Physics in the Making, edited by A. Sarlemijn and M.J. Sparnaay

CHAPTER 6

Statistical Foundations of Electrodynamic Theory

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6.1. Introduction

In 1865 James Clerk Maxwell presented his now famous paper *A Dynamical Theory of the Electromagnetic Field* in which the fundamental equations of electrodynamics are developed for the first time. In particular, this paper contains an electromagnetic theory of light. It was held by the author, who liked the use of military metaphors, to be "great guns" (Maxwell, 1865). His *Treatise on Electricity and Magnetism*, published in 1873, gives a unified treatment of the subject, in which the results of his earlier papers are contained.

Maxwell's work was difficult reading for his contemporaries. Ehrenfest considered it to be "a kind of intellectual primeval forest, almost impenetrable in its uncleared fecundity" (Ehrenfest, 1923). According to Lorentz "one feels a lack of unity in his book due to the fact that it records faithfully his gradual transition from old to new ideas" (Lorentz, 1923).

Lorentz himself contributed a great deal to a better understanding of Maxwell's writings. In his dissertation of 1875 he gave a list of optical phenomena that should be explained on the basis of Maxwell's theory. Over the years these efforts evolved into a major research programme aimed at a systematic exploration of macroscopic electrodynamics, in particular for moving polarized and magnetized media. In these investigations Lorentz often employed concepts derived from a molecular picture of matter.

At a microscopic level electrodynamics deals with the interaction of electromagnetic fields and matter consisting of particles carrying electric charges. On the one hand it describes how fields are produced by the charged particles, on the other it determines the motion of the particles under the influence of electromagnetic fields. From a fundamental point of view macroscopic electrodynamics comes about as a consequence of the combined effect of all tiny interactions on a microscopic scale. Hence, it should be feasible to show how both aspects of the microscopic interactions manifest themselves on the macroscopic level. As statistical methods are indispensible for making the transition from a microscopic to a macroscopic description, it may be said that a statistical foundation of macroscopic electrodynamics is furnished in such a way.

The details of the interaction of fields and matter on a microscopic scale are well known. In contrast, the macroscopic laws of electrodynamics in the presence of polarizable and magnetizable media have been subject to debate for a long time. Whereas the field equations, as already given by Maxwell, were firmly established theoretically and supported by a wealth of experimental data, the situation was less comfortable for the equations governing the macroscopic motion of polarizable matter under the influence of electromagnetic fields. There are two reasons for this. In the first place the theoretical derivation of the macroscopic equations of motion of matter from the microscopic laws is more involved than that of the field equations. Moreover the experimental verification of the theoretical predictions, in particular of the forces on polarizable and magnetizable matter in time-dependent fields (for instance in light fields), is quite demanding. Only in recent years has it been possible to measure these forces with sufficient accuracy.

In the following a review is given of the recent advances in the understanding of the statistical foundations of macroscopic electrodynamics. Both theoretical and experimental developments will be discussed in some detail.

6.2. Statistical Derivation of the Macroscopic Field Equations

The field equations established by Maxwell more than a century ago were purely macroscopic in nature. The atomic structure of matter, about which hardly anything was known at that time, was not taken into account. Lorentz (1902, 1904a) was the first to observe that it is possible to derive Maxwell's equations from the laws valid at a microscopic level for a set of charged point particles and their fields.

The derivation of the macroscopic field equations from microscopic theory may proceed in two steps (Mazur and Nijboer, 1953; de Groot, 1969; de Groot and Suttorp, 1972). By taking into account that the charged particles in matter are often grouped in stable entities like atoms or molecules one is first led to field equations at the so-called 'atomic level'. Subsequently, the macroscopic equations follow by applying a suitable averaging procedure. The derivation can be presented with various degrees of sophistication, by adopting either a nonrelativistic or a covariant point of view, and furthermore, by choosing either a classical or a quantum-mechanical description.

In a nonrelativistic classical approach one starts by writing the microscopic field equations, which are often called the Maxwell–Lorentz equations:

$$\nabla \cdot \boldsymbol{e} = \rho^{e}_{\text{micro}}\,, \qquad -\partial_0 \boldsymbol{e} + \nabla \wedge \boldsymbol{b} = c^{-1} \boldsymbol{j}^{e}_{\text{micro}}\,,$$
$$\nabla \cdot \boldsymbol{b} = 0\,, \qquad \partial_0 \boldsymbol{b} + \nabla \wedge \boldsymbol{e} = 0\,, \tag{2.1}$$

with ∇ and ∂_0 denoting differentiation with respect to the position $\boldsymbol{R}$ and the scaled time ct (with c the speed of light). The microscopic sources are determined by the positions $\boldsymbol{R}_j$ and the velocities $\boldsymbol{v}_j$ of the point particles with charge e_j;

$$\rho^{e}_{\text{micro}} = \sum_j e_j \delta(\boldsymbol{R}_j - \boldsymbol{R})\,,$$
$$\boldsymbol{j}^{e}_{\text{micro}} = \sum_j e_j \boldsymbol{v}_j \delta(\boldsymbol{R}_j - \boldsymbol{R})\,. \tag{2.2}$$

The field equations (2.1) determine the microscopic fields $\boldsymbol{e}$ and $\boldsymbol{b}$, if suitable boundary conditions are imposed. As is well known, Maxwell's macroscopic equations are usually written in terms of four electromagnetic fields which are generally regarded and treated as independent quantities. As Einstein (1957) once remarked, it was "Lorentz's act of intellectual liberation" that first led to a microscopic description in terms of only two independent electromagnetic fields.

If the point charges occurring in (2.2) are grouped in stable entities the source terms can be expanded in multipole series. In fact, labelling the stable groups by an index k and the individual particles by a double index ki, choosing a privileged point with coordinate vector $\boldsymbol{R}_k$ within each stable group k and making a formal Taylor expansion of the delta

functions (Cristescu and Marinescu, 1973), one obtains from the source terms (2.2):

$$\rho^e_{\text{micro}} = \rho^e - \nabla\cdot \boldsymbol{p}\,, \qquad \boldsymbol{j}^e_{\text{micro}} = \boldsymbol{j}^e + c(\partial_0 \boldsymbol{p} + \nabla \wedge \boldsymbol{m})\,. \tag{2.3}$$

The atomic charge and current densities are defined as

$$\rho^e = \sum_k \rho^e_k = \sum_k e_k \delta(\boldsymbol{R}_k - \boldsymbol{R})\,, \qquad \boldsymbol{j}^e = \sum_k \boldsymbol{j}^e_k = \sum_k e_k \boldsymbol{v}_k \delta(\boldsymbol{R}_k - \boldsymbol{R})\,, \tag{2.4}$$

with $e_k = \Sigma_i e_{ki}$ and $\boldsymbol{v}_k = \mathrm{d}\boldsymbol{R}_k/\mathrm{d}t$. Furthermore, the atomic polarization and magnetization densities are given by multipole series of which the leading terms are:

$$\boldsymbol{p} = \sum_k \boldsymbol{p}_k = \sum_k \boldsymbol{\mu}_k \delta(\boldsymbol{R}_k - \boldsymbol{R})\,, \qquad \boldsymbol{m} = \sum_k \boldsymbol{m}_k = \sum_k (\boldsymbol{\nu}_k + c^{-1}\boldsymbol{\mu}_k \wedge \boldsymbol{v}_k)\delta(\boldsymbol{R}_k - \boldsymbol{R})\,, \tag{2.5}$$

with the electric and the magnetic dipole moments

$$\boldsymbol{\mu}_k = \sum_i e_{ki}\boldsymbol{r}_{ki}, \qquad \boldsymbol{\nu}_k = \sum_i e_{ki}\boldsymbol{r}_{ki} \wedge \dot{\boldsymbol{r}}_{ki}/2c \tag{2.6}$$

that contain the relative position vectors $\boldsymbol{r}_{ki} = \boldsymbol{R}_{ki} - \boldsymbol{R}_k$ and their time derivatives.

