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Relativistic Bound-State Equations for Fermions with Instantaneous Interactions

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In this paper three types of relativistic bound-state equations for a fermion pair with instantaneous interaction are studied, viz., the instantaneous Bethe–Salpeter equation, the quasi-potential equation, and the two-particle Dirac equation. General forms for the equations describing bound states with arbitrary spin, parity, and charge parity are derived. For the special case of spinless states bound by interactions with a Coulomb-type potential the properties of the ground-state solutions of the three equations are investigated both analytically and numerically. The coupling-constant spectrum turns out to depend strongly on the spinor structure of the fermion interaction. If the latter is chosen such that the nonrelativistic limits of the equations coincide, an analogous spectrum is found for the instantaneous Bethe–Salpeter and the quasi-potential equations, whereas the two-particle Dirac equation yields qualitatively different results.

1. INTRODUCTION

The Bethe–Salpeter equation is a well-known instrument to study the properties of a pair of fermions that are strongly bound to each other [1, 2]. In this formalism the bound states are described by a wavefunction χ that depends on all components of the momentum-transfer four-vector q^μ. The explicit dependence of the wavefunction on the time component q^0 of q^μ is a peculiar feature of the relativistic Bethe–Salpeter equation, which has led to difficulties in its interpretation, since it has no counterpart in the nonrelativistic theory.

In the ladder-approximated form of the Bethe–Salpeter equation the interaction that causes the binding is represented by a potential function. The retardation effects, which are characteristic for a potential in a relativistic theory, necessitate the use of wavefunctions depending on q^0. Indeed, when the potential is chosen to be instantaneous in a privileged Lorentz frame, a reduced bound-state wavefunction may be introduced from which q^0 has been eliminated [3]. The reduced wavefunction satisfies an instantaneous Bethe–Salpeter equation which bears a closer resemblance to the familiar nonrelativistic bound-state equation.

Several alternative approaches to the bound-state problem for instantaneous binding forces have been proposed in the literature. In the quasi-potential formalism [4–6] one arrives at a bound-state equation that differs from the instantaneous

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Bethe–Salpeter equation in the neglect of some terms related to pair creation. A radical omission of all such terms leads to a two-particle Dirac equation that can be considered as a straightforward generalization of the ordinary Dirac equation of single-particle theory [7–9].

In the nonrelativistic limit all these equations should yield the same coupling-constant spectrum, at least for suitable combinations of the five possible fermion couplings. For deeply bound states the approximations that lead to the quasi-potential and two-particle Dirac equations are difficult to justify. In that case the principal motivation for the interest in these equations is their simpler structure as compared to the instantaneous Bethe–Salpeter equation. A judgment of the consequences of the various approximations can be given only when the resulting coupling-constant spectra are determined.

In the present paper the properties of bound-states of two spin $\frac{1}{2}$ particles with instantaneous interactions will be studied with the help of the three types of equations that have been mentioned. To obtain information about the bound states both analytical and numerical methods will be employed. The results will make possible a detailed comparison of the bound-state spectra so that the consequences of the various forms of the instantaneous approximations can indeed be assessed.

In Section 2 the instantaneous Bethe–Salpeter equations for states with definite spin, parity, and charge parity will be derived. Moreover the normalization and perturbation integrals for the wavefunctions will be established. Section 3 contains a discussion of the quasi-potential and the two-particle Dirac equations that follow from the Bethe–Salpeter equation by neglecting pair-creation terms.

For general interactions between the constituent fermions the bound-state equations lead to integral equations in momentum space that are difficult to solve. Much simpler equations result if the coupling constants of the interactions satisfy some special constraints, as will be shown in Section 4. Part of these equations can even be solved exactly if the interaction potential has a Coulomb form. In the remaining sections of this paper we shall concentrate therefore on the equations for spinless states bound by such potentials. In Section 5 some general properties of the coupling-constant spectrum are derived. In particular, we shall prove a number of inequalities for the bound-state mass and the coupling constants. Furthermore the nonrelativistic limit of the equations will be studied. Section 6 contains the results of a numerical investigation of the three types of equations for spinless states with negative parity. In Section 7 the conclusions of the paper are summarized.

2. The Spinor Bethe–Salpeter Equation for Instantaneous Potentials

The wavefunction $\chi(q)$ for bound states of a fermion–antifermion pair satisfies the spinor Bethe–Salpeter equation

$$(\not{q} + \tfrac{1}{2}\not{P} - 1)\,\chi(q)(\not{q} - \tfrac{1}{2}\not{P} - 1) = i\pi^{-1} \sum_{i=1}^{5} \int d^4q'\, V_i(q - q')\, \Gamma^i \chi(q')\, \Gamma^i. \quad (1)$$

Here P^μ is the momentum four-vector of the bound state with mass $2\epsilon_B$ (the mass of the constituent fermions is chosen as the unit of mass). The interactions characterized by the matrices Γ^i, with $\Gamma^S = 1$, $\Gamma^V = \gamma^\mu$, $\Gamma^T = \sigma^{\mu\nu}$, $\Gamma^A = \gamma^\mu\gamma_5$, $\Gamma^P = i\gamma_5$, are determined by the potential functions V_i, which are proportional to a coupling constant λ.

With the help of the projection operators

$$\Lambda_\mu^{\pm}(\mathbf{q}) = \tfrac{1}{2} + \tfrac{1}{2}\mu(\pm\boldsymbol{\alpha}\cdot\mathbf{q} + \gamma^0)/E \qquad (\mu = \pm 1), \tag{2}$$

with $E = (1 + q^2)^{1/2}$ and $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, the Bethe–Salpeter equation may be written as a set of four coupled equations [3]:

$$\begin{aligned}\Lambda_\mu^{+}(\mathbf{q})\,\chi(q)\,\Lambda_\nu^{-}(\mathbf{q}) &= i\pi^{-1}[(q^0 + \epsilon_B - \mu E)(q^0 - \epsilon_B - \nu E)]^{-1}\,\Lambda_\mu^{+}(\mathbf{q}) \\ &\quad\times \sum_{i=1}^{5}\int d^4q'\; V_i(q - q')\,\gamma^0\Gamma^i\chi(q')\,\Gamma^i\gamma^0\Lambda_\nu^{-}(\mathbf{q}) \qquad (\mu, \nu = \pm 1).\end{aligned} \tag{3}$$

If the potentials are instantaneous in the rest frame of P^μ, so that $V_i = V_i\,[(\mathbf{q} - \mathbf{q}')^2]$ integration over q^0 leads to an equation for the reduced wavefunction $\psi(\mathbf{q}) = \int dq^0\,\chi(q)$, viz. :

$$\begin{aligned}(\epsilon_B - \boldsymbol{\alpha}\cdot\mathbf{q} - \gamma^0)\,\psi(\mathbf{q}) + \psi(\mathbf{q})(\epsilon_B - \boldsymbol{\alpha}\cdot\mathbf{q} + \gamma^0) &= -2\sum_{i=1}^{5}\int d\mathbf{q}'\; V_i[(\mathbf{q} - \mathbf{q}')^2] \\ &\quad\times [\Lambda_+^{+}(\mathbf{q})\,\gamma^0\Gamma^i\psi(\mathbf{q}')\,\Gamma^i\gamma^0\Lambda_-^{-}(\mathbf{q}) - \Lambda_-^{+}(\mathbf{q})\,\gamma^0\Gamma^i\psi(\mathbf{q}')\,\Gamma^i\gamma^0\Lambda_+^{-}(\mathbf{q})].\end{aligned} \tag{4}$$

Multiplication by the appropriate operators (2) shows that this equation implies the algebraic constraints

$$\Lambda_\mu^{+}(\mathbf{q})\,\psi(\mathbf{q})\,\Lambda_\mu^{-}(\mathbf{q}) = 0 \qquad (\mu = \pm 1). \tag{5}$$

The general form of the reduced wavefunction $\psi(\mathbf{q})$ for bound states with angular momentum J, parity P, and charge parity C follows from its expansion in a complete set of Dirac matrices:

$$\psi(\mathbf{q}) = \psi_S + \psi_V{}^\mu\gamma_\mu + \psi_T^{\mu\nu}\sigma_{\mu\nu} + \psi_A{}^\mu\gamma_\mu\gamma_5 + \psi_P i\gamma_5\,, \tag{6}$$

by imposing the correct transformation properties with respect to rotation, space reflections, and charge conjugation. When the Pauli representation is chosen for the Dirac matrices (so that $\gamma^0 = \rho_3$, $\boldsymbol{\gamma} = i\rho_2\boldsymbol{\sigma}$) and the transformation properties of

scalar and vector spherical harmonics (see Appendix A) are employed one finds for the wavefunctions $\psi_J^{P'C'}$ of states with parity $P = (-)^J P'$ and charge parity $C = (-)^J C'$:

$$\psi_J^{++}(\mathbf{q}) = (\psi_S + i\psi_{V1}\rho_2\boldsymbol{\sigma}\cdot\hat{\mathbf{q}} + \psi_{T1}\rho_1\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\,\mathbf{Y}_J{}^M + (i\psi_{V2}\rho_2\boldsymbol{\sigma} + \psi_{T2}\rho_1\boldsymbol{\sigma} + i\psi_{A1}\rho_3\boldsymbol{\sigma}\wedge\hat{\mathbf{q}})\cdot\mathbf{Y}_J^{(e)M}, \tag{7}$$

$$\psi_J^{+-}(\mathbf{q}) = \psi_{V3}\rho_3 Y_J{}^M + i\psi_{T3}(\boldsymbol{\sigma}\wedge\hat{\mathbf{q}})\cdot\mathbf{Y}_J^{(e)M}, \tag{8}$$

$$\psi_J^{-+}(\mathbf{q}) = (\psi_P\rho_1 + i\psi_{A2}\rho_2 + \psi_{T4}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}})\,Y_J{}^M + \psi_{T5}\boldsymbol{\sigma}\cdot\mathbf{Y}_J^{(e)M}, \tag{9}$$

$$\psi_J^{--}(\mathbf{q}) = \psi_{A3}\rho_3\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\,Y_J{}^M + (\psi_{V4}\rho_2\boldsymbol{\sigma}\wedge\hat{\mathbf{q}} + i\psi_{T6}\rho_1\boldsymbol{\sigma}\wedge\hat{\mathbf{q}} + \psi_{A4}\rho_3\boldsymbol{\sigma})\cdot\mathbf{Y}_J^{(e)M}, \tag{10}$$

where $\hat{\mathbf{q}} = \mathbf{q}/|\,\mathbf{q}\,|$; the scalar structure functions ψ_i depend on q^2. For states with vanishing angular momentum J only the terms proportional to $Y_J{}^M$ show up; the structure functions multiplying $\mathbf{Y}_J^{(e)M}$ vanish then.

The conditions (5) lead to the following relations for the structure functions:

$$\psi_{V1} = q^{-1}\psi_S\,, \qquad \psi_{A1} = -q\psi_{T2}\,, \tag{11}$$

$$\psi_{V3} = \psi_{T3} = 0, \tag{12}$$

$$\psi_{T4} = q\psi_{A2}\,, \qquad \psi_{T5} = 0, \tag{13}$$

$$\psi_{A3} = 0, \qquad \psi_{A4} = -q\psi_{T6}\,. \tag{14}$$

As (12) with (8) shows, bound states with $P = (-)^J$, $C = (-)^{J+1}$ do not occur. States with $P = (-)^{J+1}$, $C = (-)^{J+1}$ are found only for $J > 0$, as follows from (14) with (10).

Sometimes a spectroscopic notation is used to denote the quantum numbers of a bound state. In this notation a state is given as ${}^{2s+1}L_J$ with quantum numbers J, $P = (-)^{L+1}$, and $C = (-)^{L+s}$. The wavefunctions (7), (9), and (10) correspond then to states ${}^3(J \pm 1)_J$, ${}^1J_J$, and ${}^3J_J$, respectively.

If the wavefunctions (7), (9), and (10) with (11), (13), and (14) are inserted into the instantaneous Bethe–Salpeter equation (4) and the coefficients of the Dirac matrices on both sides are compared one arrives at a set of coupled integral equations for the structure functions ψ_i. For the ${}^3(J \pm 1)_J$ states one finds, for instance:

$$E[(E^2q^{-1}\psi_S - \epsilon_B\psi_{T1})\,\hat{\mathbf{q}}\,Y_J{}^M + (\psi_{V2} - \epsilon_B\psi_{T2})\,\mathbf{Y}_J^{(e)M}] = -\int d\mathbf{q}'\,[(V'_V q'^{-1}\hat{\mathbf{q}}' - V'_S q\hat{\mathbf{q}})\,\psi_S Y_J{}^M + V'_V\psi_{V2}\mathbf{Y}_J^{(e)M}], \tag{15}$$

$$E[(\psi_{T1} - \epsilon_B q^{-1}\psi_S)\,\hat{\mathbf{q}}\,Y_J{}^M + (E^2\psi_{T2} - \epsilon_B\psi_{V2})\,\mathbf{Y}_J^{(e)M}] = -\int d\mathbf{q}'\,[V'_T\psi_{T1}\hat{\mathbf{q}}'\,Y_J{}^M - V'_A q q'\psi_{T2}\hat{\mathbf{q}}\wedge(\mathbf{Y}_J^{(e)M}\wedge\hat{\mathbf{q}}') + V'_T\psi_{T2}\mathbf{Y}_J^{(e)M}]. \tag{16}$$

The functions ψ_i and Y_J at the left-hand sides depend on $\mathbf{q}$, those at the right-hand

sides on $\mathbf{q}'$. The potential functions $V_i' \equiv V_i'[(\mathbf{q}-\mathbf{q}')^2]$ are linear combinations of the potentials V_j with coefficient matrix:

$$M_{ij} = \begin{pmatrix} 1 & 4 & 12 & -4 & -1 \\ 1 & -2 & 0 & -2 & 1 \\ 1 & 0 & -4 & 0 & -1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & -4 & 12 & 4 & -1 \end{pmatrix}. \tag{17}$$

When the equations (15) and (16) are multiplied by $(Y_J^M\hat{\mathbf{q}})^*$ or by $(\mathbf{Y}_J^{(e)M})^*$ and summed over M an equivalent set of four scalar equations is obtained by using the identities (A9)-(A15) of Appendix A. These equations are collected in Appendix B, together with those for the other (P, C) combinations. For low values of J equivalent sets of equations have been given before in the cartesian tensor formalism [10; see also 11].

