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COVARIANT DERIVATION OF THE MAXWELL EQUATIONS

MULTIPOLE EXPANSION OF THE POLARIZATION TENSOR TO ALL ORDERS

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Synopsis

The Maxwell equations are derived in covariant manner from the microscopic equations for the electromagnetic field in the presence of point charges. The polarization tensor is given as an expansion to all orders in the atomic electromagnetic moments, defined in atomic rest frames.

§ 1. *Introduction.* The problem of deriving Maxwell's field equations from Lorentz' microscopic field equations is solved in this paper in a manifestly covariant way. First a covariant series expansion in terms of internal atomic quantities is given in the observer's Lorentz frame (§ 3). Then the proper electromagnetic multipole moments are defined in atomic frames, in which the atoms are momentarily at rest (§ 4). With the help of the Lorentz transformation from the reference frame to the atomic frames (§ 5) we find the Maxwell equations with polarizations given as series expansions in the multipole moments to all orders (§§ 6, 7, 11). Some special cases are considered (§§ 8, 9, 10) and previous work is discussed (§ 12).

§ 2. *The sub-atomic field equations.* The microscopic equations for the electromagnetic fields $\boldsymbol{e}$ and $\boldsymbol{b}$, produced at a time t and a position $\boldsymbol{R}$ by a set of point particles with charges $e_j (j = 1, 2, \ldots)$, positions $\boldsymbol{R}_j(t)$ and velocities $\mathrm{d}\boldsymbol{R}_j(t)/\mathrm{d}t$ read as follows (if we use ∂_0 and $\boldsymbol{\nabla}$ to indicate differentiations with respect to ct and $\boldsymbol{R}$):

$$\boldsymbol{\nabla} \cdot \boldsymbol{e} = \sum_j e_j \, \delta(\boldsymbol{R}_j - \boldsymbol{R}), \tag{1}$$

$$-\partial_0 \boldsymbol{e} + \boldsymbol{\nabla} \wedge \boldsymbol{b} = \sum_j e_j (\mathrm{d}\boldsymbol{R}_j / c \, \mathrm{d}t) \, \delta(\boldsymbol{R}_j - \boldsymbol{R}), \tag{2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0, \tag{3}$$

$$\partial_0 \boldsymbol{b} + \boldsymbol{\nabla} \wedge \boldsymbol{e} = 0. \tag{4}$$

We can cast these equations into covariant form by introducing the notations $R^\alpha(\alpha = 0, 1, 2, 3)$ for $(ct, \boldsymbol{R})$, R_j^α for $(ct_j, \boldsymbol{R}_j)$, ∂_α for $(\partial_0, \boldsymbol{\nabla})$ $f^{\alpha\beta}(\alpha, \beta = 0, 1, 2, 3)$ for the antisymmetric field tensor with components $(f^{01}, f^{02}, f^{03}) = \boldsymbol{e}$ and $(f^{23}, f^{31}, f^{12}) = \boldsymbol{b}$. In this way equations (1) and (2) are the cases $\alpha = 0$ and $\alpha = 1, 2, 3$ of

$$\partial_\beta f^{\alpha\beta} = \sum_j e_j \int_{-\infty}^{+\infty} (\mathrm{d}R_j^\alpha/\mathrm{d}R_j^0)\, \delta(\boldsymbol{R}_j - \boldsymbol{R})\, \delta(R_j^0 - R^0)\, \mathrm{d}R_j^0. \tag{5}$$

Considering R_j^0 as a function $R_j^0(s)$ of an arbitrary parameter s for each particle j, we can write (5) as

$$\partial_\beta f^{\alpha\beta} = \sum_j e_j \int_{-\infty}^{+\infty} \{\mathrm{d}R_j^\alpha(s)/\mathrm{d}s\}\, \delta^{(4)}\{R_j(s) - R\}\, \mathrm{d}s, \tag{6}$$

where $\delta^{(4)}\{R_j(s) - R\}$ is the four-dimensional delta function. The parameters s, which are integration variables, may be chosen independently for each trajectory $(j = 1, 2, \ldots)$.

The field equations (3) and (4) may be written as

$$\partial_\beta f^{*\alpha\beta} = 0, \tag{7}$$

where $f^{*\alpha\beta}$ has components $(f^{*01}, f^{*02}, f^{*03}) = -\boldsymbol{b}$ and $(f^{*23}, f^{*31}, f^{*12}) = \boldsymbol{e}$.

The covariant equations (6) and (7) give the fields as measured in the time-space reference frame $(ct, \boldsymbol{R})$.

§ 3. *The atomic series expansion.* Let us now suppose that the charged point particles (in practice electrons and nuclei) are grouped into stable entities (such as for instance atoms, ions, molecules, free electrons), which for the sake of brevity will be referred to as "atoms" We shall replace the numbering j by a double indexing ki, where k numbers the atoms and i their constituent particles. It is convenient to split up the fields according to

$$f^{\alpha\beta} = \sum_k f_k^{\alpha\beta}, \tag{8}$$

where the field $f_k^{\alpha\beta}$ must satisfy the equations

$$\partial_\beta f_k^{\alpha\beta} = \sum_i e_{ki} \int_{-\infty}^{+\infty} \{\mathrm{d}R_{ki}^\alpha(s)/\mathrm{d}s\}\, \delta^{(4)}\{R_{ki}(s) - R\}\, \mathrm{d}s, \tag{9}$$

which follows from (6).

Let us now choose a privileged point $R_k(s)$ describing the motion of atom k as a whole, and introduce internal atomic parameters $r_{ki}(s)$ by means of

$$R_{ki}(s) = R_k(s) + r_{ki}(s). \tag{10}$$

Introducing (10) into (9) and expanding the delta function in powers of

r_{ki}, we obtain

$$\partial_\beta f_k^{\alpha\beta} = \sum_i e_{ki} \sum_{n=0}^{\infty} (-1)^n (n!)^{-1} \int_{-\infty}^{+\infty} (\mathrm{d}R_k^\alpha/\mathrm{d}s + \mathrm{d}r_{ki}^\alpha/\mathrm{d}s)(r_{ki} \cdot \partial)^n \, \delta^{(4)}(R_k - R) \, \mathrm{d}s, \quad (11)$$

where the dot now indicates the scalar product of two four-vectors. The first term at the right-hand side with the term $n = 0$ of the series expansion is

$$j_k^\alpha/c = e_k \int_{-\infty}^{+\infty} (\mathrm{d}R_k^\alpha/\mathrm{d}s) \, \delta^{(4)}(R_k - R) \, \mathrm{d}s, \quad (12)$$

where $e_k = \sum_i e_{ki}$ is the atomic charge. Now, j_k^α represents the atomic charge-current density vector $(c\rho_k, \boldsymbol{j}_k)$ with components given by

$$\rho_k = e_k \, \delta(\boldsymbol{R}_k - \boldsymbol{R}), \quad (13)$$

$$\boldsymbol{j}_k = e_k(\mathrm{d}\boldsymbol{R}_k/\mathrm{d}t) \, \delta(\boldsymbol{R}_k - \boldsymbol{R}). \quad (14)$$