The multipole series for $\boldsymbol{p}$ and $\boldsymbol{m}$ come about by a formal Taylor expansion. This Taylor expansion of the sources is equivalent to an expansion of the Green function occurring in the solutions of the field equations. The multipole series which represent the solutions can usually be truncated after a few terms if the observer's position is sufficiently far away from the 'atoms'. The occurrence of infinite series can be avoided altogether by inserting an additional integration over an auxiliary parameter (Irving and Kirkwood, 1950; Power and Thirunamachandran, 1971; Healey, 1977, 1978, 1982a; Craig and Thirunamachandran, 1984).

As (2.5) shows, the magnetization $\boldsymbol{m}$ contains contributions arising from electric dipole moments in motion. The curl of these terms, which shows up in (2.3) on a par with the current density, is called the Röntgen

current in the literature (Röntgen, 1888, 1890). Although one would expect similar terms with moving magnetic dipole moments in the polarization $\boldsymbol{p}$, these are not found in (2.5). They do appear, however, in a covariant theory, as will be discussed below.

The source terms (2.3) depend on the properties of the stable groups in the system, in particular on their charges and their multipole moments. The latter are often related to the fields in a simple way, e.g. by a linear relationship. Under those circumstances it is convenient to introduce the auxiliary quantities $\boldsymbol{d} = \boldsymbol{e} + \boldsymbol{p}$, $\boldsymbol{h} = \boldsymbol{b} - \boldsymbol{m}$ and write the field equations in the familar form due to Maxwell. It should be emphasized, however, that such a rewriting has the disadvantage that it tends to obscure the physical difference between the fields and their material sources.

Covariant classical derivations of the macroscopic field equations have been given by several authors, with varying degrees of rigour (for reviews see de Groot, 1969; de Groot and Suttorp, 1972). As a starting-point one writes the Maxwell–Lorentz equations in covariant tensorial form

$$\partial_\beta f^{\alpha\beta} = c^{-1} j^\alpha_{\text{micro}} , \qquad \partial_\alpha f_{\beta\gamma} + \partial_\beta f_{\gamma\alpha} + \partial_\gamma f_{\alpha\beta} = 0 , \tag{2.7}$$

with a field tensor $f^{\alpha\beta}$ and a charge-current density four-vector:

$$c^{-1} j^\alpha_{\text{micro}} = \sum_j e_j \int u_j^\alpha(s_j) \delta^{(4)}[R_j(s_j) - R] \, \mathrm{d}s_j , \tag{2.8}$$

where s_j is a suitable parametrization along the world line of particle j and $u_j^\alpha = \mathrm{d}R_j^\alpha / \mathrm{d}s_j$.

The covariant atomic field equations follow from (2.7) and (2.8) by making a formal Taylor expansion of the delta function, as before. In this way we get

$$c^{-1} j^\alpha_{\text{micro}} = c^{-1} j^\alpha + \partial_\beta m^{\alpha\beta} , \tag{2.9}$$

with j^α giving the charge-current density of the stable atoms or molecules and $m^{\alpha\beta}$ representing the antisymmetric polarization tensor on the atomic level. If only dipole contributions are retained the components of $m^{\alpha\beta}$ read

$$\begin{aligned} \boldsymbol{p} &= \sum_k (\hat{\boldsymbol{\mu}}_k - c^{-1} \hat{\boldsymbol{\nu}}_k \wedge \boldsymbol{v}_k) \delta(\boldsymbol{R}_k - \boldsymbol{R}) , \\ \boldsymbol{m} &= \sum_k (\hat{\boldsymbol{\nu}}_k + c^{-1} \hat{\boldsymbol{\mu}}_k \wedge \boldsymbol{v}_k) \delta(\boldsymbol{R}_k - \boldsymbol{R}) . \end{aligned} \tag{2.10}$$

The dipole moments $\hat{\boldsymbol{\mu}}_k$ and $\hat{\boldsymbol{\nu}}_k$ are Lorentz contracted moments, defined in terms of relative positions and velocities in the instantaneous atomic rest frame. As (2.10) shows, dipoles in motion contribute to the polarization and the magnetization vectors in a symmetric way. The asymmetric results (2.5) are an artifact of the nonrelativistic theory.

In the quantum-mechanical derivations of the atomic field equations we may again distinguish between nonrelativistic and relativistically covariant treatments. In the former one may either use a Schrödinger-type description, with Coulomb interactions and external field couplings (de Groot and Suttorp, 1972; Babiker et al., 1973), or adopt the well-known formulation of quantum electrodynamics with nonrelativistic sources (Brittin, 1957; Schram, 1960; Healey, 1982a; Craig and Thirunamachandran, 1984). Extensions of these treatments so as to include spin effects have led to the use of the Breit Hamiltonian (de Groot and Suttorp, 1972) and "semirelativistic" quantum electrodynamics (Crowther and ter Haar, 1971a, b). Finally, an approach using the full formalism of covariant quantum electrodynamics, with material Dirac fields in second quantization, has been presented (Babiker et al., 1974; Babiker, 1975); however, in that treatment the nuclei are assumed to be fixed, so that effects of multipoles in motion are discarded a priori. In all other quantum-mechanical treatments mentioned here the Röntgen current, which is connected to electric dipoles in motion, is found to have the usual form. On the other hand, a contribution of magnetic dipoles in motion to the electric polarization has been obtained only by Crowther and ter Haar (1971a, b) and de Groot and Suttorp (1972). In the former treatment the terms arising from the spin magnetic moments in motion are only partly found; this is a consequence of the choice of the position operator for spin particles made there. Up to now a complete covariant derivation of atomic field equations in the framework of quantum electrodynamics, with the inclusion of all effects of particle motion, is not available in the literature.

Having discussed the field equations at the atomic level we now consider the second step in the derivation of the macroscopic Maxwell equations: the averaging procedure that smooths the wildly varying physical variables at the atomic level and results in slowly varying macroscopic quantities. In Lorentz's original treatment (Lorentz 1902, 1904a) an averaging over small spatial regions was introduced to arrive at the macroscopic fields. Similar averaging procedures (sometimes modified to averaging over small space-time regions so as to preserve covariance) have been employed repeatedly since then (Dällenbach, 1919; Rosenfeld,

1951; Russakoff, 1970; van Kranendonk and Sipe, 1977). As a modification of spatial averaging a truncation of the Fourier integrals that represent the physical quantities has also been considered (Robinson, 1971, 1973).

In modern statistical physics one generally employs ensemble averaging techniques to arrive at macroscopic quantities. Mazur and Nijboer (1953) were the first to apply this method to derive the macroscopic field equations, albeit in a classical nonrelativistic context. In a relativistic description the retardation of the fields should be taken into account. It implies that a simple phase space formulation in terms of the coordinates and momenta of the particles, all taken at the same time, no longer suffices. Instead, retarded distribution functions should be introduced (de Groot and Vlieger, 1965; de Groot, 1969; de Groot and Suttorp, 1972). With the use of these functions the Maxwell equations

$$\partial_\beta F^{\alpha\beta} = c^{-1} J^\alpha + \partial_\beta M^{\alpha\beta}\,, \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{2.11}$$

for the averaged fields $F^{\alpha\beta}$ and the average source terms J^α and $M^{\alpha\beta}$ follow straightforwardly.