In the following we shall study in particular the equations for spinless bound states; for 0^{++} states these equations are:

$$\frac{E}{q}\left(\frac{E^2}{q}\psi_S - \epsilon_B\psi_{T1}\right) = \int d\mathbf{q}'\left(V_S' - \frac{\mathbf{q}\cdot\mathbf{q}'}{q^2q'^2}V_V'\right)\psi_S, \tag{18}$$

$$E\left(\psi_{T1} - \frac{\epsilon_B}{q}\psi_S\right) = -\int d\mathbf{q}'\frac{\mathbf{q}\cdot\mathbf{q}'}{qq'}V_T'\psi_{T1}, \tag{19}$$

while for 0^{-+} states one has

$$E(E^2\psi_{A2} - \epsilon_B\psi_P) = -\int d\mathbf{q}'\,(V_A' - \mathbf{q}\cdot\mathbf{q}'V_T')\,\psi_{A2}, \tag{20}$$

$$E(\psi_P - \epsilon_B\psi_{A2}) = -\int d\mathbf{q}'\,V_P'\psi_P. \tag{21}$$

The Bethe–Salpeter amplitude $\chi(q)$ satisfies a normalization condition [12–14]. By making use of the q^0 dependence that follows from (3) one may derive the normalization condition for the reduced wavefunction $\psi(\mathbf{q})$; it reads:

$$N \equiv \pi^2\epsilon_B^{-1}\int d\mathbf{q}\,\mathrm{Tr}(\psi^\dagger\Lambda_+{}^+\psi\Lambda_-{}^- - \psi^\dagger\Lambda_-{}^+\psi\Lambda_+{}^-) = 1. \tag{22}$$

Likewise, the perturbation integral [15] for $\lambda\partial\epsilon_B{}^2/\partial\lambda$ in terms of χ may be rewritten in a form that contains $\psi(\mathbf{q})$ only:

$$\lambda\,\partial\epsilon_B{}^2/\partial\lambda = -2\pi^2N^{-1}\int d\mathbf{q}\,\mathrm{Tr}[(E-\epsilon_B)\,\psi^\dagger\Lambda_+{}^+\psi\Lambda_-{}^- + (E+\epsilon_B)\,\psi^\dagger\Lambda_-{}^+\psi\Lambda_+{}^-]. \tag{23}$$

In particular, for 0^{++} and 0^{-+} bound states the normalization conditions are:

$$N \equiv 2\pi\epsilon_B^{-1} \int d\mathbf{q}\, Eq^{-1}\, \mathrm{Re}(\psi_S^*\psi_{T1}) = 1, \tag{24}$$

$$N \equiv 2\pi\epsilon_B^{-1} \int d\mathbf{q}\, E\, \mathrm{Re}(\psi_P^*\psi_{A2}) = 1, \tag{25}$$

while the perturbation integrals for these cases read:

$$\lambda\, \partial\epsilon_B^2/\partial\lambda = -2\pi N^{-1} \int d\mathbf{q}\, [q^{-2}E^3 \mid \psi_S \mid^2 + E \mid \psi_{T1} \mid^2 - 2q^{-1}E\epsilon_B\, \mathrm{Re}(\psi_S^*\psi_{T1})], \tag{26}$$

$$\lambda\, \partial\epsilon_B^2/\partial\lambda = -2\pi N^{-1} \int d\mathbf{q}\, [E \mid \psi_P \mid^2 + E^3 \mid \psi_{A2} \mid^2 - 2E\epsilon_B\, \mathrm{Re}(\psi_P^*\psi_{A2})]. \tag{27}$$

Both the normalization and perturbation integrals have to be finite for physical bound states. As a consequence the structure functions must satisfy the constraints:

$$\lim_{q\to 0} q^{1/2}\psi_S = 0, \qquad \lim_{q\to 0} q^{3/2}\psi_{T1} = 0, \tag{28}$$

$$\lim_{q\to\infty} q^2\psi_S = 0, \qquad \lim_{q\to\infty} q^2\psi_{T1} = 0 \tag{29}$$

for 0^{++} states and

$$\lim_{q\to 0} q^{3/2}\psi_P = 0, \qquad \lim_{q\to 0} q^{3/2}\psi_{A2} = 0, \tag{30}$$

$$\lim_{q\to\infty} q^2\psi_P = 0, \qquad \lim_{q\to\infty} q^3\psi_{A2} = 0 \tag{31}$$

for 0^{-+} states.

The integrals in (22) or (24), (25) do not have definite signs. When they are negative the wavefunction cannot be normalized by adjusting a multiplicative constant: ghost states are showing up in that case [2]. For nonghost states the perturbation expressions (23) or (26), (27) for $\lambda\partial\epsilon_B^2/\partial\lambda$ are negative, as one would expect for physical bound states.

3. Quasi-Potential and Two-Particle Dirac Equations

In the previous section bound-state equations for a fermion and an antifermion bound by an instantaneous potential have been derived from the Bethe–Salpeter equation. The general form (4) of these equations shows that the particle and antiparticle states of the fermion–antifermion pair are treated on an equal footing; at the right-hand side of (4) the contributions of these states can easily be distinguished with the help of the projection operators $\Lambda_\pm$. When the binding forces are rather weak the particle states are expected to play a dominant role. Accordingly the first

term at the right-hand side of (4) is then more important than the second. If the latter is neglected altogether one arrives at the equation:

$$(\epsilon_B - \boldsymbol{\alpha}\cdot\mathbf{q} - \gamma^0)\,\psi(\mathbf{q}) + \psi(\mathbf{q})(\epsilon_B - \boldsymbol{\alpha}\cdot\mathbf{q} + \gamma^0)$$
$$= -2\sum_{i=1}^{5}\int d\mathbf{q}'\; V_i[(\mathbf{q}-\mathbf{q}')^2]\,\Lambda_+(\mathbf{q})\,\gamma^0\Gamma^i\psi(\mathbf{q}')\,\Gamma^i\gamma^0\Lambda_-(\mathbf{q}). \tag{32}$$

The way in which this equation has been obtained here from the instantaneous Bethe–Salpeter equation suggests that it is applicable only for states bound by weak potentials. However, equations of this type have been derived along different lines as well, viz., by using the quasi-potential formalism [4–6], so that they are supposed to have a much wider range of validity.

When (32) is multiplied by projectors $\Lambda_\pm$, the wavefunction $\psi(\mathbf{q})$ of the quasi-potential equation is found to satisfy, in addition to the constraints (5), the relation

$$\Lambda_-(\mathbf{q})\,\psi(\mathbf{q})\,\Lambda_+(\mathbf{q}) = 0. \tag{33}$$

As a consequence the structure functions ψ_i defined in (7)–(10) with (11)–(14) fulfill the supplementary conditions:

$$\psi_{T1} = q^{-1}E\psi_S\,, \qquad \psi_{T2} = E^{-1}\psi_{V2}\,, \tag{34}$$
$$\psi_{A2} = E^{-1}\psi_P\,, \tag{35}$$
$$\psi_{T6} = -E^{-1}\psi_{V4}\,. \tag{36}$$

The reduced equations for the structure functions that describe ${}^3(J\pm1)_J$, ${}^1J_J$, and ${}^3J_J$ states follow straightforwardly by insertion of (7)–(10) with (11)–(14) and (34)–(36) into (32). The results have been compiled in Appendix C. For 0^{++} and 0^{-+} bound states one finds in particular:

$$\frac{E^2}{q^2}(E-\epsilon_B)\,\psi_S = \frac{1}{2}\int d\mathbf{q}'\left[V'_S - \frac{\mathbf{q}\cdot\mathbf{q}'}{q^2q'^2}(V'_V + EE'V'_T)\right]\psi_S\,, \tag{37}$$

$$(E-\epsilon_B)\,\psi_P = -\frac{1}{2}\int d\mathbf{q}'\left(V'_P + \frac{1}{EE'}V'_A - \frac{\mathbf{q}\cdot\mathbf{q}'}{EE'}V'_T\right)\psi_P\,. \tag{38}$$

The normalization and perturbation integrals for the quasi-potential wavefunctions follow directly from those for the wavefunctions of the instantaneous Bethe–Salpeter equation by using the constraints (33) or (34)–(36) in (22), (23), or (24)–(27). For 0^{++} states one gets then

$$N = 2\pi\epsilon_B^{-1}\int d\mathbf{q}\; E^2q^{-2}\,|\,\psi_S\,|^2 = 1, \tag{39}$$

$$\lambda\,\partial\epsilon_B{}^2/\partial\lambda = -4\pi N^{-1}\int d\mathbf{q}\; E^2q^{-2}(E-\epsilon_B)\,|\,\psi_S\,|^2, \tag{40}$$

while for 0^{-+} states the results are:

$$N = 2\pi\epsilon_B^{-1} \int d\mathbf{q} \mid \psi_P \mid^2 = 1, \tag{41}$$

$$\lambda \, \partial\epsilon_B{}^2/\partial\lambda = -4\pi N^{-1} \int d\mathbf{q} \, (E - \epsilon_B) \mid \psi_P \mid^2. \tag{42}$$

In contrast to the normalization integrals for the instantaneous Bethe–Salpeter equation those found here are necessarily positive: the quasi-potential equation is free from ghost states. Correspondingly the derivative $\lambda\partial\epsilon_B{}^2/\partial\lambda$ is negative definite for all solutions. The finiteness of the normalization and perturbation integrals is guaranteed if the structure functions ψ_S and ψ_P fulfill the asymptotic conditions (28)–(31).

An alternative approach to describe a fermion–antifermion pair bound by an instantaneous potential consists in postulating the validity of a two-particle Dirac equation for the bound-state wavefunction [7–9]. In the present notation it may be written as:

$$\begin{aligned}(\epsilon_B - \boldsymbol{\alpha} \cdot \mathbf{q} - \gamma^0) \, \psi(\mathbf{q}) + \psi(\mathbf{q})(\epsilon_B - \boldsymbol{\alpha} \cdot \mathbf{q} + \gamma^0) \\ = -2 \sum_{i=1}^{5} \int d\mathbf{q}' \; V_i[(\mathbf{q} - \mathbf{q}')^2] \, \gamma^0 \Gamma^i \psi(\mathbf{q}') \, \Gamma^i \gamma^0.\end{aligned} \tag{43}$$

Formally one can arrive at such an equation by omitting in the quasi-potential equation (32) the projectors $\Lambda_{\pm}^{\pm}$ at the right-hand side. As a consequence the Dirac wavefunction $\psi(\mathbf{q})$ does not have to satisfy constraints like (5) or (33); its general form is given by (7)–(10) for states with definite spin, parity, and charge parity.

The scalar equations that can be deduced from (43) for general values of the bound-state spin J have been collected in Appendix D. The 0^{++} and 0^{-+} equations are found to be:

$$\epsilon_B \psi_S - q\psi_{T1} = -\int d\mathbf{q}' \; V_S' \psi_S \,, \tag{44}$$

$$\epsilon_B \psi_{V1} - \psi_{T1} = \int d\mathbf{q}' \; \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}' V_V' \psi_{V1} \,, \tag{45}$$

$$\epsilon_B \psi_{T1} - q\psi_S - \psi_{V1} = \int d\mathbf{q}' \; \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}' V_T' \psi_{T1} \tag{46}$$

and

$$\epsilon_B \psi_P - q\psi_{T4} - \psi_{A2} = \int d\mathbf{q}' \; V_P' \psi_P \,, \tag{47}$$

$$\epsilon_B \psi_{A2} - \psi_P = \int d\mathbf{q}' \; V_A' \psi_{A2} \,, \tag{48}$$

$$\epsilon_B \psi_{T4} - q\psi_P = -\int d\mathbf{q}' \; \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}' V_T' \psi_{T4} \,, \tag{49}$$

respectively; $\hat{\mathbf{q}}$ stands for $\mathbf{q}/\mid \mathbf{q} \mid$ as before.

The wavefunction of the two-particle Dirac equation may conveniently be normalized by requiring

$$N = \int d\mathbf{q}\ \mathrm{Tr}(\psi^\dagger\psi) = 1, \tag{50}$$

so that ghost states are excluded a priori. The perturbation integral for $\lambda\partial\epsilon_B/\partial\lambda$ is found to have the form:

$$\lambda\ \partial\epsilon_B/\partial\lambda = -\tfrac{1}{2}N^{-1}\int d\mathbf{q}\ \mathrm{Tr}[\psi^\dagger(\boldsymbol{\alpha}\cdot\mathbf{q} + \gamma^0 - \epsilon_B)\ \psi + \psi^\dagger\psi(\boldsymbol{\alpha}\cdot\mathbf{q} - \gamma^0 - \epsilon_B)]. \tag{51}$$

For 0^{++} states the normalization and perturbation integrals become:

$$N = \pi^{-1}\int d\mathbf{q}\ (|\ \psi_S\ |^2 + |\ \psi_{V1}\ |^2 + |\ \psi_{T1}\ |^2) = 1, \tag{52}$$

$$\lambda\ \partial\epsilon_B/\partial\lambda = (\pi N)^{-1}\int d\mathbf{q}\ [\epsilon_B(|\ \psi_S\ |^2 + |\ \psi_{V1}\ |^2 + |\ \psi_{T1}\ |^2)$$

$$- 2\ \mathrm{Re}(q\psi_S^*\psi_{T1} + \psi_{V1}^*\psi_{T1})], \tag{53}$$

while the corresponding integrals for 0^{-+} states follow by replacing the indices S, $V1$, $T1$ by $T4$, $A2$, P, respectively. The perturbation integrals are convergent if the asymptotic behavior of the structure functions is consistent with the conditions

$$\lim_{q\to 0} q^{3/2}\psi_i = 0, \tag{54}$$

$$\lim_{q\to\infty} q^{3/2}\psi_i = 0, \qquad \lim_{q\to\infty} q^4\ \mathrm{Re}(\psi_j^*\psi_k) = 0, \tag{55}$$

with $i = S, V1, T1$ (or $T4, A2, P$) and $j, k = S, T1$ (or $T4, P$).

Up to now the bound-state equations were considered in momentum space. All equations discussed so far can in principle be studied in position space as well. However, the instantaneous Bethe–Salpeter and quasi-potential equations have a rather complicated form in position space owing to the occurrence of factors $E = (1 + q^2)^{1/2}$ in the Λ projectors. In contrast, the two-particle Dirac equation gets a simpler form by transforming to the position-space representation. When the wavefunction $\tilde{\psi}(\mathbf{r}) = \int d\mathbf{q}\ \exp(i\mathbf{q}\cdot\mathbf{r})\ \psi(\mathbf{q})$ is expanded in Dirac matrices in a way analogous to (7)–(10) (viz., by replacing $\hat{\mathbf{q}}$, $Y_J^M(\hat{\mathbf{q}})$, and $\mathbf{Y}_J^{e(M)}(\hat{\mathbf{q}})$ by $-i\hat{\mathbf{r}}$, $Y_J^M(\hat{\mathbf{r}})$, and $-i\,\mathbf{Y}_J^{(e)M}(\hat{\mathbf{r}})$, respectively, and writing the coefficients as $\tilde{\psi}_i(\mathbf{r})$ instead of $\psi_i(\mathbf{q})$), one easily obtains from the position-space version of (43) a set of coupled differential equations for $\tilde{\psi}_i(\mathbf{r})$ [7–9, 16–20]. For 0^{++} and 0^{-+} states these equations read:

$$\epsilon_B\tilde{\psi}_S + \left(\frac{d}{dr} + \frac{2}{r}\right)\tilde{\psi}_{T1} = -\tilde{V}'_S\tilde{\psi}_S\,, \tag{56}$$

$$\epsilon_B\tilde{\psi}_{V1} - \tilde{\psi}_{T1} = \tilde{V}'_V\tilde{\psi}_{V1}\,, \tag{57}$$

$$\epsilon_B\tilde{\psi}_{T1} - \frac{d}{dr}\tilde{\psi}_S - \tilde{\psi}_{V1} = \tilde{V}'_T\tilde{\psi}_{T1}\,, \tag{58}$$

and

$$\epsilon_B \tilde{\psi}_P - \tilde{\psi}_{A2} + \left(\frac{d}{dr} + \frac{2}{r}\right) \tilde{\psi}_{T4} = \tilde{V}'_P \tilde{\psi}_P , \tag{59}$$

$$\epsilon_B \tilde{\psi}_{A2} - \tilde{\psi}_P = \tilde{V}'_A \tilde{\psi}_{A2} , \tag{60}$$

$$\epsilon_B \tilde{\psi}_{T4} - \frac{d}{dr} \tilde{\psi}_P = -\tilde{V}'_T \tilde{\psi}_{T4} , \tag{61}$$

where the potentials are defined as $\tilde{V}'_i(\mathbf{r}) = \int d\mathbf{q} \exp(i\mathbf{q} \cdot \mathbf{r})\, V'_i(\mathbf{q})$. It should be noted that Eqs. (56)-(61) can not be found by directly taking the Fourier transform of (44)-(49), since the structure functions $\tilde{\psi}_i(\mathbf{r})$ are not defined as the Fourier transforms of $\psi_i(\mathbf{q})$.

