alpha = c^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_i e_i(r_i\cdot\partial)^n F^{\alpha\beta}(X)\left(\frac{\mathrm{d}X_\beta}{\mathrm{d}s} + \frac{\mathrm{d}r_{i\beta}}{\mathrm{d}s}\right), \qquad (166)$$

where the relative positions r_i^α are defined by (108). This expression may be written in the form

$$\mathfrak{f}^\alpha = c^{-1}eF^{\alpha\beta}(X)u_\beta$$
$$+c^{-1}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_i e_i(r_i\cdot\partial)^{n-1}(r_i^\gamma u_\beta - r_{i\beta}u^\gamma)\partial_\gamma F^{\alpha\beta}(X)$$
$$+c^{-1}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}s}\{\sum_i e_i(r_i\cdot\partial)^{n-1}F^{\alpha\beta}(X)r_{i\beta}\}$$
$$+c^{-1}\sum_{n=1}^{\infty}\frac{n}{(n+1)!}\sum_i e_i(r_i\cdot\partial)^{n-1}\left(\frac{\mathrm{d}r_{i\beta}}{\mathrm{d}s}r_i^\gamma - \frac{\mathrm{d}r_i^\gamma}{\mathrm{d}s}r_{i\beta}\right)\partial_\gamma F^{\alpha\beta}(X), \qquad (167)$$

where $e = \sum_i e_i$ is the total charge of the composite particle. At the right-hand side one recognizes the covariant electric and magnetic multipole moments (9) with $p^\alpha/\sqrt{-p^2}$ as the time-like unit vector n^α. Therefore by using the homogeneous field equations (13) and omitting the subscript $(n) = (p/\sqrt{-p^2})$ of the covariant multipoles, one may write the total force on the composite particle as:

$$\mathfrak{f}^\alpha = c^{-1}eF^{\alpha\beta}u_\beta$$
$$+\tfrac{1}{2}\sum_{n=1}^{\infty}(c^{-1}\mu^{\alpha_1\ldots\alpha_n}u^{\alpha_{n+1}} - c^{-1}\mu^{\alpha_1\ldots\alpha_{n-1}\alpha_{n+1}}u^{\alpha_n} + \nu^{\alpha_1\ldots\alpha_{n+1}})\partial^\alpha\partial_{\alpha_1}\ldots\partial_{\alpha_{n-1}}F_{\alpha_n,\alpha_{n+1}}$$
$$+c^{-1}\sum_{n=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}s}(\mu^{\alpha_1\ldots\alpha_n}\partial_{\alpha_1}\ldots\partial_{\alpha_{n-1}}F^\alpha_{.\alpha_n}) \qquad (168)$$

(with $\partial_{\alpha_1} \dots \partial_{\alpha_{n-1}} \equiv 1$ for $n = 1$). This result is the generalization of (111) to the case of arbitrarily varying fields, where all multipoles are needed.

The multipoles employed here were defined with respect to the normal unit vector $p^\alpha/\sqrt{-p^2}$. However, since quadratic field terms are neglected throughout again, one might as well use the multipole moments defined with respect to the normal unit vector $c^{-1}u^\alpha$, as follows from (124). The expression (168) may be written in compact form by introducing the abbreviation (11):

$$\mathfrak{m}^{\alpha_1\dots\alpha_{n+1}} \equiv c^{-1}(\mu^{\alpha_1\dots\alpha_n}u^{\alpha_{n+1}} - \mu^{\alpha_1\dots\alpha_{n-1}\alpha_{n+1}}u^{\alpha_n}) + \nu^{\alpha_1\dots\alpha_{n+1}}. \tag{169}$$

The electric multipole moment occurring in the last term of (168) may be expressed in terms of this quantity. If only terms of zero order in the fields are considered, one has

$$\mathfrak{m}^{\alpha_1\dots\alpha_{n+1}}u_{\alpha_{n+1}} = -c\mu^{\alpha_1\dots\alpha_n}, \tag{170}$$

as follows from (9) with (109), (121) and (124). The expression (168) becomes in this way

$$\mathfrak{f}^\alpha = c^{-1}eF^{\alpha\beta}u_\beta + \tfrac{1}{2}\sum_{n=1}^{\infty}(\partial^\alpha_{.\alpha_1\dots\alpha_{n-1}}F_{\alpha_n,\alpha_{n+1}})\mathfrak{m}^{\alpha_1\dots\alpha_{n+1}} - c^{-2}\sum_{n=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}s}\{(\partial_{\alpha_1\dots\alpha_{n-1}}F^\alpha_{.\alpha_n})\mathfrak{m}^{\alpha_1\dots\alpha_{n+1}}u_{\alpha_{n+1}}\} \tag{171}$$

(with $\partial_{\alpha_1\dots\alpha_n} \equiv \partial_{\alpha_1} \dots \partial_{\alpha_n}$), which is the generalization of (128) to all multipole orders.

In a similar way one may find an expression for the total torque (107), starting from its Taylor expansion

$$\mathfrak{d}^{\alpha\beta} = c^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_i e_i(r_i\cdot\partial)^n(r_i^\alpha F^{\beta\gamma} - r_i^\beta F^{\alpha\gamma})\left(\frac{\mathrm{d}X_\gamma}{\mathrm{d}s} + \frac{\mathrm{d}r_{i\gamma}}{\mathrm{d}s}\right). \tag{172}$$

We find then, as a generalization of (112),

$$\begin{aligned}\mathfrak{d}^{\alpha\beta} = {} & c^{-1}\sum_{n=1}^{\infty} n\mu^{\alpha_1\dots\alpha_{n-1}\alpha}\partial_{\alpha_1\dots\alpha_{n-1}}F^{\beta\gamma}u_\gamma \\ & + c^{-1}\sum_{n=2}^{\infty}(n-1)\frac{\mathrm{d}}{\mathrm{d}s}(\mu^{\alpha_1\dots\alpha_{n-2}\alpha\gamma})\partial_{\alpha_1\dots\alpha_{n-2}}F^\beta_{.\gamma} + \sum_{n=1}^{\infty}\nu^{\alpha_1\dots\alpha_{n-1}\alpha\gamma}\partial_{\alpha_1\dots\alpha_{n-1}}F^\beta_{.\gamma} \\ & - \sum_{n=2}^{\infty}(n-1)\nu^{\alpha_1\dots\alpha_{n-2}\alpha\gamma\alpha_{n-1}}\partial_{\alpha_1\dots\alpha_{n-1}}F^\beta_{.\gamma} - (\alpha, \beta).\end{aligned} \tag{173}$$

The symbol (α, β) indicates terms of the same structure as written down explicitly, but with the indices α and β interchanged. Instead of the multipole

moments defined with respect to $p^\alpha/\sqrt{-p^2}$, we may use those defined with respect to $c^{-1}u^\alpha$. If one employs (169–170) and the homogeneous field equations one gets, limiting oneself to terms linear in the fields:

$$\begin{aligned}\mathfrak{d}^{\alpha\beta} = {} & \sum_{n=1}^{\infty}(\partial_{\alpha_1\dots\alpha_{n-1}}F^\alpha_{.\alpha_n})\mathfrak{m}^{\alpha_1\dots\alpha_{n+1}}\Delta^\beta_{\alpha_{n+1}} - \sum_{n=2}^{\infty}(n-1)(\partial_{\alpha_1\dots\alpha_{n-1}}F^\alpha_{.\alpha_n})\mathfrak{m}^{\beta\alpha_1\dots\alpha_n} \\ & + c^{-2}\sum_{n=2}^{\infty}(n-1)\frac{\mathrm{d}}{\mathrm{d}s}\{(\partial_{\alpha_1\dots\alpha_{n-2}}F^\alpha_{.\alpha_{n-1}})\mathfrak{m}^{\beta\alpha_1\dots\alpha_n}u_{\alpha_n}\} - (\alpha, \beta),\end{aligned} \tag{174}$$

which is the generalization of (129) to all multipole orders.

The equations of motion (125) and (126) are now specified by the expressions (168) and (173), or (171) and (174), for the total force and torque exerted on a composite particle in an arbitrarily varying external field.

f. *A set of composite particles in a field*

In this subsection the equations governing the behaviour of a system of composite particles in an external field will be derived. The composite particles will be labelled by an index k, their constituent particles by the double index ki.

It is convenient to consider the law of mass conservation before studying the equation of motion. This conserved mass will be the *rest* mass of the composite particle, not the quantity m (120), since the latter is not conserved in general. (These two quantities are indeed different since m includes contributions from the intra-atomic field, as follows from its definition (120) with (65) and (62).) The rest mass flow density of the system of composite particles is defined as (cf. the analogous definition for the electric four-current density (5))

$$c\sum_k m_k^{\text{rest}}\int u_k^\alpha\delta^{(4)}(X_k - R)\mathrm{d}s_k, \tag{175}$$

where the rest mass of composite particle k is $m_k^{\text{rest}} \equiv \sum_i m_{ki}$. It obeys the conservation law

$$\partial_\alpha\left\{c\sum_k m_k^{\text{rest}}\int u_k^\alpha\delta^{(4)}(X_k - R)\mathrm{d}s_k\right\} = 0. \tag{176}$$

The equations of motion for particle k have the same form as (125) and (126). From these equations one may obtain local balance equations of energy–momentum and angular momentum by multiplying them with the four-dimensional delta function $\delta^{(4)}\{X_k(s_k) - R\}$, integrating over s_k and summing over k. Then one obtains, after a partial integration

$$c\partial_\beta \left\{ \sum_k \int m_k u_k^\alpha u_k^\beta \delta^{(4)}(X_k - R)\mathrm{d}s_k \right\} = c \sum_k \int \mathfrak{f}_k^\alpha \delta^{(4)}(X_k - R)\mathrm{d}s_k$$

$$- c^{-1}\partial_\beta \left\{ \sum_k \int (s_k^{\alpha\gamma}\mathfrak{f}_{k\gamma}/m_k + \mathfrak{d}_k^{\alpha\gamma}u_{k\gamma})u_k^\beta \delta^{(4)}(X_k - R)\mathrm{d}s_k \right\}, \quad (177)$$

$$c\partial_\gamma \left\{ \sum_k \int s_k^{\alpha\beta} u_k^\gamma \delta^{(4)}(X_k - R)\mathrm{d}s_k \right\}$$

$$= c \sum_k \int \{\Delta_{k\gamma}^\alpha \Delta_{k\varepsilon}^\beta \mathfrak{d}_k^{\gamma\varepsilon} + c^{-2}(s_k^{\alpha\gamma}u_k^\beta - s_k^{\beta\gamma}u_k^\alpha)\mathfrak{f}_{k\gamma}/m_k\}\delta^{(4)}(X_k - R)\mathrm{d}s_k. \quad (178)$$

The total force and torque $\mathfrak{f}_k^\alpha$ and $\mathfrak{d}_k^{\alpha\beta}$ have the forms (106) and (107). The field occurring in these expressions is now the combined field of the other particles l ($\neq k$) and the external field:

$$f^{\alpha\beta} + F_\mathrm{e}^{\alpha\beta} = \sum_{l(\neq k)} f_l^{\alpha\beta} + F_\mathrm{e}^{\alpha\beta}, \quad (179)$$

so that one has

$$\mathfrak{f}_k^\alpha = c^{-1} \sum_i e_{ki} F_\mathrm{e}^{\alpha\beta}(R_{ki})u_{ki\beta} + c^{-1} \sum_i \sum_{l(\neq k)} e_{ki} f_l^{\alpha\beta}(R_{ki})u_{ki\beta}, \quad (180)$$

$$\mathfrak{d}_k^{\alpha\beta} = c^{-1} \sum_i e_{ki} r_{ki}^\alpha F_\mathrm{e}^{\beta\gamma}(R_{ki})u_{ki\gamma} + c^{-1} \sum_i \sum_{l(\neq k)} e_{ki} r_{ki}^\alpha f_l^{\beta\gamma}(R_{ki})u_{ki\gamma} - (\alpha, \beta), \quad (181)$$

where (108) and the notation u_{ki}^α for $\mathrm{d}R_{ki}^\alpha/\mathrm{d}s_k$ have been employed (since s_k is the proper time of the central point of k and not of the particle ki one should note that u_{ki}^α is *not* the four-velocity of particle ki.)

The external field $F_\mathrm{e}^{\alpha\beta}$ changes slowly over the dimension of a particle, whereas the fields $f_l^{\alpha\beta}$ may change rapidly at the position of particle k. Therefore we may expand the terms with the external fields in (180) and (181) and retain only the charge and electric and magnetic dipole moments, just as in subsection c. We then obtain as the external field contributions $\mathfrak{f}_{k\mathrm{e}}^\alpha$ and $\mathfrak{d}_{k\mathrm{e}}^{\alpha\beta}$ to the total force and torque (cf. (128–129)):

$$\mathfrak{f}_{k\mathrm{e}}^\alpha = c^{-1} e_k F_\mathrm{e}^{\alpha\beta}(X_k)u_{k\beta} + \tfrac{1}{2}\{\partial^\alpha F_\mathrm{e}^{\beta\gamma}(X_k)\}\mathfrak{m}_{k\beta\gamma} - c^{-2} \frac{\mathrm{d}}{\mathrm{d}s_k} \{F_\mathrm{e}^{\alpha\beta}(X_k)\mathfrak{m}_{k\beta\gamma} u_k^\gamma\}, \quad (182)$$

$$\mathfrak{d}_{k\mathrm{e}}^{\alpha\beta} = F_\mathrm{e}^{\alpha\gamma}(X_k)\mathfrak{m}_{k\gamma}^{\cdot\varepsilon} \Delta_{k\varepsilon}^\beta - (\alpha, \beta), \quad (183)$$

where $\mathfrak{m}_k^{\alpha\beta}$ was given in terms of the dipole moments μ_k^α and $\nu_k^{\alpha\beta}$ by (11). (These dipole moments may be defined with respect to the unit vector $p^\alpha/\sqrt{-p^2}$ or $c^{-1}u^\alpha$; the difference between these two cases consist in terms quadratic in the fields.)

We now turn to the contributions in (180) and (181) with the fields $f_l^{\alpha\beta}$ due to the other composite particles. These fields have been found in the preceding chapter. They read for particle l:

$$f_l^{\alpha\beta}(R) = \int \hat{f}_l^{\alpha\beta}(R)\mathrm{d}s_l = \int \{\hat{f}_{+l}^{\alpha\beta}(R) + \hat{f}_{-l}^{\alpha\beta}(R)\}\mathrm{d}s_l, \quad (184)$$

where the partial 'plus field' is

$$\hat{f}_{+l}^{\alpha\beta}(R) \equiv - \sum_j \frac{e_{lj}}{4\pi}(u_{lj}^\alpha \partial^\beta - u_{lj}^\beta \partial^\alpha)\delta\{(R - R_{lj})^2\} \quad (185)$$

and where the partial 'minus field' $\hat{f}_{-l}^{\alpha\beta}$ has the same form except for an extra factor $\varepsilon(R - R_{lj})$. Since we chose the proper time s_l as the parameter along the world line lj, the four-vector u_{lj}^α is equal to $\mathrm{d}R_{lj}^\alpha/\mathrm{d}s_l$ (not the four-velocity of particle lj, as explained below formula (181).)

If the observer's point R^α is sufficiently far away from the sources we may make a multipole expansion of the field (184). One then obtains for the fields (v. (14–19))

$$f_l^{\alpha\beta} = f_{l(\mathrm{e})}^{\alpha\beta} + f_{l(\mathrm{m})}^{\alpha\beta}, \quad (186)$$

where

$$f_{l(\mathrm{e})}^{\alpha\beta} = \int (\hat{f}_{+l(\mathrm{e})}^{\alpha\beta} + \hat{f}_{-l(\mathrm{e})}^{\alpha\beta})\mathrm{d}s_l, \quad (187)$$

$$f_{l(\mathrm{m})}^{\alpha\beta} = \int (\hat{f}_{+l(\mathrm{m})}^{\alpha\beta} + \hat{f}_{-l(\mathrm{m})}^{\alpha\beta})\mathrm{d}s_l. \quad (188)$$

The partial plus fields are here

$$\hat{f}_{+l(e)}^{\alpha\beta} = - \frac{e_l}{4\pi}(u_l^\alpha \partial^\beta - u_l^\beta \partial^\alpha)\delta\{(R - X_l)^2\}, \quad (189)$$

$$\hat{f}_{+l(\mathrm{m})}^{\alpha\beta} = - \sum_{n=1}^{\infty} \frac{(-1)^n c}{4\pi} \Big\{ \mathfrak{m}_l^{\alpha_1 \ldots \alpha_n \alpha} \partial^\beta_{\cdot\alpha_1 \ldots \alpha_n}$$

$$- c^{-2} \left(\frac{\mathrm{d}}{\mathrm{d}s_l} - u_l \cdot \partial \right) \mathfrak{m}_l^{\alpha_1 \ldots \alpha_n \alpha} u_{l\alpha_n} \partial^\beta_{\cdot\alpha_1 \ldots \alpha_{n-1}} \Big\} \delta\{(R - X_l)^2\} - (\alpha, \beta), \quad (190)$$

where (170) has been used and where the differential operator $\mathrm{d}/\mathrm{d}s_l$ acts on $\mathfrak{m}_l$ and u_l. Similar expressions (i.e. with an additional factor $\varepsilon(R - X_l)$) may be written for the partial minus fields.

The contributions of the interatomic fields to the total force and torque (180) and (181) acting on composite particle k is specified if one inserts the interatomic fields (184) with (185). If the atoms are sufficiently far apart one may perform a double multipole expansion, i.e. an expansion of the form (171) and (174) and moreover the expansion (186) with (187–190) of the interatomic field in terms of multipoles of its sources. If the atoms are near each other such a double expansion is not justified. Therefore we write in the general case the force and torque as sums of long range and short range contributions indicated by L and S:

$$\mathfrak{f}_k^\alpha = \mathfrak{f}_k^{\mathrm{L}\alpha} + \mathfrak{f}_k^{\mathrm{S}\alpha}, \tag{191}$$

$$\mathfrak{d}_k^{\alpha\beta} = \mathfrak{d}_k^{\mathrm{L}\alpha\beta} + \mathfrak{d}_k^{\mathrm{S}\alpha\beta}. \tag{192}$$

Here the long range contributions are

$$\mathfrak{f}_k^{\mathrm{L}\alpha} = \mathfrak{f}_{k\mathrm{e}}^\alpha + \mathfrak{f}_{k(ee)}^{\mathrm{L}\alpha} + \mathfrak{f}_{k(e\mathfrak{m})}^{\mathrm{L}\alpha} + \mathfrak{f}_{k(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha}, \tag{193}$$

$$\mathfrak{d}_k^{\mathrm{L}\alpha\beta} = \mathfrak{d}_{k\mathrm{e}}^{\alpha\beta} + \mathfrak{d}_{k(ee)}^{\mathrm{L}\alpha\beta} + \mathfrak{d}_{k(e\mathfrak{m})}^{\mathrm{L}\alpha\beta} + \mathfrak{d}_{k(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha\beta}. \tag{194}$$

The external field terms $\mathfrak{f}_{k\mathrm{e}}^\alpha$ and $\mathfrak{d}_{k\mathrm{e}}^\alpha$ were given already in (182) and (183). The next terms contain the contributions from the charges of the various composite particles. They get the form

$$\mathfrak{f}_{k(ee)}^{\mathrm{L}\alpha} = \sum_{l(\neq k)} \int \hat{\mathfrak{f}}_{k;l(ee)}^{\mathrm{L}\alpha}\, \mathrm{d}s_l, \tag{195}$$

$$\mathfrak{d}_{k(ee)}^{\mathrm{L}\alpha\beta} = 0, \tag{196}$$

with

$$\hat{\mathfrak{f}}_{k;l(ee)}^{\mathrm{L}\alpha} = c^{-1} e_k \hat{f}_{l(e)}^{\alpha\beta}(X_k) u_{k\beta}. \tag{197}$$

The plus field contribution in

$$\hat{\mathfrak{f}}_{k;l(ee)}^{\mathrm{L}\alpha} = \hat{\mathfrak{f}}_{+k;l(ee)}^{\mathrm{L}\alpha} + \hat{\mathfrak{f}}_{-k;l(ee)}^{\mathrm{L}\alpha} \tag{198}$$

follows by inserting in (197) the partial plus field (189). One finds

$$\hat{\mathfrak{f}}_{+k;l(ee)}^{\mathrm{L}\alpha} = -\frac{c^{-1}}{4\pi} e_k e_l (u_l^\alpha \partial_k^\beta - u_l^\beta \partial_k^\alpha) u_{k\beta}\, \delta(X_{kl}^2), \tag{199}$$

where the abbreviations $X_{kl}^\alpha \equiv X_k^\alpha - X_l^\alpha$ and $\partial_{k\alpha} \equiv \partial/\partial X_k^\alpha$ have been employed.