In discussing the relative merits of the space–time and ensemble averaging it has been argued (Russakoff, 1970) that the latter gives insufficient smoothing if it is applied to crystalline solids. In fact, if the lattice points are kept fixed in performing the ensemble averaging in this case it is indeed true that the resulting macroscopic quantities vary wildly. However, whether such an ensemble choice, with fixed lattice points, is justified will depend on the physical properties in which one is interested. If a description with smoothly varying fields is needed the chosen ensemble is clearly not general enough; in that case an ensemble should be chosen in which the lattice points move around as well (possibly with strong correlations in their movements).

Averaging procedures in the quantum-mechanical theories can be discussed along similar lines. The ensemble averaging makes use of density operators (see e.g. Schram, 1960) or alternatively of Wigner distribution functions (de Groot and Suttorp, 1972).

In closing this section it may be remarked that the multipole expansion employed in deriving the atomic field equations is related to the multipole expansion of the Hamiltonian for charged particles interacting with electromagnetic fields (Göppert-Mayer, 1931; Power and Zienau, 1959; Fiutak, 1963; Atkins and Woolley, 1970; Woolley, 1971). In the electric dipole approximation a canonical transformation or, in quantum mech-

anics, a unitary transformation can be used to replace a coupling of the particle momentum and the vector potential by a coupling of the electric dipole moment and the electric field. This change from a 'velocity'- to a 'length'-form of the interaction has sometimes led to a debate in connexion with the derivation of the Maxwell equations (Mandel, 1979; Healey, 1980, 1982b; Power and Thirunamachandran, 1980, 1982; Haller, 1982). It should be clear that the use of such a transformation cannot change the physical contents of the theory, at least if no further approximations are made.

6.3. Composite Particles in Electromagnetic Fields

The field equations determine the fields that are generated by material sources consisting of charged particles, which may be grouped in stable entities. The description of the physical system of field and matter is complete only if the equations giving the motion of the charged particles (or of the stable groups) in the presence of the fields are added. The latter are governed by the electrodynamic force first given by Lorentz (1892) and carrying his name. The force acting on a composite particle in an electromagnetic field is found by adding the Lorentz forces on the individual constituents. As in the preceding section the details of the derivation of the equation of motion for a composite particle depend on the scope of the chosen theoretical framework.

In a nonrelativistic classical theory the equation of motion for a point particle with label j, carrying a charge e_j and a mass m_j, moving in external fields $\boldsymbol{E}$ and $\boldsymbol{B}$ reads:

$$m_j \dot{\boldsymbol{v}}_j = e_j(\boldsymbol{E} + c^{-1}\boldsymbol{v}_j \wedge \boldsymbol{B}) , \tag{3.1}$$

where the fields are to be taken at the position $\boldsymbol{R}_j$ of the particle with velocity $\boldsymbol{v}_j$. An equation of motion for a composite particle consisting of constituents j follows from (3.1) by introducing the centre of mass, with position vector $\boldsymbol{R}$, and expanding the external fields around this point. For slowly varying fields the resulting multipole series may be truncated at the dipole level. The resulting equation of motion is (de Groot and Suttorp, 1972):

$$\begin{aligned} m\dot{\boldsymbol{v}} = {} & e(\boldsymbol{E} + c^{-1}\boldsymbol{v} \wedge \boldsymbol{B}) + (\nabla \boldsymbol{E}) \cdot \boldsymbol{\mu} + (\nabla \boldsymbol{B}) \cdot (\boldsymbol{v} + c^{-1}\boldsymbol{\mu} \wedge \boldsymbol{v}) \\ & + c^{-1}\, \mathrm{d}(\boldsymbol{\mu} \wedge \boldsymbol{B})/\mathrm{d}t . \end{aligned} \tag{3.2}$$

Here m and e are the total mass and the total charge of the composite particle. Furthermore ∇ denotes a differentiation with respect to $\boldsymbol{R}$ and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla$ is the total time derivative, with $\boldsymbol{v}$ the particle velocity. An alternative form of (3.2) follows by inserting an integration over a delta function and performing some partial integrations:

$$m\dot{\boldsymbol{v}} = \int [(\rho^e - \nabla'\cdot\boldsymbol{p})\boldsymbol{E} + c^{-1}(\boldsymbol{j}^e + \partial\boldsymbol{p}/\partial t + c\nabla'\wedge\boldsymbol{m})\wedge\boldsymbol{B}]\,\mathrm{d}\boldsymbol{R}'\,, \tag{3.3}$$

with atomic charge-current densities, polarization and magnetization as defined in (2.4) and (2.5). Eu (1986) has expressed doubts as to the validity of the equation of motion (3.2) on account of the fact that its derivation would not resemble that of the atomic field equations closely enough. However, the equivalent form (3.3) clearly shows that the equation of motion can indeed be interpreted simply in terms of the (expanded) source terms occurring in the atomic field equations. It should be added here that the expression for the force on a composite particle as put forward by Eu (1986) is not correct; for instance, a term containing the time derivative of the electric dipole moment, as included in the last term of (3.2), is missing. In fact, the reasoning by which the expression for the force is obtained by Eu (1986) is not even sufficient to lead to an unambiguous result.

The expression for the force on a nonrelativistic composite particle as given in (3.2) is not symmetric in the electric and magnetic dipole moments. Magnetic dipole moments in motion are not coupled to the gradient of the electric field. Furthermore, a counterpart to the last term of (3.2) is missing. Both these features disappear if lowest order relativistic effects are taken into account. In the corresponding "semirelativistic" theory (de Groot and Suttorp, 1972; cf. also Coleman and Vleck, 1968) the equation of motion reads

$$\begin{aligned} m\dot{\boldsymbol{v}} = {} & e(\boldsymbol{E} + c^{-1}\boldsymbol{v}\wedge\boldsymbol{B}) + (\nabla\boldsymbol{E})\cdot(\boldsymbol{\mu} - c^{-1}\boldsymbol{\nu}\wedge\boldsymbol{v}) \\ & + (\nabla\boldsymbol{B})\cdot(\boldsymbol{\nu} + c^{-1}\boldsymbol{\mu}\wedge\boldsymbol{v}) + c^{-1}\,\mathrm{d}(\boldsymbol{\mu}\wedge\boldsymbol{B} - \boldsymbol{\nu}\wedge\boldsymbol{E})/\mathrm{d}t\,. \end{aligned} \tag{3.4}$$

Indeed, a "magnetodynamic" effect determined by the time derivative of the vector product of the magnetic dipole moment and the electric field now shows up on a par with the "electrodynamic" effect, which was already present in (3.2). The precise form of the forces on a magnetic

dipole, in particular the status of the magnetodynamic effect, has been the subject of much debate (Shockley and James, 1967; Penfield and Haus, 1967, 1969; Costa de Beauregard, 1967; Suttorp and de Groot, 1970; Casimir, 1972; de Groot and Suttorp, 1972). For instance, an attempt has been made to obtain this force by considerations based on a "magnetic charge" dipole (Tellegen, 1962; Pao and Hutter, 1975; Pao, 1976); for a recent discussion see Haus (1982).