A salient feature of Eqs. (56)–(61) is that some of them are purely algebraic, while the others are differential equations of first order only. The sets of coupled equations are equivalent to single second-order differential equations:

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\left[\{\epsilon_B - \tilde{V}'_T - (\epsilon_B - \tilde{V}'_V)^{-1}\}^{-1} \frac{d}{dr} \tilde{\psi}_S\right] + (\epsilon_B + \tilde{V}'_S)\, \tilde{\psi}_S = 0, \tag{62}$$

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\left[(\epsilon_B + \tilde{V}'_T)^{-1} \frac{d}{dr} \tilde{\psi}_P\right] + [\epsilon_B - \tilde{V}'_P - (\epsilon_B - \tilde{V}'_A)^{-1}]\, \tilde{\psi}_P = 0. \tag{63}$$

Alternatively one may write down second-order equations for $\tilde{\psi}_{T1}$ and $\tilde{\psi}_{T4}$, by eliminating $\tilde{\psi}_S$ and $\tilde{\psi}_P$, respectively. The potentials are seen to occur in the combinations $\epsilon_B \pm \tilde{V}'_i$, The solution of the bound state equations may therefore possess singularities at finite values of $\mathbf{r}$, even if the potentials are regular there. In fact, the position of these singularities is found by putting $\epsilon_B \pm \tilde{V}'_i(\mathbf{r}) = 0$.

The normalization and perturbation integrals as given in (50) and (51) can be translated to coordinate space in a straightforward way. For 0^{++} states in particular one gets then, on a par with (52) and (53):

$$N = \pi^{-1}(2\pi)^{-3} \int d\mathbf{r}\, (|\, \tilde{\psi}_S\,|^2 + |\, \tilde{\psi}_{V1}\,|^2 + |\, \tilde{\psi}_{T1}\,|^2), \tag{64}$$

$$\lambda\, \partial\epsilon_B/\partial\lambda = (\pi N)^{-1}\, (2\pi)^{-3} \int d\mathbf{r}\, [\epsilon_B(|\, \tilde{\psi}_S\,|^2 + |\, \tilde{\psi}_{V1}\,|^2 + |\, \tilde{\psi}_{T1}\,|^2) - 2\, \mathrm{Re}\{(d\tilde{\psi}^*_S/dr + \tilde{\psi}^*_{V1})\, \tilde{\psi}_{T1}\}]. \tag{65}$$

The integrals for 0^{-+} states have a similar form, with S, $V1$, $T1$ replaced by $T4$, $A2$, P; in the last term $(d\tilde{\psi}^*_S/dr)\, \tilde{\psi}_{T1}$ goes over in $(d\tilde{\psi}^*_P/dr)\, \tilde{\psi}_{T4}$.

4. Bound-State Equations for Interactions with $V'_i \neq 0$

The coupled equations in momentum space derived in the preceding sections are in general difficult to solve. Simpler uncoupled equations arise if the interactions between the constituent particles are such that only one of the potentials V'_i differs from zero, while the other V'_j (with $j \neq i$) vanish identically.

The instantaneous Bethe–Salpeter equations (18)–(21) for spinless bound states reduce to a single scalar equation when the constituent particles interact only via pure S', A', or P' potentials; in fact one finds in these cases:

$$\frac{E}{q^2}(E^2 - \epsilon_B{}^2)\,\psi_S = \int d\mathbf{q}'\; V'_S\psi_S\,, \tag{66}$$

$$E(E^2 - \epsilon_B{}^2)\,\psi_{A2} = -\int d\mathbf{q}'\; V'_A\psi_{A2}\,, \tag{67}$$

$$\frac{1}{E}(E^2 - \epsilon_B{}^2)\,\psi_P = -\int d\mathbf{q}'\; V'_P\psi_P\,. \tag{68}$$

The first equation describes 0^{++} states, the second and third 0^{-+} states. From (24) and (25) it follows that the solutions of these equations correspond to nonghost states with positive norm; the derivatives $\lambda\partial\epsilon_B{}^2/\partial\lambda$, as given by (26) and (27), are negative definite. For 0^{++} states with pure V' or T' coupling and for 0^{-+} states bound by pure T' interactions equations are found that are analogous to (67), (68), and (66), respectively, apart from an extra factor $\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}'$ in the integrands; alternatively one may write these as vectorial equations.

The instantaneous Bethe–Salpeter equations (66)–(68) may be compared to the quasi-potential and two-particle Dirac equations for the same type of interactions. From (37) and (38) with (35) one obtains for the three cases:

$$\frac{E^2}{q^2}(E - \epsilon_B)\,\psi_S = \frac{1}{2}\int d\mathbf{q}'\; V'_S\psi_S\,, \tag{69}$$

$$E^2(E - \epsilon_B)\,\psi_{A2} = -\frac{1}{2}\int d\mathbf{q}'\; V'_A\psi_{A2}\,, \tag{70}$$

$$(E - \epsilon_B)\,\psi_P = -\frac{1}{2}\int d\mathbf{q}'\; V'_P\psi_P\,, \tag{71}$$

while (44)–(49) give

$$\frac{\epsilon_B}{1 - \epsilon_B{}^2}(E^2 - \epsilon_B{}^2)\,\psi_S = -\int d\mathbf{q}'\; V'_S\psi_S\,, \tag{72}$$

$$\frac{\epsilon_B}{\epsilon_B{}^2 - q^2}(E^2 - \epsilon_B{}^2)\,\psi_{A2} = -\int d\mathbf{q}'\; V'_A\psi_{A2}\,, \tag{73}$$

$$\frac{1}{\epsilon_B}(E^2 - \epsilon_B{}^2)\,\psi_P = -\int d\mathbf{q}'\; V'_P\psi_P\,. \tag{74}$$

All these equations are free from ghost-state solutions. The sign of the derivative $\lambda\partial\epsilon_B/\partial\lambda$, which may be derived from the general expressions (40), (42), and (53) is negative for all equations except for Eq. (72) and (73). The two-particle Dirac equation for 0^{++} states bound by pure S' interaction has only solutions with an anomalous positive sign for $\lambda\partial\epsilon_B/\partial\lambda$, while this derivative may have either sign for 0^{-+} states with pure A' coupling.

The equations in coordinate space that correspond to (72)–(74) are readily found from (62)–(63); they read:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\tilde{\psi}_S + (\epsilon_B{}^2 - 1)(1 + \epsilon_B^{-1}\tilde{V}'_S)\,\tilde{\psi}_S = 0, \tag{75}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\tilde{\psi}_P + \left(\epsilon_B{}^2 - 1 - \frac{\tilde{V}'_A}{\epsilon_B - \tilde{V}'_A}\right)\tilde{\psi}_P = 0, \tag{76}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\tilde{\psi}_P + (\epsilon_B{}^2 - 1 - \epsilon_B\tilde{V}'_P)\,\tilde{\psi}_P = 0. \tag{77}$$

These equations are seen to be of Schrödinger form, with an effective potential that is a function of the potential $\tilde{V}'_i$. In (76) this function may cause a singular behavior of the wavefunction even for regular potentials $\tilde{V}'_A$; no such singularities are present in (75) and (77).

5. General Properties of the Equations for Spinless Bound States with Coulomb-Type Potentials

In this section and the next one the bound-state equations will be studied for the special case of binding potentials of the form $V'_i[(\mathbf{q} - \mathbf{q}')^2] = \frac{1}{4}\pi^{-1}\Lambda'_i(\mathbf{q} - \mathbf{q}')^{-2}$. Such potentials may be considered as the instantaneous limit of the retarded potentials that occur in the Bethe–Salpeter equation for fermions bound by the exchange of massless bosons; the latter equation has been investigated in detail in the literature [2, 21–28].

The equations for spinless bound states have been given in (18)–(21), (37)–(38), and (44)–(49). In the following some general properties of these bound-state equations will be derived, while in the next section the results of a numerical analysis will be presented.

a. *Integral Identities for the Wavefunctions and Inequalities for the Bound-state Mass*

Coulomb-type potentials are homogeneous functions of the momentum transfer q^2; this feature may be employed to derive an integral identity for the reduced wavefunctions $\psi(\mathbf{q})$. Let us consider as an example the instantaneous equations (20), (21) for 0^{-+} states bound by a mixture of A' and P' potentials. If (20) is acted upon by the operator $\mathbf{q}\cdot\partial/\partial\mathbf{q} - 1$ and a partial integration is performed at the right-hand side the result is:

$$\left(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}} - 1\right)[(1 + q^2)^{3/2}\,\psi_{A2} - \epsilon_B(1 + q^2)^{1/2}\,\psi_P]$$

$$= -\frac{\Lambda_A}{4\pi}\int d\mathbf{q}'\,\frac{1}{(\mathbf{q} - \mathbf{q}')^2}\,\mathbf{q}'\cdot\frac{\partial}{\partial\mathbf{q}'}\,\psi_{A2}(\mathbf{q}'). \tag{78}$$

Multiplying this equation by ψ^*_{A2}, integrating over $\mathbf{q}$, and using (20) again one gets

$$\int d\mathbf{q} \left\{\left[\left(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}} - 1\right)(1+q^2)^{3/2}\right] |\psi_{A2}|^2 - \epsilon_B\left[\left(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}} - 1\right)(1+q^2)^{1/2}\right]\psi^*_{A2}\psi_P\right\}$$
$$= \epsilon_B \int d\mathbf{q}\,(1+q^2)^{1/2}\left(\psi^*_{A2}\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}}\psi_P - \psi^*_P\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}}\psi_{A2}\right). \tag{79}$$

When (21) is treated in an analogous fashion an equation with the same right-hand side as (79) is found. Subtracting these one arrives at the following integral condition on ψ_{A2} and ψ_P :

$$\int d\mathbf{q} \left\{\left[\left(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}} - 1\right)(1+q^2)^{3/2}\right] |\psi_{A2}|^2 \right.$$
$$\left. + \left[\left(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{q}} - 1\right)(1+q^2)^{1/2}\right]\left[|\psi_P|^2 - 2\epsilon_B\,\mathrm{Re}(\psi^*_{A2}\psi_P)\right]\right\} = 0. \tag{80}$$

This identity is especially interesting for pure P' coupling. In that case one has $\psi_{A2} = \epsilon_B\psi_P/E^2$, so that (80) becomes

$$\int d\mathbf{q}\, E^{-3}[E^2(1-2\epsilon_B{}^2) + \epsilon_B{}^2]\,|\psi_P|^2 = 0; \tag{81}$$

alternatively, this relation may be obtained directly from (68). Since the integrand is positive definite for all $\epsilon_B{}^2 < \frac{1}{2}$ one must conclude that the instantaneous Bethe–Salpeter equation for 0^{-+} states bound by pure P' interactions has only loosely bound state solutions with $\epsilon_B{}^2 > \frac{1}{2}$. This statement will be corroborated by the numerical results presented in the next section. In [21] the instantaneous Bethe–Salpeter equation with pure P' coupling is solved analytically for $\epsilon_B{}^2 = 0$. The solution does not fulfill the conditions (31), however, so that the integral (27) diverges; the physical interpretation of such solutions is questionable.

A relation analogous to (80) or (81) can be derived from the instantaneous Bethe–Salpeter equation (66) that describes 0^{++} states with pure S' coupling; it reads:

$$\int d\mathbf{q}\,[3q^{-2}E(1-\epsilon_B{}^2) + E^{-1}\epsilon_B{}^2]\,|\psi_S|^2 = 0. \tag{82}$$

Since the integrand is positive for all q it follows that the instantaneous Bethe–Salpeter equation (66) has no bound-state solutions at all.

Identities similar to (81) and (82) follow for 0^{++} and 0^{-+} states with pure T' coupling. In the former case only weakly bound-state solutions exist, while in the latter case no bound states are found. The integral conditions that may be obtained for the wave-functions of the quasi-potential and the two-particle Dirac equations are not given here explicitly, since they do not lead to equally strong conclusions about the bound-state spectrum. They can be used, however, to transform the integrands of the perturbation integrals. In some cases a definite conclusion about the sign of the latter can be

reached in this way; in particular it is found that the solutions of the two-particle Dirac equation for 0^{++} states with $V' - T'$ coupling and for 0^{-+} states with $A' - P'$ coupling have a negative derivative $\lambda \partial \epsilon_B / \partial \lambda$ (this could not be inferred from (53) or (73)).

b. *Inequalities and Exact Results for the Coupling Constants*

The general form of the integral operators occurring in the bound-state equations can be used to obtain inequalities for the coupling constants. Such an inequality may be derived in the case of the instantaneous Bethe–Salpeter equations (20)–(21) that describe 0^{-+} states bound through P', A', and T' interactions by introducing first functions φ_A and φ_P instead of ψ_A and ψ_P in such a way that the equations for φ_i contain a symmetric integral kernel. This is achieved by writing $\varphi_i = \sum_{j=A,P} T_{ij}\psi_j$ with a matrix

$$T_{ij}(\mathbf{q}) = \frac{1}{E^{1/2}} \begin{pmatrix} E^2 & -\epsilon_B \\ 0 & (E^2 - \epsilon_B{}^2)^{1/2} \end{pmatrix}. \tag{83}$$

The bound-state equations in terms of φ_i get the form:

$$\varphi_i(\mathbf{q}) = \sum_j \int d\mathbf{q}'\, M_{ij}(\mathbf{q}, \mathbf{q}')\, \varphi_j(\mathbf{q}'), \tag{84}$$

with the kernel

$$M_{ij}(\mathbf{q}, \mathbf{q}') = -\tfrac{1}{4}\pi^{-1}E^{-3/2}E'^{-3/2}(E^2 - \epsilon_B{}^2)^{-1/2}\,(E'^2 - \epsilon_B{}^2)^{-1/2}\,(\mathbf{q} - \mathbf{q}')^{-2}$$
$$\times \begin{pmatrix} (E^2 - \epsilon_B{}^2)^{1/2} & 0 \\ \epsilon_B & E^2 \end{pmatrix} \begin{pmatrix} \Lambda_A - \mathbf{q}\cdot\mathbf{q}'\Lambda_T & 0 \\ 0 & \Lambda_P \end{pmatrix} \begin{pmatrix} (E'^2 - \epsilon_B{}^2)^{1/2} & \epsilon_B \\ 0 & E'^2 \end{pmatrix}, \tag{85}$$

so that indeed $M_{ij}(\mathbf{q}, \mathbf{q}') = M_{ji}(\mathbf{q}', \mathbf{q})$. The functions $\varphi_i(\mathbf{q})$ are elements of a Hilbert space with norm $\sum_i \int d\mathbf{q} \mid \varphi_i(\mathbf{q})|^2$, the finiteness of which is guaranteed by the conditions (30)–(31).