In the same way one finds for the last terms in (193) and (194):

$$\mathfrak{f}_{k(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha} = \sum_{l(\neq k)} \int \hat{\mathfrak{f}}_{k;l(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha}\, \mathrm{d}s_l, \tag{200}$$

$$\mathfrak{d}_{k(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha\beta} = \sum_{l(\neq k)} \int \hat{\mathfrak{d}}_{k;l(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha\beta}\, \mathrm{d}s_l, \tag{201}$$

with – in view of (171) and (174) – the partial forces and torques:

$$\hat{\mathfrak{f}}_{k;l(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha} = \tfrac{1}{2} \sum_{n=1}^{\infty} \{\partial_{k.\alpha_1\ldots\alpha_{n-1}}^{\alpha} \hat{f}_{l(\mathfrak{m})\alpha_n,\alpha_{n+1}}(X_k)\} \mathfrak{m}_k^{\alpha_1\ldots\alpha_{n+1}} - c^{-2} \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s_k} [\{\partial_{k\alpha_1\ldots\alpha_{n-1}} \hat{f}_{l(\mathfrak{m}).\alpha_n}^{\alpha}(X_k)\} \mathfrak{m}_k^{\alpha_1\ldots\alpha_{n+1}} u_{k\alpha_{n+1}}], \tag{202}$$

$$\begin{aligned} \hat{\mathfrak{d}}_{k;l(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha\beta} = {} & \sum_{n=1}^{\infty} \{\partial_{k\alpha_1\ldots\alpha_{n-1}} \hat{f}_{l(\mathfrak{m}).\alpha_n}^{\alpha}(X_k)\} \mathfrak{m}_k^{\alpha_1\ldots\alpha_{n+1}} \Delta_{k\alpha_{n+1}}^{\beta} \\ & - \sum_{n=2}^{\infty} (n-1) \{\partial_{k\alpha_1\ldots\alpha_{n-1}} \hat{f}_{l(\mathfrak{m}).\alpha_n}^{\alpha}(X_k)\} \mathfrak{m}_k^{\beta\alpha_1\ldots\alpha_n} \\ & + c^{-2} \sum_{n=2}^{\infty} (n-1) \frac{\mathrm{d}}{\mathrm{d}s_k} [\{\partial_{k\alpha_1\ldots\alpha_{n-2}} \hat{f}_{l(\mathfrak{m}).\alpha_{n-1}}^{\alpha}(X_k)\} \mathfrak{m}_k^{\beta\alpha_1\ldots\alpha_n} u_{k\alpha_n}] - (\alpha, \beta) \end{aligned} \tag{203}$$

(with $\partial_{k\alpha_1\ldots\alpha_{n-1}} \equiv 1$ for $n = 1$). The plus field contributions in particular follow by inserting (190):

$$\begin{aligned} \hat{\mathfrak{f}}_{+k;l(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha} = {} & \frac{c}{4\pi} \sum_{n,m=1}^{\infty} (-1)^m \left\{\mathfrak{m}_k^{\alpha_1\ldots\alpha_{n+1}} \partial_{k\alpha_n} + c^{-2} \left(\frac{\mathrm{d}}{\mathrm{d}s_k} + u_k \cdot \partial_k\right) \mathfrak{m}_k^{\alpha_1\ldots\alpha_{n+1}} u_{k\alpha_n}\right\} \\ & \left\{\mathfrak{m}_l^{\beta_1\ldots\beta_{m+1}} \partial_{k\beta_m} - c^{-2} \left(\frac{\mathrm{d}}{\mathrm{d}s_l} + u_l \cdot \partial_l\right) \mathfrak{m}_l^{\beta_1\ldots\beta_{m+1}} u_{l\beta_m}\right\} \\ & \partial_{k\alpha_1\ldots\alpha_{n-1}\beta_1\ldots\beta_{m-1}} (\partial_k^\alpha g_{\alpha_{n+1}\beta_{m+1}} - \partial_{k\alpha_{n+1}} g_{\beta_{m+1}}^{\alpha}) \delta(X_{kl}^2), \end{aligned} \tag{204}$$

$$\begin{aligned} \hat{\mathfrak{d}}_{+k;l(\mathfrak{m}\mathfrak{m})}^{\mathrm{L}\alpha\beta} = {} & \frac{c}{4\pi} \sum_{n,m=1}^{\infty} (-1)^{m+1} \left\{\Delta_{k\alpha_{n+1}}^{\alpha} \mathfrak{m}_k^{\alpha_1\ldots\alpha_{n+1}} \partial_{k\alpha_{n-1}} - (n-1) \mathfrak{m}_k^{\alpha\alpha_1\ldots\alpha_n} \partial_{k\alpha_{n-1}} \right. \\ & \left. - c^{-2}(n-1) \left(\frac{\mathrm{d}}{\mathrm{d}s_k} + u_k \cdot \partial_k\right) \mathfrak{m}_k^{\alpha\alpha_1\ldots\alpha_n} u_{k\alpha_{n-1}}\right\} \\ & \left\{\mathfrak{m}_l^{\beta_1\ldots\beta_{m+1}} \partial_{k\beta_m} - c^{-2} \left(\frac{\mathrm{d}}{\mathrm{d}s_l} + u_l \cdot \partial_l\right) \mathfrak{m}_l^{\beta_1\ldots\beta_{m+1}} u_{l\beta_m}\right\} \\ & \partial_{k\alpha_1\ldots\alpha_{n-2}\beta_1\ldots\beta_{m-1}} (\partial_k^\beta g_{\alpha_n\beta_{m+1}} - g_{\beta_{m+1}}^{\beta} \partial_{k\alpha_n}) \delta(X_{kl}^2) - (\alpha, \beta). \end{aligned} \tag{205}$$

The cross-terms $(e\mathfrak{m})$ have similar form.

The short range terms in (191) and (192) follow from (180) and (181). One finds:

$$\mathfrak{f}_k^{\mathrm{S}\alpha} = \sum_{l(\neq k)} \int \hat{\mathfrak{f}}_{k;l}^{\mathrm{S}\alpha}\, \mathrm{d}s_l, \tag{206}$$

$$\mathfrak{d}_k^{\mathrm{S}\alpha\beta} = \sum_{l(\neq k)} \int \hat{\mathfrak{d}}_{k;l}^{\mathrm{S}\alpha\beta} \mathrm{d}s_l, \tag{207}$$

with

$$\hat{\mathfrak{f}}^{\mathrm{S}\alpha}_{k;l} = c^{-1} \sum_i e_{ki}\, \hat{f}_l^{\alpha\beta}(R_{ki}) u_{ki\beta} - \hat{\mathfrak{f}}^{\mathrm{L}\alpha}_{k;l(ee)} - \mathfrak{f}^{\mathrm{L}\alpha}_{k;l(em)} - \hat{\mathfrak{f}}^{\mathrm{L}\alpha}_{k;l(mm)}, \tag{208}$$

$$\hat{\mathfrak{d}}^{\mathrm{S}\alpha\beta}_{k;l} = c^{-1} \sum_i e_{ki} \{ r^{\alpha}_{ki}\, \hat{f}_l^{\beta\gamma}(R_{ki}) - r^{\beta}_{ki}\, \hat{f}_l^{\alpha\gamma}(R_{ki}) \} u_{ki\gamma} - \hat{\mathfrak{d}}^{\mathrm{L}\alpha\beta}_{k;l(em)} - \hat{\mathfrak{d}}^{\mathrm{L}\alpha\beta}_{k;l(mm)}. \tag{209}$$

Explicitly the plus field parts become with (184) and (185):

$$\mathfrak{f}^{\mathrm{S}\alpha}_{+k;l} = \frac{c^{-1}}{4\pi} \sum_{i,j} e_{ki} e_{lj} (u_{ki}\cdot u_{lj}\, \partial^{\alpha}_{ki} - u^{\alpha}_{lj}\, u_{ki}\cdot\partial_{ki}) \delta(R^2_{ki,lj})$$
$$- \hat{\mathfrak{f}}^{\mathrm{L}\alpha}_{+k;l(ee)} - \mathfrak{f}^{\mathrm{L}\alpha}_{+k;l(em)} - \mathfrak{f}^{\mathrm{L}\alpha}_{+k;l(mm)}, \tag{210}$$

$$\hat{\mathfrak{d}}^{\mathrm{S}\alpha\beta}_{+k;l} = \frac{c^{-1}}{4\pi} \sum_{i,j} e_{ki} e_{lj}\, r^{\alpha}_{ki} (u_{ki}\cdot u_{lj}\, \partial^{\beta}_{ki} - u^{\beta}_{lj}\, u_{ki}\cdot\partial_{ki}) \delta(R^2_{ki,lj}) - (\alpha, \beta)$$
$$- \hat{\mathfrak{d}}^{\mathrm{L}\alpha\beta}_{+k;l(em)} - \hat{\mathfrak{d}}^{\mathrm{L}\alpha\beta}_{+k;l(mm)}. \tag{211}$$

In this way expressions have been found for the force and torque that occur in the balance equations of energy–momentum and angular momentum (177) and (178) for a system of composite particles in an external field. They will form the starting point for the statistical considerations of the next chapter.

APPENDIX I

Some properties of the tensor $\boldsymbol{\Omega}$

In this appendix we collect a number of properties of the tensor $\boldsymbol{\Omega}(\boldsymbol{\beta})$, which is defined as

$$\boldsymbol{\Omega}(\boldsymbol{\beta}) = \mathbf{U} + \frac{\gamma^{-1}-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta} = \mathbf{U} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}\boldsymbol{\beta}, \tag{A1}$$

where $\gamma = (1-\beta^2)^{-\frac{1}{2}}$ and $\mathbf{U}$ is the unit tensor. Its frequent use in this chapter stems from the fact that it describes the effect of a Lorentz contraction. In fact, if $\boldsymbol{a}$ is an arbitrary vector one has from (A1):

$$\boldsymbol{\Omega}\cdot\boldsymbol{a} = \boldsymbol{a}_{\perp} + \sqrt{1-\beta^2}\,\boldsymbol{a}_{//}, \tag{A2}$$

where $\boldsymbol{a}_{//} = \boldsymbol{\beta\beta}\cdot\boldsymbol{a}/\beta^2$ and $\boldsymbol{a}_{\perp} = \boldsymbol{a} - \boldsymbol{a}_{//}$ are the components of $\boldsymbol{a}$ orthogonal and parallel to $\boldsymbol{\beta}$ respectively. The longitudinal component of the vector $\boldsymbol{a}$ is thus seen to be subjected to a Lorentz contraction.

The inverse tensor $\boldsymbol{\Omega}^{-1}(\boldsymbol{\beta})$, which obeys the relation that its product with $\boldsymbol{\Omega}(\boldsymbol{\beta})$ is the unit tensor, is:

$$\boldsymbol{\Omega}^{-1}(\boldsymbol{\beta}) = \mathbf{U} + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta} = \mathbf{U} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\boldsymbol{\beta}, \tag{A3}$$

as may be checked directly. The tensor $\boldsymbol{\Omega}^{-1}$ occurs in the formulae for the Lorentz transformation (with transformation velocity $c\boldsymbol{\beta}$) connecting the frames $(ct_1, \boldsymbol{R}_1)$ and $(ct_2, \boldsymbol{R}_2)$:

$$\begin{aligned} ct_2 &= \gamma c t_1 + \gamma\boldsymbol{\beta}\cdot\boldsymbol{R}_1, \\ \boldsymbol{R}_2 &= \boldsymbol{\Omega}^{-1}\cdot\boldsymbol{R}_1 + \gamma\boldsymbol{\beta} c t_1. \end{aligned} \tag{A4}$$

On the contrary the transformation formulae for an antisymmetric tensor $\mathbf{A}$ with components $\boldsymbol{X} = (A^{01}, A^{02}, A^{03})$ and $\boldsymbol{Y} = (A^{23}, A^{31}, A^{12})$ contain the tensor $\boldsymbol{\Omega}$. They read

$$\begin{aligned} \boldsymbol{X}_2 &= \gamma(\boldsymbol{\Omega}\cdot\boldsymbol{X}_1 - \boldsymbol{\beta}\wedge\boldsymbol{Y}_1), \\ \boldsymbol{Y}_2 &= \gamma(\boldsymbol{\Omega}\cdot\boldsymbol{Y}_1 + \boldsymbol{\beta}\wedge\boldsymbol{X}_1). \end{aligned} \tag{A5}$$

From (A1) and (A3) one derives the properties

$$\boldsymbol{\Omega}^{-1} = \boldsymbol{\Omega}+\gamma\boldsymbol{\beta}\boldsymbol{\beta}, \tag{A6}$$

$$\boldsymbol{\Omega}\cdot\boldsymbol{\beta} = \gamma^{-1}\boldsymbol{\beta}, \tag{A7}$$

$$\boldsymbol{\Omega}^{-1}\cdot\boldsymbol{\beta} = \gamma\boldsymbol{\beta}, \tag{A8}$$

$$\boldsymbol{\beta}\wedge\boldsymbol{\Omega}\cdot\boldsymbol{a} = \boldsymbol{\beta}\wedge\boldsymbol{a}, \tag{A9}$$

$$\boldsymbol{\Omega}\cdot(\boldsymbol{a}\wedge\boldsymbol{b}) = \gamma^{-1}(\boldsymbol{\Omega}^{-1}\cdot\boldsymbol{a})\wedge(\boldsymbol{\Omega}^{-1}\cdot\boldsymbol{b}), \tag{A10}$$

$$\boldsymbol{\Omega}^{-1}\cdot(\boldsymbol{a}\wedge\boldsymbol{b}) = \gamma(\boldsymbol{\Omega}\cdot\boldsymbol{a})\wedge(\boldsymbol{\Omega}\cdot\boldsymbol{b}), \tag{A11}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two arbitrary vectors. The squares of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{-1}$ have the forms

$$\boldsymbol{\Omega}^2 = \mathbf{U}-\boldsymbol{\beta}\boldsymbol{\beta}, \tag{A12}$$

$$\boldsymbol{\Omega}^{-2} = \mathbf{U}+\gamma^2\boldsymbol{\beta}\boldsymbol{\beta}. \tag{A13}$$

The tensor $\boldsymbol{\Omega}^2$ occurs in the relation

$$\boldsymbol{\Omega}^2\cdot\partial_0(\boldsymbol{\beta}\gamma) = \gamma\partial_0\boldsymbol{\beta}. \tag{A14}$$

It may be proved from (A12) if use is made of the identity

$$\boldsymbol{\beta}\cdot\partial_0\boldsymbol{\beta} = \gamma^{-3}\partial_0\gamma, \tag{A15}$$

which follows from the definition of γ. Finally the determinant values of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{-1}$ are

$$|\boldsymbol{\Omega}| = \gamma^{-1}, \qquad |\boldsymbol{\Omega}^{-1}| = \gamma, \tag{A16}$$

as follows from (A1) and (A3).

APPENDIX II

The connexion between the covariant and the atomic multipole moments

The covariant multipole moments, depending on a certain time-like unit vector and defined in (9), contain quantities in the observer's frame $(ct, \boldsymbol{R})$. They still depend on the velocities of the atoms. Therefore we introduced atomic multipole moments (27, 28), defined in momentary rest frames; they are thus independent of the atomic velocities. The connexion between these atomic multipole moments and the covariant multipole moments (with the four-velocity chosen as the time-like unit vector) will be obtained in this appendix by studying the Lorentz transformation between the observer's frame and the momentary rest frames.

The covariant multipole moments contain the internal parameters r^{α}_{ki} (3) that fulfil the relation (2) with the four-velocity $c^{-1}u^{\alpha}_k(s_k)$ chosen as the unit vector $n^{\alpha}_k(s_k)$ i.e.:

$$r_{ki\alpha}(s_k)u^{\alpha}_k(s_k) \equiv -r^0_{ki}(s_k)u^0_k(s_k)+\boldsymbol{r}_{ki}(s_k)\cdot\boldsymbol{u}_k(s_k) = 0. \tag{A17}$$

This covariant condition means that in the atomic rest frame (where $\boldsymbol{u}^{(0)}_k(s_k) = 0$ at the moment $t = t_0$, or correspondingly for $s_k = s_{k0}$) the vectors $\boldsymbol{r}^{(0)}_{ki}$ become purely spatial ($r^{(0)0}_{ki}$ then vanishes according to the condition) and thus constitute at that moment the atomic parameters, which we want to employ for the characterization of the atoms.

The frame in which the atom as a whole is momentarily at rest must have a velocity $\boldsymbol{v}$ equal to $(\mathrm{d}\boldsymbol{R}_k/\mathrm{d}t)_{t=t_0}$ at the moment $t = t_0$ with respect to the reference frame $(ct, \boldsymbol{R})$ of the observer. Time–space coordinates of the reference frame $(ct, \boldsymbol{R})$ and the atomic rest frame $(ct^{(0)}, \boldsymbol{R}^{(0)})$ are connected by the Lorentz transformation (A4):

$$\begin{aligned} ct &= \gamma ct^{(0)}+\gamma\boldsymbol{\beta}\cdot\boldsymbol{R}^{(0)},\\ \boldsymbol{R} &= \boldsymbol{\Omega}^{-1}\cdot\boldsymbol{R}^{(0)}+\gamma\boldsymbol{\beta}ct^{(0)}, \end{aligned} \tag{A18}$$

where the tensor (A3) has been used.