The expression for the magnetodynamic effect as contained in (3.4) is only an approximation: the full result (still up to order c^{-2} only) contains the angular momentum s of the composite particle as well:

$$-c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\boldsymbol{v}-\frac{e}{mc}\,\boldsymbol{s}\right)\wedge\boldsymbol{E}\right]. \tag{3.5}$$

Hence the magnetodynamic effect depends only on the "anomalous" part of the magnetic dipole moment. However, for an atom or molecule the anomalous magnetic moment is much bigger than the normal one, since in the latter the total mass of the composite particle comes into play.

In contrast with the above the normal part of the magnetic moment is certainly important if the constituent particles carry an intrinsic spin. Under these circumstances it is preferable to use a quantum-mechanical description. In the nonrelativistic regime the quantum results for the equation of motion coincide with the classical ones (if the so-called Weyl representation is chosen, see de Groot and Suttorp, 1972). If semirelativistic terms are retained, however, for instance by using the Breit Hamiltonian, one finds that the magnetodynamic effect is mostly determined by the orbital magnetic moment. The spin magnetic moment drops out in first approximation since its anomalous part is very small. Hence the last term of (3.4) in this case should read

$$-c^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}_{\mathrm{orb}}\wedge\boldsymbol{E}\right). \tag{3.6}$$

It is indeed remarkable that the force on a magnetic dipole not only depends on its strength and orientation, but also on its origin. The same feature shows up if the balance equation of the internal energy and of the angular momentum of a composite particle are investigated in the semirelativistic approximation. Most discussions in the past have overlooked this possibility.

Up to now a purely covariant theory for composite particles has been worked out only for the classical case. An essential step in the theory is

the choice of the subsidiary condition that defines the centre of the particle. If the antisymmetric tensor $s^{\alpha\beta}$ representing the angular momentum is chosen to be orthogonal to the four-velocity u^α of the composite particle (i.e. $u_\alpha s^{\alpha\beta} = 0$) helical solutions may occur even in the field-free case. Such a strange feature is absent if the condition $p_\alpha s^{\alpha\beta} = 0$, with p^α the particle momentum, is adopted (Pyrce, 1948; Dixon, 1964, 1970a, b, 1974; Suttorp and de Groot, 1970; de Groot and Suttorp, 1972; Souriau, 1974). The covariant equations of motion and of spin then become:

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = f^\alpha , \qquad \frac{\mathrm{d}s^{\alpha\beta}}{\mathrm{d}s} = d^{\alpha\beta} - u^\alpha p^\beta + u^\beta p^\alpha , \tag{3.7}$$

with

$$\begin{aligned} p^\alpha &= mu^\alpha + m^{-1}c^{-2}s^{\alpha\beta}f_\beta + c^{-2}d^{\alpha\beta}u_\beta , \\ f^\alpha &= c^{-1}eF^{\alpha\beta}u_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})m_{\beta\gamma} - c^{-2}\frac{\mathrm{d}}{\mathrm{d}s}(F^{\alpha\beta}m_{\beta\gamma}u^\gamma) , \\ d^{\alpha\beta} &= F^{\alpha\gamma}m_{\gamma\varepsilon}\Delta^{\beta\varepsilon} - F^{\beta\gamma}m_{\gamma\varepsilon}\Delta^{\alpha\varepsilon} . \end{aligned} \tag{3.8}$$

Here s is the proper time along the central world line of the particle, $F^{\alpha\beta}$ is the antisymmetric external field tensor and $m^{\alpha\beta}$ is the antisymmetric dipole tensor of the particle. Furthermore, the tensor $\Delta^{\alpha\beta} = g^{\alpha\beta} + c^{-2}u^\alpha u^\beta$, with $g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ the metric tensor, projects a four-vector on the space orthogonal to u^α.

The equations (3.7)–(3.8) simplify considerably if the dipole tensor $m^{\alpha\beta}$ is proportional to the angular momentum tensor $s^{\alpha\beta}$. An alternative way to arrive at the equations pertinent to this particular case has been presented by van Dam and Ruijgrok (1980). More complicated equations arise, on the other hand, if radiation reaction terms are included as well (van Weert, 1974, 1975a, b; Teitelboim et al., 1980; Rowe and Rowe, 1987).

6.4. Macroscopic Forces on Polarizable Matter in Nonrelativistic and Semirelativistic Theory

The macroscopic forces exerted by electromagnetic fields on polarizable matter are the result of the forces experienced by the composite particles that are the constituents of the medium at the atomic level. In the following the macroscopic forces on dielectrics, which can be treated

in a purely nonrelativistic theory and those on magnetizable matter, in which semirelativistic effects play a role, will be discussed. The purely relativistic formulation will be considered in the following section.

To derive the forces on nonrelativistic dielectrics one starts from the equation of motion of a neutral atom or molecule with an electric dipole moment in the presence of external electromagnetic fields and of other particles. By averaging the atomic equation of motion one obtains the macroscopic balance equation for a dielectric (de Groot and Suttorp, 1972):

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} = -\nabla\cdot(\varrho \boldsymbol{v}\boldsymbol{v} + \boldsymbol{P}^K) + \boldsymbol{F}^{\mathrm{L}} + \boldsymbol{F}^{\mathrm{S}}\,, \tag{4.1}$$

with ϱ the mass density, $\boldsymbol{v}$ the hydrodynamic velocity and $\boldsymbol{P}^K$ the kinetic pressure tensor. Furthermore, $\boldsymbol{F}^{\mathrm{L}}$ results from the contributions of the external fields and of the long-range dipole interaction, while $\boldsymbol{F}^{\mathrm{S}}$ arises from short-range interactions. Upon introducing the macroscopic Maxwell fields $\boldsymbol{E}, \boldsymbol{B}$ instead of the external fields and using the Irving–Kirkwood expansion (Irving and Kirkwood, 1950) to rewrite the correlation and short-range terms as a divergence one arrives at the final form of the (nonrelativistic) balance equation for a dielectric:

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} = -\nabla\cdot(\varrho \boldsymbol{v}\boldsymbol{v} + \boldsymbol{P}) + \boldsymbol{F}\,, \tag{4.2}$$

with $\boldsymbol{P}$ the macroscopic pressure tensor (consisting of kinetic and potential parts) and $\boldsymbol{F}$ the macroscopic force density:

$$\boldsymbol{F} = (\nabla \boldsymbol{E})\cdot \boldsymbol{P} + c^{-1}\varrho \frac{\mathrm{d}}{\mathrm{d}t}(v\boldsymbol{P}\wedge \boldsymbol{B})\,, \tag{4.3}$$

with $v = \varrho^{-1}$ the volume per unit mass, $\boldsymbol{P}$ the macroscopic polarization and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla$ the 'material time derivative'.

Incidentally, it may be remarked that some time ago doubts were expressed (Miller, 1971) on the convergence of the expansion as presented in the paper by Irving and Kirkwood (1950). However, the convergence of the expansion is warranted as long as only short-range functions are expanded; it is an easy matter to rewrite the expressions in Irving and Kirkwood's (1950) paper in terms of such functions only, so that the criticism is not justified. Likewise, the expansions used above are fastly convergent, at least for fluid systems that are not too near to the

critical point. For crystalline solids an alternative expression for the correlation part of the pressure should be used, which resembles that given by Miller (1971).