Equation (11) can now be written as

$$\partial_\beta f_k^{\alpha\beta} = j_k^\alpha/c + \partial_\beta \sum_i e_{ki} \sum_{n=1}^{\infty} (-1)^n (n!)^{-1} \int_{-\infty}^{+\infty} (\mathrm{d}R_k^\alpha/\mathrm{d}s) \, r_{ki}^\beta \cdot$$
$$\cdot (r_{ki} \cdot \partial)^{n-1} \, \delta^{(4)}(R_k - R) \, \mathrm{d}s +$$
$$+ \partial_\beta \sum_i e_{ki} \sum_{n=0}^{\infty} (-1)^n (n!)^{-1} \int_{-\infty}^{+\infty} (\mathrm{d}r_{ki}^\alpha/\mathrm{d}s) \, r_{ki}^\beta (r_{ki} \cdot \partial)^{n-1} \, \delta^{(4)}(R_k - R) \, \mathrm{d}s. \quad (15)$$

Let us subtract a term of similar structure, but with $r_{ki}^\alpha(\mathrm{d}R_k^\beta/\mathrm{d}s)$, from the second term at the right-hand side and add it to the last. Then after a partial integration the equation takes the form

$$\partial_\beta f_k^{\alpha\beta} = j_k^\alpha/c + \partial_\beta m_k^{\alpha\beta}, \quad (16)$$

with

$$m_k^{\alpha\beta} = \sum_i e_{ki} \sum_{n=1}^{\infty} [(-1)^n (n!)^{-1} \int_{-\infty}^{+\infty} \{(\mathrm{d}R_k^\alpha/\mathrm{d}s) \, r_{ki}^\beta - r_{ki}^\alpha(\mathrm{d}R_k^\beta/\mathrm{d}s)\} \cdot$$
$$\cdot (r_{ki} \cdot \partial)^{n-1} \, \delta^{(4)}(R_k - R) \, \mathrm{d}s +$$
$$+ (-1)^n (n-1)(n!)^{-1} \int_{-\infty}^{+\infty} \{r_{ki}^\alpha(\mathrm{d}r_{ki}^\beta/\mathrm{d}s) - (\mathrm{d}r_{ki}^\alpha/\mathrm{d}s) \, r_{ki}^\beta\}(r_{ki} \cdot \partial)^{n-2} \, \delta^{(4)}(R_k - R) \, \mathrm{d}s]. \quad (17)$$

The last expression is an antisymmetric four-tensor, which we shall call the polarization tensor. If we denote its components by $(m_k^{10}, m_k^{20}, m_k^{30}) = \boldsymbol{p}_k$ and $(m_k^{23}, m_k^{31}, m_k^{12}) = \boldsymbol{m}_k$, then equations (16) read for $\alpha = 0$ and $\alpha = 1, 2, 3$ respectively

$$\boldsymbol{\nabla} \cdot \boldsymbol{e}_k = \rho_k - \boldsymbol{\nabla} \cdot \boldsymbol{p}_k, \quad (18)$$

$$-\partial_0 \boldsymbol{e}_k + \boldsymbol{\nabla} \wedge \boldsymbol{b}_k = \boldsymbol{j}_k/c + \partial_0 \boldsymbol{p}_k + \boldsymbol{\nabla} \wedge \boldsymbol{m}_k, \quad (19)$$

which are already of the form of Maxwell's inhomogeneous equations. However, the polarization tensor (17) (which contains the atomic parameters $\boldsymbol{r}_{ki}$, measured in the $(ct, \boldsymbol{R})$-frame) must still be expressed in terms of the proper atomic multipole moments.

§ 4. *The atomic multipole moments.* We wish to characterize the atomic internal electromagnetic structure by means of parameters defined in a Lorentz frame in which the atom as a whole is at rest. Such a frame must have a velocity $\boldsymbol{v}$ equal to $(\mathrm{d}\boldsymbol{R}_k/\mathrm{d}t)_{t=t_0}$ with respect to the reference frame $(ct, \boldsymbol{R})$ of the observer. Time-space coordinates of the reference frame $(ct, \boldsymbol{R})$ and the atomic frame $(ct', \boldsymbol{R}')$ are connected by the Lorentz transformation

$$ct = \gamma ct' + \gamma \boldsymbol{\beta}\cdot\boldsymbol{R}', \tag{20}$$

$$\boldsymbol{R} = \boldsymbol{\Omega}^{-1}\cdot\boldsymbol{R}' + \gamma\boldsymbol{\beta}ct', \tag{21}$$

where (with U the unit three-tensor),

$$\boldsymbol{\beta} = \boldsymbol{v}/c,\ \gamma = (1-\beta^2)^{-\frac{1}{2}},\ \boldsymbol{\Omega}^{-1} = \mathsf{U} + (\gamma - 1)\,\boldsymbol{\beta}\boldsymbol{\beta}/\beta^2. \tag{22}$$

In the Lorentz transformation we introduced the three-tensor $\boldsymbol{\Omega}^{-1}$. It is the inverse of a tensor

$$\boldsymbol{\Omega} = \mathsf{U} + (\gamma^{-1} - 1)\,\boldsymbol{\beta}\boldsymbol{\beta}/\beta^2, \tag{23}$$

a useful quantity, which is related to the Lorentz contraction, as we shall see.

Since the atom suffers accelerations, at every moment t_0 one needs a different atomic rest frame. Every atomic frame is therefore only a *momentary* rest frame: only for $t = t_0$ does the atomic velocity $\mathrm{d}\boldsymbol{R}'_k/\mathrm{d}t'$ vanish.

In the atomic frame the atom is characterized by internal parameters r'_{ki}, which at the moment $t' = t'_0$ are purely spatial vectors, i.e. $r'^0_{ki} = 0$. At that moment $t' = t'_0$ we have thus for the scalar product of the four-vectors r'_{ki} and $\mathrm{d}R'_{ki}/\mathrm{d}t'$

$$r'^0_{ki}\,\mathrm{d}R'^0_k/\mathrm{d}t' - \boldsymbol{r}'_{ki}\cdot\mathrm{d}\boldsymbol{R}'_k/\mathrm{d}t' = 0. \tag{24}$$

(We shall see in the next section how the r'_{ki} are related to the r_{ki}).

The atomic multipole moments are certain useful combinations of the atomic internal parameters; the electric 2^n-pole moment is

$$\boldsymbol{\mu}_k^{(n)} = (n!)^{-1} \sum_i e_{ki}(\boldsymbol{r}'_{ki})^n, \qquad (n = 1, 2 \ldots) \tag{25}$$

and the magnetic 2^n-pole moment

$$\boldsymbol{\nu}_k^{(n)} = n\{(n+1)!\}^{-1} \sum_i e_{ki}(\boldsymbol{r}'_{ki})^n \wedge \dot{\boldsymbol{r}}'_{ki}/c, \qquad (n = 1, 2, \ldots) \tag{26}$$

where the powers indicate polyads of three-vectors and where

$$\boldsymbol{r}'_{ki} = \{\boldsymbol{r}'_{ki}(t')\}_{t'=t_0'} \quad \text{and} \quad \dot{\boldsymbol{r}}'_{ki} = (\mathrm{d}\boldsymbol{r}'_{ki}/\mathrm{d}t')_{t'=t_0'}.$$

We shall now discuss the transformation of quantities defined in the reference frame $(ct, \boldsymbol{R})$ to quantities defined in the atomic momentary rest frame, in order to be able to express the polarization tensor in terms of the atomic multipole moments as defined here.