The norm $\| M \|$ of the operator (85) may be estimated by writing first

$$\| M \| \leqslant \| M_{P'} \| + \| M_{A'} \| + \| M_{T'} \| \tag{86}$$

(with $M_{i'}$ the operator for pure i' coupling) and using then the theorem [29]:

$$\| M \| \leqslant \sup_{(\mathbf{q},i)} \sum_j \int d\mathbf{q}' \mid M_{ij}(\mathbf{q}, \mathbf{q}')| \, \frac{g_j(\mathbf{q}')}{g_i(\mathbf{q})}, \tag{87}$$

which is valid for any choice of the positive definite functions $g_i(\mathbf{q})$. If these are chosen for the A' and P' cases as

$$g_i(\mathbf{q}) = (q^2 + a^2)^{-1} \sum T_{ij}(\mathbf{q})\, c_j \tag{88}$$

with $c_A = 1$, $c_P = \epsilon_B$, and as $g_i'(\mathbf{q}) = q^{-1}g_i(\mathbf{q})$ for the T' case, one finds upon evaluating the integrals and taking the limit $a \to 0$:

$$\| M \| \leqslant \tfrac{1}{4}\pi^2 \sup_{(q)} qE^{-1}(E^2 - \epsilon_B{}^2)^{-1}\,(E^2 \mid \Lambda_P \mid + \mid \Lambda_A \mid + q^2 \mid \Lambda_T \mid). \tag{89}$$

Since one must have $\| M \| \geqslant 1$ in order to fulfill the bound-state equation this inequality implies a lower bound for the coupling constants. If (89) is weakened somewhat by inserting an extra factor E^2 in front of $\mid \Lambda_A \mid$ one arrives at the simple relation:

$$(\mid \Lambda_P \mid + \mid \Lambda_A \mid)[\tfrac{1}{2}\epsilon_B^{-1}(1 - \epsilon_B{}^2)^{-1/2}\,\theta(\epsilon_B{}^2 - \tfrac{1}{2}) + \theta(\tfrac{1}{2} - \epsilon_B{}^2)] + \mid \Lambda_T \mid \geqslant 4\pi^{-2}. \tag{90}$$

With the help of (89) one may prove in the same way as in [29] that the spectrum of bound-state energies $2\epsilon_B$ is discrete for all coupling constants satisfying $\mid \Lambda_P \mid + \mid \Lambda_A \mid + \mid \Lambda_T \mid < 4\pi^{-2}$. For pure A' coupling the spectrum is discrete for all Λ_A, while for pure T' coupling we know already that no bound-state solutions exist.

Bounds for the coupling constants of the 0^{++} instantaneous Bethe–Salpeter equation are obtained along similar lines; one finds an inequality of the same form as (90), with Λ_T, Λ_A, and Λ_P replaced by Λ_S, Λ_V, and Λ_T, respectively. The spectral properties are completely analogous to those of the 0^{-+} equations given above.

When the same majorization technique is applied to the quasi-potential equations (37)–(38) one gets for 0^{-+} states

$$(\mid \Lambda_P \mid + \mid \Lambda_A \mid)\,\tfrac{1}{2}(1 - \epsilon_B{}^2)^{-1/2} + \mid \Lambda_T \mid [1 + (1 - \epsilon_B{}^2)^{1/2}]^{-1} \geqslant 4\pi^{-2}, \tag{91}$$

while for 0^{++} states an analogous relation, with P, A, $T \to T$, V, S, is found. The bound-state energy spectra are discrete if $\sum_i \mid \Lambda_i \mid < 8\pi^{-2}$ (in the case of 0^{++} states with pure V' and 0^{-+} states with pure A' coupling the discreteness of the spectrum can be established for all Λ_i).

One may try to use the above methods for the two-particle Dirac equations in momentum space as well. In general, however, the integral kernel that occurs in these equations cannot be brought in a symmetric form by a suitable redefinition of the wavefunction. This can be achieved only for 0^{++} states with pure S' or T' and 0^{-+} states with pure P' or T' couplings. The resulting bounds are of no importance, since in these cases the two-particle Dirac equation can be solved exactly. To find the solutions it is more convenient to introduce the position-space representation, as in (62)–(63) and (75)–(77), with $\hat{V}_i = \tfrac{1}{2}\pi\Lambda_i r^{-1}$.

For 0^{++} states with pure S' coupling Eq. (75) reduces to a Schrödinger equation with Coulomb potential the eigenvalues of which are $\Lambda_S = -4\pi^{-1}(n + 1)\,\epsilon_B(1 - \epsilon_B{}^2)^{-1/2}$, with nonnegative integer n. The modulus of Λ_S is seen to increase for decreasing binding energies $2(1 - \epsilon_B)$; this anomalous behavior could be inferred already quite generally in the preceding section. Even worse, infinitesimally weak binding energies ($\epsilon_B \to 1$) are found here to be realized only for a coupling constant tending to infinity. Hence one must conclude that the two-particle Dirac equation is pathological for 0^{++} states with S' binding. Similar conclusions can be drawn for the two-particle Dirac equation describing 0^{-+} states with pure T' coupling. In fact, if

(63) is transformed by means of (59)–(61) to an equation for $\tilde{\psi}_{T4}$, one arrives again at a Schrödinger equation, with solution $\Lambda_T = -4\pi^{-1}(n+1)\,\epsilon_B(1-\epsilon_B{}^2)^{-1/2}$, $n \geqslant 1$.

The Dirac equation (77) for 0^{-+} states with pure P' interaction is likewise an ordinary Schrödinger equation, with solution $\Lambda_P = -4\pi^{-1}(n+1)\,\epsilon_B^{-1}(1-\epsilon_B{}^2)^{1/2}$, $n \geqslant 0$. Its vectorial counterpart, describing 0^{++} states with pure T' interaction, has the same spectrum (with $n \geqslant 1$), as follows by rewriting (62) in terms of $\tilde{\psi}_{T1}$. No anomalous behavior as found above occurs in these cases.

For 0^{++} states with pure V' and for 0^{-+} states with pure A' coupling the Dirac equation becomes a Schrödinger equation with a potential of the form $a/(r+b)$; for the latter case the equation has been written down in (76). For negative b (corresponding to $\Lambda_A > 0$) this potential is singular on a sphere around the origin. Bound states for such singular potentials have been studied recently in connection with bag models [20]. For the nonsingular case, with $\Lambda_A < 0$, the coupling constant is determined by the zero points of a confluent hypergeometric function [30], at least for S-wave solutions. For 0^{-+} states with pure A' interaction one finds in particular:

$$U[1 + \tfrac{1}{4}\pi\epsilon_B^{-1}(1-\epsilon_B{}^2)^{-1/2}\,\Lambda_A,\, 2,\, -\pi\epsilon_B^{-1}(1-\epsilon_B{}^2)^{1/2}\,\Lambda_A] = 0. \tag{92}$$

This transcendental equation for Λ_A can easily be solved numerically; the results will be given in the next section. An analogous equation for 0^{++} states with pure V' coupling is not available; in that case one needs a P-wave solution of the Schrödinger equation.

The equations (62) and (63) still have the form of a Schrödinger equation for 0^{++} states with $V' - T'$ couplings and for 0^{-+} states with $P' - A'$ interactions. Since both cases are quite analogous we shall limit ourselves to a consideration of the 0^{-+} states for which the Dirac equation is:

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\frac{d\tilde{\psi}_P}{dr} + [\epsilon_B{}^2 - 1 - V(r)]\,\tilde{\psi}_P = 0, \tag{93}$$

with the effective potential

$$V(r) = \frac{1}{2}\,\pi\left[\epsilon_B\Lambda_P\,\frac{1}{r} + \epsilon_B^{-1}\Lambda_A\,\frac{1}{r - \tfrac{1}{2}\pi\epsilon_B^{-1}\Lambda_A}\right]. \tag{94}$$

When Λ_A is positive this potential is singular for $r = \frac{1}{2}\pi\epsilon_B^{-1}\Lambda_A$, while no such singularity shows up for $\Lambda_A < 0$. As a consequence the character of the bound-state solutions will change radically as Λ_A goes through zero.

For the nonsingular case a coupling-constant inequality can be found by comparing the equation to an exactly solvable Schrödinger equation. In fact, since one has for $\Lambda_A < 0$:

$$V(r) \geqslant \tfrac{1}{2}\pi\epsilon_B(\Lambda_P + \epsilon_B^{-2}\Lambda_A)\,r^{-1}, \tag{95}$$

the following bound for the coupling constants can be established:

$$\Lambda_P + \epsilon_B^{-2}\Lambda_A \leqslant -4\pi^{-1}\epsilon_B^{-1}(1-\epsilon_B{}^2)^{1/2}. \tag{96}$$

Another inequality for the coupling constants of the mixed $P' - A'$ equation follows, in the special case $\Lambda_A < 0$ and $\Lambda_P > 0$, by requiring ϵ_B to be larger than the minimum of the potential $V(r)$; one finds in this way:

$$\Lambda_P + \epsilon_B^{-2}[1 - (1 - \epsilon_B^2)^{1/2}]^2 \Lambda_A < 0. \tag{97}$$

From the general properties of the effective potential (94) one may prove with the theorem given in [31] that the spectrum of bound-state energies is discrete for all coupling constants satisfying the inequalities $\Lambda_A < 0$ and $\Lambda_P + \epsilon_B^{-2}\Lambda_A < 0$.

c. *Nonrelativistic Limit*

The properties of weakly bound states with small binding energies and couplings can be derived from the nonrelativistic limit of the relevant equations. This limit follows by regarding both $1 - \epsilon_B^2$, Λ_i, and q^2 as small quantities.

The nonrelativistic limit of the instantaneous Bethe–Salpeter equation for 0^{-+} states may be studied conveniently in the representation (83)–(85). In fact, let us try to solve (84) in the nonrelativistic approximation by assuming φ_A to be small compared to φ_P. The second equation contained in (84) then becomes:

$$\varphi_P(\mathbf{q}) = -\frac{1}{4\pi}(\Lambda_A + \Lambda_P) \int d\mathbf{q}' \frac{\varphi_P(\mathbf{q}')}{(E^2 - \epsilon_B^2)^{1/2}(E'^2 - \epsilon_B^2)^{1/2}(\mathbf{q} - \mathbf{q}')^2} \tag{98}$$

of which the eigenvalue spectrum is

$$\Lambda_A + \Lambda_P = -4\pi^{-1}(n + 1)(1 - \epsilon_B^2)^{1/2} \qquad (n = 0, 1,...). \tag{99}$$

The contribution of T' couplings to the binding strength turns out to be negligible in comparison to that of the other coupling types. The first equation of (84) determines φ_A as

$$\varphi_A(\mathbf{q}) = \Lambda_A(\Lambda_A + \Lambda_P)^{-1}(E^2 - \epsilon_B^2)^{1/2} \varphi_P(\mathbf{q}), \tag{100}$$

so that indeed $|\varphi_A| \ll |\varphi_P|$, at least if Λ_A/Λ_P is not too close to -1. It should be remarked that the above derivation of the nonrelativistic limit breaks down in the case of pure T' coupling. In that case the instantaneous Bethe–Salpeter equation does not possess solutions describing 0^{-+} states.

For 0^{++} states the nonrelativistic limit of the instantaneous Bethe–Salpeter equation may be discussed in an analogous way. The eigenvalue spectrum has the form (99), with (A', P') replaced by (V', T'), while n must be greater than 1. In this case the S' interaction does not contribute.

The quasi-potential equations (37)–(38) are single integral equations, the nonrelativistic limit of which is easily established. For 0^{-+} states one gets

$$(E - \epsilon_B)\psi_P = -\frac{1}{8}\pi^{-1}(\Lambda_A + \Lambda_P) \int d\mathbf{q}' \frac{\psi_P(\mathbf{q}')}{(\mathbf{q} - \mathbf{q}')^2} \tag{101}$$

with a spectrum

$$\Lambda_A + \Lambda_P = -4\,(2)^{1/2}\pi^{-1}(n+1)(1-\epsilon_B)^{1/2} \qquad (n = 0, 1,...). \tag{102}$$

As expected the spectra (99) and (102) coincide for $\epsilon_B \sim 1$. For pure T' coupling the general inequality (91) implies that a nonrelativistic limit does not exist. The 0^{++} quasi-potential equation leads to results analogous to (101)–(102).

The nonrelativistic spectrum of the two-particle Dirac equation can be derived from its position space representation (62)–(63). We shall limit the discussion to 0^{-+} states again; for 0^{++} states one may proceed analogously. The potential (94) occurring in the equation for 0^{-+} states bound by a mixture of A' and P' interactions may be written approximately as:

$$V(r) = \tfrac{1}{2}\pi(\Lambda_A + \Lambda_P)\, r^{-1} \tag{103}$$

for $\Lambda_A < 0$. With this approximation the eigenvalue spectrum is found to coincide with (99). For pure T' coupling the two-particle Dirac equation has no nonrelativistic limit: the coupling constant tends to infinity in that case. When T' and A' or P' couplings are present simultaneously the two-particle Dirac equation (63) has no longer a simple Schrödinger form; the behavior of the solutions in the nonrelativistic limit cannot be deduced easily in that case.

6. Numerical Analysis of the Equations for 0^{-+} States Bound by P' and A' Couplings

The study of the general properties of the three types of bound-state equations in the preceding section leads to the conclusion that for 0^{++} with $V' - T'$ and for 0^{-+} states with $P' - A'$ coupling a consistent picture is obtained : the sign of the derivative $\lambda\partial\epsilon_B{}^2/\partial\lambda$ is negative (at least for the nonghost states), the nonrelativistic limits coincide and the general spectral properties are similar, although not completely the same. In contrast, the equations for 0^{++} states with pure S' and for 0^{-+} states with pure T' couplings have a pathological character: in particular it has been found that the nonrelativistic limit does not exist in these cases. For general mixtures of couplings the situation is less clear; in the nonrelativistic limit it was found, however, that the instantaneous Bethe–Salpeter and quasi-potential equations are insensitive to the strength of the S' and the T' components in the 0^{++} and 0^{-+} equations, respectively: the binding is dominated by the $V' - T'$ and $P' - A'$ components.

The structure of the 0^{++} $V' - T'$ and 0^{-+} $P' - A'$ equations is quite analogous. In fact, the former may be considered as the vectorial counterparts of the latter. It is reasonable, therefore, to concentrate a numerical investigation on the 0^{-+} equations with a mixture of P' and A' couplings. Inspection of the matrix (17) shows that such a mixture of P' and A' interactions includes purely vectorial binding forces as a special case, viz., for $\Lambda_A = -\frac{1}{2}\Lambda_P$.