Since the atom suffers accelerations, at every moment t_0 one needs a different atomic rest frame. Every atomic frame is therefore only a *momentary* rest frame: only for $t = t_0$ does the atomic velocity $\mathrm{d}\boldsymbol{R}^{(0)}_k/\mathrm{d}t^{(0)}$ vanish. The transformation which connects the reference frame to the momentary atomic rest frame (in which the atom is at rest at time t_0) is therefore determined by the transformation velocity:

$$\boldsymbol{\beta}_k = \frac{\boldsymbol{v}_k}{c} = \frac{1}{c}\left(\frac{\mathrm{d}\boldsymbol{R}_k}{\mathrm{d}t}\right)_{t=t_0} = \left(\frac{\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s_k}{\mathrm{d}R_k^0/\mathrm{d}s_k}\right)_{s_k=s_{k0}}, \tag{A19}$$

with s_{k0} corresponding to t_0. For the atomic position four-vector R_k^α this Lorentz transformation reads

$$\begin{aligned} R_k^0(s_k) &= \gamma_k R_k^{(0)0}(s_k) + \gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{R}_k^{(0)}(s_k), \\ \boldsymbol{R}_k(s_k) &= \boldsymbol{\Omega}_k^{-1} \cdot \boldsymbol{R}_k^{(0)}(s_k) + \gamma_k \boldsymbol{\beta}_k R_k^{(0)0}(s_k). \end{aligned} \tag{A20}$$

Differentiation with respect to s gives

$$\begin{aligned} \frac{\mathrm{d}R_k^0}{\mathrm{d}s_k} &= \gamma_k \frac{\mathrm{d}R_k^{(0)0}}{\mathrm{d}s_k} + \gamma_k \boldsymbol{\beta}_k \cdot \frac{\mathrm{d}\boldsymbol{R}_k^{(0)}}{\mathrm{d}s_k}, \\ \frac{\mathrm{d}\boldsymbol{R}_k}{\mathrm{d}s_k} &= \boldsymbol{\Omega}_k^{-1} \cdot \frac{\mathrm{d}\boldsymbol{R}_k^{(0)}}{\mathrm{d}s_k} + \gamma_k \boldsymbol{\beta}_k \frac{\mathrm{d}R_k^{(0)0}}{\mathrm{d}s_k}. \end{aligned} \tag{A21}$$

Similarly for the internal quantities one has the Lorentz transformation

$$\begin{aligned} r_{ki}^0(s_k) &= \gamma_k r_{ki}^{(0)0}(s_k) + \gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}_{ki}^{(0)}(s_k), \\ \boldsymbol{r}_{ki}(s_k) &= \boldsymbol{\Omega}_k^{-1} \cdot \boldsymbol{r}_{ki}^{(0)}(s_k) + \gamma_k \boldsymbol{\beta}_k r_{ki}^{(0)0}(s_k), \end{aligned} \tag{A22}$$

and thus for the derivatives

$$\begin{aligned} \frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k} &= \gamma_k \frac{\mathrm{d}r_{ki}^{(0)0}}{\mathrm{d}s_k} + \gamma_k \boldsymbol{\beta}_k \cdot \frac{\mathrm{d}\boldsymbol{r}_{ki}^{(0)}}{\mathrm{d}s_k}, \\ \frac{\mathrm{d}\boldsymbol{r}_{ki}}{\mathrm{d}s_k} &= \boldsymbol{\Omega}_k^{-1} \cdot \frac{\mathrm{d}\boldsymbol{r}_{ki}^{(0)}}{\mathrm{d}s_k} + \gamma_k \boldsymbol{\beta}_k \frac{\mathrm{d}r_{ki}^{(0)0}}{\mathrm{d}s_k}. \end{aligned} \tag{A23}$$

In the momentary atomic rest frame the atomic velocity vanishes: $\mathrm{d}\boldsymbol{R}_k^{(0)}/\mathrm{d}s_k = 0$ for $s_k = s_{k0}$. Furthermore, as a consequence of the orthogonality relation (A17) the internal parameter r_{ki}^α becomes purely space-like: $r_{ki}^{(0)\alpha} = (0, \boldsymbol{r}_{ki}^{(0)})$ for $s_k = s_{k0}$. These relations can be formulated conveniently with the help of a coordinate frame in which the atom is at rest all the time. This frame, which will be called the *permanent atomic rest frame* (denoted by a prime) is a succession of Lorentz frames, not a Lorentz frame itself; it coincides at time t_0 with the momentary atomic rest frame (denoted by (0)). The permanent rest frame is connected with the reference frame by a Lorentz

transformation with velocity

$$\boldsymbol{\beta}_k(s_k) = \frac{\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s_k}{\mathrm{d}R_k^0/\mathrm{d}s_k}. \tag{A24}$$

With the help of $\boldsymbol{\beta}_k(s_k)$ we define $\boldsymbol{\Omega}_k(s_k)$ analogous to (A1) with $\gamma_k(s_k) = \{1-\beta_k^2(s_k)\}^{-\frac{1}{2}}$. In the permanent rest frame the atomic velocity vanishes identically: $(\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s_k)' = 0$ for all s_k; moreover $(r_{ki}^0)' = 0$ for all s_k. From (A24) it follows that

$$\begin{aligned} \frac{\mathrm{d}R_k^0}{\mathrm{d}s_k} &= c\gamma_k(s_k), \\ \frac{\mathrm{d}\boldsymbol{R}_k}{\mathrm{d}s_k} &= c\gamma_k(s_k)\boldsymbol{\beta}_k(s_k). \end{aligned} \tag{A25}$$

Furthermore, (A22) may be written with the help of quantities in the permanent rest frame:

$$\begin{aligned} r_{ki}^0(s_k) &= \gamma_k(s_k)\boldsymbol{\beta}_k(s_k) \cdot \boldsymbol{r}'_{ki}(s_k), \\ \boldsymbol{r}_{ki}(s_k) &= \boldsymbol{\Omega}_k^{-1}(s_k) \cdot \boldsymbol{r}'_{ki}(s_k). \end{aligned} \tag{A26}$$

Finally (A23) becomes

$$\begin{aligned} \frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k} &= \gamma_k(s_k)\left(\frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k}\right)' + \gamma_k(s_k)\boldsymbol{\beta}_k(s_k) \cdot \left(\frac{\mathrm{d}\boldsymbol{r}_{ki}}{\mathrm{d}s_k}\right)', \\ \frac{\mathrm{d}\boldsymbol{r}_{ki}}{\mathrm{d}s_k} &= \boldsymbol{\Omega}_k^{-1}(s_k) \cdot \left(\frac{\mathrm{d}\boldsymbol{r}_{ki}}{\mathrm{d}s_k}\right)' + \gamma_k(s_k)\boldsymbol{\beta}_k(s_k)\left(\frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k}\right)'. \end{aligned} \tag{A27}$$

Since internal quantities are to be defined in the atomic rest frame, but external quantities (atomic positions, velocities etc.) in the reference frame, one needs a few more consequences of the preceding transformation formulae. In the first place we want an expression for the second derivative of $\boldsymbol{R}_k^{(0)}(s_k)$ with respect to s_k. According to the Lorentz transformation one has

$$\left(\frac{\mathrm{d}^2\boldsymbol{R}_k}{\mathrm{d}s_k^2}\right)' = \boldsymbol{\Omega}_k^{-1}(s_k) \cdot \frac{\mathrm{d}^2\boldsymbol{R}_k}{\mathrm{d}s_k^2} - \gamma_k(s_k)\boldsymbol{\beta}_k(s_k)\frac{\mathrm{d}^2R_k^0}{\mathrm{d}s_k^2}. \tag{A28}$$

The second derivatives at the right-hand side follow from (A25):

$$\begin{aligned} \frac{\mathrm{d}^2\boldsymbol{R}_k}{\mathrm{d}s_k^2} &= c^2\gamma_k\partial_0(\gamma_k\boldsymbol{\beta}_k), \\ \frac{\mathrm{d}^2R_k^0}{\mathrm{d}s_k^2} &= c^2\gamma_k\partial_0\gamma_k. \end{aligned} \tag{A29}$$

Substituting these expressions and using (A6) and (A15) one finds:

$$\left(\frac{\mathrm{d}^2\boldsymbol{R}_k}{\mathrm{d}s_k^2}\right)' = c^2\gamma_k\boldsymbol{\Omega}_k\cdot\partial_0(\gamma_k\boldsymbol{\beta}_k). \tag{A30}$$

A second result can be obtained from the invariant condition (A17), which reads in the momentary atomic rest frame

$$r_{ki}^{(0)0}\frac{\mathrm{d}R_k^{(0)0}}{\mathrm{d}s_k} - \boldsymbol{r}_{ki}^{(0)}\cdot\frac{\mathrm{d}\boldsymbol{R}_k^{(0)}}{\mathrm{d}s_k} = 0. \tag{A31}$$

Differentiating this relation with respect to s and taking into account $r_{ki}^{0'} = 0$ and $(\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s_k)' = 0$, one finds in the permanent atomic rest frame

$$\left(\frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k}\right)' = c^{-1}\boldsymbol{r}'_{ki}(s_k)\cdot(\mathrm{d}^2\boldsymbol{R}_k/\mathrm{d}s_k^2)', \tag{A32}$$

which, with (A30), becomes finally

$$\left(\frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k}\right)' = c\gamma_k\boldsymbol{r}'_{ki}\cdot\boldsymbol{\Omega}_k\cdot\partial_0(\gamma_k\boldsymbol{\beta}_k). \tag{A33}$$

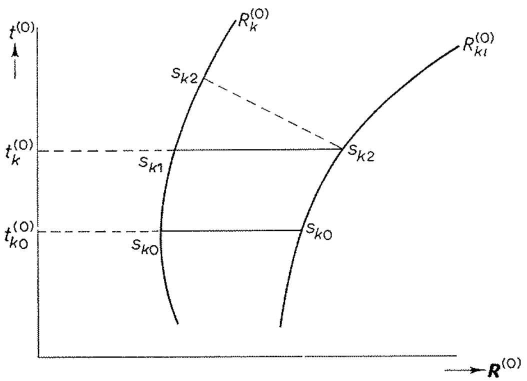

Fig. 1. World lines of atom k and constituent particle ki in the momentary atomic rest frame.

In the definition (28) of the magnetic multipole moments a time derivative of the internal coordinates occurs which is the limit of the difference of two purely spatial vectors divided by the corresponding time difference (cf. fig. 1):

$$\dot{\boldsymbol{r}}'_{ki} = \lim_{t_k{}^{(0)}\to t_{ko}{}^{(0)}}\frac{\boldsymbol{R}_{ki}^{(0)}(s_{k2})-\boldsymbol{R}_k^{(0)}(s_{k1})-\boldsymbol{r}_{ki}^{(0)}(s_{ko})}{t_k^{(0)}-t_{ko}^{(0)}}$$

$$= \lim_{s_{k1}\to s_{ko}}\frac{\boldsymbol{R}_{ki}^{(0)}(s_{k2})-\boldsymbol{R}_k^{(0)}(s_{k1})-\boldsymbol{r}_{ki}^{(0)}(s_{ko})}{s_{k1}-s_{ko}}. \tag{A34}$$

The values s_{k1} and s_{k2} of the parameter s_k are related by (cf. fig. 1)

$$R_{ki}^{(0)0}(s_{k2}) = R_k^{(0)0}(s_{k1}), \tag{A35}$$

or, with the splitting (3)

$$R_k^{(0)0}(s_{k2})+r_{ki}^{(0)0}(s_{k2}) = R_k^{(0)0}(s_{k1}). \tag{A36}$$

A Taylor expansion of the left- and right-hand sides with respect to $s_{k2}-s_{ko}$ and $s_{k1}-s_{ko}$ respectively gives

$$s_{k2}-s_{ko} = \left(1+c^{-1}\frac{\mathrm{d}r_{ki}^{(0)0}}{\mathrm{d}s_k}\right)^{-1}(s_{k1}-s_{ko})+\ldots. \tag{A37}$$

With the help of this relation, expression (A34) becomes after expansion of the numerator around s_{ko}:

$$\dot{\boldsymbol{r}}'_{ki} = \frac{\mathrm{d}\boldsymbol{r}_{ki}^{(0)}}{\mathrm{d}s_k}\left(1+c^{-1}\frac{\mathrm{d}r_{ki}^{(0)0}}{\mathrm{d}s_k}\right)^{-1}. \tag{A38}$$

This expression is derived for $s_k = s_{ko}$. Introducing quantities measured in the permanent rest frame we get a relation valid for all s_k:

$$\dot{\boldsymbol{r}}'_{ki} = \left(\frac{\mathrm{d}\boldsymbol{r}_{ki}}{\mathrm{d}s_k}\right)'\left\{1+c^{-1}\left(\frac{\mathrm{d}r_{ki}^0}{\mathrm{d}s_k}\right)'\right\}^{-1}. \tag{A39}$$

With the use of (A33) this can be written in the form

$$\left(\frac{\mathrm{d}\boldsymbol{r}_{ki}}{\mathrm{d}s_k}\right)' = \dot{\boldsymbol{r}}'_{ki}\{1+\gamma_k\boldsymbol{r}'_{ki}\cdot\boldsymbol{\Omega}_k\cdot\partial_0(\gamma_k\boldsymbol{\beta}_k)\}, \tag{A40}$$

which gives $(\mathrm{d}\boldsymbol{r}_{ki}/\mathrm{d}s_k)'$ in terms of the internal quantities $\boldsymbol{r}'_{ki}$ and $\dot{\boldsymbol{r}}'_{ki}$ occurring in the multipole moments which characterize the atomic structure.

With the help of the results that followed from the Lorentz transformation, we now write the connexion between the covariant multipole moments (9) with the choice $c^{-1}\mathrm{d}R_k^\gamma/\mathrm{d}s_k$ for n^α and the atomic multipole moments (27–28). To that end we substitute into (9) the transformation formulae (A25–A27), using also (A33) and (A40). Then one obtains (omitting the index $(n) = (c^{-1}\mathrm{d}R_k/\mathrm{d}s_k)$) for the covariant electric multipole moment

$$\mu_k^{0\ldots0i_{m+1}\ldots i_n} = \{(\boldsymbol{\Omega}_k^{-1})^{n-m}(\gamma_k\boldsymbol{\beta}_k)^m : \boldsymbol{\mu}_k^{(n)}\}^{i_{m+1}\ldots i_n},$$

$$(n = 1, 2, \ldots;\ m = 0, 1, \ldots, n), \tag{A41}$$

where the indices $\alpha_1, \ldots, \alpha_m$ have been chosen as zero and the indices $\alpha_{m+1}, \ldots, \alpha_n$ as $i_{m+1}\ldots i_n$, which can take the values 1, 2, 3. The symmetrical character of the covariant electric multipole moment could be used to write the zeros first. The covariant magnetic multipole moment has components

$$\begin{aligned}
\nu_k^{0\ldots 0 i_{m+1}\ldots i_{n-1} ij} = \Big\{ & \gamma_k(\boldsymbol{\Omega}_k^{-1})^{n-m-1}(\gamma_k\boldsymbol{\beta}_k)^m \vdots \boldsymbol{\nu}_k^{(n)}\cdot\boldsymbol{\Omega}_k \\
& + \frac{n(n+2)}{n+1}\gamma_k^2(\boldsymbol{\Omega}_k^{-1})^{n-m-1}(\gamma_k\boldsymbol{\beta}_k)^m\partial_0(\gamma_k\boldsymbol{\beta}_k)\cdot\boldsymbol{\Omega}_k \vdots \boldsymbol{\nu}_k^{(n+1)}\cdot\boldsymbol{\Omega}_k \\
& + n\gamma_k^2(\boldsymbol{\Omega}_k^{-1})^{n-m-1}(\gamma_k\boldsymbol{\beta}_k)^m\partial_0(\gamma_k\boldsymbol{\beta}_k)\cdot\boldsymbol{\Omega}_k \vdots \boldsymbol{\mu}_k^{(n+1)}\wedge\boldsymbol{\beta}_k\Big\}^{i_{m+1}\ldots i_{n-1}p},
\end{aligned}$$

$$(n = 1, 2, \ldots;\ m = 0, \ldots, n-1;\ i, j, p = 1, 2, 3 \text{ cycl.}), \quad \text{(A42)}$$

$$\begin{aligned}
\nu_k^{0\ldots 0 i_{m+1}\ldots i_{n-1} i0} = \Big\{ & -\gamma_k(\boldsymbol{\Omega}_k^{-1})^{n-m-1}(\gamma_k\boldsymbol{\beta}_k)^m \boldsymbol{\nu}_k^{(n)}\wedge\boldsymbol{\beta}_k \\
& - \frac{n(n+2)}{n+1}\gamma_k^2(\boldsymbol{\Omega}_k^{-1})^{n-m-1}(\gamma_k\boldsymbol{\beta}_k)^m\partial_0(\gamma_k\boldsymbol{\beta}_k)\cdot\boldsymbol{\Omega}_k \vdots \boldsymbol{\nu}_k^{(n+1)}\wedge\boldsymbol{\beta}_k \\
& + n\gamma_k^2(\boldsymbol{\Omega}_k^{-1})^{n-m-1}(\gamma_k\boldsymbol{\beta}_k)^m\partial_0(\gamma_k\boldsymbol{\beta}_k)\cdot\boldsymbol{\Omega}_k \vdots \boldsymbol{\mu}_k^{(n+1)}\cdot\boldsymbol{\Omega}_k\Big\}^{i_{m+1}\ldots i_{n-1}i},
\end{aligned}$$

$$(n = 1, 2, \ldots;\ m = 0, \ldots, n-1), \quad \text{(A43)}$$

where we used (A6–A10) and the fact that the covariant magnetic multipole moment is symmetric in its first $n-1$ indices and antisymmetric in its last two indices.

From these connexions it is apparent that in the absence of acceleration the purely space-like components of the covariant electromagnetic multipole moments (the case $m = 0$) in the rest frame coincide with the atomic electromagnetic moments; the mixed space-time components vanish in that frame. If accelerations are present this needs no longer be the case.

APPENDIX III

On equations of motion with explicit radiation damping

In the main text we carried out a programme to obtain equations of motion for a composite particle in an external field. The basis of these equations consisted in the microscopic balance equation (61) with (62–63). Essential steps were the definition of a centre of energy and the derivation of the equations of motion and spin in the linear field approximation.

Part of this programme – but indeed only part of it – may be accomplished as well on the basis of a different form, namely (III.A26), of the balance equation, derived in the appendix of the preceding chapter:

$$\partial_\beta t^{*\alpha\beta} = f^{*\alpha} \quad \text{(A44)}$$

with the (symmetric) energy–momentum tensor $t^{*\alpha\beta}$ (III.A25), which consists of a material part and a field part that accounts for the interactions due to the plus fields of the particles:

$$\begin{aligned}
t^{*\alpha\beta} = {} & c\sum_i m_i \int u_i^\alpha u_i^\beta \delta^{(4)}(R_i - R)\mathrm{d}s_i \\
& + \frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int (u_i^\alpha u_j^\beta + u_i^\beta u_j^\alpha)\delta\{(R_i-R_j)^2\}\delta^{(4)}(R_i-R)\mathrm{d}s_i\,\mathrm{d}s_j \\
& + \frac{1}{8\pi}\sum_{i,j(i\neq j)} e_i e_j \int\int_{\lambda=0}^{1} [2u_i\cdot u_j(R_i-R_j)^\alpha(R_i-R_j)^\beta\delta'\{(R_i-R_j)^2\} \\
& + \{u_i^\alpha(R_i-R_j)^\beta + u_i^\beta(R_i-R_j)^\alpha\}u_j^\gamma\partial_\gamma\delta\{(R_i-R_j)^2\}] \\
& \delta^{(4)}\{R_j+\lambda(R_i-R_j)-R\}\mathrm{d}s_i\,\mathrm{d}s_j\,\mathrm{d}\lambda. \quad \text{(A45)}
\end{aligned}$$

This tensor has the special property that it vanishes outside the domain enclosed by the world lines. The force density $f^{*\alpha}$ was found to be

$$f^{*\alpha} = f^\alpha + \sum_{i,j}\int f_{-j}^{\alpha\beta} u_{i\beta}\delta^{(4)}(R_i - R)\mathrm{d}s_i \quad \text{(A46)}$$

(v. (III.A3)). It contains, just as (63), the Lorentz forces due to the external field, and moreover those due to the minus fields, generated by the constituent particles j (including the particle i itself).

The first step in the derivation is the application of the general definition of the centre of energy as given in section 3*b* to the particular case of a composite particle described by the energy–momentum tensor $t^{*\alpha\beta}$ (A45) and acted upon by the force density $f^{*\alpha}$ (A46). To that end we have to check whether all assumptions used there are indeed justified for the present case.