§ 5. *The Lorentz transformation from the reference frame to the momentary atomic rest frame.* In the result of the atomic series expansion of § 3 internal quantities r_{ki} occur, which are measured in the reference frame $(ct, \boldsymbol{R})$. However, as explained in the preceding section we wish to characterize the structure of the atoms by means of quantities which are connected to the purely spatial internal parameters $\boldsymbol{r}'_{ki}$, which are measured in the momentary atomic rest frame. Therefore we must find the connexion between the two kinds of internal parameters.

We now choose such a parametrization along the particle trajectories that for all ki and all values of s one has

$$r^0_{ki}(s)\, \mathrm{d}R^0_k(s)/\mathrm{d}s - \boldsymbol{r}_{ki}(s)\cdot \mathrm{d}\boldsymbol{R}_k(s)/\mathrm{d}s = 0. \tag{27}$$

In fact this means that the parametrization along the trajectory of the privileged point of atom k *induces* a parametrization along the trajectories of the constituent particles ki through the "perpendicular" projection expressed by (27) *).

The condition (27), which is a covariant condition, has as a consequence that by transformation to the momentary atomic rest frame r'_{ki} becomes purely spatial and thus equal to the atomic parameters $\boldsymbol{r}'_{ki}$ of the preceding section.

The Lorentz transformation with transformation velocity given by

$$\boldsymbol{\beta}_k = \boldsymbol{v}_k/c = \frac{1}{c}\left(\frac{\mathrm{d}\boldsymbol{R}_k}{\mathrm{d}t}\right)_{t=t_0} = \left(\frac{\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s}{\mathrm{d}R^0_k/\mathrm{d}s}\right)_{s=s_0}, \tag{28}$$

(with s_0 corresponding to t_0) reads

$$R^0_k(s) = \gamma_k R'^0_k(s) + \gamma_k \boldsymbol{\beta}_k\cdot \boldsymbol{R}'_k(s), \tag{29}$$

$$\boldsymbol{R}_k(s) = \boldsymbol{\Omega}_k^{-1}\cdot \boldsymbol{R}'_k(s) + \gamma_k \boldsymbol{\beta}_k R'^0_k(s). \tag{30}$$

Differentiation with respect to s gives

$$\mathrm{d}R^0_k/\mathrm{d}s = \gamma_k\, \mathrm{d}R'^0_k/\mathrm{d}s + \gamma_k \boldsymbol{\beta}_k\cdot \mathrm{d}\boldsymbol{R}'_k/\mathrm{d}s, \tag{31}$$

$$\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s = \boldsymbol{\Omega}_k^{-1}\cdot \mathrm{d}\boldsymbol{R}'_k/\mathrm{d}s + \gamma_k \boldsymbol{\beta}_k\, \mathrm{d}R'^0_k/\mathrm{d}s. \tag{32}$$

*) A possible many-valuedness of the parametrization s thus induced gives no difficulties if an appropriate prescription is imposed for the way of integration over s.

Similarly for the internal quantities one has the Lorentz transformations

$$r^0_{ki}(s) = \gamma_k r'^0_{ki}(s) + \gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}'_{ki}(s), \tag{33}$$

$$\boldsymbol{r}_{ki}(s) = \boldsymbol{\Omega}_k^{-1} \cdot \boldsymbol{r}'_{ki}(s) + \gamma_k \boldsymbol{\beta}_k r'^0_{ki}(s), \tag{34}$$

and for the derivatives

$$\mathrm{d}r^0_{ki}/\mathrm{d}s = \gamma_k \, \mathrm{d}r'^0_{ki}/\mathrm{d}s + \gamma_k \boldsymbol{\beta}_k \cdot \mathrm{d}\boldsymbol{r}'_{ki}/\mathrm{d}s, \tag{35}$$

$$\mathrm{d}\boldsymbol{r}_{ki}/\mathrm{d}s = \boldsymbol{\Omega}_k^{-1} \cdot \mathrm{d}\boldsymbol{r}'_{ki}/\mathrm{d}s + \gamma_k \boldsymbol{\beta}_k \, \mathrm{d}r'^0_{ki}/\mathrm{d}s. \tag{36}$$

Now in the momentary atomic rest frame both the temporal component $r'^0_{ki}(s)$ and the atomic velocity which is proportional to $\mathrm{d}\boldsymbol{R}'_k(s)/\mathrm{d}s$ (cf. (28)) vanish at the moment for which $s = s_0$. Thus, (31) and (32) become

$$\mathrm{d}R^0_k/\mathrm{d}s = \gamma_k \, \mathrm{d}R'^0_k/\mathrm{d}s, \tag{37}$$

$$\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s = \gamma_k \boldsymbol{\beta}_k \, \mathrm{d}R'^0_k/\mathrm{d}s, \tag{38}$$

and (33) and (34) get the form

$$r^0_{ki}(s) = \gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}'_{ki}(s), \tag{39}$$

$$\boldsymbol{r}_{ki}(s) = \boldsymbol{\Omega}_k^{-1} \cdot \boldsymbol{r}'_{ki}(s) \tag{40}$$

Since we measure internal quantities in the atomic frame, but external quantities (atomic positions, velocities etc.) in the reference frame, we need a few more consequences of the preceding transformation formulae. In the first place we want an expression for the second derivative of $\boldsymbol{R}'_k(s)$ with respect to s. According to the Lorentz transformation one has

$$\mathrm{d}^2\boldsymbol{R}'_k(s)/\mathrm{d}s^2 = \boldsymbol{\Omega}_k^{-1} \cdot \mathrm{d}^2\boldsymbol{R}_k(s)/\mathrm{d}s^2 - \gamma_k \boldsymbol{\beta}_k \, \mathrm{d}^2R^0_k(s)/\mathrm{d}s^2. \tag{41}$$

Furthermore from (37) and (38) one has

$$\mathrm{d}\boldsymbol{R}_k/\mathrm{d}s = \boldsymbol{\beta}_k \, \mathrm{d}R^0_k/\mathrm{d}s, \tag{42}$$

and by differentiation of this relation

$$\mathrm{d}^2\boldsymbol{R}_k(s)/\mathrm{d}s^2 = \boldsymbol{\beta}_k \, \mathrm{d}^2R^0_k(s)/\mathrm{d}s^2 + \partial_0 \boldsymbol{\beta}_k \{\mathrm{d}R^0_k(s)/\mathrm{d}s\}^2. \tag{43}$$

Substituting (43) into (41), and using also

$$\boldsymbol{\Omega}_k^{-1} \cdot \boldsymbol{\beta}_k = \gamma_k \boldsymbol{\beta}_k, \tag{44}$$

which follows from (22), one obtains

$$\mathrm{d}^2\boldsymbol{R}'_k(s)/\mathrm{d}s^2 = \boldsymbol{\Omega}_k^{-1} \cdot \partial_0 \boldsymbol{\beta}_k \{\mathrm{d}R^0_k(s)/\mathrm{d}s\}^2. \tag{45}$$

With the identity, which can be checked by using (22) and (23),

$$\boldsymbol{\Omega}_k^{-1} \cdot \partial_0 \boldsymbol{\beta}_k = \gamma_k^{-1} \boldsymbol{\Omega}_k \cdot \partial_0 (\boldsymbol{\beta}_k \gamma_k), \tag{46}$$

one obtains finally from (45)

$$\mathrm{d}^2\boldsymbol{R}'_k(s)/\mathrm{d}s^2 = \gamma_k^{-1}\,\boldsymbol{\Omega}_k\cdot\partial_0(\beta_k\gamma_k)\{\mathrm{d}R^{0}_k(s)/\mathrm{d}s\}^2. \tag{47}$$

