Exact Solutions of the Spinor Bethe-Salpeter Equation for Tightly Bound States.

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Summary. — Exact solutions are obtained for the spinor Bethe-Salpeter equation that describes tightly bound states of spin-$\frac{1}{2}$ fermions with massless-boson exchange. The corresponding coupling constants form a discrete spectrum that depends continuously on the parameters characterizing the type of fermion-boson interaction.

1. – Introduction.

The Bethe-Salpeter equation that describes bound states in a covariant way has been the subject of an extensive literature ([1]). As a result the properties of the scalar equation are rather well understood by now. Much less is known however about the more complex spinor equation.

The solvability of the spinor Bethe-Salpeter equation in the ladder approximation depends largely on the strength and type of the interaction between the constituents. In fact, when strong binding is considered, with binding energy equal to the sum of the masses of the constituent particles, symmetry arguments may be employed to simplify the equation. In that case exact solutions have been obtained recently for several *ad hoc* interaction potentials, *viz.* square-well potentials ([2]) and relativistic harmonic potentials ([3]). From

([1]) E. E. SALPETER and H. A. BETHE: *Phys. Rev.*, **84**, 1232 (1951); for a review, see N. NAKANISHI: *Suppl. Prog. Theor. Phys.*, **43**, 1 (1969).
([2]) R. F. KEAM: *Journ. Math. Phys.*, **10**, 594 (1969).
([3]) M. BÖHM, H. JOOS and M. KRAMMER: *Nucl. Phys.*, **51** B, 397 (1973); D. ZUM WINKEL: *Journ Math. Phys.*, **16**, 93 (1975).

the point of view of quantum field theory, however, potentials arising from boson exchange are more interesting. When these bosons have negligible mass the momentum-space Bethe-Salpeter equation for tightly bound states may be reduced to a set of second-order differential equations. For special couplings at the fermion-boson vertices some solutions of these equations have been found by GOLDSTEIN (4), BASTAI *et al.* (5), KUMMER (6) and KEAM (7). It is the purpose of the present paper to extend the results of these authors by deriving new sets of solutions.

On the basis of the general formalism of the Bethe-Salpeter equation in momentum-space the coupled differential equations will be derived for the structure functions that describe tightly bound states with total angular momentum $J=0$ and parity $\eta_P=\pm 1$ (Sect. **2** and **3**). The coupled equations for 0^+-states are then studied in Sect. **4** by considering their Euler transform, inserting power series expansions for the transformed structure functions and solving the ensuing recursion relations for the coefficients of the power series. As a result a discrete spectrum of coupling constants is obtained that corresponds to a set of symmetric normalizable solutions of the Bethe-Salpeter equation. The symmetric solutions of BASTAI *et al.*, KUMMER and KEAM are shown to be special cases of these results. In Sect. **5** finally symmetric solutions for 0^--states are investigated; they are found to be nonnormalizable (as Goldstein's solutions are). The Appendices A and B contain a discussion of the easily solvable equations that arise if bound states with suitably chosen linear combinations of fermion-boson interactions are considered. In the third Appendix the normalization integrals for the solutions obtained in the text are evaluated by making use of Meyer functions.

2. – The Bethe-Salpeter equation for bound states of fermions.

The four-point function $G_P(p, q)$, which is the momentum-space representation of the vacuum expectation value of a time-ordered product of four constituent fermion fields

$$(1)\quad \langle 0|T[\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\bar{\psi}_\gamma(y_1)\psi_\delta(y_2)]|0\rangle = \\ = (2\pi)^{-6}\int \mathrm{d}^4P\,\mathrm{d}^4p\,\mathrm{d}^4q \exp[-iP\cdot(X-Y)-ip\cdot x+iq\cdot y]\,G_{P\alpha\beta\gamma\delta}(p, q)$$

(4) J. S. GOLDSTEIN: *Phys. Rev.*, **91**, 1516 (1953).
(5) A. BASTAI, L. BERTOCCHI, G. FURLAN and M. TONIN: *Nuovo Cimento*, **30**, 1532 (1963).
(6) W. KUMMER: *Nuovo Cimento*, **31**, 219 (1964).
(7) R. F. KEAM: *Journ. Math. Phys.*, **12**, 515 (1971).

with (*) $X = \frac{1}{2}(x_1 + x_2)$, $Y = \frac{1}{2}(y_1 + y_2)$, $x = x_1 - x_2$, $y = y_1 - y_2$, satisfies the integral relation

$$(2)\qquad G_{P\alpha\beta\gamma\delta}(p, q) = G^0_{P\alpha\beta\gamma\delta}(p, q) + \\ + (2\pi)^2 i\int \mathrm{d}^4p'\, \mathrm{d}^4q'\, G^0_{P\alpha\beta\alpha'\beta'}(p, p')\, V_{\alpha'\beta'\alpha''\beta''}(p', q')\, G_{P\alpha''\beta''\gamma\delta}(q', q)\,.$$