To begin with, we notice that the tensor $t^{*\alpha\beta}$ is symmetric and that the force $f^{*\alpha}$ has finite support. Furthermore the integrals

$$p^{*\alpha} \equiv -c^{-1}\int t^{*\alpha\beta} n_\beta \, \mathrm{d}^3\Sigma \tag{A47}$$

for the total momentum (which is assumed to be time-like with positive time-component) and

$$s^{*\alpha\beta} \equiv -c^{-1}\int \{(R^\alpha - X^{*\alpha})t^{*\beta\gamma} - (R^\beta - X^{*\beta})t^{*\alpha\gamma}\} n_\gamma \, \mathrm{d}^3\Sigma \tag{A48}$$

for the inner angular momentum are both convergent *sensu stricto* because of the finiteness of the support of the tensor $t^{*\alpha\beta}$. The proof on the uniqueness of the centre of energy construction is now complicated by the difficulty that the force density $f^{*\alpha}$ (A46) cannot be made arbitrarily small by varying the external fields, due to the occurrence of the minus fields. In this case one has to *assume* that the centre of energy construction leads to a unique central point.

By making use of the balance equation (A44) one may derive by applying Gauss's theorem (in a straightforward manner since $t^{*\alpha\beta}$ has 'local character') the equations of motion and spin, which are analogous to (96) and (102):

$$\frac{\mathrm{d}p^{*\alpha}}{\mathrm{d}s^*} = \mathfrak{f}^{*\alpha}, \tag{A49}$$

$$\frac{\mathrm{d}s^{*\alpha\beta}}{\mathrm{d}s^*} = \mathfrak{d}^{*\alpha\beta} - (u^{*\alpha}p^{*\beta} - u^{*\beta}p^{*\alpha}). \tag{A50}$$

The total force and the total torque occurring here have the same form as (106) and (107) but with the sum of the external field and the total minus field instead of the external field *tout court*:

$$\mathfrak{f}^{*\alpha}(s^*) = c^{-1}\sum_i e_i[F^{\alpha\beta}\{R_i(s^*)\} + \sum_j f^{\alpha\beta}_{-j}\{R_i(s^*)\}]\frac{\mathrm{d}R_{i\beta}}{\mathrm{d}s^*}, \tag{A51}$$

$$\mathfrak{d}^{*\alpha\beta}(s^*) = c^{-1}\sum e_i\{R_i^\alpha(s_i^*) - X^{*\alpha}(s_i^*)\}[F^{\beta\gamma}\{R_i(s^*)\} + \sum_j f^{\beta\gamma}_{-j}\{R_i(s^*)\}]\frac{\mathrm{d}R_{i\gamma}}{\mathrm{d}s^*} - (\alpha, \beta). \tag{A52}$$

Since the minus fields are finite at the world lines of the particles (v. (III.111)) we may develop them into a multipole series just as the external field. If one retains the charge and dipole contributions one finds for the total force (and torque) quantities that are the sum of expressions $\mathfrak{f}^\alpha(s^*)$ (115) (and $\mathfrak{d}^{\alpha\beta}(s^*)$ (116)) with $X^{*\alpha}$, $u^{*\alpha}$, $\mathfrak{m}^{*\alpha\beta}$ and $p^{*\alpha}$ depending on s^* instead of X^α, u^α, $\mathfrak{m}^{\alpha\beta}$ and p^α depending on s, and minus field contributions of similar form:

$$\mathfrak{f}^{*\alpha}(s^*) = \mathfrak{f}^\alpha(s^*) + \mathfrak{f}^\alpha_-(s^*), \tag{A53}$$

$$\mathfrak{d}^{*\alpha\beta}(s^*) = \mathfrak{d}^{\alpha\beta}(s^*) + \mathfrak{d}^{\alpha\beta}_-(s^*), \tag{A54}$$

with

$$\mathfrak{f}^\alpha_-(s^*) = c^{-1}ef^{\alpha\beta}_-(X^*)u^*_\beta + \tfrac{1}{2}\{\partial^\alpha f^{\beta\gamma}_-(X^*)\}\mathfrak{m}^*_{\beta\gamma} + \frac{\mathrm{d}}{\mathrm{d}s^*}\left\{f^{\alpha\beta}_-(X^*)\mathfrak{m}^*_{\beta\gamma}\frac{p^{*\gamma}}{u^*_\varepsilon p^{*\varepsilon}}\right\}, \tag{A55}$$

$$\mathfrak{d}^{\alpha\beta}_-(s^*) = f^{\alpha\gamma}_-(X^*)\mathfrak{m}^{*\cdot\beta}_\gamma + u^{*\alpha}f^{\beta\gamma}_-(X^*)\mathfrak{m}^*_{\gamma\varepsilon}\frac{p^{*\varepsilon}}{u^*_\zeta p^{*\zeta}} - (\alpha, \beta), \tag{A56}$$

where $f^{\alpha\beta}_-$ is the total minus field $\sum_j f^{\alpha\beta}_{-j}$. This total minus field may be developed into a multipole series in terms of its sources by using the formulae (17) with (18–19). Retaining only the charges and dipoles, we then get (suppressing the asterisks for convenience from now on):

$$f^{\alpha\beta}_-(R) = -\frac{e}{4\pi}\int (u^\alpha\partial^\beta - u^\beta\partial^\alpha)\delta\{(R-X)^2\}\varepsilon(R-X)\mathrm{d}s + \frac{c}{4\pi}(\mathfrak{m}^{\gamma\alpha}\partial^\beta\partial_\gamma - \mathfrak{m}^{\gamma\beta}\partial^\alpha\partial_\gamma)\delta\{(R-X)^2\}\varepsilon(R-X)\mathrm{d}s. \tag{A57}$$

In order to evaluate (A55) and (A56) we have to calculate this minus field and its derivative at the position of the world line. The minus field due to the charge at the position of the world line has been found already in (III.111) of the preceding chapter:

$$f^{\alpha\beta}_{-(e)} = \frac{e}{6\pi}c^{-4}(u^\alpha\dot{a}^\beta - u^\beta\dot{a}^\alpha). \tag{A58}$$

where a^α is the four-acceleration $\mathrm{d}u^\alpha/\mathrm{d}s$. Its derivative at the position of the world line will be calculated in the next appendix. The result is (cf. (A77)):

$$\partial^\gamma f^{\alpha\beta}_{-(e)} = \frac{e}{12\pi}c^{-4}\{\ddot{a}^\alpha g^{\beta\gamma} + c^{-2}a\cdot\dot{a}g^{\alpha\gamma}u^\beta + c^{-2}a^2g^{\alpha\gamma}a^\beta - 2c^{-2}u^\alpha a^\beta\dot{a}^\gamma - 4c^{-2}u^\alpha\dot{a}^\beta a^\gamma + u^\gamma(3c^{-4}a^2u^\alpha a^\beta - 3c^{-2}u^\alpha\ddot{a}^\beta - 2c^{-2}a^\alpha\dot{a}^\beta)\} - (\alpha, \beta). \tag{A59}$$

It is possible to derive in a similar way expressions for the dipole minus field and its derivative. Since the results are rather lengthy[1] we shall give here only those terms which are independent of the first and higher derivatives of the velocity. These terms (which will be derived in the next appendix) read

$$f^{\alpha\beta}_{-(\mathrm{m})} = \frac{c^{-3}}{6\pi}\left(\overset{(3)}{\mathfrak{m}}{}^{\alpha\beta}+2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\alpha\gamma}u_\gamma u^\beta-2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\beta\gamma}u_\gamma u^\alpha\right), \tag{A60}$$

$$\partial^\gamma f^{\alpha\beta}_{-(\mathrm{m})} = -\frac{c^{-5}}{12\pi}\left(\overset{(4)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon g^{\beta\gamma}+\overset{(4)}{\mathfrak{m}}{}^{\alpha\gamma}u^\beta+\overset{(4)}{\mathfrak{m}}{}^{\alpha\beta}u^\gamma+6c^{-2}\overset{(4)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon u^\beta u^\gamma\right)-(\alpha,\beta), \tag{A61}$$

where numbers above the symbols indicate the number of differentiations with respect to the proper time. If the expressions (A58–A61) are introduced into (A55) and (A56), one obtains the total minus field contribution of the force and torque exerted on the composite particle. If again only terms independent of the first and higher derivatives of the velocity are written down, one gets

$$\begin{aligned}\mathfrak{f}^\alpha_- = {} & -\frac{ec^{-4}}{6\pi}\overset{(3)}{\mathfrak{m}}{}^{\alpha\beta}u_\beta+\frac{c^{-5}}{12\pi}\left(\overset{(4)}{\mathfrak{m}}{}^{\alpha\beta}\mathfrak{m}_{\beta\gamma}u^\gamma\right. \\ & \left.+\mathfrak{m}^{\alpha\beta}\overset{(4)}{\mathfrak{m}}{}_{\beta\gamma}u^\gamma-u^\alpha\overset{(4)}{\mathfrak{m}}{}^{\beta\gamma}\mathfrak{m}_{\beta\gamma}-6c^{-2}u^\alpha\overset{(4)}{\mathfrak{m}}{}^{\beta\gamma}u_\gamma u^\varepsilon\mathfrak{m}_{\beta\varepsilon}\right) \\ & +\frac{c^{-3}}{6\pi}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(\overset{(3)}{\mathfrak{m}}{}^{\alpha\beta}+2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon u^\beta-2c^{-2}u^\alpha\overset{(3)}{\mathfrak{m}}{}^{\beta\varepsilon}u_\varepsilon\right)\mathfrak{m}_{\beta\gamma}\frac{p^\gamma}{u_\zeta p^\zeta}\right\},\end{aligned} \tag{A62}$$

$$\begin{aligned}\mathfrak{d}^{\alpha\beta}_- = {} & \frac{c^{-3}}{6\pi}\left[\overset{(3)}{\mathfrak{m}}{}^{\alpha\gamma}\dot{\mathfrak{m}}_\gamma{}^\beta+2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon u^\gamma\dot{\mathfrak{m}}_\gamma{}^\beta\right. \\ & \left.+u^\alpha\left\{\left(\overset{(3)}{\mathfrak{m}}{}^{\beta\gamma}+2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\beta\varepsilon}u_\varepsilon u^\gamma-2c^{-2}u^\beta\overset{(3)}{\mathfrak{m}}{}^{\gamma\varepsilon}u_\varepsilon\right)\mathfrak{m}_{\gamma\zeta}\frac{p^\zeta}{u_\vartheta p^\vartheta}-2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\gamma\varepsilon}u_\varepsilon\dot{\mathfrak{m}}_\gamma{}^\beta\right\}\right] \\ & -(\alpha,\beta).\end{aligned} \tag{A63}$$

The equations of motion (A49) and (A50) are now completely specified if one substitutes (A53) and (A54) with (A62) and (A63). They contain explicitly terms that describe the radiation damping. Owing to this fact these equations do not reduce, for the field-free case, to such simple forms as derived in the main text. The problem of their solution requires the consideration of runaway solutions (just as in chapter III).

[1] S. Emid and J. Vlieger, Physica **52**(1971)329.

APPENDIX IV

The minus field of a charged dipole particle

In this appendix expressions will be derived for the minus field of a charged dipole particle at the position of the particle itself in a way similar to that of section 2e of the preceding chapter. We start from formulae (14–16) which give the general expressions for the retarded field. Combining it with the expression for the advanced field, one gets

$$\begin{aligned}f^{\alpha\beta}_{\mathrm{r,a}} \equiv f^{\alpha\beta}_{\mathrm{r,a}(e)}+f^{\alpha\beta}_{\mathrm{r,a}(\mathrm{m})} = {} & -\frac{e}{2\pi}\int(u^\alpha\partial^\beta-u^\beta\partial^\alpha)\delta\{(R-X)^2\}\theta\{\pm(R-X)\}\mathrm{d}s \\ & +\frac{c}{2\pi}\int(\mathfrak{m}^{\gamma\alpha}\partial^\beta\partial_\gamma-\mathfrak{m}^{\gamma\beta}\partial^\alpha\partial_\gamma)\delta\{(R-X)^2\}\theta\{\pm(R-X)\}\mathrm{d}s,\end{aligned} \tag{A64}$$

with e the charge and $\mathfrak{m}^{\alpha\beta}$ the covariant dipole moment (11) of the particle. Performing the integration, one finds

$$f^{\alpha\beta}_{\mathrm{r,a}(e)} = \mp\frac{e}{4\pi}\partial^\alpha\left\{\left(\frac{u^\beta}{u\cdot r}\right)\Big|_{\mathrm{r,a}}\right\}-(\alpha,\beta), \tag{A65}$$

$$f^{\alpha\beta}_{\mathrm{r,a}(\mathrm{m})} = \pm\frac{c}{4\pi}\partial^\alpha\partial_\gamma\left\{\left(\frac{\mathfrak{m}^{\gamma\beta}}{u\cdot r}\right)\Big|_{\mathrm{r,a}}\right\}-(\alpha,\beta), \tag{A66}$$

with $r^\alpha = R^\alpha - X^\alpha$, where the suffixes r and a at the bar indicate that one should take the retarded and advanced expressions. The differentiations with respect to R may be performed by making use of the equation (III.94) that follows from the light-cone equation.

The charge minus field that may be calculated from (A65) has been obtained already in section 2e of the preceding chapter. There we have found for the minus field at the world line (v. (III.111)):

$$f^{\alpha\beta}_{-(e)} = \frac{e}{6\pi}c^{-4}(u^\alpha\ddot{a}^\beta-u^\beta\ddot{a}^\alpha). \tag{A67}$$

We are also interested (in view of the equation of motion) in the space–time derivative of the minus field, i.e. in $\partial^\gamma f^{\alpha\beta}_{-(e)}$. By making use of the projection operator $\Delta^{\alpha\beta} \equiv g^{\alpha\beta}+c^{-2}u^\alpha u^\beta$, one may split this derivative into two parts:

$$\partial^\gamma f^{\alpha\beta}_{-(e)} = \Delta^\gamma_\varepsilon\partial^\varepsilon f^{\alpha\beta}_{-(e)}-c^{-2}u^\gamma u^\varepsilon\partial_\varepsilon f^{\alpha\beta}_{-(e)}, \tag{A68}$$

namely into a part which gives the derivative in a direction orthogonal to the four-velocity and a part that specifies the derivative in a direction parallel to u^α. The latter part may be calculated directly from (A67), since

$$u^\varepsilon \partial_\varepsilon f^{\alpha\beta}_{-(e)} = \frac{\mathrm{d}}{\mathrm{d}s} f^{\alpha\beta}_{-(e)} = \frac{e}{6\pi} c^{-4}(a^\alpha \dot{a}^\beta + u^\alpha \ddot{a}^\beta - a^\beta \dot{a}^\alpha - u^\beta \ddot{a}^\alpha). \qquad \text{(A69)}$$

To evaluate the former part one has to consider the minus field at a position $R^\alpha = X^\alpha(s_1) + \varepsilon n^\alpha$ (v. (III.101)) with fixed s_1 and space-like unit vector n^α orthogonal to $u^\alpha(s_1)$ and then take the derivative with respect to εn^α. To be able to perform this programme one has to push the series expansions in ε as given in (III.103–109) one step further. The extension of (III.103) with one more term leads to an extension of (III.104) of the form:

$$\begin{aligned} s_{\mathrm{r,a}} - s_1 = {} & \mp c^{-1}\varepsilon[1 - \tfrac{1}{2}c^{-2}a\cdot n\varepsilon + \{\tfrac{3}{8}c^{-4}(a\cdot n)^2 - \tfrac{1}{24}c^{-4}a^2 \pm \tfrac{1}{6}c^{-3}\dot{a}\cdot n\}\varepsilon^2 \\ & + \{-\tfrac{5}{16}c^{-6}(a\cdot n)^3 + \tfrac{5}{48}c^{-6}a\cdot n a^2 \mp \tfrac{1}{3}c^{-5}a\cdot n\dot{a}\cdot n \\ & - \tfrac{1}{24}c^{-4}\ddot{a}\cdot n \pm \tfrac{1}{24}c^{-5}a\cdot\dot{a}\}\varepsilon^3 + \ldots]. \qquad \text{(A70)} \end{aligned}$$

Furthermore one has to calculate the extensions of (III.105–108). This leads to the series expansions:

$$\begin{aligned} u(s)\cdot r(s) = {} & \mp c\varepsilon[1 + \tfrac{1}{2}c^{-2}a\cdot n\varepsilon + \{-\tfrac{1}{8}c^{-4}(a\cdot n)^2 + \tfrac{1}{8}c^{-4}a^2 \mp \tfrac{1}{3}c^{-3}\dot{a}\cdot n\}\varepsilon^2 \\ & + \{\tfrac{1}{16}c^{-6}(a\cdot n)^3 - \tfrac{3}{16}c^{-6}a\cdot n a^2 \pm \tfrac{1}{3}c^{-5}a\cdot n\dot{a}\cdot n \\ & + \tfrac{1}{8}c^{-4}\ddot{a}\cdot n \mp \tfrac{1}{6}c^{-5}a\cdot\dot{a}\}\varepsilon^3 + \ldots], \qquad \text{(A71)} \end{aligned}$$

$$\begin{aligned} a(s)\cdot r(s) = {} & \varepsilon\{a\cdot n \mp c^{-1}(\dot{a}\cdot n \mp \tfrac{1}{2}c^{-1}a^2)\varepsilon \\ & + (\mp\tfrac{5}{6}c^{-3}a\cdot\dot{a} - \tfrac{1}{2}c^{-4}a^2 a\cdot n + \tfrac{1}{2}c^{-2}\ddot{a}\cdot n \pm \tfrac{1}{2}c^{-3}a\cdot n\dot{a}\cdot n)\varepsilon^2 + \ldots\}, \qquad \text{(A72)} \end{aligned}$$

$$\begin{aligned} r^\alpha(s)u^\beta(s) - r^\beta(s)u^\alpha(s) = {} & \varepsilon(n^\alpha u^\beta + (\mp c^{-1}n^\alpha a^\beta - \tfrac{1}{2}c^{-2}u^\alpha a^\beta)\varepsilon \\ & + (\pm\tfrac{1}{2}c^{-3}a\cdot n n^\alpha a^\beta + \tfrac{1}{2}c^{-2}n^\alpha\dot{a}^\beta + \tfrac{1}{2}c^{-4}a\cdot n u^\alpha a^\beta \pm \tfrac{1}{3}c^{-3}u^\alpha\dot{a}^\beta)\varepsilon^2 \\ & + [\{\mp\tfrac{3}{8}c^{-5}(a\cdot n)^2 \pm \tfrac{1}{24}c^{-5}a^2 - \tfrac{1}{6}c^{-4}\dot{a}\cdot n\}n^\alpha a^\beta - \tfrac{1}{2}c^{-4}a\cdot n n^\alpha\dot{a}^\beta \\ & \mp\tfrac{1}{6}c^{-3}n^\alpha\ddot{a}^\beta + \{-\tfrac{1}{2}c^{-6}(a\cdot n)^2 + \tfrac{1}{24}c^{-6}a^2 \mp \tfrac{1}{6}c^{-5}\dot{a}\cdot n\}u^\alpha a^\beta \\ & \mp\tfrac{1}{2}c^{-5}a\cdot n u^\alpha\dot{a}^\beta - \tfrac{1}{8}c^{-4}u^\alpha\ddot{a}^\beta - \tfrac{1}{12}c^{-4}a^\alpha\dot{a}^\beta]\varepsilon^3 + \ldots) - (\alpha, \beta), \qquad \text{(A73)} \end{aligned}$$

$$\begin{aligned} r^\alpha(s)a^\beta(s) - r^\beta(s)a^\alpha(s) = {} & \varepsilon(n^\alpha a^\beta \pm c^{-1}u^\alpha a^\beta + (\mp c^{-1}n^\alpha\dot{a}^\beta \mp \tfrac{1}{2}c^{-3}a\cdot n u^\alpha a^\beta \\ & - c^{-2}u^\alpha\dot{a}^\beta)\varepsilon + [\pm\tfrac{1}{2}c^{-3}a\cdot n n^\alpha\dot{a}^\beta + \tfrac{1}{2}c^{-2}n^\alpha\ddot{a}^\beta + c^{-4}a\cdot n u^\alpha\dot{a}^\beta \\ & \pm\tfrac{1}{2}c^{-3}u^\alpha\ddot{a}^\beta \pm \tfrac{1}{2}c^{-3}a^\alpha\dot{a}^\beta + \{\pm\tfrac{3}{8}c^{-5}(a\cdot n)^2 \mp \tfrac{1}{24}c^{-5}a^2 \\ & + \tfrac{1}{6}c^{-4}\dot{a}\cdot n\}u^\alpha a^\beta]\varepsilon^2 + \ldots) - (\alpha, \beta). \qquad \text{(A74)} \end{aligned}$$