A second result can be obtained from the invariant condition (27), which reads in the atomic frame

$$r'^{0}_{ki}\,\mathrm{d}R'^{0}_k/\mathrm{d}s - \boldsymbol{r}'_{ki}\cdot\mathrm{d}\boldsymbol{R}'_k/\mathrm{d}s = 0. \tag{48}$$

Differentiating this relation with respect to s and taking into account $(r'^{0}_{ki})_{s=s_0} = 0$ and $(\mathrm{d}\boldsymbol{R}'_k/\mathrm{d}s)_{s=s_0} = 0$, one finds for $s = s_0$

$$\mathrm{d}r'^{0}_{ki}(s)/\mathrm{d}s = \boldsymbol{r}'_{ki}(s)\,.\,\{\mathrm{d}^2\boldsymbol{R}'_k(s)/\mathrm{d}s^2\}\{\mathrm{d}R'^{0}_k(s)/\mathrm{d}s\}^{-1}, \tag{49}$$

which, with (37) and (47), becomes finally

$$\mathrm{d}r'^{0}_{ki}(s)/\mathrm{d}s = \boldsymbol{r}'_{ki}\cdot\boldsymbol{\Omega}_k\cdot\partial_0(\beta_k\gamma_k)\,\mathrm{d}R^{0}_k(s)/\mathrm{d}s. \tag{50}$$

In the magnetic multipole moment (26) a time derivative occurs which

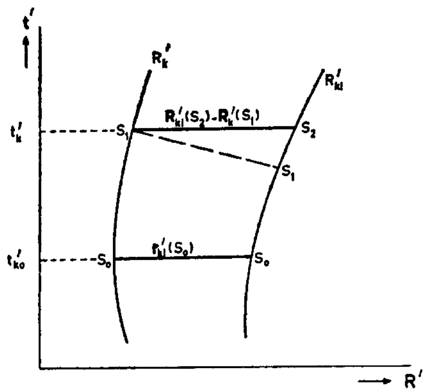


Fig. 1. World lines of atom k and constituent particle ki in the momentary rest frame.

is the limit of the difference of two purely spatial vectors divided by the corresponding time difference (cf. fig. 1):

$$\dot{\boldsymbol{r}}'_{ki} = \lim_{t_k'\to t_{k0}'} \frac{\boldsymbol{R}'_{ki}(s_2) - \boldsymbol{R}'_k(s_1) - \boldsymbol{r}'_{ki}(s_0)}{t'_k - t'_{k0}} =$$

$$= \lim_{s_1\to s_0} \frac{\boldsymbol{R}'_{ki}(s_2) - \boldsymbol{R}'_k(s_1) - \boldsymbol{r}'_{ki}(s_0)}{s_1 - s_0}\left[\left\{\frac{\mathrm{d}t'_k(s)}{\mathrm{d}s}\right\}_{s=s_0}\right]^{-1}. \tag{51}$$

The values s_1 and s_2 of the parameter s are related by (cf. fig. 1)

$$R'^{0}_{ki}(s_2) = R'^{0}_k(s_1), \tag{52}$$

or, with the splitting (10),

$$R'^{0}_k(s_2) + r'^{0}_{ki}(s_2) = R'^{0}_k(s_1). \tag{53}$$

Taylor expansion of the left- and right-hand sides with respect to $s_2 - s_0$ and $s_1 - s_0$ respectively gives

$$s_2 - s_0 = \left\{1 + \frac{\mathrm{d}r_{ki}'^{0}(s)/\mathrm{d}s}{\mathrm{d}R_k'^{0}(s)/\mathrm{d}s}\right\}^{-1} (s_1 - s_0) + \dots \tag{54}$$

With the help of this relation, expression (51) becomes after expansion of the numerator around s_0:

$$\dot{\boldsymbol{r}}_{ki}' = c\{\mathrm{d}\boldsymbol{r}_{ki}'(s)/\mathrm{d}s\}\{\mathrm{d}R_k'^{0}(s)/\mathrm{d}s + \mathrm{d}r_{ki}'^{0}(s)/\mathrm{d}s\}^{-1}. \tag{55}$$

Using (37) and (50) this can be written in the form

$$\mathrm{d}\boldsymbol{r}_{ki}'(s)/\mathrm{d}s = (\gamma_k c)^{-1}\dot{\boldsymbol{r}}_{ki}'\{1 + \gamma_k \boldsymbol{r}_{ki}' \,.\, \boldsymbol{\Omega}_k\cdot\partial_0(\boldsymbol{\beta}_k\gamma_k)\}\, \mathrm{d}R_k^0(s)/\mathrm{d}s, \tag{56}$$

which gives the derivative of $\boldsymbol{r}_{ki}'(s)$ in terms of the internal quantities $\boldsymbol{r}_{ki}'$ and $\dot{\boldsymbol{r}}_{ki}'$, which occur in the atomic multipole moments.

§ 6. *Calculation of the polarization tensor.* With the help of the results of the preceding section we now want to obtain expressions for the polarization vectors $\boldsymbol{p}_k$ and $\boldsymbol{m}_k$, involving the atomic multipole moments defined in § 4. Let us substitute into (17) the transformation formulae (39), (40), (42), (35) and (36) using also (50) and (56). The integral over s can now be evaluated with the help of the relation $\{\mathrm{d}R_k^0(s)/\mathrm{d}s\}\,\mathrm{d}s = \mathrm{d}R_k^0$. Employing an identity which is valid for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$(\boldsymbol{\Omega}_k^{-1}\cdot\boldsymbol{a}) \wedge (\boldsymbol{\Omega}_k^{-1}\cdot\boldsymbol{b}) = \gamma_k\boldsymbol{\Omega}_k\cdot(\boldsymbol{a}\wedge\boldsymbol{b}), \tag{57}$$

the electric polarization $\boldsymbol{p}_k$ can be cast into the form:

$$\begin{aligned}\boldsymbol{p}_k = \sum_i e_{ki} \sum_{n=1}^{\infty} \Bigg(&\sum_{p=0}^{n-1}\sum_{q=0}^{p} \binom{n-1}{p}\binom{p}{q} (-1)^{n-1}(n!)^{-1}\cdot \\ &\cdot\partial_0^{p-q}\{\boldsymbol{\Omega}_k\cdot\boldsymbol{r}_{ki}'(\gamma_k\boldsymbol{\beta}_k\cdot\boldsymbol{r}_{ki}')^p\,(\boldsymbol{r}_{ki}'\cdot\boldsymbol{\Omega}_k^{-1}\cdot\boldsymbol{\nabla})^{n-1-p}\}\,\partial_0^q\delta(\boldsymbol{R}_k - \boldsymbol{R}) + \\ &+\sum_{p=0}^{n-2}\sum_{q=0}^{p}\binom{n-2}{p}\binom{p}{q}(-1)^n(n-1)(n!)^{-1}\,\partial_0^{p-q}[\boldsymbol{\Omega}_k\cdot\boldsymbol{r}_{ki}'(\gamma_k\boldsymbol{\beta}_k\cdot\boldsymbol{r}_{ki}')^p\cdot \\ &\cdot(\boldsymbol{r}_{ki}'\cdot\boldsymbol{\Omega}_k^{-1}\cdot\boldsymbol{\nabla})^{n-2-p}\{\gamma_k\boldsymbol{r}_{ki}'\cdot\boldsymbol{\Omega}_k\cdot\partial_0(\boldsymbol{\beta}_k\gamma_k)\}]\,\partial_0^q\,\delta(\boldsymbol{R}_k - \boldsymbol{R}) + \\ &+\sum_{p=0}^{n-2}\sum_{q=0}^{p}\binom{n-2}{p}\binom{p}{q}(-1)^n(n-1)(n!)^{-1}\,\partial_0^{p-q}[\boldsymbol{\beta}_k\wedge\{\boldsymbol{\Omega}_k\cdot(\boldsymbol{r}_{ki}'\wedge\dot{\boldsymbol{r}}_{ki}'/c)\}\cdot \\ &\cdot(\gamma_k\boldsymbol{\beta}_k\cdot\boldsymbol{r}_{ki}')^p\,(\boldsymbol{r}_{ki}'\cdot\boldsymbol{\Omega}_k^{-1}\cdot\boldsymbol{\nabla})^{n-2-p}\{1+\gamma_k\boldsymbol{r}_{ki}'\cdot\boldsymbol{\Omega}_k\cdot\partial_0(\boldsymbol{\beta}_k\gamma_k)\}]\,\partial_0^q\delta(\boldsymbol{R}_k - \boldsymbol{R})\Bigg).\end{aligned} \tag{58}$$

Here the electric polarization is written in terms of the internal variables $\boldsymbol{r}_{ki}'$ and $\dot{\boldsymbol{r}}_{ki}'$, measured in the atomic rest frame, and the external variables $\boldsymbol{\beta}_k$, γ_k, $\boldsymbol{\Omega}_k$, $\boldsymbol{\Omega}_k^{-1}$ and their derivatives; the position vector $\boldsymbol{R}_k$ enters only in the delta function, as it should be.

The internal variables $\boldsymbol{r}'_{ki}$ occur partly in combinations like $\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki}$. The tensor $\boldsymbol{\Omega}_k$ can be interpreted in terms of a Lorentz contraction, since for every vector $\boldsymbol{a}$ one has

$$\boldsymbol{\Omega}_k \cdot \boldsymbol{a} = \boldsymbol{\Omega}_k \cdot (\boldsymbol{a}_{//} + \boldsymbol{a}_{\perp}) = \boldsymbol{a}_{\perp} + (1 - \beta_k^2)^{\frac{1}{2}} \boldsymbol{a}_{//}, \tag{59}$$

where $\boldsymbol{a}$ is split into a part parallel with and a part perpendicular to the transformation velocity $c\boldsymbol{\beta}_k$. In view of this property of $\boldsymbol{\Omega}_k$ we shall eliminate in (58) the tensor $\boldsymbol{\Omega}_k^{-1}$ by means of the identity:

$$\boldsymbol{\Omega}_k^{-1} = \boldsymbol{\Omega}_k + \gamma_k \boldsymbol{\beta}_k \boldsymbol{\beta}_k, \tag{60}$$

which is a consequence of (22) and (23). With the help of this equation one obtains from (58), upon splitting the third term and replacing p by $p - 1$ in the second term, n by $n + 1$ in the first part of the third term and p by $p - 1$ in the second part of the third term:

$$\begin{aligned}
\boldsymbol{p}_k = {} & \sum_i e_{ki} \sum_n \sum_{pqm} ((-1)^{n-1}\{n(n-1-p-m)!\,(p-q)!\,q!m!\}^{-1}\, \partial_0^{p-q}\{\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki} \cdot \\
& \cdot (\gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}'_{ki})^{p+m} (\boldsymbol{r}'_{ki} \cdot \boldsymbol{\Omega}_k \cdot \boldsymbol{\nabla})^{n-1-p-m}\}\, \partial_0^q\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\} + \\
& + (-1)^n \{n(n-1-p-m)!\,(p-q-1)!\,q!\,m!\}^{-1} \cdot \\
& \cdot \partial_0^{p-q-1}[\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki} (\gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}'_{ki})^{p+m-1} \cdot \\
& \cdot \{\gamma_k \boldsymbol{r}'_{ki} \cdot \boldsymbol{\Omega}_k \cdot \partial_0(\boldsymbol{\beta}_k \gamma_k)\} (\boldsymbol{r}'_{ki} \cdot \boldsymbol{\Omega}_k \cdot \boldsymbol{\nabla})^{n-1-p-m}]\, \partial_0^q\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\} + \\
& + (-1)^{n-1}\{(n+1)(n-1-p-m)!\,(p-q)!\,q!\,m!\}^{-1} \cdot \\
& \cdot \partial_0^{p-q}[\boldsymbol{\beta}_k \wedge \{\boldsymbol{\Omega}_k \cdot (\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki}/c)\} \cdot \\
& \cdot (\gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}'_{ki})^{p+m} (\boldsymbol{r}'_{ki} \cdot \boldsymbol{\Omega}_k \cdot \boldsymbol{\nabla})^{n-1-p-m}]\, \partial_0^q\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\} + \\
& + (-1)^n\{n(n-1-p-m)!\,(p-q-1)!\,q!\,m!\}^{-1}\, \partial_0^{p-q-1}[\boldsymbol{\beta}_k \wedge \{\boldsymbol{\Omega}_k \cdot (\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki}/c)\} \cdot \\
& \cdot (\gamma_k \boldsymbol{\beta}_k \cdot \boldsymbol{r}'_{ki})^{p+m-1} \{\gamma_k \boldsymbol{r}'_{ki} \cdot \boldsymbol{\Omega}_k \cdot \partial_0(\boldsymbol{\beta}_k \gamma_k)\} (\boldsymbol{r}'_{ki} \cdot \boldsymbol{\Omega}_k \cdot \boldsymbol{\nabla})^{n-1-p-m}] \cdot \\
& \cdot \partial_0^q\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\}).
\end{aligned} \tag{61}$$

The indices are bounded as a result of the occurrence of factorials in the denominators; thus in the first term for instance one has $0 \leqslant q \leqslant p$ and $0 \leqslant m \leqslant n - 1 - p$. Next we want to regroup the terms in a different way by carrying out one of the time derivatives in the first and third term. After a final relabelling of indices $p + m \to p$ and $q + m \to q$ (61) becomes

$$\begin{aligned}
\boldsymbol{p}_k = {} & \sum_n \sum_{pqm} (-1)^{n-1}(n-1)!\{(n-1-p)!\,(p-q)!\,(q-m)!\,m!\}^{-1} \cdot \\
& \cdot \partial_0^{p-q-1}[(\gamma_k \boldsymbol{\beta}_k)^{p-q} \vdots\, \partial_0\{(\underline{\boldsymbol{\mu}}_k^{(n)} + \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(n)}) \vdots\, \gamma_k^p (\boldsymbol{\beta}_k \gamma_k)^q\, \boldsymbol{\nabla}^{n-1-p}\}] \cdot \\
& \cdot \partial_0^{q-m}\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\} + \\
& + \sum_n \sum_{\substack{pqm \\ p \geqslant q+1}} (-1)^n (n-1)!\{n(n-1-p)!\,(p-q)!\,(q-m)!\,m!\}^{-1} \cdot \\
& \cdot \partial_0^{p-q-1}\{\partial_0(\gamma_k \boldsymbol{\beta}_k)^{p-q} \vdots (\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(n)}) \vdots \gamma_k^p (\boldsymbol{\beta}_k \gamma_k)^q\, \boldsymbol{\nabla}^{n-1-p}\}\, \partial_0^{q-m}\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\},
\end{aligned} \tag{62}$$

where we introduced the abbreviations (cf. (25), (26)):

$$\underline{\boldsymbol{\mu}}_k^{(n)} = \boldsymbol{\Omega}_k^n \vdots \boldsymbol{\mu}_k^{(n)} = (n!)^{-1} \sum_i e_{ki}(\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki})^n, \tag{63}$$

$$\underline{\boldsymbol{\nu}}_k^{(n)} = \boldsymbol{\Omega}_k^n \vdots \boldsymbol{\nu}_k^{(n)} = n\{(n+1)!\}^{-1} \sum_i e_{ki}(\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki})^{n-1}\{\boldsymbol{\Omega}_k \cdot (\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki}/c)\}. \tag{64}$$