The form of the equation of motion as given in (4.2) is not unique, since terms may be shifted from the pressure tensor to the force density. Only the combination occurring in (4.2) has a physical meaning. In particular, one may arrive at a different expression for the force density in a quite natural way (Suttorp and de Groot, 1981a):

$$\hat{\boldsymbol{F}} = \boldsymbol{F} + \nabla\cdot(\tfrac{1}{5}\boldsymbol{PP} + \tfrac{1}{10}P^2\boldsymbol{U}), \tag{4.4}$$

with $\boldsymbol{U}$ the unit tensor. Correspondingly, the pressure tensor $\hat{\boldsymbol{P}}$ associated with this force density contains an extra term equal to the tensor of which the divergence appears in the last term of (4.4).

The balance equation (4.2) can be written in the form of a conservation law by employing an identity that follows from the Maxwell equations for a dielectric:

$$(\nabla\boldsymbol{E})\cdot\boldsymbol{P} = \nabla\cdot[\boldsymbol{DE} + \boldsymbol{BB} - \tfrac{1}{2}(E^2 + B^2)\boldsymbol{U}] - c^{-1}\frac{\partial}{\partial t}(\boldsymbol{D}\wedge\boldsymbol{B}). \tag{4.5}$$

With the help of this relation and the continuity equation the force density (4.3) can be written as the sum of a time derivative and a divergence. As a result the balance equation (4.2) becomes a conservation law:

$$\frac{\partial(\varrho\boldsymbol{v} + c^{-1}\boldsymbol{E}\wedge\boldsymbol{B})}{\partial t} = -\nabla\cdot[\varrho\boldsymbol{vv} + \boldsymbol{P} - \boldsymbol{DE} - \boldsymbol{BB} - c^{-1}\boldsymbol{vP}\wedge\boldsymbol{B} + \tfrac{1}{2}(E^2 + B^2)\boldsymbol{U}]. \tag{4.6}$$

The time derivative at the left-hand side contains the sum of the material momentum density $\varrho\boldsymbol{v}$ and the field momentum density which for fields in a dielectric is found to be $c^{-1}\boldsymbol{E}\wedge\boldsymbol{B}$. At the right-hand side the divergence of a total momentum flow occurs; the latter is the sum of a material part and a field part as well.

The pressure $\boldsymbol{P}$ occurring in (4.2) has the property that it is isotropic and diagonal for a fluid dielectric in (local) equilibrium; in that case one has: $\boldsymbol{P} = p\boldsymbol{U}$, as can be proved from the Gibbs relation. On the other hand, the pressure tensor $\hat{\boldsymbol{P}}$ contains an anisotropic part even in this

special case. Both P and $\hat{P}$ are symmetric, for fluid and solid dielectrics at rest (or with a velocity small compared to c), if the polarization is parallel to the electric field and if the angular momenta of the molecules have relaxed to their equilbrium value. This can be proved by considering the balance equation of angular momentum.

The Gibbs relation can be used to connect the pressure p for a dielectric fluid in equilibrium to the pressure p_0 for a fluid with the same temperature and density in the absence of electromagnetic fields:

$$p - p_0 = \int_0^E \left(\boldsymbol{P} + v \frac{\partial \boldsymbol{P}}{\partial v} \right) \cdot \mathrm{d}\boldsymbol{E} \,. \tag{4.7}$$

For a dielectric fluid in local equilibrium the momentum balance equation (4.2) may thus be written

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} = -\nabla \cdot (\varrho \boldsymbol{v}\boldsymbol{v} + p_0 \boldsymbol{U}) + \boldsymbol{F}^{\mathrm{H}} \,, \tag{4.8}$$

with the force density

$$\boldsymbol{F}^{\mathrm{H}} = \boldsymbol{F} - \nabla(p - p_0) \,. \tag{4.9}$$

For a linear medium, with $\boldsymbol{P} = \kappa \boldsymbol{E}$, it reduces to

$$\boldsymbol{F}^{\mathrm{H}} = -\tfrac{1}{2} E^2 \nabla \kappa - \tfrac{1}{2} \nabla \left(v \frac{\partial \kappa}{\partial v} E^2 \right) + c^{-1} \varrho \frac{\mathrm{d}}{\mathrm{d}t} (v \boldsymbol{P} \wedge \boldsymbol{B}) \,. \tag{4.10}$$

This is the well-known expression first obtained by Korteweg (1880) and Helmholtz (1881, 1882) for the static case. It should be noted here that strictly speaking the linear law $\boldsymbol{P} = \kappa \boldsymbol{E}$ connects quantities in the local rest frame; we have assumed that the hydrodynamic velocity is small compared to the velocity of light.

Both (4.3) and (4.10) are correct expressions for the force density felt by a dielectric fluid in an electromagnetic field. However, each is associated with its own pressure, which is defined in a different way in the two cases. The paradoxical situation that one may arrive at different expressions for the force density has led to much debate in the past. Although its explanation is by now well known (see Mazur and Prigogine, 1953; Mazur and de Groot, 1956; Landau and Lifshitz, 1960; de Groot and Mazur, 1962; de Groot and Suttorp, 1972; Pavlov, 1978; Lahoz,

1980) it can still lead to erroneous statements (see, for instance, Gingras, 1980; Novak 1980a). In recent years several statistical treatments have reconfirmed the validity of the Helmholtz expression for the force density, both for the case of polar dielectrics (Høye and Stell, 1980) and for the general case (Lai et al., 1981, 1982; cf. Lai et al., 1986). In the latter papers various simplifying assumptions have been made, however, for instance on the validity of the Clausius–Mossotti relation and on the isotropy of the two-particle distribution functions in the presence of a field. As a consequence the correlation part of the pressure tensor (as defined in de Groot and Suttorp, 1972) could be expressed in terms of the macroscopic polarization alone; this is not possible for a general dielectric fluid.

Several experiments have been carried out to check the consequences of the momentum balance equation for a dielectric, either in the form (4.2) or (4.8). Light scattering experiments carried out by Hakim and Higham (1962) have confirmed the changes of pressure in a static dielectric fluid as predicted by (4.7). Electro-optical effects, which might change the interpretation of these experiments, have been shown to be negligible by Brevik (1979). The latter author concludes, however, that the experimental findings show that (4.2) is inappropriate. As stated above both (4.2) and (4.8) can describe a dielectric fluid.

Further experiments supporting the description of a dielectric given above have been performed with time-dependent fields (Goetz, 1955; Goetz and Zahn, 1958; Zahn, 1962; James, 1968; Walker and Lahoz, 1975; Walker et al., 1975). Whereas the former three papers are concerned with the forces in time-dependent electric fields, the experiments described in the latter three papers concentrate on the forces exerted by a magnetic field on a dielectric with a varying polarization, as given by the last two terms of (4.3) or (4.10).