Numerical estimates for the coupling constants Λ_A and Λ_P that give rise to 0^{-+} bound states with mass $2\epsilon_B$ can be found by making use of a variational principle of Rayleigh–Ritz type [26, 32–36]. For the *instantaneous Bethe-Salpeter equations* (20), (21), with $V_i' = \frac{1}{4}\pi^{-1}\Lambda_i(\mathbf{q}-\mathbf{q}')^{-2}$, one may derive such a principle by first writing the equations in coordinate space as:

$$\tilde{\chi}_A(\mathbf{r}) = -\tfrac{1}{2}\pi\Lambda_A r^{-1}\tilde{\psi}_A(\mathbf{r}), \tag{104}$$

$$\tilde{\chi}_P(\mathbf{r}) = -\tfrac{1}{2}\pi\Lambda_P r^{-1}\tilde{\psi}_P(\mathbf{r}). \tag{105}$$

Here $\tilde{\psi}_A$, $\tilde{\psi}_P$ are the Fourier transforms of ψ_A, ψ_P, while $\tilde{\chi}_A$, $\tilde{\chi}_P$ are the transforms of the functions $\chi_A = E(E^2\psi_A - \epsilon_B\psi_P)$ and $\chi_P = E(\psi_P - \epsilon_B\psi_A)$, respectively. Upon multiplying the first equation by $\tilde{\psi}_A^*$, the second by $\tilde{\psi}_P^*$, adding the results, and integrating over $\mathbf{r}$ one gets:

$$(2\pi)^3 \int d\mathbf{q}\, [\psi_A^* E(E^2\psi_A - \epsilon_B\psi_P) + \psi_P^* E(\psi_P - \epsilon_B\psi_A)]$$

$$= -\tfrac{1}{2}\pi \int d\mathbf{r}\, r^{-1}(\Lambda_A \mid \tilde{\psi}_A \mid^2 + \Lambda_P \mid \tilde{\psi}_P \mid^2), \tag{106}$$

where at the left-hand side the momentum-space representation has been introduced again, so that χ_A and χ_P could be eliminated. When this equation is varied at constant ϵ_B by writing $\psi_i + \delta\psi_i$ and $\Lambda_i + \delta\Lambda_i$ one finds the relation

$$\delta\Lambda_A \int d\mathbf{r}\, r^{-1} \mid \tilde{\psi}_A \mid^2 + \delta\Lambda_P \int d\mathbf{r}\, r^{-1} \mid \tilde{\psi}_P \mid^2 = 0 \tag{107}$$

for functions $\tilde{\psi}_A$ and $\tilde{\psi}_P$ that satisfy the bound-state equations. Hence the sum $\Lambda = \mid \Lambda_A \mid + \mid \Lambda_P \mid$ of coupling constants can be determined by evaluating, for fixed partial strengths $\rho_i = \Lambda_i/\Lambda$ and fixed ϵ_B, the values of the quotient

$$\frac{1}{\Lambda} = -\frac{\int d\mathbf{r}\, r^{-1}(\rho_A \mid \tilde{\psi}_A \mid^2 + \rho_P \mid \tilde{\psi}_P \mid^2)}{16\pi^2 \int d\mathbf{q}\, [E^3 \mid \psi_A \mid^2 - 2E\epsilon_B \operatorname{Re}(\psi_P^*\psi_A) + E \mid \psi_P \mid^2]} \tag{108}$$

that are stationary with respect to variations of ψ_i and $\tilde{\psi}_i$.

A second independent variational principle for Λ^{-1} follows from (104) and (105) by multiplication with $\Lambda_A^{-1} r\tilde{\chi}_A^*$ and $\Lambda_P^{-1} r\tilde{\chi}_P^*$, respectively, addition and integration over $\mathbf{r}$; this procedure yields the variational quotient;

$$\frac{1}{\Lambda} = -\frac{4\pi^4 \int d\mathbf{q}\, E^{-1}(E^2 - \epsilon_B^2)^{-1}\, [\mid \chi_A \mid^2 - 2\epsilon_B \operatorname{Re}(\chi_P^*\chi_A) + E^2 \mid \chi_P \mid^2]}{\int d\mathbf{r}\, r(\rho_A^{-1} \mid \tilde{\chi}_A \mid^2 + \rho_P^{-1} \mid \tilde{\chi}_P \mid^2)}, \tag{109}$$

which depends on the functions χ_i and $\tilde{\chi}_i$ only.

To obtain the deepest bound states (the "ground states") for given strengths of the coupling constants one must determine from either (108) or (109) the greatest value of Λ^{-1} that yields (for fixed values of ρ_A and ρ_P) a bound state with given ϵ_B. This can

be achieved by writing ψ_i or χ_i as a linear combination of trial functions and solving the ensuing generalized matrix eigenvalue problem of the type $A \cdot x = \Lambda^{-1} B \cdot x$. For positive-definite matrices B_{ij} a generalization of the Hylleraas–Undheim theorem [37] states that the greatest eigenvalue changes in a monotonous way when the matrix size increases by including a greater number of trial functions. If the trial functions are part of a complete set (in the space of functions satisfying the appropriate boundary conditions) this eigenvalue will converge monotonously to an eigenvalue of the original integral equations. Inspection of the denominators of (108) and (109) now shows that in general only the ψ variational principle will lead to a Λ^{-1} value with the convenient properties just described. The χ variational principle has a positive definite denominator only if ρ_A and ρ_P have the same sign. For that reason the ψ principle has been used mainly for the numerical work; part of the eigenvalues have been checked by employing the χ principle as well.

A useful set of trial functions for ψ_A and ψ_P in momentum space is

$$\psi_n(\mathbf{q}) = \left(1 + \frac{q^2}{a^2}\right)^{-n-\nu} \qquad (n = 1, 2,\ldots), \tag{110}$$

with $a^2 = 1 - \epsilon_B{}^2$. The solutions of the instantaneous Bethe–Salpeter equation in the nonrelativistic limit are indeed linear combinations of functions of this form, with $\nu_A = \nu_P = 1$, as follows from (98) and (100) with (83). In general ν_A and ν_P will be different from 1; the constraints (31) imply $\nu_A > \frac{1}{2}$ and $\nu_P > 0$.

To evaluate the variational quotient (108) we also need the position-space representations of the trial functions; they have the form [38]:

$$\begin{aligned}\tilde{\psi}_n(\mathbf{r}) &= 4\pi r^{-1} \int_0^\infty dq\, q \sin(qr)\, \psi_n(\mathbf{q}) \\ &= 16\pi^{3/2}(\tfrac{1}{2}a)^{n+\nu+3/2}\, [\Gamma(n+\nu)]^{-1}\, r^{n+\nu-3/2} K_{n+\nu-3/2}(ar).\end{aligned} \tag{111}$$

Information about an optimal choice for the parameters ν_A and ν_P can be obtained by considering the asymptotic behavior for large q of the equations (104) and (105) in momentum space. The Fourier transform of $r^{-1}\tilde{\psi}_n(r)$ is found to be [38]:

$$\begin{aligned}&\text{F.T.}[r^{-1}\tilde{\psi}_n(\mathbf{r})] \\ &\quad = a\pi^{-1/2}\Gamma(n+\nu-\tfrac{1}{2})[\Gamma(n+\nu)]^{-1}\, {}_2F_1(n+\nu-\tfrac{1}{2}, 1, \tfrac{3}{2}; -q^2/a^2).\end{aligned} \tag{112}$$

A comparison of the asymptotic terms in the momentum-space versions of (104) and (105) now shows that for $0 < \nu_P < \frac{1}{2}$ one must have $\nu_A = \nu_P + 1$ and

$$\Lambda_P = -\frac{4}{\pi} \frac{\nu_P}{\operatorname{tg}(\pi\nu_P)}; \tag{113}$$

if $\nu_P > \frac{1}{2}$ neither ν_A nor Λ_P can be related to ν_P. It should be noted that this asymptotic analysis makes sense only if large values of q^2 really play an important role in the bound-state wavefunction. In particular, it is not suitable to discuss nonrelativistic bound states since then q^2 is of order $1 - \epsilon_B{}^2$ and hence small compared to 1;

accordingly the factor $E = (1 + q^2)^{1/2}$ in the bound-state equation may be replaced by unity in the nonrelativistic limit, as has been done in the preceding section.

For values of Λ_P in the range $-4\pi^{-2} < \Lambda_P < 0$ the relation (113) can be employed to determine ν_P in a self-consistent way: a choice of ν_P gives, with the help of the variational principle, a value of Λ_P from which ν_P can be found again by means of (113). If Λ_P is positive this scheme does not work. In that case ν_P and ν_A have been chosen such that Λ has the smallest possible value for a given number of trial functions. Bound-state solutions with $\Lambda_P < -4\pi^{-2}$ have not been found; the stationary values of the variational quotient in that region turned out to depend strongly on the number of trial functions.

The integrals that result upon substituting the trial functions (110) and (111) in (108) are readily evaluated. The numerator becomes an integral over the product of two modified Bessel functions and a power of r, which may be expressed in terms of gamma functions [38]. The denominator is found to be a linear combination of integrals

$$\int_0^\infty dq \, \frac{q^2(1 + q^2)^{1/2}}{(1 + a^{-2}q^2)^{2\nu+p+1}}$$

$$= \frac{1}{4}\pi^{1/2} \frac{\Gamma(p + 2\nu - 1)}{\Gamma(p + 2\nu + \frac{1}{2})} {}_2F_1\left(p + 2\nu + 1, \frac{3}{2}, p + 2\nu + \frac{1}{2}; -\frac{1 - a^2}{a^2}\right), \quad (114)$$

with $p = 1, 2, \ldots$. When the hypergeometric functions for $p = 1$ and 2 have been calculated numerically one can use recursion relations to obtain the integrals for $p \geqslant 3$.

The stationary values of the Rayleigh–Ritz quotient are found now by solving the associated generalized matrix eigenvalue problem. A suitable method for this purpose is the so-called QZ procedure [39], which is a generalization of the QR algorithm. The values of Λ^{-1} given by this procedure turn out to converge rapidly with increasing matrix size. Matrices of dimension $\leqslant 25$ were in all cases sufficient to reach an accuracy of four digits; often the obtained accuracy was much better.

The results of the numerical work on the instantaneous Bethe–Salpeter equation have been collected in Tables I–IV and in Figs. 1–3. For pure P' and pure A' interactions the coupling constants that correspond to the states with deepest binding are given by the curves with labels 0.0 in Fig. 1 and Fig. 2, respectively. It is seen that the qualitative behavior of these curves is quite different. For pure P' coupling the solution breaks off at a critical value $\epsilon_{Bc}^2 = 0.789$, corresponding to $\Lambda_P = -4\pi^{-2}$. This could be expected since it has been proved on general grounds in the preceding section that no bound states with $\epsilon_B{}^2 < 0.5$ exist in this case. A similar discontinuous behavior of the coupling constant for a finite value of $\epsilon_B{}^2$ has been found in a numerical analysis of the covariant Bethe–Salpeter equation for a fermion pair with massless-boson exchange [26]; the instantaneous approximation clearly has not resulted in a qualitative change of the coupling-constant spectrum. For pure A' coupling, on the other hand, the coupling constant is a continuous function of $\epsilon_B{}^2$ in the whole interval (0, 1); for $\epsilon_B{}^2 = 0$ in particular Λ_A has a finite value.

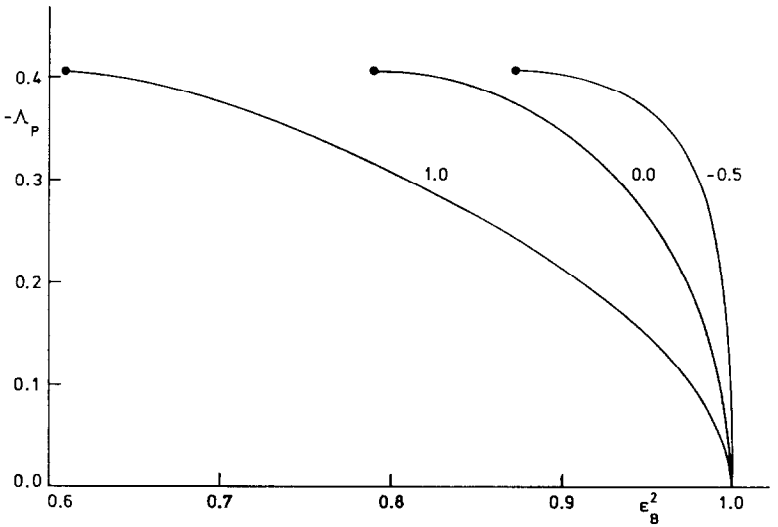

FIG. 1. The coupling constant Λ_P of the instantaneous Bethe–Salpeter equation as a function of the bound-state mass $2\epsilon_B$, for $\rho = \Lambda_A/\Lambda_P = 1$, 0, and -0.5.

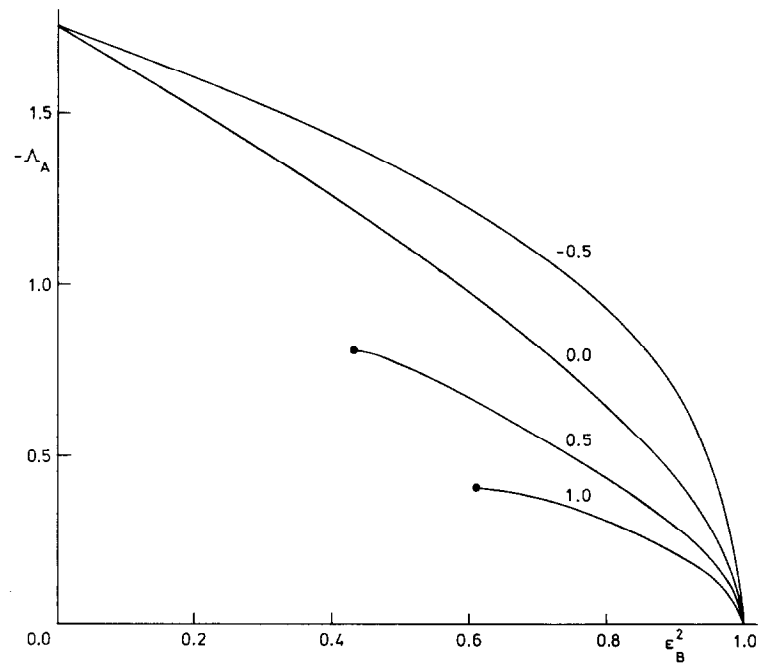

FIG. 2. The coupling constant Λ_A of the instantaneous Bethe–Salpeter equation for $\rho^{-1} = 1$, 0.5, 0, and -0.5.

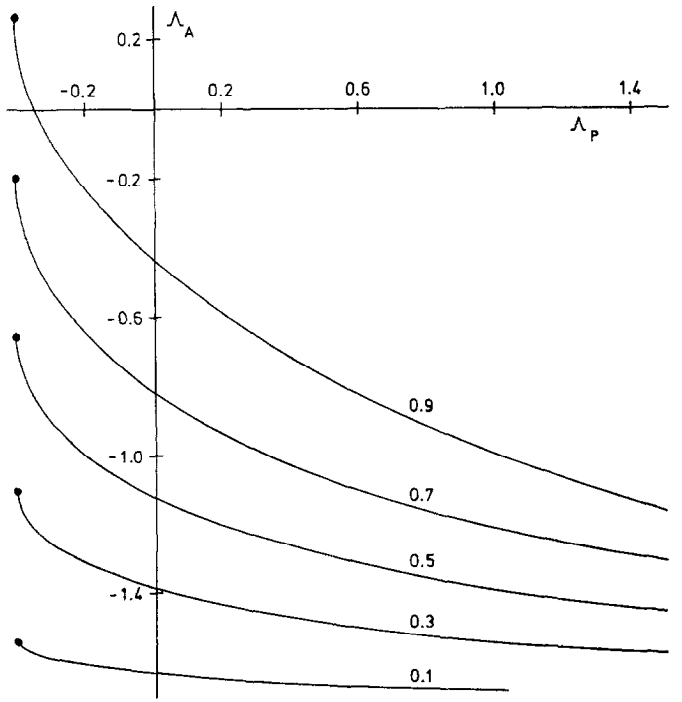

FIG. 3. The coupling-constant spectrum of the instantaneous Bethe–Salpeter equation for $\epsilon_B^2 = 0.1, 0.3, 0.5, 0.7$, and 0.9.