The multipole moments $\boldsymbol{\mu}_k^{(n)}$ and $\boldsymbol{\nu}_k^{(n)}$ occur only in these combinations with the tensor $\boldsymbol{\Omega}_k$; this can be explained in terms of a Lorentz contraction of the internal variables, as was shown above (59).

In connection with (62) a remark about the tensor notation must be made: the outer product sign in front of $\boldsymbol{\nu}_k^{(n)}$ stands for an outer product with the vector $\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki}$ contained in the dyadic tensor $\boldsymbol{\nu}_k^{(n)}$; the contractions take place with the vector $\boldsymbol{r}'_{ki}$ in $\boldsymbol{\nu}_k^{(n)}$. Furthermore the product $\partial_0^{-1}\partial_0$, which occurs in the first term as $p = q$, is to be considered as unity.

Up to now we obtained an expression for $\boldsymbol{p}_k$ only. From (17) it can be deduced that $\boldsymbol{m}_k$ can be obtained from $\boldsymbol{p}_k$ by replacing the non-contracted vectors in (58) in the following way:

$$\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki} \rightarrow -\boldsymbol{\beta}_k \wedge (\boldsymbol{\Omega}_k \cdot \boldsymbol{r}'_{ki}), \tag{65}$$

$$\boldsymbol{\beta}_k \wedge \{\boldsymbol{\Omega}_k \cdot (\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki})\} \rightarrow \boldsymbol{\Omega}_k \cdot (\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki}). \tag{66}$$

In (62) one must make the following transformations:

$$\underline{\boldsymbol{\mu}}_k^{(n)} \rightarrow -\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\mu}}_k^{(n)}, \tag{67}$$

$$\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(n)} \rightarrow \underline{\boldsymbol{\nu}}_k^{(n)}. \tag{68}$$

Thus the expressions for $\boldsymbol{p}_k$ and $\boldsymbol{m}_k$ show a remarkable analogy. Furthermore the leading terms in both expressions are symmetric with respect to an interchange of electric and magnetic multipole moments. The second term in (62) and in the corresponding expression for $\boldsymbol{m}_k$ is due to a relativistic correction (§ 8).

§ 7. *The polarization tensor expressed with spatial derivatives of multipole densities.* The expressions for $\boldsymbol{p}_k$ and $\boldsymbol{m}_k$ of the previous section contain time derivatives of the delta function. Now these derivatives can be expressed in terms of spatial derivatives with the help of the relation:

$$\partial_0 \delta\{\boldsymbol{R}_k(t) - \boldsymbol{R}\} = -\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla}\, \delta\{\boldsymbol{R}_k(t) - \boldsymbol{R}\}. \tag{69}$$

Using a generalized "chain rule" of differentiation the higher order time derivatives of the delta function can be reduced in a similar way to a sum of spatial derivatives each with a factor in front of it. In fact, eq. (62) for $\boldsymbol{p}_k$ contains the linear combination:

$$D(q)\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) \equiv \sum_{m=0}^{q} \{m!(q-m)!\}^{-1}\, \partial_0^{q-m}\{(\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^m\, \delta(\boldsymbol{R}_k - \boldsymbol{R})\}. \tag{70}$$

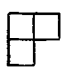
From this definition it follows that

$$D(q) = 0 \text{ if } q < 0, D(q) = 1 \text{ if } q = 0, D(q) = 0 \text{ if } q = 1. \tag{71}$$

For $q \geqslant 2$ the result of the application of $D(q)$ to the delta function can be formulated in a convenient way with the help of diagrams. A diagram is here a set of squares arranged in columns of decreasing height. The interpretation of these diagrams is apparent from the examples

$$\rightarrow (\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})\, \delta(\boldsymbol{R}_k - \boldsymbol{R}),$$

$$\rightarrow (\partial_0 \boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})\, \delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{72}$$

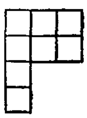


$$\rightarrow (\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})(\partial_0 \boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})\, \delta(\boldsymbol{R}_k - \boldsymbol{R}).$$

Each diagram is provided with a numerical factor

$$\prod_i \{(n_i - 1)(n_i!)^{-1}\} \prod_j \{(m_j - m_{j+1})!\}^{-1}, \tag{73}$$

where n_i is the height of the column i and m_j is the width of the row j. Thus e.g.



$$\rightarrow (3 \times 1 \times 1)(4!\, 2!\, 2!\, 2!)^{-1}\, (\partial_0^{(3)} \boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})(\partial_0 \boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})^2\, \delta(\boldsymbol{R}_k - \boldsymbol{R}). \tag{74}$$

The numerical factor of diagrams with a column of height 1 vanishes.

Now the prescription for $D(q)\, \delta(\boldsymbol{R}_k - \boldsymbol{R})$ can be formulated as: write all diagrams with q squares, interpret with the correct factor (73) in front of it and take the sum. As a consequence of the preceding remark $D(q)$ will contain no diagrams with a column of height 1, and therefore no terms with $\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla}\, \delta(\boldsymbol{R}_k - \boldsymbol{R})$. The expression for $\boldsymbol{p}_k$ (62) now becomes:

$$\begin{aligned}
\boldsymbol{p}_k(\boldsymbol{R}, t) &= \sum_n \sum_{pq} (-1)^{n-1} (n-1)!\, \{(n-1-p)!\, (p-q)!\}^{-1}\, \partial_0^{p-q-1} [(\gamma_k \boldsymbol{\beta}_k)^{p-q} \vdots \cdot \\
&\cdot \partial_0 \{(\underline{\boldsymbol{\mu}}_k^{(n)} + \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(n)}) \vdots \gamma_k^p (\boldsymbol{\beta}_k \gamma_k)^q\, \boldsymbol{\nabla}^{n-1-p}\}]\, D(q)\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) + \\
&+ \sum_n \sum_{\substack{pq \\ p \geqslant q+1}} (-1)^n (n-1)!\, \{n(n-1-p)!\, (p-q)!\}^{-1}\, \partial_0^{p-q-1} \{\partial_0 (\gamma_k \boldsymbol{\beta}_k)^{p-q} \vdots \cdot \\
&\cdot (\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(n)}) \vdots \gamma_k^p (\boldsymbol{\beta}_k \gamma_k)^q\, \boldsymbol{\nabla}^{n-1-p}\}\, D(q)\, \delta(\boldsymbol{R}_k - \boldsymbol{R}).
\end{aligned} \tag{75}$$