To analyze the experiments by Walker and Lahoz (1975) and Walker et al. (1975), in which the torque on a dielectric cylinder in a time-dependent radial electric field and a static axial magnetic field is measured, we start from the balance equation for the angular momentum associated with the bulk cylinder motion:

$$\frac{\partial(\boldsymbol{R} \wedge \varrho \boldsymbol{v})}{\partial t} = -\boldsymbol{R} \wedge [\nabla \cdot (\varrho \boldsymbol{v}\boldsymbol{v} + \boldsymbol{P})] + \boldsymbol{R} \wedge \boldsymbol{F} \,. \tag{4.11}$$

Assuming the pressure tensor to be symmetric we may write the first term at the right-hand side as a divergence. Upon integrating over the cylindri-

cal volume of the sample we use the relation (Liénard, 1923; de Groot and Suttorp, 1972) determining the pressure tensor at the boundary of a dielectric:

$$\boldsymbol{n} \cdot \boldsymbol{P} = \boldsymbol{n}[p_0 - \tfrac{1}{2}(\boldsymbol{P} \cdot \boldsymbol{n})^2] , \tag{4.12}$$

with $\boldsymbol{n}$ the normal to the boundary and p_0 the pressure of the surrounding atmosphere. Using the cylinder symmetry of the system we finally obtain

$$\frac{\partial}{\partial t} \int (\boldsymbol{R} \wedge \varrho \boldsymbol{v}) \, \mathrm{d}\boldsymbol{R} = \int (\boldsymbol{R} \wedge \boldsymbol{F}) \, \mathrm{d}\boldsymbol{R} . \tag{4.13}$$

It should be noted that the pressure tensor has dropped out completely. Hence the oscillatory motion of the sample in a time-dependent field is determined by the torque of the force density.

In the experiments described by Walker and Lahoz (1975) and Walker et al. (1975) the contribution of the term $\partial \boldsymbol{P}/\partial t \wedge \boldsymbol{B}$ is confirmed (within 10% experimental error). A similar conclusion had been reached several years before by James (1968). In later experiments (Walker and Walker, 1976, 1977a, b) the contributions of the term $\boldsymbol{P} \wedge \partial \boldsymbol{B}/\partial t$ are considered as well. Further experiments to check the force on a dielectric in a magnetic field have been described in Lahoz and Graham (1979a). The interpretation of the experiments has been discussed by Israel (1977), Brevik (1979) and Lorrain (1980).

The radiation force on a mirror immersed in a dielectric fluid has been measured in several experiments by Jones and collaborators (Jones, 1951; Jones and Richards, 1954; Jones and Leslie, 1978). In the early experiments light from a tungsten lamp was used to determine the radiation pressure at normal incidence. It was found that for fixed intensity the pressure is proportional to the refractive index n. In the experiment by Jones and Leslie (1978) the higher intensity of a laser source was employed to improve the accuracy from 1% to 0.05% and also to determine the pressure due to polarized light for oblique incidence. The earlier findings for normal incidence were confirmed, while for oblique incidence the pressure turned out to be independent of the polarization.

The theoretical interpretation of the radiation pressure experiments can be based on the balance equation (4.2) with (4.3) or alternatively on the conservation law (4.6). Since only the average of the radiation pressure over a time interval that is long compared with the oscillation period of the light field can be measured in the experiments, quickly oscillating

terms, like the last term of (4.3) or the time derivative of the field momentum density in (4.6), yield no contribution. As has been remarked before (de Groot and Suttorp, 1972; Brevik, 1979; Suttorp and de Groot, 1981b, c; Lai et al., 1984) this means in particular that the experiments cannot lead to statements about the precise form of the field momentum density in polarized matter, although the opposite has often been claimed. The actual source of the radiation pressure is the momentum flow (the pressure tensor); apart from the field contribution the material momentum flow should also be considered here. For oblique incidence at an angle θ the radiation pressure on an ideal mirror is (Suttorp and de Groot, 1981b, c; Casimir, 1983; Lai et al., 1984)

$$p^{\mathrm{rad}} = 2In\cos^2\theta\,, \tag{4.14}$$

independent of the polarization direction of the incident light with intensity I. For mirrors with finite reflectivity the radiation does depend on the polarization direction.

To interpret (4.14) the following reasoning might be attempted. In the dielectric fluid light propagates with the velocity c/n. If the momentum density in the light beam is $\boldsymbol{g}$, the radiation pressure for normal incidence must be $2(c/n)g$, with $g = |\boldsymbol{g}|$; for oblique incidence this should be multiplied by the purely geometrical factor $\cos^2\theta$. Comparison with (4.14) then shows that g equals In^2/c, or in terms of the Maxwell fields: $\boldsymbol{g} = c^{-1}\boldsymbol{D}\wedge\boldsymbol{B}$, at least after time averaging. In this way the radiation pressure experiments would determine the field momentum density, in contrast with our statements above. However, the reasoning is incorrect, since the presence of the medium is neglected completely. Moreover it has tacitly been assumed that the momentum flow in the direction of the beam follows directly by multiplying the momentum density and the phase velocity. Sometimes the reasoning is refined by introducing several types of momentum densities, e.g. the density of pseudo-momentum (Arnaud, 1972, 1974, 1976; Gordon, 1973; Burt and Peierls, 1973; Kastler, 1974; Joyce, 1974, 1975; Peierls, 1976, 1977; Wong and Young, 1977; Jones, 1978). Such subtle notions tend to obscure an issue, which in principle is not that complicated.

If a light beam enters a dielectric fluid the surface experiences a 'radiation tension': it shows an outward bulge towards the incoming light. This effect has been demonstrated by Ashkin and Dziedzic (1973) for a laser pulse entering water. In particular, the time development of the local surface curvature at the position of the (tiny) laser spot has been

measured. These dynamical features can be understood by solving the equation of motion (4.2), with a surface tension term included (Lai and Young, 1976; Brevik, 1979). As discussed above for the radiation pressure, reasonings based on (pseudo-)momentum densities have also been employed to understand the radiation tension effect (Gordon, 1973; Peierls, 1976; Wong and Young, 1977); however, these have occasionally led to the prediction of an inward instead of an outward bulge, corresponding to a pressure instead of a tension effect.

We now turn to a discussion of the forces on materials with both polarization and magnetization. In this case the starting point is the semirelativistic equation of motion for a composite particle in the electromagnetic field, as given by (3.4)–(3.6). The macroscopic balance equation for such materials again has the form (4.2). However both the pressure P and the macroscopic force $\boldsymbol{F}$ now contain contributions from the interacation with the magnetic dipoles; these depend on the (orbital or spin) type of the dipole. The force density is (de Groot and Suttorp, 1972)

$$\boldsymbol{F} = (\nabla \boldsymbol{E}) \cdot \boldsymbol{P} + (\nabla \boldsymbol{B}) \cdot \boldsymbol{M} + c^{-1} \varrho \frac{\mathrm{d}}{\mathrm{d}t} [v(\boldsymbol{P} \wedge \boldsymbol{B} - \boldsymbol{M}_{\mathrm{orb}} \wedge \boldsymbol{E})] , \qquad (4.15)$$

with $\boldsymbol{M}$ the macroscopic magnetization and $\boldsymbol{M}_{\mathrm{orb}}$ the magnetization due to the orbital magnetic dipoles.