TABLE I

Coupling Constants $\Lambda = |\Lambda_A| + |\Lambda_P|$, with $\Lambda_A < 0$, $\Lambda_P < 0$, of the Instantaneous Bethe–Salpeter Equation as a Function of the Bound-State Mass Parameter ϵ_B^2 and the Coupling-Mixture Coefficient $\rho = \Lambda_A/\Lambda_P$

ϵ_B^2 \ ρ	0.1	0.2	0.5	1	2	5	10
0.95	0.2718	0.2770	0.2866	0.2937	0.2979	0.2992	0.2989
0.9	0.3638	0.3773	0.4039	0.4239	0.4356	0.4388	0.4377
0.85	0.4164	0.4397	0.4885	0.5269	0.5491	0.5543	0.5519
0.8	0.4420	0.4754	0.5510	0.6143	0.6506	0.6577	0.6540
0.7				0.7513	0.8341	0.8476	0.8381
0.6					1.001	1.024	1.007
0.5					1.150	1.196	1.167
0.4						1.366	1.323
0.3						1.538	1.475
0.2						1.715	1.626
0.1						1.902	1.777
0.0						2.104	1.929

TABLE II

Coupling Constant Λ of the Instantaneous Bethe–Salpeter Equation for $\Lambda_A \geqslant 0$, $\Lambda_P < 0$

ϵ_B^2 \ ρ	0.0	−0.1	−0.2	−0.5
0.95	0.2652	0.3134	0.3674	0.5540
0.9	0.3470	0.3989	0.4522	0.6041
0.85	0.3890	0.4368	0.4829	
0.8	0.4048			

TABLE III

Coupling Constants Λ of the Instantaneous Bethe–Salpeter Equation for $\Lambda_A < 0$, $\Lambda_P \geqslant 0$

ϵ_B^2 \ ρ	−0.5	−1	−2	−5	−10	$-\infty$
0.95	3.896	1.763	0.7751	0.4379	0.3615	0.2979
0.9	4.015	2.006	1.042	0.6279	0.5239	0.4346
0.85	4.124	2.182	1.233	0.7767	0.6540	0.5459
0.8	4.223	2.324	1.386	0.9037	0.7672	0.6443
0.7	4.399	2.554	1.632	1.119	0.9637	0.8188
0.6	4.554	2.741	1.831	1.303	1.135	0.9753
0.5	4.695	2.903	2.000	1.465	1.291	1.120
0.4	4.823	3.045	2.150	1.612	1.434	1.257
0.3	4.943	3.175	2.285	1.748	1.568	1.388
0.2	5.055	3.294	2.409	1.874	1.694	1.514
0.1	5.161	3.404	2.523	1.992	1.814	1.635
0.0	5.261	3.507	2.631	2.104	1.929	1.754

TABLE IV

Critical Values of $\rho = \Lambda_A/\Lambda_P$ for the Instantaneous Bethe–Salpeter (IBS) and the Quasi-Potential (QP) Equation

ϵ_B^2	ρ_c (IBS)	ρ_c (QP)	ϵ_B^2	ρ_c (IBS)	ρ_c (QP)
0.0	4.327	1.682	0.6	1.063	−1.102
0.1	3.790	0.598	0.7	0.505	−1.356
0.2	3.251	0.134	0.8	−0.062	−1.600
0.3	2.709	−0.232	0.9	−0.642	−1.839
0.4	2.164	−0.548	1.0	−1.24	−2.07
0.5	1.615	−0.835			

The other curves in Figs. 1 and 2 represent the coupling constants for some mixtures of P' and A' interactions. The discontinuous character of the Λ_P coupling constant is not affected by the addition of a not too strong A' interaction. In particular, for $\rho \equiv \Lambda_A/\Lambda_P = -0.5$ (corresponding to a pure V coupling) the curve breaks off at a critical value $\epsilon_{Bc}^2 = 0.878$. From Fig. 2 it is found that a relatively weak admixture of P' coupling to the pure A' interaction is enough to introduce the characteristic discontinuity in the curves.

For pure V coupling the spectrum of the instantaneous Bethe–Salpeter equation has been investigated recently by Kellett [10]. The numerical results in that paper are obtained by inserting a set of trial functions in the momentum-space equations and solving the (nonhermitean) matrix eigenvalue problem that arises after numerical evaluation of the momentum-space integrals. In this way values for the coupling constants of the deepest bound states are found that turn out to be systematically higher than those presented here. The reason for this discrepancy is not clear; it is probably due to the use in [10] of trial functions with a too simple asymptotic behavior for large q.

A better insight in the qualitative aspects of the coupling-constant spectrum for the deepest bound states may be gained with the help of Fig. 3, where curves of fixed ϵ_B^2 in the $\Lambda_P - \Lambda_A$ plane have been drawn. It is seen that all curves break off when Λ_P reaches the value $-4\pi^{-2}$. For $\epsilon_B^2 = 0$ this happens at $\rho_c = 4.327$ and for $\epsilon_B^2 \to 1$ one finds $\rho_c \to -1.24$. The critical values of ρ for intermediate ϵ_B^2 are given in Table IV; ρ_c is found to be an almost linear function of ϵ_B^2. In the Λ_A direction the coupling constants are bound as well: for all $\epsilon_B^2 \geq 0$ one finds $\Lambda_A \geq -1.754$, the equality sign being fulfilled for $\epsilon_B^2 = 0$ (independent of ρ).

In view of these general spectral properties one may distinguish several intervals in the range of possible ρ values. For positive ρ both the A' and P' couplings contribute to the binding potential that is attractive if $\Lambda_A < 0$, $\Lambda_P < 0$. If $\rho \geq 4.327$ (so that A' coupling dominates heavily) bound states with any ϵ_B^2 in the range (0, 1) may be produced. For $0 \leq \rho < 4.327$ (corresponding to a more important P' coupling) the solutions exhibit the P' discontinuity discussed above: only bound states with ϵ_B^2 above a critical value (that depends on ρ) are realized.

When ρ is negative the picture is more complicated as may be expected since now the A' and P' forces are opposing each other, one being attractive, the other repulsive. Bound states come about as a result of the balancing of the interactions. If $\rho < -1.24$ the A' coupling wins: bound states of all ϵ_B^2 are found for interactions with $\Lambda_A < 0$ and $\Lambda_P > 0$. For $-1.24 \leq \rho < 0$ the balancing of the two interactions is delicate: the coupling constants that give bound states are found to lie on two branches, with different signs for Λ_A and Λ_P. The branch with $\Lambda_A < 0$, $\Lambda_P > 0$ gives bound states for all ϵ_B^2. However, if $-1 < \rho < 0$ the coupling constants remain finite for $\epsilon_B^2 \to 1$, so that weak binding forces are not enough to produce loosely bound states; these solutions clearly have no nonrelativistic limit. For $-1.24 < \rho < -1$ the solutions with $\Lambda_A < 0$, $\Lambda_P > 0$ behave normally in the nonrelativistic limit. The solutions on the other branch, with $\Lambda_A > 0$, $\Lambda_P < 0$ have just the opposite properties: normal nonrelativistic behavior is found on this branch for $-1 < \rho < 0$. The dominating P'

coupling then gives rise to the well-known discontinuity in the spectrum. The change-over of the normal nonrelativistic behavior from one branch to the other at $\rho = -1$ could be inferred already from (99) since it implies that $\Lambda_A + \Lambda_P$ must be negative in the nonrelativistic limit.

The coupling constants of the *quasi-potential equation* (38) may likewise be obtained by determining the stationary values of a Rayleigh–Ritz quotient:

$$\frac{1}{\Lambda} = -\frac{\int d\mathbf{r}\, r^{-1}(\rho_A \mid \tilde{\psi}_A \mid^2 + \rho_P \mid \tilde{\psi}_P \mid^2)}{32\pi^2 \int d\mathbf{q}\, (E - \epsilon_B) \mid \psi_P \mid^2}, \tag{115}$$

where $\tilde{\psi}_A$ is the Fourier transform of $\psi_A = E^{-1}\psi_P$. This variational principle, which is the counterpart of (108), follows directly by multiplying (38) with ψ_P^*, integrating over $\mathbf{q}$, and introducing the position-space representation to transform the resulting double integral over $\mathbf{q}$ and $\mathbf{q}'$ to a single integral over $\mathbf{r}$. Since the denominator in (115) is again positive definite the eigenvalues of the quasi-potential equation are approximated monotonously when a linear combination of an increasing number of trial functions is inserted in the variational quotient.

As trial functions for ψ_P we shall take once more functions of the form (110). The asymptotic form of the quasi-potential equation for large q^2 can be used to determine ν. For $-8\pi^{-2} < \Lambda_P < 0$ a value of ν, with $0 < \nu < \frac{1}{2}$, follows in a self-consistent way from

$$\Lambda_P = -\frac{8}{\pi}\frac{\nu}{\operatorname{tg}(\pi\nu)} \tag{116}$$

(cf. (113)), while for $\Lambda_P > 0$ the parameter ν is fixed by requiring Λ to be minimal for a fixed number of trial functions. For $\Lambda_P < -8\pi^{-2}$ convergent stationary values of the variational quotient have not been found.

To evaluate the numerator of (115) analytical expressions for both $\tilde{\psi}_P$ and $\tilde{\psi}_A =$ F.T. $(E^{-1}\psi_P)$ must be available. For that reason we have chosen $a = 1$ in the trial functions. Owing to this choice a large number of trial functions will be needed to approximate the nonrelativistic solutions of the quasi-potential equation. Hence the variational principle will lead to slowly converging values of Λ when $\epsilon_B{}^2$ approaches 1. In practice, however, it turns out that 15 trial functions are enough to determine Λ in four digits up to $\epsilon_B{}^2 = 0.95$.

The numerical results as given in Tables V–VII and Figs. 4–6 show that the spectrum of the quasi-potential equation has the same qualitative features as that of the instantaneous Bethe–Salpeter equation. In particular, from the curves in Figs. 4 and 5 one finds that for dominating P' coupling the bound-state solution breaks off at a finite value of $\epsilon_B{}^2$, whereas no such discontinuity occurs if the A' coupling dominates. In Fig. 6 the curves of fixed $\epsilon_B{}^2$ in the $\Lambda_P - \Lambda_A$ plane have been drawn. All these curves break off when Λ_P becomes $-8\pi^{-2}$; for $\epsilon_B{}^2 = 0$ and $\epsilon_B{}^2 \to 1$ this happens when ρ reaches the values 1.682 and -2.07, respectively (see Table IV for intermediate values of $\epsilon_B{}^2$). The coupling constant Λ_A has no uniform lower bound. Just as for the instantaneous Bethe–Salpeter equation the domain of ρ values may be divided into

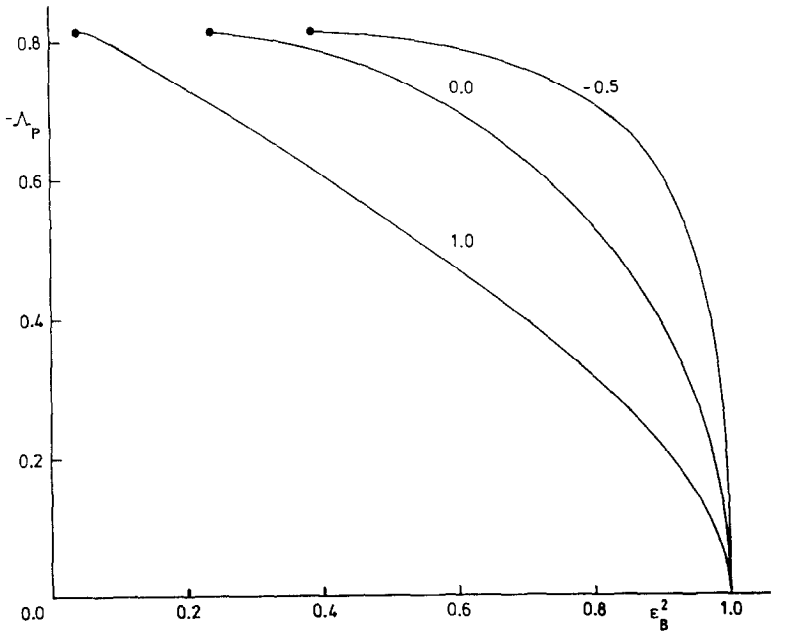

FIG. 4. Solutions of the quasi-potential equations for $\rho = 1, 0$ and -0.5.

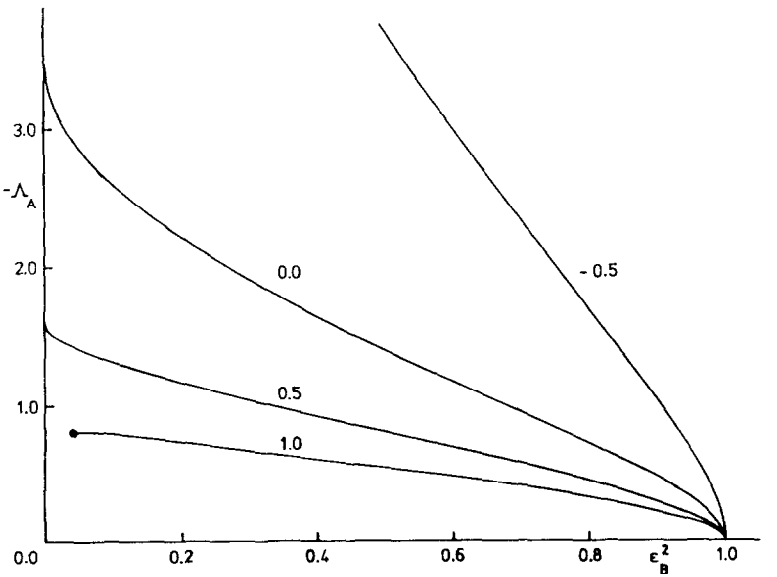

FIG. 5. Solutions of the quasi-potential equation for $\rho^{-1} = 1, 0.5, 0$, and -0.5.