Analogously the expression for $\boldsymbol{m}_k$ becomes (remembering (67) and (68)):

$$\begin{aligned}
\boldsymbol{m}_k(\boldsymbol{R}, t) &= \sum_n \sum_{pq} (-1)^{n-1} (n-1)!\, \{(n-1-p)!\, (p-q)!\}^{-1}\, \partial_0^{p-q-1} [(\gamma_k \boldsymbol{\beta}_k)^{p-q} \vdots \cdot \\
&\cdot \partial_0 \{(\underline{\boldsymbol{\nu}}_k^{(n)} - \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\mu}}_k^{(n)}) \vdots \gamma_k^p (\boldsymbol{\beta}_k \gamma_k)^q\, \boldsymbol{\nabla}^{n-1-p}\}]\, D(q)\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) + \\
&+ \sum_n \sum_{\substack{pq \\ p \geqslant q+1}} (-1)^n (n-1)!\, \{n(n-1-p)!\, (p-q)!\}^{-1}\, \partial_0^{p-q-1} \{\partial_0 (\gamma_k \boldsymbol{\beta}_k)^{p-q} \vdots \cdot \\
&\cdot \underline{\boldsymbol{\nu}}_k^{(n)} \vdots \gamma_k^p (\boldsymbol{\beta}_k \gamma_k)^q\, \boldsymbol{\nabla}^{n-1-p}\}\, D(q)\, \delta(\boldsymbol{R}_k - \boldsymbol{R}).
\end{aligned} \tag{76}$$

Here the final results are reached in a form with spatial derivatives of multipole densities; time derivatives also occur but since the nabla operators only act on the delta function they might be written in front of the expressions.

§ 8. *The polarization tensor in the atomic frame.* In the momentary atomic rest frame (ct', $\boldsymbol{R}'$) the polarization tensor gets a particularly simple form, since in the leading terms of (75) and (76) only the term with $p = 0$, $q = 0$ remains; the other term in $\boldsymbol{p}_k$ disappears altogether whereas in $\boldsymbol{m}_k$ only the part with $q = 0$ is left over. In this way we obtain

$$\boldsymbol{p}'_k(\boldsymbol{R}', t') = \sum_{n=1}^{\infty} (-1)^{n-1} \boldsymbol{\mu}_k^{(n)} \vdots (\boldsymbol{\nabla}')^{n-1} \delta(\boldsymbol{R}'_k - \boldsymbol{R}'), \tag{77}$$

$$\boldsymbol{m}'_k(\boldsymbol{R}', t') = \sum_{n=1}^{\infty} (-1)^{n-1} \boldsymbol{\nu}_k^{(n)} \vdots (\boldsymbol{\nabla}')^{n-1} \delta(\boldsymbol{R}'_k - \boldsymbol{R}') +$$
$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{n-1} (-1)^n (n-1)! \{n(n-1-p)!\}^{-1} (\partial'_0 \boldsymbol{\beta}'_k)^p \vdots \boldsymbol{\nu}_k^{(n)} \vdots (\boldsymbol{\nabla}')^{n-1-p} \delta(\boldsymbol{R}'_k - \boldsymbol{R}'). \tag{78}$$

It should be noted that in $\boldsymbol{m}'_k$ a term appears which depends on the acceleration $\partial'_0 \boldsymbol{\beta}'_k$; it represents a special relativistic effect.

§ 9. *The polarization in the nonrelativistic approximation.* We shall consider two kinds of nonrelativistic limiting cases of our general formalism. In the first conception we shall consider all atomic multipole moments as constants without considering whether they contain factors c^{-1}. Then, neglecting terms of order c^{-2} or higher, since now $\gamma_k \simeq 1$ and $\boldsymbol{\Omega}_k \simeq U$, and since only terms with $p = 0$ and $q = 0$ remain, we find from (75) and (76)

$$\boldsymbol{p}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty} (-1)^{n-1} (\boldsymbol{\mu}_k^{(n)} + \boldsymbol{\beta}_k \wedge \boldsymbol{\nu}_k^{(n)}) \vdots \boldsymbol{\nabla}^{n-1} \delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{79}$$

$$\boldsymbol{m}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty} (-1)^{n-1} (\boldsymbol{\nu}_k^{(n)} - \boldsymbol{\beta}_k \wedge \boldsymbol{\mu}_k^{(n)}) \vdots \boldsymbol{\nabla}^{n-1} \delta(\boldsymbol{R}_k - \boldsymbol{R}). \tag{80}$$

These formulae show a symmetry in the sense that $\boldsymbol{p}_k$ contains terms due to moving magnetic multipoles $\boldsymbol{\nu}_k^{(n)}$, just as $\boldsymbol{m}_k$ contains contributions from moving electric multipoles $\boldsymbol{\mu}_k^{(n)}$.

In the second place we may consider the nonrelativistic limiting case, taking into account the fact that the magnetic multipoles $\boldsymbol{\nu}_k^{(n)}$ contain a factor c^{-1}. Then one is left with

$$\boldsymbol{p}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty} (-1)^{n-1} \boldsymbol{\mu}_k^{(n)} \vdots \boldsymbol{\nabla}^{n-1} \delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{81}$$

$$\boldsymbol{m}_k(\boldsymbol{R}, t) \simeq \sum_{n=1}^{\infty} (-1)^{n-1} (\boldsymbol{\nu}_k^{(n)} - \boldsymbol{\beta}_k \wedge \boldsymbol{\mu}_k^{(n)}) \vdots \boldsymbol{\nabla}^{n-1} \delta(\boldsymbol{R}_k - \boldsymbol{R}). \tag{82}$$

These are the formulae found in the nonrelativistic expansion to all orders [8]. Now no terms with moving magnetic moments appear.

§ 10. *The polarization tensor to lowest multipole orders.* The polarization tensor $(\boldsymbol{p}, \boldsymbol{m})$ as given by (75) and (76) contains the atomic multipole moments, which are defined in the momentary rest frame. Let us consider the case in which only one of these multipole moments is different from zero. Then in all Lorentz frames only terms with this particular multipole moment will occur. Since the polarization tensor is a covariant quantity these terms with only one multipole moment form a covariant tensor too. This means that $(\boldsymbol{p}, \boldsymbol{m})$ consists of a sum of terms which are all antisymmetric tensors, and each of them contains one multipole moment only.