As before the balance equation (4.2), with the force density (4.15), can be written as a conservation law, since the field equations for a polarizable and magnetizable material imply the identity:

$$(\nabla \boldsymbol{E}) \cdot \boldsymbol{P} + (\nabla \boldsymbol{B}) \cdot \boldsymbol{M} = \nabla \cdot [\boldsymbol{DE} + \boldsymbol{BH} - (\tfrac{1}{2} E^2 + \tfrac{1}{2} B^2 - \boldsymbol{M} \cdot \boldsymbol{B}) U] - c^{-1} \frac{\partial (\boldsymbol{D} \wedge \boldsymbol{B})}{\partial t} . \qquad (4.16)$$

Insertion in (4.2) with (4.15) yields the momentum conservation law:

$$\frac{\nabla [\varrho \boldsymbol{v} + c^{-1} \boldsymbol{E} \wedge (\boldsymbol{B} - \boldsymbol{M}_{\mathrm{orb}})]}{\partial t} = -\nabla \cdot [\varrho \boldsymbol{vv} + P - \boldsymbol{DE} - \boldsymbol{BH} - c^{-1} \boldsymbol{v} (\boldsymbol{P} \wedge \boldsymbol{B} - \boldsymbol{M}_{\mathrm{orb}} \wedge \boldsymbol{E}) + (\tfrac{1}{2} E^2 + \tfrac{1}{2} B^2 - \boldsymbol{M} \cdot \boldsymbol{B}) U] . \qquad (4.17)$$

The field momentum density for fields in a polarizable and magnetizable

medium is therefore given by $c^{-1}\boldsymbol{E} \wedge (\boldsymbol{B} - \boldsymbol{M}_{\text{orb}})$; clearly the spin magnetic moments do not contribute to the field momentum. In contrast, both spin and orbital magnetization are present in the field momentum flow.

To check the magnetostatic terms in (4.15) Lahoz and Walker (1975) have repeated with improved accuracy old experiments in which the height difference of a paramagnetic fluid in a U-shaped container, with one of its legs situated in a magnetic field, is measured; theory is confirmed within an experimental error of a few percent.

Experiments that are sensitive to the dynamic terms in (4.15) have been carried out by Lahoz and Graham (1979b, 1982). In these experiments a similar configuration to that described above (4.11) was used, with the difference that now ferrite samples were employed. The measurements can be analyzed along the same lines as in (4.11)–(4.13). The experimental results were found to be consistent with the force density (4.15). In fact, as the last term of (4.15) makes no contribution in the case of a ferrite sample a null result is expected and was indeed measured. In a theoretical discussion (Lahoz and Graham, 1981) it has been concluded that the magnetodynamic effect does not exist. However, this is not true: only orbital magnetic moments contribute to the effect, since only these are nearly completely anomalous.

6.5. Relativistic Energy-Momentum Laws

The nonrelativistic and semirelativistic theories discussed so far are sufficient to interpret the experiments on the electromagnetic forces in material media that have been carried out up to now. In spite of this fact there has been a lively debate over the years on the correct form of the relativistic energy-momentum laws for polarizable and magnetizable matter in an electromagnetic field. These laws can be written as

$$\partial_\beta T^{\alpha\beta} = 0\,, \tag{5.1}$$

with $T^{\alpha\beta}$ the total energy-momentum tensor, which is the sum of a material and a field-dependent part

$$T^{\alpha\beta} = T^{\alpha\beta}_{(\text{m})} + T^{\alpha\beta}_{(\text{f})}\,. \tag{5.2}$$

The discussions have often concentrated on the relative merits of the proposals for the field energy-momentum tensor put forward by Minkowski (1908, 1910) and Abraham (1909, 1910) on the basis of formal

grounds such as form invariance and symmetry. However it was already recognized at a fairly early date (Lorentz, 1904b; Einstein and Laub, 1908; Abraham, 1914) that it should be possible to obtain the macroscopic energy-momentum laws from a consideration of microscopic 'electron theory'.

A derivation of the covariant macroscopic laws of energy and momentum may start from the equations (3.7)–(3.8). As these are classical equations for a composite particle consisting of charged point particles without structure, spin effects are left out; the magnetization thus arises from orbital magnetic dipoles only. Upon employing a covariant averaging procedure (de Groot and Suttorp, 1972) these equations indeed lead to a conservation law (5.1) with an energy-momentum tensor (5.2) containing a field-dependent part:

$$T^{\alpha\beta}_{(\mathrm{f})} = -F^{\alpha\gamma}H^{\cdot\beta}_{\gamma} - \tfrac{1}{4}g^{\alpha\beta}F_{\gamma\varepsilon}F^{\gamma\varepsilon} + c^{-2}(F^{\alpha\gamma}M_{\gamma\varepsilon}U^{\varepsilon} - \Delta^{\alpha\gamma}M_{\gamma\varepsilon}F^{\varepsilon\zeta}U_{\zeta})U^{\beta}\,, \tag{5.3}$$

with $H^{\alpha\beta} = F^{\alpha\beta} - M^{\alpha\beta}$ (cf (2.11)), $g^{\alpha\beta}$ the metric tensor, U^{α} the hydrodynamic four-velocity and $\Delta^{\alpha\beta}$ the projector on the space orthogonal to U^{α}. The material part of the energy-momentum tensor is found as a statistical expression involving correlation functions.

In the local rest frame, for which $U^{\alpha} = (c, \boldsymbol{0})$, the field energy-momentum tensor (5.3) has the components:

$$\begin{aligned} T^{00}_{(\mathrm{f})} &= \tfrac{1}{2}(E^2 + B^2)\,, \\ T^{0i}_{(\mathrm{f})} &= T^{i0}_{(\mathrm{f})} = (\boldsymbol{E} \wedge \boldsymbol{H})^i\,, \\ T^{ij}_{(\mathrm{f})} &= -E^iD^j - H^iB^j + (\tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 - \boldsymbol{M}\cdot\boldsymbol{B})g^{ij}\,. \end{aligned} \tag{5.4}$$

For a dielectric, with $\boldsymbol{M} = 0$, these expressions agree with those of Lorentz (1904b) and Einstein and Laub (1908). The momentum density and momentum flow are the same as found in the nonrelativistic and the semirelativistic theory (see (4.6) and (4.17) with $\boldsymbol{M}_{\mathrm{orb}} = \boldsymbol{M}$); the energy density and the energy flow likewise agree with those of the simpler theories. Minkowski (1908, 1910) postulated a field momentum density in a medium equal to $c^{-1}\boldsymbol{D} \wedge \boldsymbol{B}$, which, in the case of a dielectric, already clearly disagrees with the findings of the nonrelativistic formulation. Whereas Abraham (1909, 1910) adopted $c^{-1}\boldsymbol{E} \wedge \boldsymbol{H}$ as the field momen-

tum density he required the momentum flow to be a symmetric tensor even in an anisotropic crystalline medium.

An alternative form for (5.1) is

$$\partial_\beta T^{\alpha\beta}_{(\mathrm{m})} = F^\alpha \,, \tag{5.5}$$

with the four-vector F^α representing the force density

$$F^\alpha = \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma}) M_{\beta\gamma} - c^{-2}\partial_\beta [U^\beta (F^{\alpha\gamma} M_{\gamma\varepsilon} U^\varepsilon - \Delta^{\alpha\gamma} M_{\gamma\varepsilon} F^{\varepsilon\zeta} U_\zeta)] \,. \tag{5.6}$$

In the nonrelativistic and the semirelativistic limits the space part of this four-vector reduces to (4.3) and (4.15), respectively (with $\boldsymbol{M}_{\mathrm{orb}} = \boldsymbol{M}$ for the present case).