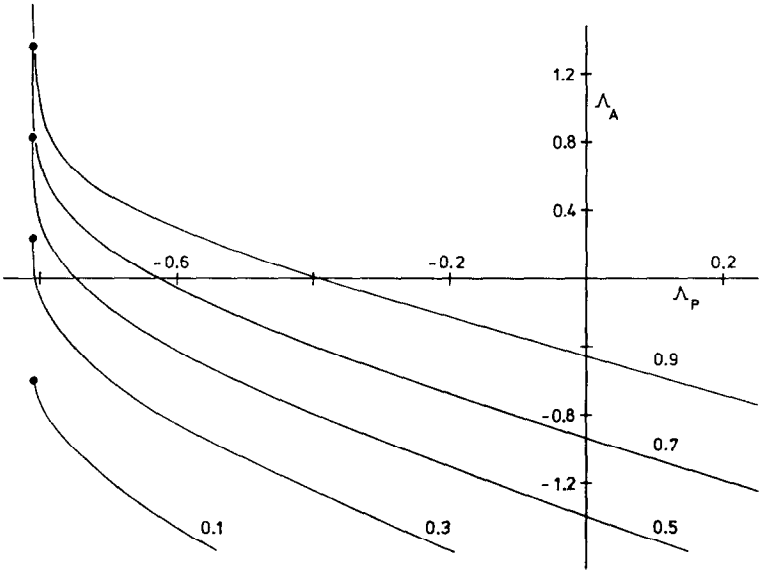

FIG. 6. The coupling-constant spectrum of the quasi-potential equation for $\epsilon_B{}^2 = 0.1$, 0.3, 0.5, 0.7, and 0.9.

TABLE V

Coupling Constants Λ of the Quasi-Potential Equation for $\Lambda_A < 0$, $\Lambda_P < 0$

$\epsilon_B{}^2$ \ p	0.1	0.2	0.5	1	2	5	10
0.95	0.2820	0.2843	0.2893	0.2941	0.2986	0.3030	0.3050
0.9	0.3950	0.4010	0.4137	0.4258	0.4374	0.4484	0.4533
0.85	0.4791	0.4893	0.5112	0.5322	0.5521	0.5711	0.5793
0.8	0.5477	0.5626	0.5948	0.6257	0.6552	0.6831	0.6953
0.7	0.6563	0.6815	0.7371	0.7915	0.8433	0.8922	0.9134
0.6	0.7394	0.7758	0.8583	0.9406	1.020	1.094	1.126
0.5	0.8035	0.8516	0.9643	1.080	1.192	1.297	1.342
0.4	0.8509	0.9107	1.057	1.213	1.366	1.508	1.568
0.3	0.8814	0.9524	1.136	1.343	1.547	1.735	1.814
0.2	0.8916	0.9722	1.195	1.467	1.743	1.991	2.094
0.1				1.581	1.971	2.310	2.447
0.0					2.382	3.026	3.257

TABLE VI

Coupling Constants Λ of the Quasi-Potential Equation for $\Lambda_A \geqslant 0$, $\Lambda_P < 0$

$\epsilon_B{}^2$ \ ρ	0	−0.1	−0.2	−0.5	−1
0.95	0.2791	0.3367	0.4062	0.7288	1.527
0.9	0.3877	0.4627	0.5502	0.9054	1.557
0.85	0.4667	0.5515	0.6477	1.001	1.577
0.8	0.5297	0.6204	0.7205	1.062	1.593
0.7	0.6265	0.7223	0.8236	1.139	1.612
0.6	0.6974	0.7934	0.8914	1.183	1.620
0.5	0.7495	0.8428	0.9360	1.207	
0.4	0.7857	0.8747	0.9625	1.216	
0.3	0.8064	0.8903	0.9723		

TABLE VII

Coupling Constants Λ of the Quasi-Potential Equation for $\Lambda_A < 0$, $\Lambda_P \geqslant 0$

$\epsilon_B{}^2$ \ ρ	−2	−5	−10	$-\infty$
0.95	0.9900	0.4706	0.3790	0.3073
0.9	1.541	0.7109	0.5693	0.4590
0.85	2.043	0.9216	0.7340	0.5890
0.8	2.528	1.120	0.8877	0.7095
0.7	3.494	1.503	1.182	0.9381
0.6	4.490	1.887	1.475	1.163
0.5	5.549	2.287	1.777	1.393
0.4	6.703	2.716	2.098	1.637
0.3	7.998	3.191	2.453	1.904
0.2	9.523	3.745	2.864	2.211
0.1	11.50	4.456	3.388	2.600
0.0	16.24	6.140	4.624	3.507

five separate intervals (with boundaries $\rho = 1.682, 0, -1, -2.07$) that are characterized by a different qualitative behavior of the coupling constants as a function of $\epsilon_B{}^2$. The discussion is closely analogous to that given before and will not be repeated.

The coupling constants of the *two-particle Dirac equations* for 0^{-+} states with $P' - A'$ interactions can be obtained from a variational principle that is based on the momentum-space representation (47)–(49). A more convenient starting point, however, is given by the position-space equation (63) or (93) with (94).

Since the effective potential (94) becomes singular for positive Λ_A, the two-particle Dirac equation will have a spectrum comparable to that of the instantaneous Bethe–

Salpeter and the quasi-potential equations only for the case $\Lambda_A < 0$, to which we shall confine ourselves in the following. Some numerical work on the singular case has been reported in [20].

The eigenvalues $1 - \epsilon_B{}^2$ of the equation (93) may be determined from the stationary values of the variational quotient

$$1 - \epsilon_B{}^2 = \frac{\int d\mathbf{r}\, \tilde{\psi}_P^*[d^2/dr^2 + 2r^{-1}\, d/dr - V(r)]\, \tilde{\psi}_P}{\int d\mathbf{r}\, |\, \tilde{\psi}_P\, |^2}. \tag{117}$$

TABLE VIII

Coupling Constants Λ of the Two-Particle Dirac Equation for $\Lambda_A \leqslant 0$, $\Lambda_P < 0$

$\epsilon_B{}^2$ \ ρ	0	0.1	0.2	0.5	1	2	5	10
0.95	0.2921	0.2912	0.2911	0.2930	0.2972	0.3039	0.3133	0.3185
0.9	0.4244	0.4217	0.4214	0.4258	0.4366	0.4540	0.4792	0.4939
0.85	0.5349	0.5294	0.5286	0.5360	0.5544	0.5848	0.6305	0.6581
0.8	0.6366	0.6275	0.6260	0.6365	0.6634	0.7086	0.7793	0.8239
0.7	0.8335	0.8137	0.8098	0.8265	0.8727	0.9535	1.089	1.181
0.6	1.040	1.003	0.9948	1.018	1.086	1.211	1.434	1.599
0.5	1.273	1.209	1.195	1.224	1.319	1.498	1.840	2.112
0.4	1.559	1.450	1.426	1.461	1.588	1.835	2.339	2.777
0.3	1.945	1.755	1.715	1.755	1.922	2.260	2.994	3.694
0.2	2.546	2.189	2.121	2.166	2.389	2.858	3.948	5.098
0.1	3.820	2.979	2.849	2.893	3.213	3.918	5.681	7.775

TABLE IX

Coupling Constants Λ of the Two-Particle Dirac Equation for $\Lambda_A < 0$, $\Lambda_P \geqslant 0$

$\epsilon_B{}^2$ \ ρ	-2	-5	-10	$-\infty$
0.95	1.695	0.5279	0.4108	0.3259
0.9		0.9000	0.6682	0.5148
0.85		1.334	0.9358	0.6992
0.8		1.898	1.236	0.8926
0.7		4.019	2.018	1.336
0.6		15.38	3.281	1.903
0.5			5.865	2.681
0.4			15.58	3.834
0.3				5.742
0.2				9.545
0.1				20.93

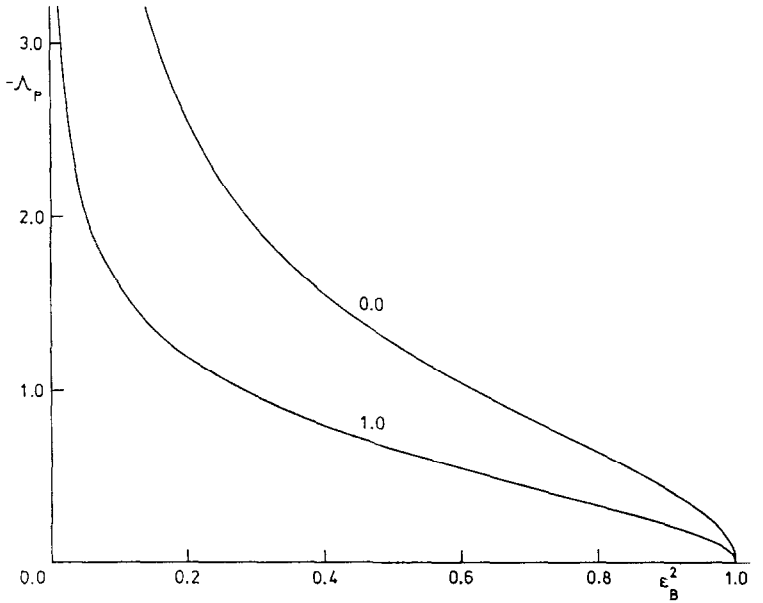

FIG. 7. Λ_P-curves for the two-particle Dirac equation with $\rho = 1$ and 0.

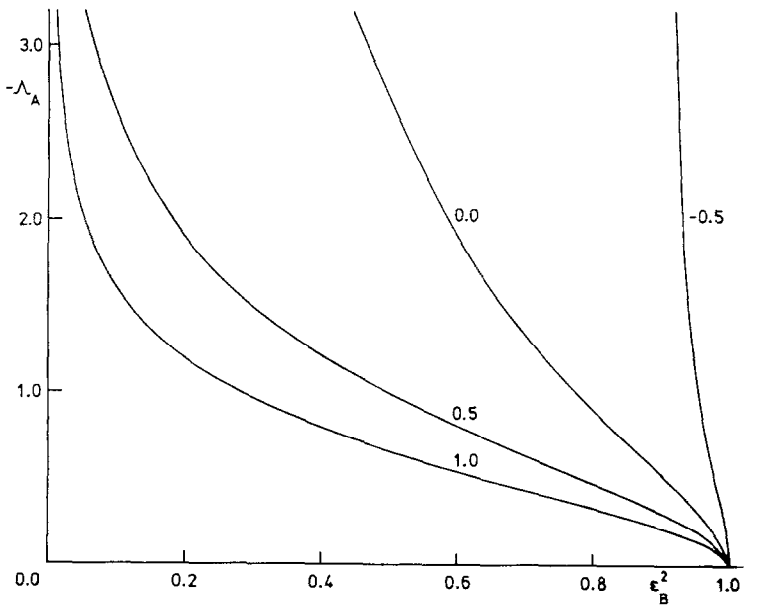

FIG. 8. Λ_A-curves for the two-particle Dirac equation with $\rho^{-1} = 1, 0.5, 0$, and -0.5.

As trial functions linear combinations of the Coulomb wavefunctions

$$\tilde{\psi}_n(\mathbf{r}) = r^{n-1}e^{-ar} \qquad (n = 1, 2,...) \tag{118}$$

are employed. The parameter $a = (1 - \epsilon_B{}^2)^{1/2}$ is found in a self-consistent way by successive approximations: each time the result of the variational procedure is used to obtain an improved value for a.

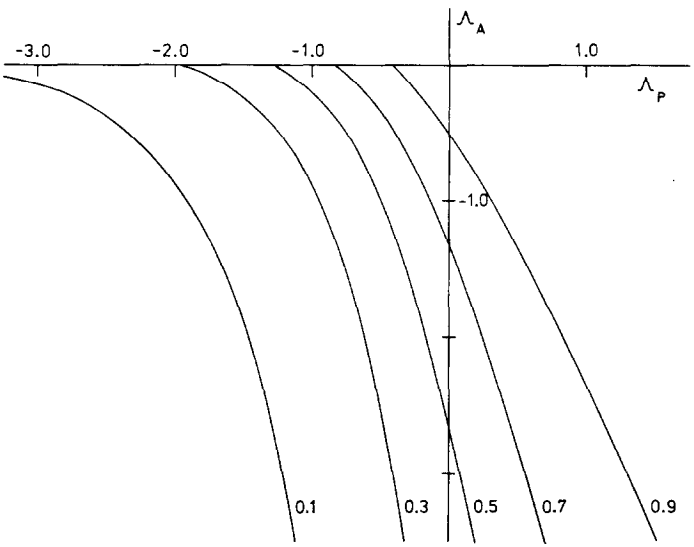

FIG. 9. The coupling-constant spectrum of the two-particle Dirac equation for $\epsilon_B{}^2 = 0.1$, 0.3, 0.5, 0.7, and 0.9.

Numerical values for the coupling constants Λ_P and $\Lambda_A < 0$ as a function of $\epsilon_B{}^2$ are given in Tables VIII–IX and in Figs. 7–9. For pure P' interactions one recovers the coupling constants of the Schrödinger equation for the Coulomb problem, viz., $\Lambda_P = -4\pi^{-1}\epsilon_B{}^{-1}(1 - \epsilon_B{}^2)^{1/2}$. For pure A' interactions the results agree with those following from the confluent hypergeometric function in (92). In neither of these cases the solutions display the characteristic discontinuity that we encountered for the instantaneous Bethe–Salpeter and quasi-potential equations (see Figs. 7 and 8). For mixed $A' - P'$ interactions with positive $\rho = \Lambda_A/\Lambda_P$ bound states of any $\epsilon_B{}^2$ in the interval (0, 1) can be formed; for $\epsilon_B{}^2 \to 0$ the coupling constants tend to $-\infty$. If ρ is negative a singularity of a new type shows up: for $-\infty < \rho < -1$ the coupling constants become infinite already for a positive value of $\epsilon_B{}^2$, so that deep binding (with $\epsilon_B{}^2 \simeq 0$) is no longer possible; for $-1 < \rho < 0$ no bound-state solutions have been found.

7. Conclusion

In this paper we have studied three types of instantaneous equations for bound states of a fermion pair, viz., the "instantaneous Bethe–Salpeter equation," which follows straightforwardly from the corresponding covariant equation by insertion of an instantaneous potential, the "quasi-potential equation," which is obtained when some pair-creation terms are supposed to be negligible, and the "two-particle Dirac equation," which is a direct generalization of the ordinary Dirac equation.

The consequences of the approximations that led to the instantaneous equations can be estimated by investigating the coupling-constant spectra. To that end the general equations for bound states of arbitrary spin, parity, and charge parity have been applied in this paper to the special case of spinless states bound by interactions with a Coulomb-type potential. The spectral properties are found to depend strongly on the type of fermion coupling, as characterized by the labels S', V', T', A', P'. This can be anticipated already by studying the nonrelativistic limit. In this limit neither of the equations has a solution that describes a 0^{++} state bound by pure S' coupling or a 0^{-+} state with pure T' interaction. If a mixture of couplings is present, however, nonrelativistic solutions do exist and all three equations give identical values for the coupling constants; the V', T' and A', P' components turn out to give the dominating binding forces in nonrelativistic 0^{++} and 0^{-+} states, respectively.

The properties of the instantaneous equations for states with strong binding are in accordance with the nonrelativistic results. For 0^{++} states with S' coupling and 0^{-+} states with T' interaction the three equations have a completely different coupling-constant spectrum. Whereas the instantaneous Bethe–Salpeter equation has no solutions at all for these cases, the two-particle Dirac equation predicts bound states with masses that increase when the interaction becomes stronger; the quasipotential equation possesses solutions only if the coupling constants are larger than a critical value.

A more consistent picture is expected for 0^{++} states bound by V' or T' couplings and for 0^{-+} states with A', P' interactions. For the latter case this has been checked by a numerical analysis of the coupling constants that correspond to the ground-state solutions. The results, as presented in Figs. 1–9, show that indeed the coupling constants Λ_A and Λ_P of the instantaneous Bethe–Salpeter and the quasi-potential equation have a similar qualitative behavior as a function of the bound-state mass $2\epsilon_B$, although the numerical values are rather different for deeply bound states. A peculiar feature shared by both these equations is the occurrence of a lower bound on the coupling constant Λ_P. As a consequence the ground-state solutions for dominating P' interactions break off at finite values of ϵ_B (see Figs. 1 and 4). Such a behavior is also displayed by the covariant Bethe–Salper equation for 0^{-+} states with pure P' coupling, as has been shown before.