If these expressions are substituted into (A65) and half the difference of the retarded and advanced field is taken, one finds for the minus field in the neighbourhood of the world line

$$\begin{aligned} f^{\alpha\beta}_{-(e)} = {} & \frac{ec^{-4}}{12\pi}\{2u^\alpha\dot{a}^\beta + (c^{-2}a\cdot\dot{a}n^\alpha u^\beta + c^{-2}a^2 n^\alpha a^\beta \\ & - 2c^{-2}\dot{a}\cdot n u^\alpha a^\beta - 4c^{-2}a\cdot n u^\alpha\dot{a}^\beta - n^\alpha\ddot{a}^\beta)\varepsilon + \ldots\} - (\alpha, \beta). \qquad \text{(A75)} \end{aligned}$$

(Indeed the limit $\varepsilon \to 0$ gives back (A67).) The derivative of the minus field in a direction orthogonal to the four-velocity follows by taking the derivative with respect to εn^α:

$$\begin{aligned} \Delta^\gamma_\delta \partial^\delta f^{\alpha\beta}_{-(e)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Delta^\gamma_\delta \frac{\partial}{\partial n_\delta} f^{\alpha\beta}_{-(e)} = {} & \frac{ec^{-4}}{12\pi}\Delta^\gamma_\delta(c^{-2}a\cdot\dot{a}g^{\alpha\delta}u^\beta + c^{-2}a^2 g^{\alpha\delta}a^\beta \\ & - 2c^{-2}\dot{a}^\delta u^\alpha a^\beta - 4c^{-2}a^\delta u^\alpha\dot{a}^\beta - g^{\alpha\delta}\ddot{a}^\beta) - (\alpha, \beta). \qquad \text{(A76)} \end{aligned}$$

Combining this relation with (A69) according to (A68) one finds

$$\begin{aligned} \partial^\gamma f^{\alpha\beta}_{-(e)} = {} & \frac{e}{12\pi}c^{-4}\{\ddot{a}^\alpha g^{\beta\gamma} + c^{-2}a\cdot\dot{a}g^{\alpha\gamma}u^\beta + c^{-2}a^2 g^{\alpha\gamma}a^\beta - 2c^{-2}u^\alpha a^\beta\dot{a}^\gamma \\ & - 4c^{-2}u^\alpha\dot{a}^\beta a^\gamma + u^\gamma(3c^{-4}a^2 u^\alpha a^\beta - 3c^{-2}u^\alpha\ddot{a}^\beta - 2c^{-2}a^\alpha\dot{a}^\beta)\} - (\alpha, \beta), \qquad \text{(A77)} \end{aligned}$$

which is the expression (A59). This formula shows that the derivative of the minus field, just as the minus field itself at the world line of the particle, disappears if no accelerations are present, i.e. if the particle moves uniformly.

We now turn to the dipole field given by (A66). It is possible to calculate (in an analogous fashion as used for the charge field) the general expression for the field and its derivative at the world line. The results are lengthy[1] and will not be reproduced here. Much more simple results are obtained if one retains right from the beginning only those terms which are independent of the first and higher derivatives of the velocity. Then formula (A66) becomes upon using (III.94)

$$\begin{aligned} f^{\alpha\beta}_{\mathrm{r,a}(\mathfrak{m})} = \pm\frac{c}{4\pi}\frac{1}{(u\cdot r)^2}\Bigg\{ & \frac{\ddot{\mathfrak{m}}^{\alpha\gamma}r_\gamma r^\beta}{u\cdot r} - \frac{2\dot{\mathfrak{m}}^{\alpha\gamma}u_\gamma r^\beta}{u\cdot r} - \frac{2\dot{\mathfrak{m}}^{\alpha\gamma}r_\gamma u^\beta}{u\cdot r} - \frac{3\dot{\mathfrak{m}}^{\alpha\gamma}r_\gamma r^\beta c^2}{(u\cdot r)^2} \\ & + \dot{\mathfrak{m}}^{\alpha\beta} + \frac{2\mathfrak{m}^{\alpha\gamma}u_\gamma u^\beta}{u\cdot r} + \frac{3\mathfrak{m}^{\alpha\gamma}u_\gamma r^\beta c^2}{(u\cdot r)^2} + \frac{3\mathfrak{m}^{\alpha\gamma}r_\gamma u^\beta c^2}{(u\cdot r)^2} + \frac{3\mathfrak{m}^{\alpha\gamma}r_\gamma r^\beta c^4}{(u\cdot r)^3} \\ & - \frac{\mathfrak{m}^{\alpha\beta}c^2}{u\cdot r}\Bigg\}\Bigg|_{\mathrm{r,a}} - (\alpha, \beta). \qquad \text{(A78)} \end{aligned}$$

[1] Harish-Chandra, Proc. Roy. Soc. **A185**(1946)269; J. R. Ellis, J. Math. Phys. **7**(1966) 1185; S. Emid and J. Vlieger, Physica **52**(1971)329.

Its value near the world line, i.e. at a position $R^\alpha = X^\alpha(s)+\varepsilon n^\alpha$ follows by making use of formulae of the type (A70–A74). In particular, since we limit ourselves here to terms without accelerations, the equation (A70) now reduces to

$$s_{\mathrm{r,a}}-s_1 = \mp c^{-1}\varepsilon. \tag{A79}$$

Furthermore one finds then

$$u(s)\cdot r(s) = \mp c\varepsilon, \tag{A80}$$

$$r^\alpha(s) = \varepsilon n^\alpha \pm c^{-1}\varepsilon u^\alpha \tag{A81}$$

and expressions for $\mathfrak{m}^{\alpha\beta}(s)$ and its derivatives that follow immediately from the Taylor expansion around s_1. If these expressions are inserted into (A78) one obtains for the minus field in the neighbourhood of the world line up to order ε:

$$f^{\alpha\beta}_{-(\mathrm{m})} = \frac{c^{-3}}{6\pi}(\overset{(3)}{\mathfrak{m}}{}^{\alpha\beta}+2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\alpha\gamma}u_\gamma u^\beta-2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\beta\gamma}u_\gamma u^\alpha)$$
$$-\frac{\varepsilon}{12\pi}c^{-5}(\overset{(4)}{\mathfrak{m}}{}^{\alpha\gamma}u_\gamma n^\beta+\overset{(4)}{\mathfrak{m}}{}^{\alpha\gamma}n_\gamma u^\beta-\overset{(4)}{\mathfrak{m}}{}^{\beta\gamma}u_\gamma n^\alpha-\overset{(4)}{\mathfrak{m}}{}^{\beta\gamma}n_\gamma u^\alpha). \tag{A82}$$

From this formula one finds directly, by letting tend ε to zero, for the field at the position of the particle

$$f^{\alpha\beta}_{-(\mathrm{m})} = \frac{c^{-3}}{6\pi}(\overset{(3)}{\mathfrak{m}}{}^{\alpha\beta}+2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\alpha\gamma}u_\gamma u^\beta-2c^{-2}\overset{(3)}{\mathfrak{m}}{}^{\beta\gamma}u_\gamma u^\alpha) \tag{A83}$$

and for the derivative of the field in the direction orthogonal to the four-velocity,

$$\Delta^\gamma_\delta\partial^\delta f^{\alpha\beta}_{-(\mathrm{m})} = -\frac{c^{-5}}{12\pi}\Delta^\gamma_\delta(\overset{(4)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon g^{\beta\delta}+\overset{(4)}{\mathfrak{m}}{}^{\alpha\delta}u^\beta-\overset{(4)}{\mathfrak{m}}{}^{\beta\varepsilon}u_\varepsilon g^{\alpha\delta}-\overset{(4)}{\mathfrak{m}}{}^{\beta\delta}u^\alpha). \tag{A84}$$

Furthermore it follows from (A83) that the derivative in the direction of the four-velocity is

$$u^\varepsilon\partial_\varepsilon f^{\alpha\beta}_{-(\mathrm{m})} = \frac{c^{-3}}{6\pi}(\overset{(4)}{\mathfrak{m}}{}^{\alpha\beta}+2c^{-2}\overset{(4)}{\mathfrak{m}}{}^{\alpha\gamma}u_\gamma u^\beta-2c^{-2}\overset{(4)}{\mathfrak{m}}{}^{\beta\gamma}u_\gamma u^\alpha), \tag{A85}$$

where again accelerations have been neglected. Then, according to (A68), we find from (A84) and (A85)

$$\partial^\gamma f^{\alpha\beta}_{-(\mathrm{m})} = -\frac{c^{-5}}{12\pi}(\overset{(4)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon g^{\beta\gamma}+\overset{(4)}{\mathfrak{m}}{}^{\alpha\gamma}u^\beta+\overset{(4)}{\mathfrak{m}}{}^{\alpha\beta}u^\gamma+6c^{-2}\overset{(4)}{\mathfrak{m}}{}^{\alpha\varepsilon}u_\varepsilon u^\beta u^\gamma)-(\alpha,\beta), \tag{A86}$$

which is the final result for the derivative of the dipole minus field at the position of the particle (at least for the terms without accelerations).

APPENDIX V

Semi-relativistic equations of motion for a composite particle

1. *The semi-relativistic approximation*

In the relativistic treatment of a composite particle in an external field equations of motion and of angular momentum have been found. They contain, as compared to the corresponding non-relativistic treatment, a number of relativistic effects. Amongst these figure in particular terms of order c^{-2}, which contain an explicit factor c^{-1} and the magnetic dipole moment, which is itself of order c^{-1}. Hence these are the terms which would survive in the so-called *semi-relativistic* approximation (v. section 2*e*) in which one considers the magnetic dipole moment to be an atomic parameter of order c^0 and subsequently retains only terms up to order c^{-1}. It is possible to give a theory in which all these semi-relativistic effects are included, without going through the complete relativistic treatment. In this way some insight is gained about the origin of these terms, in particular about the so-called magnetodynamic effect, that contains the vector product of the magnetic dipole moment and the electric field. In the following we shall carry out this programme by first developing the theory up to order c^{-2} (which includes the use of some notions of relativity) and then taking the semi-relativistic limit of the resulting equations.

2. *The momentum and energy equations*

a. *The equation of motion*

In non-relativistic theory the equation of motion for a particle in an electromagnetic field contains the time derivative of mass times velocity. In a theory which includes all effects up to order c^{-2}, again the time derivative of the momentum appears. However, the momentum now contains not only the ordinary rest mass, but also a mass which is equivalent to the kinetic energy. The equation of motion for constituent particle i of the composite particle reads then

$$\frac{\mathrm{d}}{\mathrm{d}t}\{m_i(1+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{R}}_i^2)\dot{\boldsymbol{R}}_i\} = e_i\{\boldsymbol{e}_t(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{b}_t(\boldsymbol{R}_i,t)\}, \tag{A87}$$

where e_i is the charge, m_i the mass, $\boldsymbol{R}_i(t)$ the position and $\dot{\boldsymbol{R}}_i(t)$ the velocity of the particle i. The right-hand side is the Lorentz force, just as in non-relativistic theory. The total electromagnetic fields $\boldsymbol{e}_t$ and $\boldsymbol{b}_t$ contain the fields due to the other particles j ($\neq i$) and due to external sources. The electric and magnetic fields are needed up to order c^{-2} and c^{-1} respectively so as to describe all effects up to order c^{-2}.

Since we are interested in the motion of the composite particle as a whole, we shall define now a privileged point $\boldsymbol{X}$ which describes the position of the atom as a whole. In non-relativistic theory the centre of mass has been chosen as such a point. In relativity one should like to include beside the rest masses also internal kinetic and potential energies in the definition. This would lead to an energy centre, defined, up to order c^{-2}, as

$$\boldsymbol{X}^* = \frac{\sum_i (m_i + \frac{1}{2}c^{-2}m_i\dot{\boldsymbol{R}}_i^2 + c^{-2}\sum_{j(\neq i)} e_i e_j/8\pi|\boldsymbol{R}_i - \boldsymbol{R}_j|)\boldsymbol{R}_i}{\sum_i (m_i + \frac{1}{2}c^{-2}m_i\dot{\boldsymbol{R}}_i^2 + c^{-2}\sum_{j(\neq i)} e_i e_j/8\pi|\boldsymbol{R}_i - \boldsymbol{R}_j|)}. \tag{A88}$$

Such a definition, however, is still not convenient, since in this way the privileged point $\boldsymbol{X}^*$ would depend on the velocity of the observer with respect to the composite particle, and is hence not invariant. To overcome this drawback, let us consider the coordinate frame in which the composite particle is momentarily at rest, i.e. a frame which moves with the velocity $\dot{\boldsymbol{X}}^*$ with respect to the observer. If we now determine the energy centre in this momentary rest frame, we find a point $\boldsymbol{X}$ (different from $\boldsymbol{X}^*$). By repeating this procedure for each time t one obtains a world line of energy centres $\boldsymbol{X}(t)$ [1]. From this construction we shall now derive the relation that determines the privileged point $\boldsymbol{X}$ in terms of the positions $\boldsymbol{R}_i$.

[1] One may ask whether this construction is the c^{-2}-limit of the relativistic definition of the centre of energy. In the relativistic case one takes the energy centre in space-like surfaces. As a weight function one uses the time-time component t^{00} of the energy–momentum tensor (62). If one evaluates up to order c^{-2} the expression $c^{-2}\int t^{00}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}$ one finds indeed the numerator of (A88) (v. problem 7). What remains to be checked is that the relativistic construction, in which one takes space-like surfaces normal to the total momentum p^α, reduces to the present one, in which different space-like surfaces are employed, namely surfaces orthogonal to the four-velocity corresponding to $\boldsymbol{X}^*$. From (124) it follows that the space-components of c times the normal unit vector i.e.: $cp^\alpha/\sqrt{-p^2}$ differs from the space-components of $u^\alpha \equiv \mathrm{d}X^\alpha/\mathrm{d}s$ by terms which are of order c^{-2} and smaller. In the course of this appendix it will turn out that the velocity $\dot{\boldsymbol{X}}$ differs from the velocity $\dot{\boldsymbol{X}}^*$ by terms which are also of order c^{-2} and smaller. Hence c times the space components of the normal unit vector of the relativistic definition differs by terms which are of order c^{-2} (and smaller) from $\dot{\boldsymbol{X}}^*$, which is used here. Since the construction given here will show that not the precise form (A88) of $\boldsymbol{X}^*$ is relevant, but only the fact that its c^0-terms give the non-relativistic centre of mass, it follows that the construction given here is indeed the c^{-2}-approximation to the relativistic definition.

Let us consider the points of the world lines $\boldsymbol{R}_i$, $\boldsymbol{X}$ and $\boldsymbol{X}^*$ which have the same time coordinate t' in the coordinate frame (indicated by primes) which moves with the velocity $\dot{\boldsymbol{X}}^*(t^*)$ with respect to the observer (see fig. 2). In

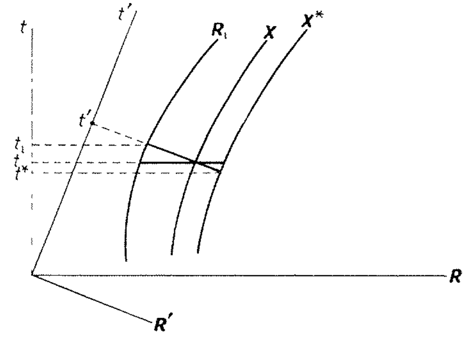

Fig. 2. The construction of the semi-relativistic energy centre.

the observer's frame these world points have the time components t_i, t and t^* respectively. The relative position vector in the primed frame

$$\boldsymbol{r}_i'(t) \equiv \boldsymbol{R}_i'(t_i) - \boldsymbol{X}'(t) \tag{A89}$$

of particle i with respect to the energy centre $\boldsymbol{X}$ fulfils the relation

$$\sum_i \left\{ m_i + \tfrac{1}{2}c^{-2}m_i\dot{\boldsymbol{R}}_i'^2(t_i) + c^{-2}\sum_{j(\neq i)} \frac{e_i e_j}{8\pi|\boldsymbol{r}_i'(t) - \boldsymbol{r}_j'(t)|} \right\} \boldsymbol{r}_i'(t) = 0. \tag{A90}$$

In fact this is the defining formula of an energy centre up to order c^{-2} (cf. (A88)). We now want to find what follows from this relation for the relative positions

$$\boldsymbol{r}_i(t) \equiv \boldsymbol{R}_i(t) - \boldsymbol{X}(t), \tag{A91}$$

in the observer's frame. From the fact that the world points $(\boldsymbol{R}_i(t_i), t_i)$ and $(\boldsymbol{X}(t), t)$ have the same time components in the primed frame, it follows, with the help of a Lorentz transformation, that, up to order c^{-2}, one has

$$t_i - t = c^{-2}\dot{\boldsymbol{X}}^*(t^*)\cdot\boldsymbol{r}_i'(t), \tag{A92}$$

where (A89) has been used. In the same way it follows that $t^* - t$ is of order c^{-2}, so that (A92) may be written as

$$t_i - t = c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i'(t), \tag{A93}$$

up to order c^{-2}. The first terms of a Taylor expansion of $\boldsymbol{R}_i(t_i)$ around $\boldsymbol{R}_i(t)$

become with the help of (A93):

$$\boldsymbol{R}_i(t_i) = \boldsymbol{R}_i(t)+c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i'(t)\dot{\boldsymbol{R}}_i(t). \qquad \text{(A94)}$$

The same Lorentz transformation which has led to (A92) also yields with the notation (A89):

$$\boldsymbol{R}_i(t_i)-\boldsymbol{X}(t) = \boldsymbol{r}_i'(t)+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i'(t)\dot{\boldsymbol{X}}^*(t), \qquad \text{(A95)}$$

where in the transformation velocity $\dot{\boldsymbol{X}}^*(t)$ the time t has been written instead of t^*, since the difference between these times is of order c^{-2} only. From (A91), (A94) and (A95) follows the relation

$$\boldsymbol{r}_i(t) = \boldsymbol{r}_i'(t)+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i'(t)\dot{\boldsymbol{X}}^*(t)-c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i'(t)\dot{\boldsymbol{R}}_i(t). \qquad \text{(A96)}$$