The field energy-momentum tensor $T^{\alpha\beta}_{(\mathrm{f})}$ is generally asymmetric; this is not surprising, since it represents only part of the total energy-momentum tensor and is thus associated with an 'open' system. However, it should be noted that in the present case even the total energy-momentum tensor is generally asymmetric. The reason for this asymmetry is that the composite particles in the medium carry an angular momentum $s^{\alpha\beta}$. The balance equation for the corresponding macroscopic 'inner' angular momentum density $S^{\alpha\beta}$ contains the asymmetric part of $T^{\alpha\beta}$ as a source term:

$$\partial_\gamma (S^{\alpha\beta} U^\gamma) = -\partial_\gamma J^{\alpha\beta\gamma} + T^{\alpha\beta} - T^{\beta\alpha} \,, \tag{5.7}$$

with $J^{\alpha\beta\gamma}$ the tensor that determines the inner angular momentum flow. Since the orbital angular momentum balance equation reads:

$$\partial_\gamma (R^\alpha T^{\beta\gamma} - R^\beta T^{\alpha\gamma}) = T^{\beta\alpha} - T^{\alpha\beta} \,, \tag{5.8}$$

the total angular momentum is indeed conserved, as expected. Of course the total energy-momentum tensor can be symmetrized in the way described by Belinfante (1939) and Rosenfeld (1940) on account of the validity of (5.7).

As in the nonrelativistic theory the form (5.5) of the energy-momentum laws is by no means unique. On the basis of a covariant Gibbs relation one can derive the difference between the material pressure (and also the material energy density) in the presence and in the absence of fields. For fluids in equilibrium one obtains a "Helmholtz" material

energy-momentum tensor:

$$F^{\mathrm{H},\alpha\beta}_{(\mathrm{m})} = (\varrho + c^{-2}u^{\mathrm{H}}_{v})U^{\alpha}U^{\beta} + p^{\mathrm{H}}\Delta^{\alpha\beta}\,, \tag{5.9}$$

with u^{H}_{v} and p^{H} the internal energy and the pressure in the absence of electromagnetic fields. Correspondingly, the field energy-momentum tensor in the Helmholtz picture is, in the local rest frame, defined by its components:

$$\begin{aligned}
T^{\mathrm{H},00}_{(\mathrm{f})} &= \tfrac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H} + T\frac{\partial\kappa}{\partial T}E^2 + T\frac{\partial\chi}{\partial T}B^2\right),\\
T^{\mathrm{H},0i}_{(\mathrm{f})} &= T^{\mathrm{H},i0}_{(\mathrm{f})} = (\boldsymbol{E}\wedge\boldsymbol{H})^i\,,\\
T^{\mathrm{H},ij}_{(\mathrm{f})} &= -E^iD^j - H^iB^j\\
&\quad + \tfrac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H} + v\frac{\partial\kappa}{\partial v}E^2 + v\frac{\partial\chi}{\partial v}B^2\right)g^{ij}\,,
\end{aligned} \tag{5.10}$$

where linear laws $\boldsymbol{P} = \kappa\boldsymbol{E}$ and $\boldsymbol{M} = \chi\boldsymbol{B}$ in the rest frame have been assumed. The Helmholtz form of the balance equation of energy and momentum for a fluid in equilibrium is:

$$\partial_\beta T^{\mathrm{H},\alpha\beta}_{(\mathrm{m})} = F^{\mathrm{H},\alpha} = -\partial_\beta T^{\mathrm{H},\alpha\beta}_{(\mathrm{f})}\,, \tag{5.11}$$

which is a covariant generalization of (4.8) and (4.10).

The quest for the covariant energy-momentum laws of matter in electromagnetic fields has a long history, details of which can be found in several reviews (Brevik, 1970a, b, 1979; de Groot and Suttorp, 1972; Skobel'tsyn, 1973; Robinson, 1975; Novak 1980a). As remarked above the search for the "correct" form of the field energy-momentum tensor is not a well-posed problem as long as the material part of the tensor is left unspecified. From this fact, which has been emphasized repeatedly in the recent literature (see for instance Pitteri, 1973; Mikura, 1976; Israel, 1977, 1978; Horibata, 1977; Kranyš, 1979, 1980, 1982; Israel and Stewart, 1980; Maugin, 1980), it should not be construed, however, that all expressions for the field energy-momentum tensor are equally useful and equivalent. For instance, even nonrelativistic theory already shows (see (4.6)) that it would be artificial to choose $c^{-1}\boldsymbol{D}\wedge\boldsymbol{B}$ as the field momentum density (as Minkowski proposed); this choice would correspond to a

material momentum density different from $\varrho \boldsymbol{v}$ in the nonrelativistic limit. Likewise, a symmetrized field momentum flow, as proposed by Abraham, leads to a rather unconventional choice for the material pressure in an anisotropic solid.

To derive the macroscopic energy-momentum laws we started (de Groot and Suttorp, 1972) from the microscopic equations of motion. Although several people have followed the same strategy since then (Israel, 1973, 1977, 1978; Peierls, 1976, 1977; Maugin, 1978a, b, c, d; Israel and Stewart, 1980; Lai et al., 1982; Eu 1986) many authors have continued to try and use purely macroscopic arguments, for instance variational principles (Mikura, 1976; Dewar, 1977), mathematical identities based on Maxwell's equations only (Ginzburg, 1973; Ginzburg and Ugarov, 1976; Ginzburg, 1979) or various 'Gedanken' experiments (Skobel'tsyn, 1973; Costa de Beauregard, 1975; Lai, 1980, 1984; Brevik, 1982). Sometimes methods of general classical field theory for Proca fields have been employed (Novak, 1980a, b).

In several papers arguments based on thermodynamics are introduced in order to find the energy-momentum tensor. The first authors to try and use relativistic thermodynamics in this context were Kluitenberg and de Groot (1954, 1955a, b). By postulating a form for the relativistic Gibbs relation and the symmetry of the material part of the energy-momentum tensor they could deduce a field energy-momentum tensor which is closely related to (5.3). A similar reasoning was used in a macroscopic semirelativistic treatment by de Groot and Mazur (1962). As these authors are well aware, the Gibbs relation for material media in motion should be formulated in terms of rest frame quantities (for instance $\boldsymbol{E}' = \boldsymbol{E} + c^{-1}\boldsymbol{v} \wedge \boldsymbol{B}$ and $\boldsymbol{P}' = \boldsymbol{P} - c^{-1}\boldsymbol{v} \wedge \boldsymbol{M}$ in semirelativistic theory), since only these quantities satisfy relations independent of the state of motion; for linear media one has $\boldsymbol{P}' = \kappa \boldsymbol{E}'$, but not $\boldsymbol{P} = \kappa \boldsymbol{E}$. This point seems to have been missed in a recent paper by Eu and Oppenheim (1986).

A closely related method starts from the principle of virtual power (Chu et al., 1966; Penfield and Haus, 1967; Robinson, 1975; Maugin, 1981), which is also based on an assumption about the general form of the thermodynamic laws for media in motion. It has been criticized (Brevik, 1970b, 1973; Lo Surdo, 1973; Cavalleri, 1973; Cavalleri et al., 1975; see also Haus et al., 1972, 1974) for its use of quantities in the local rest frame; however, these should appear in a proper formulation of the thermodynamic laws for the reasons explained above.

6.6. Conclusion

In the past few decades our understanding of the foundations of macroscopic electrodynamics has greatly improved. The derivation of the macroscopic Maxwell equations from microscopic dynamics has been refined so as to include the effects of relativistic motion, particle spin and quantum electrodynamics. The macroscopic balance equations of momentum and of energy have been derived systematically by starting from the equations of motion for composite particles in external fields. The resulting expression for the ponderomotive force density felt by polarized and magnetized matter has been confirmed by experiments in which the force due to time-dependent electromagentic fields were measured with a high accuracy.

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