The two-particle Dirac equation yields a coupling-constant spectrum with somewhat different properties. The characteristic discontinuity in the solution for dominating P' coupling is not found here (see Fig. 7). Moreover a singularity at a finite interparticle distance shows up in the position-space representation of the bound-state

equation, when the coupling constant Λ_A becomes positive. As a consequence the analogy with the other equations is completely lost for $\Lambda_A > 0$.

It must be concluded that the approximations that have led to the two-particle Dirac equation have brought about essential changes in the global properties of the bound-state spectrum, so that it cannot be used as a substitute for the instantaneous Bethe–Salpeter equation. The quasi-potential equation on the other hand combines the advantages of a relatively simple structure and a spectrum that is qualitatively analogous to that of the instantaneous Bethe-Salpeter equation, at least for the 0^{-+} state solutions discussed here.

APPENDIX A: VECTOR SPHERICAL HARMONICS

The vector spherical harmonics are defined in terms of the ordinary spherical harmonics $Y_L{}^M$ as:

$$\mathbf{Y}_{J.L}^{M}(\hat{\mathbf{q}}) = \sum_{M_1, M_2} Y_L^{M_1}(\hat{\mathbf{q}})\, \mathbf{e}^{M_2} C(LM_1\, 1M_2 \mid JM), \tag{A1}$$

with $\mathbf{e}_{\pm} = \mp(\mathbf{e}_x \pm i\mathbf{e}_y)/2^{1/2}$ and $\mathbf{e}_0 = \mathbf{e}_z$. They may be used to construct 2×2 matrix eigenfunctions of the z component and the square of the operator $\mathbf{J} = -i\mathbf{q} \wedge \nabla_{\mathbf{q}} + \frac{1}{2}[\boldsymbol{\sigma},]$. In fact, when one introduces the electric and magnetic spherical harmonics

$$\mathbf{Y}_J^{(e)M} = \left(\frac{J+1}{2J+1}\right)^{1/2} \mathbf{Y}_{J.J-1}^{M} + \left(\frac{J}{2J+1}\right)^{1/2} \mathbf{Y}_{J.J+1}^{M}, \tag{A2}$$

$$\mathbf{Y}_J^{(m)M} = \mathbf{Y}_J^{MM}, \tag{A3}$$

which are related by

$$\mathbf{Y}_J^{(m)M} = i\mathbf{Y}_J^{(e)M} \wedge \hat{\mathbf{q}}, \qquad \mathbf{Y}_J^{(e)M} = i\mathbf{Y}_J^{(m)M} \wedge \hat{\mathbf{q}}, \tag{A4}$$

one can prove that a complete set of eigenfunctions is given by $Y_J{}^M$, $Y_J{}^M$ $\boldsymbol{\sigma} \cdot \mathbf{q}$, $\boldsymbol{\sigma} \cdot \mathbf{Y}_J^{(e)M}$, $\boldsymbol{\sigma} \cdot \mathbf{Y}_J^{(m)M}$. The eigenfunctions may be grouped into two sets on the basis of their space-inversion properties, since one has:

$$Y_J{}^M(-\hat{\mathbf{q}}) = (-)^J\, Y_J{}^M(\hat{\mathbf{q}}), \tag{A5}$$

$$\mathbf{Y}_J^{(e)M}(-\hat{\mathbf{q}}) = (-)^{J+1}\, \mathbf{Y}_J^{(e)M}(\hat{\mathbf{q}}), \tag{A6}$$

$$\mathbf{Y}_J^{(m)M}(-\hat{\mathbf{q}}) = (-)^J\, \mathbf{Y}_J^{(m)M}(\hat{\mathbf{q}}). \tag{A7}$$

These eigenfunctions have been employed in Section 2 to construct bound-state wavefunctions.

To derive scalar bound-state equations identities must be used that give the contractions of the scalar and vector harmonics with their complex conjugates. By employing the coupling theorem

$$Y_{J_1}^{M_1} Y_{J_2}^{M_2} = \sum_{JM} \left[\frac{(2J_1 + 1)(2J_2 + 1)}{4\pi(2J + 1)}\right]^{1/2} C(J_1 0 J_2 0 \mid J0)\, C(J_1 M_1 J_2 M_2 \mid JM)\, Y_J^M \quad \text{(A8)}$$

and the basic contraction relation

$$\frac{4\pi}{2J + 1} \sum_M [Y_J^M(\hat{\mathbf{q}})]^*\, Y_J^M(\hat{\mathbf{q}}') = P_J(z), \quad \text{(A9)}$$

with $z = \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}'$, the following set of contraction formulas may be proved:

$$4\pi(2J + 1)^{-1} \sum_M [Y_J^M(\hat{\mathbf{q}})\, \hat{\mathbf{q}}]^* \cdot \hat{\mathbf{q}}'\, Y_J^M(\hat{\mathbf{q}}') = zP_J(z), \quad \text{(A10)}$$

$$4\pi(2J + 1)^{-1} \sum_M [\mathbf{Y}_J^{(e)M}(\hat{\mathbf{q}})]^* \cdot \mathbf{Y}_J^{(e)M}(\hat{\mathbf{q}}') = P_J^{ee}(z), \quad \text{(A11)}$$

$$4\pi(2J + 1)^{-1} \sum_M [\mathbf{Y}_J^{(m)M}(\hat{\mathbf{q}})]^* \cdot \mathbf{Y}_J^{(m)M}(\hat{\mathbf{q}}') = P_J(z), \quad \text{(A12)}$$

$$4\pi(2J + 1)^{-1} \sum_M [Y_J^M(\hat{\mathbf{q}})\, \hat{\mathbf{q}}]^* \cdot \mathbf{Y}_J^{(e)M}(\hat{\mathbf{q}}') = P_J^e(z), \quad \text{(A13)}$$

$$4\pi(2J + 1)^{-1} \sum_M [Y_J^M(\hat{\mathbf{q}})\, \hat{\mathbf{q}}]^* \cdot \mathbf{Y}_J^{(m)M}(\hat{\mathbf{q}}') = 0, \quad \text{(A14)}$$

$$4\pi(2J + 1)^{-1} \sum_M [\mathbf{Y}_J^{(e)M}(\hat{\mathbf{q}})]^* \cdot \mathbf{Y}_J^{(m)M}(\hat{\mathbf{q}}') = 0; \quad \text{(A15)}$$

here the abbreviations

$$P_J^e = [J(J + 1)]^{1/2} (2J + 1)^{-1} (P_{J-1} - P_{J+1}), \quad \text{(A16)}$$

$$P_J^{ee} = (2J + 1)^{-1} [(J + 1) P_{J-1} + JP_{J+1}] \quad \text{(A17)}$$

have been used.

APPENDIX B: Instantaneous Bethe–Salpeter Equations for General J

In this appendix the scalar equations are collected that follow from the Bethe–Salpeter equation with instantaneous interaction, as given in (4). The structure functions $\psi_i(\mathbf{q})$ have been defined in (7)–(10), with (11)–(14). The potential functions V_i' depend on $(\mathbf{q} - \mathbf{q}')^2$, the Legendre polynomials P_J and its linear combinations P_J^e and P_J^{ee}, as defined in (A16)–(A17), are functions of $z = \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}'$.

$^3(J \pm 1)_J$:

$$\frac{E}{q}\left(\frac{E^2}{q}\,\psi_S - \epsilon_B\psi_{T1}\right) = \int d\mathbf{q}'\left[P_J\left(V'_S - \frac{z}{qq'}\,V'_V\right)\psi_S - P_J{}^e\,\frac{1}{q}\,V'_V\psi_{V2}\right], \tag{B1}$$

$$E(\psi_{V2} - \epsilon_B\psi_{T2}) = -\int d\mathbf{q}'\left(P_J{}^e\,\frac{1}{q'}\,V'_V\psi_S + P_J^{ee}V'_V\psi_{V2}\right), \tag{B2}$$

$$E\left(\psi_{T1} - \frac{\epsilon_B}{q}\,\psi_S\right) = -\int d\mathbf{q}'\,(P_JzV'_T\psi_{T1} + P_J{}^eV'_T\psi_{T2}), \tag{B3}$$

$$E(E^2\psi_{T2} - \epsilon_B\psi_{V2}) = -\int d\mathbf{q}'\,(P_J{}^eV'_T\psi_{T1} - P_Jqq'V'_A\psi_{T2} + P_J^{ee}V'_T\psi_{T2}). \tag{B4}$$

For bound states with $J = 0$ the structure functions ψ_{V2} and ψ_{T2} vanish identically; in that case only the equations (B1) and (B3) (which are then equivalent to (18) and (19)) survive.

^1J_J :

$$E(E^2\psi_{A2} - \epsilon_B\psi_P) = -\int d\mathbf{q}'\,P_J(V'_A - qq'zV'_T)\,\psi_{A2}\,, \tag{B5}$$

$$E(\psi_P - \epsilon_B\psi_{A2}) = -\int d\mathbf{q}'\,P_JV'_P\psi_P\,. \tag{B6}$$

For $J = 0$ these equations reduce to (20) and (21) of the main text.

^3J_J :

$$E(\psi_{V4} + \epsilon_B\psi_{T6}) = -\int d\mathbf{q}'\,P_JV'_V\psi_{V4}\,, \tag{B7}$$

$$E(E^2\psi_{T6} + \epsilon_B\psi_{V4}) = -\int d\mathbf{q}'\,(P_JV'_T - P_J^{ee}qq'V'_A)\,\psi_{T6}\,. \tag{B8}$$

Bound states of the type ^3J_J occur only for $J > 0$.

APPENDIX C: QUASI-POTENTIAL EQUATIONS FOR GENERAL J

Equations of the quasi-potential type are obtained from the general form (32) by insertion of (7)–(10) with (11)–(14) and (34)–(36). When the same abbreviations as in Appendix B are used the results are as follows:

$^3(J \pm 1)_J$:

$$(E - \epsilon_B)\,\psi_S = \frac{1}{2}\int d\mathbf{q}'\left\{\left[\frac{q^2}{E^2}\,V'_S - P_Jz\,\frac{q}{q'E^2}\,(V'_V + EE'V'_T)\right]\psi_S - P_J{}^e\,\frac{q}{E^2}\left(V'_V + \frac{E}{E'}\,V'_T\right)\psi_{V2}\right\}, \tag{C1}$$

$$(E - \epsilon_B)\,\psi_{V2} = -\frac{1}{2}\int d\mathbf{q}'\left\{\left[P_J^{ee}\left(V'_V + \frac{1}{EE'}\,V'_T\right) - P_J\,\frac{qq'}{EE'}\,V'_A\right]\psi_{V2} + P_J{}^e\,\frac{1}{q'}\left(V'_V + \frac{E'}{E}\,V'_T\right)\psi_S\right\}. \tag{C2}$$

^1J_J :

$$(E - \epsilon_B)\,\psi_P = -\frac{1}{2}\int d\mathbf{q}'\, P_J \left(V'_P + \frac{1}{EE'}\, V'_A - \frac{qq'}{EE'}\, zV'_T\right)\psi_P\,. \tag{C3}$$

^3J_J :

$$(E - \epsilon_B)\,\psi_{V4} = -\frac{1}{2}\int d\mathbf{q}' \left[P_J \left(V'_V + \frac{1}{EE'}\, V'_T\right) - P_J^{ee}\,\frac{qq'}{EE'}\, V'_A\right]\psi_{V4}\,. \tag{C4}$$

The equations (C2) and (C4) apply only for states with $J > 0$; (C1) and (C3) reduce for $J = 0$ to (37) and (38) of the main text.

APPENDIX D: Two-Particle Dirac Equations for General J

The scalar equations in momentum space that follow from the two-particle Dirac equation (43) by insertion of the wavefunctions $\psi(\mathbf{q})$ given in (7)–(10) have the following form:

$^3(J \pm 1)_J$:

$$\epsilon_B\psi_S - q\psi_{T1} = -\int d\mathbf{q}'\, P_J V'_S\psi_S\,, \tag{D1}$$

$$\epsilon_B\psi_{V1} - \psi_{T1} = \int d\mathbf{q}'\, (P_J z V'_V\psi_{V1} + P_J{}^e V'_V\psi_{V2}), \tag{D2}$$

$$\epsilon_B\psi_{V2} - \psi_{T2} + q\psi_{A1} = \int d\mathbf{q}'\, (P_J{}^e V'_V\psi_{V1} + P_J^{ee} V'_V\psi_{V2}), \tag{D3}$$

$$\epsilon_B\psi_{T1} - q\psi_S - \psi_{V1} = \int d\mathbf{q}'\, (P_J z V'_T\psi_{T1} + P_J{}^e V'_T\psi_{T2}), \tag{D4}$$

$$\epsilon_B\psi_{T2} - \psi_{V2} = \int d\mathbf{q}'\, (P_J{}^e V'_T\psi_{T1} + P_J^{ee} V'_T\psi_{T2}), \tag{D5}$$

$$\epsilon_B\psi_{A1} + q\psi_{V2} = -\int d\mathbf{q}'\, P_J V'_A\psi_{A1}\,. \tag{D6}$$

^1J_J :

$$\epsilon_B\psi_P - q\psi_{T4} - \psi_{A2} = \int d\mathbf{q}'\, P_J V'_P\psi_P\,, \tag{D7}$$

$$\epsilon_B\psi_{A2} - \psi_P = \int d\mathbf{q}'\, P_J V'_A\psi_{A2}\,, \tag{D8}$$

$$\epsilon_B\psi_{T4} - q\psi_P = -\int d\mathbf{q}'\, (P_J z V'_T\psi_{T4} + P_J{}^e V'_T\psi_{T5}), \tag{D9}$$

$$\epsilon_B\psi_{T5} = -\int d\mathbf{q}'\, (P_J{}^e V'_T\psi_{T4} + P_J^{ee} V'_T\psi_{T5}). \tag{D10}$$

^3J_J:

$$\epsilon_B\psi_{V4} - q\psi_{A4} + \psi_{T6} = \int d\mathbf{q}'\, P_J V'_V \psi_{V4}\,, \tag{D11}$$

$$\epsilon_B\psi_{T6} + \psi_{V4} = \int d\mathbf{q}'\, P_J V'_T \psi_{T6}\,, \tag{D12}$$

$$\epsilon_B\psi_{A3} = -\int d\mathbf{q}'\, (P_{JZ} V'_A \psi_{A3} + P_J^e V'_A \psi_{A4}), \tag{D13}$$

$$\epsilon_B\psi_{A4} - q\psi_{V4} = -\int d\mathbf{q}'\, (P_J^e V'_A \psi_{A3} + P_J^{ee} V'_A \psi_{A4}). \tag{D14}$$

In particular, $^3 1_0$ states (0^{++}) are described by structure functions ψ_S, ψ_{V1} and ψ_{T1} that follow from (D1), (D2), and (D4), while for $^1 0_0$ states (0^{-+}) the equations (D7)–(D9) for ψ_P, ψ_{A2}, and ψ_{T4} apply.

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