Its inversion, up to order c^{-2}, reads

$$\boldsymbol{r}_i'(t) = \boldsymbol{r}_i(t)-\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i(t)\dot{\boldsymbol{X}}^*(t)+c^{-2}\dot{\boldsymbol{X}}^*(t)\cdot\boldsymbol{r}_i(t)\dot{\boldsymbol{R}}_i(t). \qquad \text{(A97)}$$

Apart from $\boldsymbol{r}_i'(t)$, which has now been found, the quantity $\dot{\boldsymbol{R}}_i'(t_i)$ also occurs in (A90). It may be written, up to order c^0, as

$$\dot{\boldsymbol{R}}_i'(t_i) = \dot{\boldsymbol{R}}_i(t_i)-\dot{\boldsymbol{X}}^*(t^*) = \dot{\boldsymbol{R}}_i(t)-\dot{\boldsymbol{X}}^*(t), \qquad \text{(A98)}$$

where the fact has been used that both t_i-t and t^*-t are of order c^{-2}. Inserting (A97) and (A98) into (A90) we obtain now up to order c^{-2}:

$$\sum_i \left\{m_i+\tfrac{1}{2}c^{-2}m_i(\dot{\boldsymbol{R}}_i-\dot{\boldsymbol{X}}^*)^2+c^{-2}\sum_{j(\neq i)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\right\}$$
$$(\boldsymbol{r}_i-\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}^*\cdot\boldsymbol{r}_i\dot{\boldsymbol{X}}^*+c^{-2}\dot{\boldsymbol{X}}^*\cdot\boldsymbol{r}_i\dot{\boldsymbol{R}}_i) = 0, \qquad \text{(A99)}$$

where the arguments t of all quantities have been suppressed. From (A88) and (A99) it follows that the difference between $\dot{\boldsymbol{X}}^*$ and $\dot{\boldsymbol{X}}$ is of order c^{-2}. This permits us to write (A96) and (A99) with (A91), up to order c^{-2}, as

$$\boldsymbol{r}_i = \boldsymbol{r}_i'-\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i'\dot{\boldsymbol{X}}-c^{-2}\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i'\dot{\boldsymbol{r}}_i \qquad \text{(A100)}$$

and

$$\sum_i \left(m_i\boldsymbol{r}_i+\tfrac{1}{2}c^{-2}m_i\dot{\boldsymbol{r}}_i^2\boldsymbol{r}_i+c^{-2}\sum_{j(\neq i)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\boldsymbol{r}_i+c^{-2}m_i\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\dot{\boldsymbol{r}}_i\right) = 0. \qquad \text{(A101)}$$

The last relation defines a privileged point of the composite particle in a unique way. In fact if there would exist two different points $\boldsymbol{X}$ and $\boldsymbol{X}+\Delta\boldsymbol{X}$, one would have, apart from the relation (A101) as it stands, a relation like (A101) but with $\boldsymbol{X}+\Delta\boldsymbol{X}$ and $\boldsymbol{r}_i-\Delta\boldsymbol{X}$ instead of $\boldsymbol{X}$ and $\boldsymbol{r}_i$ respectively. Then from these two relations together it follows that $\Delta\boldsymbol{X}$ is at least of order c^{-4}.

Hence it is negligible in the framework of the present treatment, in which only effects of order c^{-2} are considered. One may still ask whether the definition of the privileged point is biased by the original choice of the $(\boldsymbol{R}, t)$-frame as a starting point. Suppose in fact that one had started from an $(\boldsymbol{R}, t)$-frame which moves with a velocity $\boldsymbol{V}$ with respect to the observer to define the ancillary point $\hat{\boldsymbol{X}}^*$. Then one would have arrived at a privileged point $\hat{\boldsymbol{X}}(\hat{t})$ and relative positions

$$\hat{\boldsymbol{r}}_i(\hat{t}) = \hat{\boldsymbol{R}}_i(\hat{t})-\hat{\boldsymbol{X}}(\hat{t}), \qquad \text{(A102)}$$

which satisfy a relation like (A101) but 'circumflexed'. The Lorentz transform of the point $(\hat{\boldsymbol{X}}(\hat{t}), \hat{t})$ in the $(\hat{\boldsymbol{R}}, \hat{t})$-frame is the point $(\boldsymbol{X}(t), t)$ in the $(\boldsymbol{R}, t)$ frame. One may now ask which relation is satisfied by the relative positions $\boldsymbol{r}_i(t) = \boldsymbol{R}_i(t)-\boldsymbol{X}(t)$ with respect to the newly defined privileged point $\boldsymbol{X}(t)$. They are connected to the $\hat{\boldsymbol{r}}_i(\hat{t})$ by a relation which may be derived in the same way as (A97), and which reads

$$\hat{\boldsymbol{r}}_i(\hat{t}) = \boldsymbol{r}_i(t)-\tfrac{1}{2}c^{-2}\boldsymbol{V}\cdot\boldsymbol{r}_i(t)\boldsymbol{V}+c^{-2}\boldsymbol{V}\cdot\boldsymbol{r}_i(t)\dot{\boldsymbol{R}}_i(t). \qquad \text{(A103)}$$

Substitution of this relation and the transformation formula $\dot{\hat{\boldsymbol{X}}} = \dot{\boldsymbol{X}}-\boldsymbol{V}$ into the circumflexed (A101) gives

$$\sum_i \Big\{m_i\boldsymbol{r}_i-\tfrac{1}{2}c^{-2}m_i\boldsymbol{V}\cdot\boldsymbol{r}_i\boldsymbol{V}+c^{-2}m_i\boldsymbol{V}\cdot\boldsymbol{r}_i\dot{\boldsymbol{R}}_i+\tfrac{1}{2}c^{-2}m_i\dot{\boldsymbol{r}}_i^2\boldsymbol{r}_i$$
$$+c^{-2}\sum_{j(\neq i)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\boldsymbol{r}_i+c^{-2}m_i(\dot{\boldsymbol{X}}-\boldsymbol{V})\cdot\boldsymbol{r}_i\dot{\boldsymbol{r}}_i\Big\} = 0. \qquad \text{(A104)}$$

With $\dot{\boldsymbol{R}}_i = \dot{\boldsymbol{X}}+\dot{\boldsymbol{r}}_i$ the third term splits, such that the second part of it cancels together with the last term. Furthermore the first part of the third term and the second term are both of order c^{-4}, as follows because $\sum_i m_i\boldsymbol{r}_i$ is of order c^{-2}. As a result one is left with a relation of the same form as (A101). Since there is only one point which satisfies (A101), as was shown above, the choice of the $(\boldsymbol{R}, t)$ frame as a starting point does not cause a bias.

The relation (A101) will be used to derive an equation of motion for the atom as a whole. In fact by summation over i of equation (A87) and the use of (A91) and (A101) one gets an equation of motion with at the left-hand side the time derivative of the quantity

$$(m+\tfrac{1}{2}c^{-2}m\dot{\boldsymbol{X}}^2+\tfrac{1}{2}c^{-2}\sum_i m_i\dot{\boldsymbol{r}}_i^2)\dot{\boldsymbol{X}}+\sum_i c^{-2}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}\dot{\boldsymbol{r}}_i+\sum_i \tfrac{1}{2}c^{-2}m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{r}}_i$$
$$-c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i^2\boldsymbol{r}_i+\sum_{j(\neq i)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\boldsymbol{r}_i+m_i\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\dot{\boldsymbol{r}}_i\right). \qquad \text{(A105)}$$

This is the extension up to order c^{-2} of the non-relativistic momentum $m\dot{\boldsymbol{X}}$

(the mass of the composite particle is $m = \sum_i m_i$). If the time derivation in (A105) is performed, one obtains – amongst other terms – second derivatives. These may be rewritten with the help of the zero order equations of motion

$$m_i \ddot{\boldsymbol{R}}_i = e_i \boldsymbol{e}_t(\boldsymbol{R}_i, t), \qquad m\ddot{\boldsymbol{X}} = \sum_i e_i \boldsymbol{e}_t(\boldsymbol{R}_i, t), \tag{A106}$$

which follow from (A87). In this way (A105) becomes

$$\begin{aligned}(m+\tfrac{1}{2}c^{-2}m\dot{\boldsymbol{X}}^2+\tfrac{1}{2}c^{-2}\sum_i m_i\dot{\boldsymbol{r}}_i^2)\dot{\boldsymbol{X}}-c^{-2}\sum_i\Big[\dot{\boldsymbol{r}}_i\cdot\Big\{e_i\boldsymbol{e}_t(\boldsymbol{R}_i,t)\\ -\frac{m_i}{m}\sum_j e_j\boldsymbol{e}_t(\boldsymbol{R}_j,t)\Big\}\boldsymbol{r}_i+\sum_{j(\neq i)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\Big\{\dot{\boldsymbol{r}}_i-\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot(\dot{\boldsymbol{r}}_i-\dot{\boldsymbol{r}}_j)\boldsymbol{r}_i}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\Big\}\\ +\frac{m_i}{m}\sum_j e_j\boldsymbol{e}_t(\boldsymbol{R}_j,t)\cdot\boldsymbol{r}_i\dot{\boldsymbol{r}}_i+\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\Big\{e_i\boldsymbol{e}_t(\boldsymbol{R}_i,t)-\frac{m_i}{m}\sum_j e_j\boldsymbol{e}_t(\boldsymbol{R}_j,t)\Big\}\Big],\end{aligned} \tag{A107}$$

where the last term between the brackets may be omitted, because it is of order c^{-4}, since $\sum_i m_i\boldsymbol{r}_i$ is of order c^{-2}.

The total electric field $\boldsymbol{e}_t(\boldsymbol{R}_i, t)$, occurring in (A107), consists of the intra-atomic field, generated by the constituent particles j ($\neq i$) of the atom itself, and the field $\boldsymbol{E}(\boldsymbol{R}_i, t)$ from outside the atom. Up to order c^0 we have thus

$$\boldsymbol{e}_t(\boldsymbol{R}_i, t) = \sum_{j(\neq i)} \frac{e_j(\boldsymbol{r}_i-\boldsymbol{r}_j)}{4\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|^3} + \boldsymbol{E}(\boldsymbol{R}_i, t). \tag{A108}$$

Substitution of this expression into (A107) yields

$$\begin{aligned}(m+\tfrac{1}{2}c^{-2}m\dot{\boldsymbol{X}}^2+\tfrac{1}{2}c^{-2}\sum_i m_i\dot{\boldsymbol{r}}_i^2)\dot{\boldsymbol{X}}\\ -c^{-2}\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\Big\{\dot{\boldsymbol{r}}_i+\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{R}}_i}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\Big\}\\ -c^{-2}\sum_i e_i\Big\{\frac{\bar{\boldsymbol{s}}}{m}\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)+\dot{\boldsymbol{r}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{r}_i+\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\boldsymbol{E}(\boldsymbol{R}_i,t)\Big\},\end{aligned} \tag{A109}$$

where we introduced the non-relativistic inner angular momentum

$$\bar{\boldsymbol{s}} \equiv \sum_i m_i \boldsymbol{r}_i \wedge \dot{\boldsymbol{r}}_i. \tag{A110}$$

The time derivative of expression (A109) constitutes the left-hand side of the equation of motion for the composite particle. The right-hand side is the sum over all constituent particles i of the Lorentz forces which appear in the right-hand side of (A87). In these forces the total fields are sums of intra-atomic fields and fields from outside the atom (cf. (A108)). Up to orders c^{-2} and c^{-1} the electric and magnetic fields are given by (v. (III.72))

$$\begin{aligned}\boldsymbol{e}_t(\boldsymbol{R}_i,t)=\sum_{j(\neq i)}e_j\Big[\frac{\boldsymbol{r}_i-\boldsymbol{r}_j}{4\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}+c^{-2}\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\dot{\boldsymbol{R}}_j^2}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}-c^{-2}\frac{3(\boldsymbol{r}_i-\boldsymbol{r}_j)\{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{R}}_j\}^2}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|^5}\\ -c^{-2}\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\ddot{\boldsymbol{R}}_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}-c^{-2}\frac{\ddot{\boldsymbol{R}}_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\Big]+\boldsymbol{E}(\boldsymbol{R}_i,t),\end{aligned} \tag{A111}$$

$$\boldsymbol{b}_t(\boldsymbol{R}_i,t)=c^{-1}\sum_{j(\neq i)}e_j\frac{\dot{\boldsymbol{R}}_j\wedge(\boldsymbol{r}_i-\boldsymbol{r}_j)}{4\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}+\boldsymbol{B}(\boldsymbol{R}_i,t),$$

where (A91) has been used. With those expressions for the total electromagnetic field the right-hand side of the equation of motion for the composite particle gets the form

$$\begin{aligned}-c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\Big\{\dot{\boldsymbol{R}}_i+\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{R}}_i}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\Big\}\Big]\\ +\sum_i e_i\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\}.\end{aligned} \tag{A112}$$

The equation of motion for the composite particle up to order c^{-2} follows finally by equating the time derivative of (A109) and expression (A112):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\Big(m+\tfrac{1}{2}c^{-2}m\dot{\boldsymbol{X}}^2+\tfrac{1}{2}c^{-2}\sum_i m_i\dot{\boldsymbol{r}}_i^2+c^{-2}\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\Big)\dot{\boldsymbol{X}}\Big\}\\ =\sum_i e_i\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\}\\ +c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sum_i e_i\Big\{\dot{\boldsymbol{r}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{r}_i+\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\boldsymbol{E}(\boldsymbol{R}_i,t)+\frac{\bar{\boldsymbol{s}}}{m}\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)\Big\}\Big].\end{aligned} \tag{A113}$$

In the left-hand side one recognizes the time derivative of the velocity of the atom times its total energy divided by c^2 (its 'total mass'). At the right-hand side appears, apart from the sum of the Lorentz forces, a (total) time derivative of order c^{-2}. The last term of the quantity between brackets may be written in a compact form:

$$\sum_i e_i\frac{\bar{\boldsymbol{s}}}{m}\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)=\bar{\boldsymbol{s}}\wedge\ddot{\boldsymbol{X}} \tag{A114}$$

by using (A106) and (A108) (cf. the remark after eq. (126)).

In the case without fields $\boldsymbol{E}$ and $\boldsymbol{B}$ the right-hand side of (A113) vanishes, so that then the bracket in the left-hand side is conserved. According to equation (I.63) the time derivative of the factor between brackets (the 'total mass') is in this case at most of order c^{-4} and hence negligible. This means that the velocity $\dot{X}$ of the atom as a whole is conserved, as one would expect in the field-free case.

Let us now consider the equation of motion (A113) for a composite particle in an external field $(\boldsymbol{E}, \boldsymbol{B})$ that changes slowly over the dimensions of the particle. Then the right-hand side of the equation of motion may be expanded in terms of the relative positions $\boldsymbol{r}_i$ (A91). In that way one may obtain a series expansion containing the non-relativistic multipole moments (I.15–16) of chapter I. These non-relativistic multipole moments however are not independent of the motion of the atoms in a theory in which all terms up to order c^{-2} are taken into account. Therefore we want to introduce rest frame multipole moments, as in section 2*b* of the present chapter, which read

$$\begin{aligned} \boldsymbol{\mu}^{(n)} &= \frac{1}{n!}\sum_i e_i \boldsymbol{r}_i'^{n}, \qquad (n = 0, 1, 2, \ldots), \\ \boldsymbol{\nu}^{(n)} &= \frac{n}{(n+1)!}\sum_i e_i \boldsymbol{r}_i'^{n} \wedge \frac{\dot{\boldsymbol{r}}_i'}{c}, \qquad (n = 1, 2, \ldots). \end{aligned} \tag{A115}$$

The connexion (A100) between $\boldsymbol{r}_i$ and $\boldsymbol{r}_i'$ allows us to express the right-hand side of the equation of motion in terms of these rest frame multipole moments. If we confine ourselves to the contributions of the electric charge $\mu^{(0)} \equiv e$ and the electric and magnetic dipole moments $\boldsymbol{\mu}^{(1)} \equiv \boldsymbol{\mu}$ and $\boldsymbol{\nu}^{(1)} \equiv \boldsymbol{\nu}$, and if we make use of the homogeneous field equations for the external fields

$$\nabla\cdot\boldsymbol{B} = 0, \qquad c^{-1}\partial\boldsymbol{B}/\partial t + \nabla\wedge\boldsymbol{E} = 0, \tag{A116}$$

we obtain as the equation of motion up to order c^{-2} for a composite charged dipole particle in an external field

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(m+\tfrac{1}{2}c^{-2}mv^2+\tfrac{1}{2}c^{-2}\sum_i m_i\dot{\boldsymbol{r}}_i'^2+c^{-2}\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i'-\boldsymbol{r}_j'|}\right)\boldsymbol{v}\right\} \\ = e(\boldsymbol{E}+c^{-1}\boldsymbol{v}\wedge\boldsymbol{B})+(\nabla\boldsymbol{E})\cdot(\boldsymbol{\mu}-\tfrac{1}{2}c^{-2}\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{\mu}-c^{-1}\boldsymbol{\nu}\wedge\boldsymbol{v})+(\nabla\boldsymbol{B})\cdot(\boldsymbol{\nu}+c^{-1}\boldsymbol{\mu}\wedge\boldsymbol{v}) \\ +c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}\wedge\boldsymbol{B})-c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{v}\wedge\boldsymbol{E}-c^{-1}\boldsymbol{v}\cdot\boldsymbol{\mu}\boldsymbol{E}-c^{-1}\frac{\bar{\boldsymbol{s}}}{m}\wedge\{e\boldsymbol{E}+(\nabla\boldsymbol{E})\cdot\boldsymbol{\mu}\}\right], \end{aligned} \tag{A117}$$

where the fields depend on the position $\boldsymbol{X}$ and the time t and where we have written $\boldsymbol{v}$ for the velocity $\dot{\boldsymbol{X}}$. This equation, which is indeed the c^{-2} approximation of the relativistic equation (135) with (137), may be compared to the

non-relativistic equation (55) of chapter I. At the left-hand side three additional terms appear, which describe c^{-2} corrections to the inertial mass. At the right-hand side in the second term the Lorentz contracted electric dipole moment $\boldsymbol{\mu}_\perp+(1-\frac{1}{2}c^{-2}\boldsymbol{v}^2)\boldsymbol{\mu}_{//}$ up to order c^{-2} appears (the dipole moment has been split into components orthogonal to and parallel with the velocity) and moreover a term due to the moving magnetic dipole moment. The last time derivative contains in the first place a magnetodynamic effect with $\boldsymbol{v}\wedge\boldsymbol{E}$ analogous to the electrodynamic effect with $\boldsymbol{\mu}\wedge\boldsymbol{B}$ of the penultimate time derivative. The third contribution to the last time derivative has the same form as the first contribution but with the 'normal magnetic dipole moment' $c^{-1}e\bar{\boldsymbol{s}}/m$ instead of the (total) magnetic dipole moment $\boldsymbol{\nu}$, and with the opposite sign. Hence effectively only the 'anomalous magnetic moment' couples with the electric field (as has been found already in the relativistic treatment). For an ordinary atom the anomalous magnetic moment is nearly the same as the total magnetic moment $\boldsymbol{\nu}$, since the atomic mass m is several thousand times greater than the masses of the particles which contribute to the inner angular momentum.