The terms can, therefore, be grouped in arbitrary manner, e.g. according to their order in powers of internal coordinates $\boldsymbol{r}'_{ki}$ and $\dot{\boldsymbol{r}}'_{ki}$. Another classification groups together the electromagnetic moments of a certain multipole order. (We note that the electric and magnetic multipole moments $\boldsymbol{\mu}_k^{(n)}$ (25) and $\boldsymbol{\nu}_k^{(n)}$ (26) are of order n and $n+1$ respectively in the internal coordinates). Let us give explicitly the lowest order terms of $\boldsymbol{p}_k = \boldsymbol{p}_k^{(1)} + \boldsymbol{p}_k^{(2)} + \boldsymbol{p}_k^{(3)} + \dots$ and $\boldsymbol{m}_k = \boldsymbol{m}_k^{(1)} + \boldsymbol{m}_k^{(2)} + \boldsymbol{m}_k^{(3)} + \dots$, where (n) indicates the multipole order. From (75) and (76), using the prescription for $D(q)$, one finds:

terms with dipoles $(n = 1)$:

$$\boldsymbol{p}_k^{(1)}(\boldsymbol{R}, t) = (\underline{\boldsymbol{\mu}}_k^{(1)} + \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(1)})\, \delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{83}$$

$$\boldsymbol{m}_k^{(1)}(\boldsymbol{R}, t) = (\underline{\boldsymbol{\nu}}_k^{(1)} - \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\mu}}_k^{(1)})\, \delta(\boldsymbol{R}_k - \boldsymbol{R}); \tag{84}$$

terms with quadrupoles $(n = 2)$:

$$\begin{aligned}\boldsymbol{p}_k^{(2)}(\boldsymbol{R}, t) = &-\gamma_k \boldsymbol{\beta}_k \cdot \partial_0 \{(\underline{\boldsymbol{\mu}}_k^{(2)} + \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(2)})\, \gamma_k\}\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) - \\ &- (\underline{\boldsymbol{\mu}}_k^{(2)} + \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\nu}}_k^{(2)}) \cdot \boldsymbol{\nabla}\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) - \\ &- \tfrac{1}{2} \partial_0 (\gamma_k \boldsymbol{\beta}_k) \cdot (\underline{\boldsymbol{\nu}}_k^{(2)} \wedge \boldsymbol{\beta}_k)\, \gamma_k\, \delta(\boldsymbol{R}_k - \boldsymbol{R}),\end{aligned} \tag{85}$$

$$\begin{aligned}\boldsymbol{m}_k^{(2)}(\boldsymbol{R}, t) = &-\gamma_k \boldsymbol{\beta}_k \cdot \partial_0 \{(\underline{\boldsymbol{\nu}}_k^{(2)} - \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\mu}}_k^{(2)})\, \gamma_k\}\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) - \\ &- (\underline{\boldsymbol{\nu}}_k^{(2)} - \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\mu}}_k^{(2)}) \cdot \boldsymbol{\nabla}\, \delta(\boldsymbol{R}_k - \boldsymbol{R}) + \\ &+ \tfrac{1}{2} \partial_0 (\gamma_k \boldsymbol{\beta}_k) \cdot \underline{\boldsymbol{\nu}}_k^{(2)} \gamma_k\, \delta(\boldsymbol{R}_k - \boldsymbol{R}).\end{aligned} \tag{86}$$

(The reader is also reminded of the vector notation, where $\boldsymbol{\beta}_k \wedge \boldsymbol{\nu}_k^{(2)} :$ represents a contraction with $\boldsymbol{r}'_{ki}$ and an outer product with $\boldsymbol{r}'_{ki} \wedge \dot{\boldsymbol{r}}'_{ki}$).

§ 11. *The Maxwell equations.* The inhomogeneous "atomic field equations" for the total atomic fields $\boldsymbol{e} = \sum_k \boldsymbol{e}_k$ and $\boldsymbol{b} = \sum_k \boldsymbol{b}_k$, due to all atoms together, can now be obtained by summation over k of equations (18) and

(19):

$$\nabla \cdot \boldsymbol{e} = \rho - \nabla \cdot \boldsymbol{p}, \tag{87}$$

$$-\partial_0 \boldsymbol{e} + \nabla \wedge \boldsymbol{b} = \boldsymbol{j}/c + \partial_0 \boldsymbol{p} + \nabla \wedge \boldsymbol{m}. \tag{88}$$

(The homogeneous atomic field equations are (3) and (4)). Here we have introduced the quantities

$$\rho = \sum_k \rho_k, \boldsymbol{j} = \sum_k \boldsymbol{j}_k, \boldsymbol{p} = \sum_k \boldsymbol{p}_k, \boldsymbol{m} = \sum_k \boldsymbol{m}_k, \tag{89}$$

where $\rho_k, \boldsymbol{j}_k, \boldsymbol{p}_k$ and $\boldsymbol{m}_k$ are given by (13), (14), (75) and (76). The expressions for $\boldsymbol{p}$ and $\boldsymbol{m}$ are series expansions in terms of the atomic multipole moments $\boldsymbol{\mu}_k^{(n)}$ and $\boldsymbol{\nu}_k^{(n)}$. The solutions of (87) and (88) are thus series expansions in terms of the multipole moments. These series converge if the reference point of measurement is outside the atoms.

With the definitions of the fields

$$\boldsymbol{d} = \boldsymbol{e} + \boldsymbol{p}, \ \boldsymbol{h} = \boldsymbol{b} - \boldsymbol{m}, \tag{90}$$

one can cast the atomic field equations (87) and (88) into the form

$$\nabla \cdot \boldsymbol{d} = \rho, \tag{91}$$

$$-\partial_0 \boldsymbol{d} + \nabla \wedge \boldsymbol{h} = \boldsymbol{j}/c. \tag{92}$$

The macroscopic electromagnetic quantities are defined as averages over sets of atoms which occupy macroscopically small regions, but which on the microscopic scale contain enough atoms, such that principles of statistical mechanics may be applied to them [7]). Let us indicate these macroscopic quantities as

$$\langle \boldsymbol{e} \rangle = \boldsymbol{E}, \langle \boldsymbol{b} \rangle = \boldsymbol{B}, \langle \boldsymbol{d} \rangle = \boldsymbol{D}, \langle \boldsymbol{h} \rangle = \boldsymbol{H}, \tag{93}$$

$$\langle \rho \rangle = \varrho, \ \langle \boldsymbol{j} \rangle = \boldsymbol{J}. \tag{94}$$

Since averaging and time-space differentiations commute [7]), we can write (91), (92), (3) and (4) respectively as:

$$\nabla \cdot \boldsymbol{D} = \varrho, \tag{95}$$

$$-\partial_0 \boldsymbol{D} + \nabla \wedge \boldsymbol{H} = \boldsymbol{J}/c, \tag{96}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{97}$$

$$\partial_0 \boldsymbol{B} + \nabla \wedge \boldsymbol{E} = 0. \tag{98}$$

These are Maxwell's equations, where we have now expressed all quantities as averages over microscopic quantities in a completely covariant form with multipole expansion up to all orders.

§ 12. *Discussion of previous work.* Dällenbach [1]) was the first to generalize Lorentz's derivation of Maxwell's equations from electron theory in a covariant way. His treatment was limited to uniformly moving solids and yielded an expression for the electromagnetic moment tensor in the rest frame of the substance in terms of the atomic electric and magnetic dipole moments. Pauli[2]) transformed these formulae to an arbitrary frame, in which the substance was moving as a whole.

Kaufman[3]) generalized the nonrelativistic treatment of Mazur and Nijboer[4]) to a covariant theory. He used a relation like (27) but justified this in terms of proper times which is not possible. However, since he did not employ this interpretation he got correct expressions up to dipole moments for arbitrary motion of the atoms. He defined the dipole moments in the atomic rest frames, but introduced a slightly different magnetic moment. This would only lead to different results in a higher order multipole moment approximation.

Bacry[5]) gave a relativistic theory, but did not define the atomic moments in their rest frame. His final expressions do not contain all effects of moving multipoles.

De Groot and Vlieger[6]) derived the relativistic polarization tensor to second order in the atomic parameters $\boldsymbol{r}'_{ki}$ and $\dot{\boldsymbol{r}}'_{ki}$, but their treatment was not manifestly covariant. They introduced a relativistically covariant way of averaging[7]).

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