In the preceding all effects of order c^{-2} were taken into account. The equation of motion is simplified if we consider the *semi-relativistic approximation*. The latter has been defined in section 1 of this appendix as the approximation which results if one retains terms up to order c^{-1} only, considering the magnetic dipole moment as being of order c^0. In this way we get from (A117) the semi-relativistic equation of motion for a charged particle in a slowly varying external field:

$$\begin{aligned} m\dot{\boldsymbol{v}} = e(\boldsymbol{E}+c^{-1}\boldsymbol{v}\wedge\boldsymbol{B})+(\nabla\boldsymbol{E})\cdot(\boldsymbol{\mu}-c^{-1}\boldsymbol{\nu}\wedge\boldsymbol{v})+(\nabla\boldsymbol{B})\cdot(\boldsymbol{\nu}+c^{-1}\boldsymbol{\mu}\wedge\boldsymbol{v}) \\ +c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}\wedge\boldsymbol{B}-\boldsymbol{\nu}\wedge\boldsymbol{E}). \end{aligned} \tag{A118}$$

The dipole terms in this equation are symmetric with respect to electric and magnetic phenomena. This was not the case in the non-relativistic equation (I.55).

A semi-relativistic equation of motion has been given also by Coleman and Van Vleck[1], starting from the Darwin Hamiltonian (v. problem 6 of chapter III). They employ the point $\boldsymbol{X}^*$ (A88) as the centre of energy. This means that the relative coordinates with respect to the privileged point satisfy a relation like (A101) but without the fourth term. It follows however from the discussion leading to (A104) that such a definition is biased by the choice of the observer's frame: in other words such a definition of the centre of

[1] S. Coleman and J. H. Van Vleck, Phys. Rev. **171** (1968) 1370.

energy is not even covariant up to order c^{-2}. Using this centre of energy they find an equation of motion up to order c^{-2}, which has a form similar to (A113) but with an extra term $c^{-2}(\mathrm{d}^2/\mathrm{d}t^2)(\sum_i m_i\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\dot{\boldsymbol{r}}_i)$ at the left-hand side. Making a multipole expansion and taking the semi-relativistic limit, one arrives then at an equation like (A118) without the term $-c^{-1}(\nabla\boldsymbol{E})\cdot(\boldsymbol{v}\wedge\boldsymbol{v})$. Coleman and Van Vleck limit themselves to the case of a magnetic dipole in an electric field and thus find for the force only the term $-c^{-1}(\mathrm{d}/\mathrm{d}t)(\boldsymbol{v}\wedge\boldsymbol{E})$.

b. *The energy equation*

The energy equation is obtained by multiplying the equation of motion (A87) with $\dot{\boldsymbol{R}}_i$ and summing over i:

$$\sum_i \frac{\mathrm{d}}{\mathrm{d}t}\{m_i(1+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{R}}_i^2)\dot{\boldsymbol{R}}_i\}\cdot\dot{\boldsymbol{R}}_i = \sum_i e_i\dot{\boldsymbol{R}}_i\cdot\boldsymbol{e}_t(\boldsymbol{R}_i, t). \qquad \text{(A119)}$$

If the relative positions $\boldsymbol{r}_i$ (A91) are introduced, and the relation (A101) is used, one gets as the left-hand side the time derivative of the quantity

$$\begin{aligned}&\tfrac{1}{2}m\dot{\boldsymbol{X}}^2+\tfrac{3}{8}c^{-2}m\dot{\boldsymbol{X}}^4+\sum_i\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i^2\\ &\quad -c^{-2}\sum_i\left\{\dot{\boldsymbol{X}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i^2\boldsymbol{r}_i+\sum_{j(\neq i)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\boldsymbol{r}_i+m_i\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\dot{\boldsymbol{r}}_i\right)\right.\\ &\qquad\left.-\tfrac{3}{4}m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{X}}^2-\tfrac{3}{2}m_i(\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}})^2-\tfrac{3}{2}m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}-\tfrac{3}{8}m_i\dot{\boldsymbol{r}}_i^4\right\}. \qquad \text{(A120)}\end{aligned}$$

If the time derivative in the fourth term is performed, and the equations of motion (A106) up to order c^0 are employed, one obtains:

$$\begin{aligned}&\tfrac{1}{2}m\dot{\boldsymbol{X}}^2+\tfrac{3}{8}c^{-2}m\dot{\boldsymbol{X}}^4+\sum_i\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i^2+c^{-2}\sum_i\left[\tfrac{3}{4}m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{X}}^2+\tfrac{1}{2}m_i(\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}})^2+m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}\right.\\ &\quad+\tfrac{3}{8}m_i\dot{\boldsymbol{r}}_i^4-\sum_{j(\neq i)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\left\{\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}-\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot(\dot{\boldsymbol{r}}_i-\dot{\boldsymbol{r}}_j)\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\right\}\\ &\quad-\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}(\dot{\boldsymbol{X}}+\dot{\boldsymbol{r}}_i)\cdot\left\{e_i\boldsymbol{e}_t(\boldsymbol{R}_i,t)-\frac{m_i}{m}\sum_j e_j\boldsymbol{e}_t(\boldsymbol{R}_j,t)\right\}\\ &\qquad\left.-\frac{m_i}{m}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}\boldsymbol{r}_i\cdot\{\sum_j e_j\boldsymbol{e}_t(\boldsymbol{R}_j,t)\}\right]. \qquad \text{(A121)}\end{aligned}$$

The third term is the part of order c^0 of the internal kinetic energy in the observer's frame. We want to introduce the kinetic energy in the rest frame of the composite particle, since this is an invariant quantity. Instead of using

the total time derivative of $\boldsymbol{r}_i'$, which would be obtained from (A100), it is more convenient to use the time derivative of $\boldsymbol{r}_i'$ at constant transformation velocity. It follows from (A96) or (A97) by taking $\dot{\boldsymbol{X}}^*(t)$ constant. Moreover we want to take into account the fact that time differentiations in the observer's frame and in the rest frame differ by a factor $1-\frac{1}{2}c^{-2}\{\dot{\boldsymbol{X}}^*(t)\}^2$. In this way we are led to the introduction[1] of the quantity $\dot{\boldsymbol{r}}_i'$ by means of

$$\dot{\boldsymbol{r}}_i=\dot{\boldsymbol{r}}_i'-\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}^2\dot{\boldsymbol{r}}_i'+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{r}}_i\dot{\boldsymbol{X}}-c^{-2}\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{r}}_i\dot{\boldsymbol{R}}_i-c^{-2}\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\ddot{\boldsymbol{R}}_i, \qquad \text{(A122)}$$

where we used the fact that $\dot{\boldsymbol{X}}^*$ and $\dot{\boldsymbol{X}}$ differ by terms of order c^{-2} only. From (A122) and the equation of motion in zeroth order (A106) it follows that

$$\begin{aligned}\sum_i\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i^2&=\sum_i\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i'^2-c^{-2}\sum_i\{\tfrac{1}{2}m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{X}}^2+\tfrac{1}{2}m_i(\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}})^2\\ &\qquad+m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}+e_i\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}\dot{\boldsymbol{r}}_i\cdot\boldsymbol{e}_t(\boldsymbol{R}_i,t)\}. \qquad \text{(A123)}\end{aligned}$$

Substituting this expression into (A121) and using (A101) we obtain as the left-hand side of the energy equation the time derivative of

$$\begin{aligned}&\tfrac{1}{2}m\dot{\boldsymbol{X}}^2+\tfrac{3}{8}c^{-2}m\dot{\boldsymbol{X}}^4+\tfrac{1}{2}\sum_i m_i\dot{\boldsymbol{r}}_i'^2+c^{-2}\sum_i\left[\tfrac{1}{4}m_i\dot{\boldsymbol{r}}_i^2\dot{\boldsymbol{X}}^2+\tfrac{3}{8}m_i\dot{\boldsymbol{r}}_i^4\right.\\ &\quad-\sum_{j(\neq i)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\left\{\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}-\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot(\dot{\boldsymbol{r}}_i-\dot{\boldsymbol{r}}_j)\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\right\}-e_i\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}(\dot{\boldsymbol{X}}+2\dot{\boldsymbol{r}}_i)\cdot\boldsymbol{e}_t(\boldsymbol{R}_i,t)\\ &\qquad\left.+\frac{m_i}{m}(\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}\dot{\boldsymbol{r}}_i-\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}\boldsymbol{r}_i)\cdot\sum_j e_j\boldsymbol{e}_t(\boldsymbol{R}_j,t)\right]. \qquad \text{(A124)}\end{aligned}$$

In the right-hand side of the energy equation (A119) we now substitute the expression (A111) for the total electric field. Then we obtain

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\left(1+c^{-2}\left[\tfrac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_j+\frac{1}{2}\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{r}}_i(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{r}}_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\right.\right.\\ &\quad\left.\left.+\tfrac{1}{2}\dot{\boldsymbol{X}}^2+\frac{1}{2}\frac{\{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{X}}\}^2}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}+\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{X}}+\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{r}}_i(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{X}}}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\right]\right)\\ &\qquad+\sum_i e_i\dot{\boldsymbol{R}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t). \qquad \text{(A125)}\end{aligned}$$

The equation of energy (A119) becomes finally with (A100), (A110), (A122),

[1] Earlier in this appendix we did not need to make a difference between $\dot{\boldsymbol{r}}_i$ and $\dot{\boldsymbol{r}}_i'$ since they only occurred with factors c^{-1} or c^{-2}. The quantity $\dot{\boldsymbol{r}}_i$ introduced here is the c^{-2} approximation of (A34).

(A124) and (A125)

$$\frac{d}{dt}\Big(\tfrac12 m\dot{X}^2+\tfrac38 c^{-2}m\dot{X}^4+\sum_i(\tfrac12 m_i\dot{r}_i'^2+\tfrac14 c^{-2}m_i\dot{r}_i'^2\dot{X}^2+\tfrac38 c^{-2}m_i\dot{r}_i'^4)$$
$$+\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|r_i'-r_j'|}\Big[1+\tfrac12 c^{-2}\Big\{\dot{r}_i'\cdot\dot{r}_j'+\frac{(r_i'-r_j')\cdot\dot{r}_i'(r_i'-r_j')\cdot\dot{r}_j'}{|r_i'-r_j'|^2}+\dot{X}^2\Big\}\Big]\Big)$$
$$=\sum_i e_i\dot{R}_i\cdot E(R_i,t)+c^{-2}\frac{d}{dt}\Big[\sum_i e_i\Big\{\frac{\bar{s}}{m}\wedge E(R_i,t)$$
$$+2\dot{r}_i\cdot E(R_i,t)r_i+\dot{X}\cdot r_iE(R_i,t)\Big\}\cdot\dot{X}\Big]. \quad (A126)$$

This is the energy equation up to order c^{-2} for a composite particle (cf. the equation of motion (A113)). It shows which corrections of order c^{-2} arise as compared to the non-relativistic equation of chapter I. If the fields from outside the atom are slowly varying, we may perform a multipole expansion and introduce the multipole moments (A115). Just as for the equation of motion we shall confine ourselves to the contributions of the charge and the dipole moments. Moreover we introduce again the *semi-relativistic* approximation, as in the preceding subsection. We then obtain from (A126), using also the field equations (A116), and the notation v for the atomic velocity $\dot{X}$:

$$\frac{d}{dt}\Big(\tfrac12 mv^2+\sum_i\tfrac12 m_i\dot{r}_i'^2+\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|r_i'-r_j'|}\Big)$$
$$=ev\cdot E+v\cdot(\nabla E)\cdot(\mu-c^{-1}\nu\wedge v)+v\cdot(\nabla B)\cdot(\nu+c^{-1}\mu\wedge v)$$
$$+\Big\{\frac{d}{dt}(\mu-c^{-1}\nu\wedge v)\Big\}\cdot E-(\nu+c^{-1}\mu\wedge v)\cdot\frac{dB}{dt}+2c^{-1}\frac{d}{dt}\{(\nu\wedge v)\cdot E\}, \quad (A127)$$

which is the semi-relativistic energy equation for a charged dipole particle in an external field. (It might have been obtained by taking the limit of the relativistic equation.) As compared to the non-relativistic equation (I.67) various new terms with magnetic dipoles in motion arise here.

3. The angular momentum equation

The angular momentum equation for a composite particle is obtained by multiplying the equation of motion (A87) with the position R_i and summing over the index i that labels the constituent particles:

$$\sum_i R_i\wedge\frac{d}{dt}\{(m_i(1+\tfrac12 c^{-2}\dot{R}_i^2)\dot{R}_i\}=\sum_i e_iR_i\wedge\{e_t(R_i,t)+c^{-1}\dot{R}_i\wedge b_t(R_i,t)\}. \quad (A128)$$

The left-hand side is the time derivative of the expression:

$$\sum_i m_i(1+\tfrac12 c^{-2}\dot{R}_i^2)R_i\wedge\dot{R}_i. \quad (A129)$$

We introduce now the relative positions r_i (A91) and use the centre of energy condition (A101). This gives, up to order c^{-2}, for (A129):

$$mX\wedge\dot{X}+\bar{s}+c^{-2}(\tfrac12 m\dot{X}^2+\tfrac12\sum_i m_i\dot{r}_i^2)X\wedge\dot{X}$$
$$+c^{-2}\sum_i m_i(\tfrac12\dot{X}^2+\dot{X}\cdot\dot{r}_i+\tfrac12\dot{r}_i^2)r_i\wedge\dot{r}_i-c^{-2}(\bar{s}\wedge\dot{X})\wedge\dot{X}$$
$$-c^{-2}\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|r_i-r_j|}\Big\{r_i\wedge\dot{X}+X\wedge\dot{r}_i-\frac{(r_i-r_j)\cdot(\dot{r}_i-\dot{r}_j)}{|r_i-r_j|^2}X\wedge r_i\Big\}$$
$$-c^{-2}\sum_i m_iX\wedge(\dot{r}_i\cdot\ddot{r}_ir_i+\ddot{X}\cdot r_i\dot{r}_i+\dot{X}\cdot r_i\ddot{r}_i), \quad (A130)$$

where the non-relativistic inner angular momentum $\bar{s}$ (A110) has been introduced. The accelerations in the last three terms may be eliminated by means of the equation of motion (A106) up to order c^0. For the electric field which then appears we write expression (A108). In this way we get as the left-hand side of the angular momentum equation the time derivative of the expression

$$mX\wedge\dot{X}+\bar{s}+c^{-2}(\tfrac12 m\dot{X}^2+\tfrac12\sum_i m_i\dot{r}_i^2)X\wedge\dot{X}$$
$$+c^{-2}\sum_i m_i(\tfrac12\dot{X}^2+\dot{X}\cdot\dot{r}_i+\tfrac12\dot{r}_i^2)r_i\wedge\dot{r}_i-c^{-2}(\bar{s}\wedge\dot{X})\wedge\dot{X}$$
$$-c^{-2}\sum_i e_iX\wedge\Big\{\dot{r}_i\cdot E(R_i,t)r_i+\dot{X}\cdot r_iE(R_i,t)+\frac{\bar{s}}{m}\wedge E(R_i,t)\Big\}$$
$$-c^{-2}\sum_{i,j(i\neq j)}\frac{e_ie_j}{8\pi|r_i-r_j|}\Big\{r_i\wedge\dot{X}+X\wedge\dot{r}_i+\frac{(r_i-r_j)\cdot\dot{R}_iX\wedge(r_i-r_j)}{|r_i-r_j|^2}\Big\}. \quad (A131)$$

The right-hand side of the angular momentum equation (A128) contains the total electromagnetic field, for which we substitute the expressions (A111). In this way one finds for the right-hand side

$$-c^{-2}\sum_{i,j(i\neq j)}\frac{d}{dt}\Big[\frac{e_ie_j}{8\pi|r_i-r_j|}\Big\{R_i\wedge\dot{R}_j+\frac{(r_i-r_j)\cdot\dot{R}_jR_i\wedge(r_i-r_j)}{|r_i-r_j|^2}\Big\}\Big]$$
$$+\sum_i e_iR_i\wedge\{E(R_i,t)+c^{-1}\dot{R}_i\wedge B(R_i,t)\}. \quad (A132)$$

The angular momentum equation (A128) becomes with (A131) and (A132):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[& m\boldsymbol{X}\wedge\dot{\boldsymbol{X}}+\bar{\boldsymbol{s}}+c^{-2}\left(\tfrac{1}{2}m\dot{\boldsymbol{X}}^2+\tfrac{1}{2}\sum_i m_i\dot{\boldsymbol{r}}_i^2+\sum_{i,j(i\neq j)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\right)\boldsymbol{X}\wedge\dot{\boldsymbol{X}} \\
& +c^{-2}\sum_i m_i(\tfrac{1}{2}\dot{\boldsymbol{X}}^2+\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{r}}_i+\tfrac{1}{2}\dot{\boldsymbol{r}}_i^2)\boldsymbol{r}_i\wedge\dot{\boldsymbol{r}}_i-c^{-2}(\bar{\boldsymbol{s}}\wedge\dot{\boldsymbol{X}})\wedge\dot{\boldsymbol{X}} \\
& +c^{-2}\sum_{i,j(i\neq j)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\left\{\boldsymbol{r}_i\wedge\dot{\boldsymbol{r}}_j-\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{R}}_j\,\boldsymbol{r}_i\wedge\boldsymbol{r}_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\right\}\Bigg] \\
& =\sum_i e_i\boldsymbol{R}_i\wedge\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\} \\
& +c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i e_i\boldsymbol{X}\wedge\left\{\dot{\boldsymbol{r}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{r}_i+\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\boldsymbol{E}(\boldsymbol{R}_i,t)+\frac{\bar{\boldsymbol{s}}}{m}\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)\right\}\right].
\end{aligned}
\tag{A133}
$$

This equation still contains the position $\boldsymbol{X}$ of the composite particle. It may be eliminated with the help of the equation which results if the equation of motion (A113) is multiplied (vectorially) by $\boldsymbol{X}$. Then we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[& \bar{\boldsymbol{s}}+c^{-2}\sum_i m_i(\tfrac{1}{2}\dot{\boldsymbol{X}}^2+\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{r}}_i+\tfrac{1}{2}\dot{\boldsymbol{r}}_i^2)\boldsymbol{r}_i\wedge\dot{\boldsymbol{r}}_i-c^{-2}(\bar{\boldsymbol{s}}\wedge\dot{\boldsymbol{X}})\wedge\dot{\boldsymbol{X}} \\
& +c^{-2}\sum_{i,j(i\neq j)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\left\{\boldsymbol{r}_i\wedge\dot{\boldsymbol{r}}_j-\frac{(\boldsymbol{r}_i-\boldsymbol{r}_j)\cdot\dot{\boldsymbol{R}}_j\,\boldsymbol{r}_i\wedge\boldsymbol{r}_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^2}\right\}\Bigg] \\
& =\sum_i e_i\boldsymbol{r}_i\wedge\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\} \\
& +c^{-2}\sum_i e_i\boldsymbol{X}\wedge\left\{\dot{\boldsymbol{r}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{r}_i+\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\boldsymbol{E}(\boldsymbol{R}_i,t)+\frac{\bar{\boldsymbol{s}}}{m}\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)\right\}.
\end{aligned}
\tag{A134}
$$

The non-relativistic inner angular momentum $\bar{\boldsymbol{s}}$ (A110), which occurs here, has been defined in terms of $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$. If we eliminate these quantities in the left-hand side in favour of $\boldsymbol{r}_i'$ and $\dot{\boldsymbol{r}}_i'$, with (A100) and (A122), we obtain as inner angular momentum equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[& \sum_i m_i(1+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{r}}_i'^2)\boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_i'-\tfrac{1}{2}c^{-2}\{(\sum_i m_i\boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_i')\wedge\dot{\boldsymbol{X}}\}\wedge\dot{\boldsymbol{X}} \\
& +c^{-2}\sum_{i,j(i\neq j)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i'-\boldsymbol{r}_j'|}\left\{\boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_j'-\frac{(\boldsymbol{r}_i'-\boldsymbol{r}_j')\cdot\dot{\boldsymbol{r}}_j'\,\boldsymbol{r}_i'\wedge\boldsymbol{r}_j'}{|\boldsymbol{r}_i'-\boldsymbol{r}_j'|^2}\right\}\Bigg] \\
& =\sum_i e_i\boldsymbol{r}_i\wedge\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\} \\
& +c^{-2}\sum_i e_i\dot{\boldsymbol{X}}\wedge\left\{\dot{\boldsymbol{r}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{r}_i+\dot{\boldsymbol{X}}\cdot\boldsymbol{r}_i\boldsymbol{E}(\boldsymbol{R}_i,t)+\frac{\bar{\boldsymbol{s}}}{m}\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)\right\} \\
& +c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\{\sum_i e_i\boldsymbol{r}_i\cdot\dot{\boldsymbol{X}}\boldsymbol{r}_i\wedge\boldsymbol{E}(\boldsymbol{R}_i,t)\}.
\end{aligned}
\tag{A135}
$$

If the external fields are slowly varying we may perform a multipole expansion of the right-hand side and limit ourselves to dipole terms. Expressing the results (with the help of (A100)) in terms of the dipole moments (A115), and retaining only terms up to c^{-1} (considering magnetic dipole moments as being of order c^0) we obtain the *semi-relativistic* inner angular momentum equation:

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\sum_i m_i\boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_i')=\boldsymbol{\mu}\wedge(\boldsymbol{E}+c^{-1}\boldsymbol{v}\wedge\boldsymbol{B})+\boldsymbol{\nu}\wedge(\boldsymbol{B}-c^{-1}\boldsymbol{v}\wedge\boldsymbol{E}). \tag{A136}
$$

Here we have written $\boldsymbol{v}$ for the velocity $\dot{\boldsymbol{X}}$ of the composite particle. This equation (which may be found also from the relativistic theory namely from (136)) shows a symmetry between electric and magnetic phenomena, which was absent in the non-relativistic equation (I.79).

PROBLEMS

1. It is possible to prove directly the covariance of the dipole contributions (48) and (49) to the polarization tensor (although the manifestly covariant derivation of these formulae guarantees their covariance already). To that end consider first the Lorentz transformation from the momentary atomic rest frame $(ct^{(0)}, \boldsymbol{R}^{(0)})$ to the observer's frame $(ct, \boldsymbol{R})$ which transforms the coordinates $(ct^{(0)}, \boldsymbol{R}_k^{(0)}(t^{(0)}))$ of a point of the world line of atom k into $(ct_k, \boldsymbol{R}_k(t_k))$, where in general t_k differs from t. (The transformation velocity is $c\boldsymbol{\beta}_k(t_k) \equiv \mathrm{d}\boldsymbol{R}_k(t_k)/\mathrm{d}t_k$.) Show from the transformation formulae that

$$c(t_k - t) = \gamma_k^2(t)\boldsymbol{\beta}_k(t)\cdot\{\boldsymbol{R}_k(t) - \boldsymbol{R}\} + \ldots, \tag{P1}$$

where a Taylor expansion has been performed and where the dots stand for terms of higher order in $\boldsymbol{R}_k(t) - \boldsymbol{R}$. Prove then, again using the Lorentz transformation formulae, that

$$\boldsymbol{R}_k^{(0)}(t^{(0)}) - \boldsymbol{R}^{(0)} = \boldsymbol{\Omega}_k^{-1}(t)\cdot\{\boldsymbol{R}_k(t) - \boldsymbol{R}\} + \ldots, \tag{P2}$$

where $\boldsymbol{\Omega}^{-1}$ has been given in (A3). Show from the latter formula that

$$\delta\{\boldsymbol{R}_k^{(0)}(t^{(0)}) - \boldsymbol{R}^{(0)}\} = \gamma_k^{-1}(t)\delta\{\boldsymbol{R}_k(t) - \boldsymbol{R}\}, \tag{P3}$$

where (A16) has been used.

Show with (P1), (P3) and the transformation property (A5) of an antisymmetric tensor that one obtains the formulae (48) and (49) starting from the formulae

$$\begin{aligned} \boldsymbol{p}_k^{(0)}(\boldsymbol{R}^{(0)}, t^{(0)}) &= \boldsymbol{\mu}_k^{(1)}(t^{(0)})\delta\{\boldsymbol{R}_k^{(0)}(t^{(0)}) - \boldsymbol{R}^{(0)}\}, \\ \boldsymbol{m}_k^{(0)}(\boldsymbol{R}^{(0)}, t^{(0)}) &= \boldsymbol{\nu}_k^{(1)}(t^{(0)})\delta\{\boldsymbol{R}_k^{(0)}(t^{(0)}) - \boldsymbol{R}^{(0)}\}, \end{aligned} \tag{P4}$$

which are the atomic rest frame cases of (48) and (49).

2. Prove from (69) and (70) that, for sufficiently small forces, the world line determined by the centre of energy construction of section 3*b* is time-like. To that end consider the world line points $(R^0, \boldsymbol{X})$ and $(R^0 + \delta R^0, \boldsymbol{X} + \delta\boldsymbol{X})$ in the proper frame of p^α corresponding to the first point. The second point is a centre of energy in a plane surface Σ' with normal parallel to $p^\alpha + \delta p^\alpha$. The proper frame of $p^\alpha + \delta p^\alpha$ is connected to the proper frame of p^α by relations like (71). The point $(R^0 + \delta R^0, \boldsymbol{X} + \delta\boldsymbol{X})$ has in the new frame the coordinates

$(\hat{R}^{0\prime}, \boldsymbol{X}' + \delta\boldsymbol{X}')$. Prove that in the present case one gets

$$\begin{aligned} \hat{R}^0 &= R^0 + \delta R^0 + \boldsymbol{\varepsilon}\cdot(\boldsymbol{X} - \hat{\boldsymbol{R}}'), \\ \hat{\boldsymbol{R}} &= \hat{\boldsymbol{R}}' - \boldsymbol{\varepsilon} R^0 \end{aligned}$$

in the same notation as in (76–77). Show then with the help of the equation of motion (64) that one has the relations

$$\int [\{\boldsymbol{\varepsilon}\cdot(\boldsymbol{X} - \boldsymbol{R}) + \delta R^0\}\boldsymbol{f}(R^0, \boldsymbol{R}) + \boldsymbol{\varepsilon} t^{00}(R^0, \boldsymbol{R})]\mathrm{d}\boldsymbol{R} = 0,$$

$$\int [\delta\boldsymbol{X} t^{00}(R^0, \boldsymbol{R}) + (\boldsymbol{X} - \boldsymbol{R})\{\boldsymbol{\varepsilon}\cdot(\boldsymbol{X} - \boldsymbol{R}) + \delta R^0\} f^0(R^0, \boldsymbol{R})]\mathrm{d}\boldsymbol{R} - c\boldsymbol{\varepsilon}\wedge\boldsymbol{s} = 0,$$

where (68) has been used. From the first of these relations it follows that $\boldsymbol{\varepsilon}$ vanishes if no forces are present, so that for small forces one has

$$\boldsymbol{\varepsilon} = -\delta R^0 \frac{\int \boldsymbol{f}\mathrm{d}\boldsymbol{R}}{\int t^{00}\mathrm{d}\boldsymbol{R}}$$

up to terms linear in the forces. Prove by substitution of this expression into the second relation that the infinitesimal change $\delta\boldsymbol{X}$ of the centre of energy and the change of time $c^{-1}\delta R^0$ are related as

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}R^0} = \frac{c^{-2}}{m}\int (\boldsymbol{R} - \boldsymbol{X}) f^0 \mathrm{d}\boldsymbol{R} - \frac{c^{-3}}{m^2}\left(\int \boldsymbol{f}\mathrm{d}\boldsymbol{R}\right)\wedge\boldsymbol{s},$$

where we introduced $m = c^{-2}\int t^{00}\mathrm{d}\boldsymbol{R}$ as an abbreviation (v. (120) and (123)). This formula shows that $\mathrm{d}\boldsymbol{X}/\mathrm{d}R^0$ tends to zero if the forces tend to zero, so that the world line is time-like if the forces are sufficiently small.

Discuss the connexion between the last formula and (124) (together with (120) and (123)).

3. Consider two plane surfaces $\Sigma(s)$ and $\Sigma(s + \mathrm{d}s)$, one through the position $X^\alpha(s)$ (in the same notation as used in § 3*c*) and with normal $n^\alpha(s)$, and the other through $X^\alpha(s + \mathrm{d}s)$ and with normal $n^\alpha(s + \mathrm{d}s)$. Their equations are therefore $n^\alpha(s)\{R_\alpha - X_\alpha(s)\} = 0$ and $n^\alpha(s + \mathrm{d}s)\{R_\alpha - X_\alpha(s + \mathrm{d}s)\} = 0$. Prove now that the volume $\mathrm{d}^4 V$ of a parallelepiped with basis $\mathrm{d}^3\Sigma$ in $\Sigma(s)$ at the position R^α, with edges parallel to $n^\alpha(s)$ and top in $\Sigma(s + \mathrm{d}s)$ is

$$\mathrm{d}^4 V = -n_\alpha u^\alpha \left\{1 - \frac{\dot{n}_\beta (R - X)^\beta}{n_\gamma u^\gamma}\right\} \mathrm{d}^3\Sigma\, \mathrm{d}s,$$

where n^α, X^α and $u^\alpha \equiv \mathrm{d}X^\alpha/\mathrm{d}s$ all depend on s.

By choosing for n^α the vector $p^\alpha/\sqrt{-p^2}$ one recovers formula (94) with (95). (If one chooses for n^α the vector $c^{-1}u^\alpha$ one finds back the expression given in the footnote of chapter III after formula (129).)

4. Consider, as in section 2*a*, a composite particle consisting of point particles with charges e_i (we omit the index k, which labelled the composite particle). Prove, by making a multipole expansion around a central world line $X^\alpha(s)$, the identity

$$\sum_i e_i u_i^\alpha f(R_i) = eu^\alpha f(X)$$
$$+\sum_{n=1}^{\infty}(\mu^{\alpha_1\ldots\alpha_n}u^\alpha-\mu^{\alpha_1\ldots\alpha_{n-1}\alpha}u^{\alpha_n}+cv^{\alpha_1\ldots\alpha_n\alpha})\partial_{\alpha_1\ldots\alpha_n}f(X)$$
$$+\sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}s}\{\mu^{\alpha_1\ldots\alpha_n\alpha}\partial_{\alpha_1\ldots\alpha_n}f(X)\},$$

where f is an arbitrary function and $\mu^{\alpha_1\ldots\alpha_n}$ and $v^{\alpha_1\ldots\alpha_n}$ are the multipole moments (9); s is the proper time along the central world line X^α and $u^\alpha = \mathrm{d}X^\alpha/\mathrm{d}s$.

Apply this formula with $f(R_i) = \delta^{(4)}(R_i - R)$ and find then (7) with (5) and (10). Show furthermore that (15) and (16) are the multipole expansions of the integrand of (III.59).

Finally, by application to (106) with $f(R_i) = F^{\alpha\beta}(R_i)$ one may obtain (168) by making use of the homogeneous field equations (13).

5. Prove that, under the same circumstances as in the preceding problem, one has for an arbitrary function $f(R_i)$ the multipole expansion

$$\sum_i e_i r_i^\alpha u_i^\beta f(R_i)$$
$$=\sum_{n=1}^{\infty} n\mu^{\alpha_1\ldots\alpha_{n-1}\alpha}\partial_{\alpha_1\ldots\alpha_{n-1}}u^\beta f(X)+\sum_{n=2}^{\infty}(n-1)\frac{\mathrm{d}}{\mathrm{d}s}(\mu^{\alpha_1\ldots\alpha_{n-2}\alpha\beta})\partial_{\alpha_1\ldots\alpha_{n-2}}f(X)$$
$$+\sum_{n=1}^{\infty}\{-(n-1)v^{\alpha_1\ldots\alpha_{n-2}\alpha\beta\alpha_{n-1}}+v^{\alpha_1\ldots\alpha_{n-1}\alpha\beta}\}\partial_{\alpha_1\ldots\alpha_{n-1}}f(X).$$

Apply this formula to (107) with $f(R_i) = F^{\alpha\beta}(R_i)$ and find (173).

6. Prove from the definition (120) of m by evaluating it in the rest frame of p^α (65) with (62) that one has up to order c^{-2}:

$$m = \sum_i m_i(1+\tfrac{1}{2}c^{-2}\dot{r}_i^2)+c^{-2}\sum_{i,j(i\neq j)}\frac{e_i e_j}{8\pi|r_i - r_j|}.$$

Hint: The fields occurring in $t^{\alpha\beta}$ (62) follow from (III.72).

7. In the rest frame of p^α the definition (66) of the centre of energy reads $\boldsymbol{X} = \int \boldsymbol{R}t^{00}\mathrm{d}\boldsymbol{R}/\int t^{00}\mathrm{d}\boldsymbol{R}$. Prove that, up to order c^{-2}, this expression reduces to

$$\boldsymbol{X} = \frac{\sum_i m_i(1+\frac{1}{2}c^{-2}\dot{\boldsymbol{R}}_i^2)\boldsymbol{R}_i + c^{-2}\sum_{i,j(i\neq j)}(e_i e_j/8\pi|\boldsymbol{R}_i-\boldsymbol{R}_j|)\boldsymbol{R}_i}{\sum_i m_i(1+\frac{1}{2}c^{-2}\dot{\boldsymbol{R}}_i^2)+c^{-2}\sum_{i,j(i\neq j)}(e_i e_j/8\pi|\boldsymbol{R}_i-\boldsymbol{R}_j|)}.$$

Hint: the denominator follows along the same lines as in the preceding problem, and so does the material part of the numerator. For the field part of the numerator one finds first $\frac{1}{2}c^{-2}\int\sum_{i,j(i\neq j)}\boldsymbol{e}_i\cdot\boldsymbol{e}_j\boldsymbol{R}\mathrm{d}\boldsymbol{R}$ with $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ the Coulomb fields, due to particle i and j. With the help of a partial integration, the application of Gauss's theorem and the integration prescription of section 3*b* (according to which one has to integrate over a spherical volume of unboundedly increasing dimension) one obtains the last term of the numerator.

8. Show from the definition (68) with (62) that up to order c^{-2} the space-space components of the inner angular momentum tensor $s^{\alpha\beta}$ are given by

$$s^{kl} = \Big[\sum_i m_i \boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_i'(1+\tfrac{1}{2}c^{-2}\dot{\boldsymbol{r}}_i'^2)-\tfrac{1}{2}c^{-2}\{(\sum_i m_i\boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_i')\wedge\dot{\boldsymbol{X}}\}\wedge\dot{\boldsymbol{X}}$$
$$+c^{-2}\sum_{i,j(i\neq j)}\frac{e_i e_j}{8\pi|\boldsymbol{r}_i'-\boldsymbol{r}_j'|}\Big\{\boldsymbol{r}_i'\wedge\dot{\boldsymbol{r}}_j'-\frac{(\boldsymbol{r}_i'-\boldsymbol{r}_j')\cdot\dot{\boldsymbol{r}}_j'(\boldsymbol{r}_i'\wedge\boldsymbol{r}_j')}{|\boldsymbol{r}_i'-\boldsymbol{r}_j'|^2}\Big\}\Big]^m$$

(with $k, l, m = 1, 2, 3$ cycl.) in the same notation as in appendix V. The material part of this expression follows directly from the material part of $t^{\alpha\beta}$ (62), if one employs the defining relations(A100) and (A122) for $\boldsymbol{r}_i'$ and $\dot{\boldsymbol{r}}_i'$, and the transformation formulae (A5) for the antisymmetric tensor $s^{\alpha\beta}$. To find the field part, one should use the expressions (III.72) for the fields up to order c^{-2} or rather the expressions (III.70) with (III.83) for the fields in terms of Coulomb gauge potentials. Furthermore one should employ here the integration prescription, as explained in § 3*b* (i.e. integrating over a sphere of increasing radius around the origin).

9. Prove from (A112) and the first line of (A106) with (A108) that one may write the sum of the total Lorentz forces on a set of particles i with mass m_i, charge e_i, position $\boldsymbol{R}_i$, velocity $\dot{\boldsymbol{R}}_i$, acceleration $\ddot{\boldsymbol{R}}_i$ as

$$\sum_i e_i\{\boldsymbol{e}_t(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{b}_t(\boldsymbol{R}_i,t)\}$$
$$=-c^{-2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(\sum_{i,j(i\neq j)}\frac{e_i e_j\boldsymbol{R}_i}{8\pi|\boldsymbol{R}_i-\boldsymbol{R}_j|}\Big)-c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}(\sum_i m_i\dot{\boldsymbol{R}}_i\cdot\ddot{\boldsymbol{R}}_i\boldsymbol{R}_i)$$
$$+\sum_i e_i\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\}+c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\{\sum_i e_i\dot{\boldsymbol{R}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{R}_i\}. \quad \text{(P5)}$$

Prove by insertion of this expression into (A87) (with a summation over i) that one obtains:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\sum_i\left(m_i+\tfrac{1}{2}c^{-2}m_i\dot{\boldsymbol{R}}_i^2+c^{-2}\sum_{j(\neq i)}\frac{e_ie_j}{8\pi|\boldsymbol{R}_i-\boldsymbol{R}_j|}\right)\boldsymbol{R}_i\right\}$$
$$=\sum_i e_i\{\boldsymbol{E}(\boldsymbol{R}_i,t)+c^{-1}\dot{\boldsymbol{R}}_i\wedge\boldsymbol{B}(\boldsymbol{R}_i,t)\}+c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\{\sum_i e_i\dot{\boldsymbol{R}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{R}_i\}. \quad \text{(P6)}$$

Check that one recovers the equation of motion (A113) by employing (A91), (A101) and (A106).

Remark. At the right-hand side of (P6) two terms with the external fields $\boldsymbol{E}$ and $\boldsymbol{B}$ appear: the ordinary Lorentz force and an extra term. The latter has a form analogous to the term

$$c^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\{\sum_i e_i\dot{\boldsymbol{r}}_i\cdot\boldsymbol{E}(\boldsymbol{R}_i,t)\boldsymbol{r}_i\},$$

which appears in the semi-relativistic equation of motion (A113). This term led, via the multipole expansion, to the term

$$-c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}\wedge\boldsymbol{E})$$

of the dipole equation of motion (A117), i.e. to the 'magnetodynamic effect' associated with the total magnetic moment. Some discussions (e.g. H. A. Haus and P. Penfield Jr., Physica **42**(1969)447) about the magnetodynamic effect limit themselves to the derivation of (part of) the terms given at the right-hand side of (P5), namely the terms with $\boldsymbol{E}$ and $\boldsymbol{B}$. The problem of deriving an equation of motion which involves the definition of a proper centre of energy is not solved then. The conclusion that also inner angular momentum terms appear in the equation, so that effectively only the anomalous magnetic moment contributes, is then missed.

10. Prove that the time-component of the equation (160) reads in three-dimensional notation:

$$m^*\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\gamma^{-4}\left[e\boldsymbol{E}\cdot\gamma\boldsymbol{v}-\mathfrak{m}\cdot\left(\frac{\partial\boldsymbol{B}}{\partial t}-\boldsymbol{\beta}\wedge\frac{\partial\boldsymbol{E}}{\partial t}\right)\right.$$
$$\left.+\gamma^2\mathfrak{m}\cdot\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}-\boldsymbol{\beta}\wedge\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\right)-\gamma^2(\boldsymbol{\beta}\wedge\mathfrak{m}_{(\mathrm{a})})\cdot\left(\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}+\boldsymbol{\beta}\wedge\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)\right].$$

Compare this result with that of problem 6 of chapter VIII.