Here V is an irreducible potential function and G^0_P is a product of two two-point functions corresponding to free propagation of the constituent fermions with mass M:

$$(3)\qquad G^0_{P\alpha\beta\gamma\delta}(p, q) = -(2\pi)^{-2}\delta^{(4)}(p - q)(\not p + \tfrac{1}{2}\not P - M + i0)^{-1}_{\alpha\gamma}(\not p - \tfrac{1}{2}\not P - M + i0)^{-1}_{\delta\beta}\,.$$

A fermion-antifermion bound state with mass M_B will lead to a pole at $P^2 = M_B^2$ in the four-point function:

$$(4)\qquad G_{P\alpha\beta\gamma\delta}(p, q) = 2\pi i(P^2 - M_B^2)^{-1}\chi_{P\alpha\beta}(p)\,\bar\chi_{P\delta\gamma}(q) + \dots$$

with the bound-state wave function

$$(5)\qquad \chi_{P\alpha\beta}(q) = (2\pi)^{-2}\int \mathrm{d}^4x \exp[iq\cdot x]\langle 0|T[\psi_\alpha(\tfrac{1}{2}x)\,\bar\psi_\beta(-\tfrac{1}{2}x)]|P, M_B\rangle$$

and its adjoint

$$(6)\qquad \bar\chi_{P\alpha\beta}(q) = (2\pi)^{-2}\int \mathrm{d}^4x \exp[-iq\cdot x]\langle P, M_B|T[\bar\psi_\beta(\tfrac{1}{2}x)\,\psi_\alpha(-\tfrac{1}{2}x)]|0\rangle\,.$$

Comparison of the residues at $P^2 = M_B^2$ on both sides of (2) yields the Bethe-Salpeter equation for the fermion-antifermion bound-state wave function

$$(7)\qquad (\not q + \tfrac{1}{2}\not P - M)_{\alpha\alpha'}\chi_{P\alpha'\beta'}(q)(\not q - \tfrac{1}{2}\not P - M)_{\beta'\beta} = -i\int \mathrm{d}^4q'\, V_{\alpha\beta\alpha'\beta'}(q, q')\,\chi_{P\alpha'\beta'}(q')\,.$$

For the adjoint $\bar\chi_P$ one finds in an analogous way

$$(8)\qquad (\not q - \tfrac{1}{2}\not P - M)_{\alpha\alpha'}\bar\chi_{P\alpha'\beta'}(q)(\not q + \tfrac{1}{2}\not P - M)_{\beta'\beta} = -i\int \mathrm{d}^4q'\,\bar\chi_{P\alpha'\beta'}(q')\, V_{\beta'\alpha'\beta\alpha}(q', q)\,.$$

The transformation properties of the bound-state wave function follow from those of the constituent fields. In particular, the Lorentz-transformation

(*) Inner products of four-vectors are defined with respect to the metric tensor $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

properties of the latter imply the invariance relation

$$-i\boldsymbol{p}\wedge\nabla_{\boldsymbol{p}}\chi_P(p)+\tfrac{1}{2}[\boldsymbol{\sigma},\chi_P(p)]=0 \tag{9}$$

(with $\boldsymbol{\sigma}=\frac{1}{2}i\boldsymbol{\gamma}\wedge\boldsymbol{\gamma}$) for the wave function of a spinless bound state in its rest frame characterized by $P^\mu=(M_B, 0)$. Likewise the parity and charge conjugation properties of the constituent fields lead to the relations

$$\gamma^0\chi_{P^0,-\boldsymbol{P}}(q^0,-\boldsymbol{q})\gamma^0=\eta_P\chi_P(q)\,, \tag{10}$$

$$C\tilde{\chi}_P(-q)\,C^{-1}\qquad=\eta_C\chi_P(q)\,, \tag{11}$$

with C a unitary antisymmetric matrix satisfying $C^{-1}\gamma^\mu C=-\tilde{\gamma}^\mu$ and $\eta_P=\pm1$, $\eta_C=\pm1$ the parity and charge parity quantum numbers of the bound state.

The bound-state wave function and its adjoint satisfy a normalization condition. In fact, by multiplying (2) by $[G_P^0(p', p)]^{-1}$ from the left and by $[G_P(q, q')]^{-1}$ from the right, and integrating over p and q one gets

$$-(2\pi)^2(\not{q}+\tfrac{1}{2}\not{P}-M)_{\alpha\gamma}(\not{q}-\tfrac{1}{2}\not{P}-M)_{\delta\beta}\,\delta^{(4)}(q-q')= \tag{12}$$
$$=[G_P^{-1}(q, q')]_{\alpha\beta\gamma\delta}+(2\pi)^2 iV_{\alpha\beta\gamma\delta}(q, q')\,.$$

Differentiation with respect to P^μ and comparison of the residues at the pole $P^2=M_B^2$, with the use of (4), yields the normalization condition [8]

$$(2\pi)^3 i\int d^4q\,\mathrm{Tr}[\bar{\chi}_P(q)(\not{q}+\tfrac{1}{2}\not{P}-M)\chi_P(q)\not{P}- \tag{13}$$
$$-\bar{\chi}_P(q)\not{P}\chi_P(q)(\not{q}-\tfrac{1}{2}\not{P}-M)]=4M_B^2\,.$$

A more convenient condition is obtained by employing an auxiliary relation found by differentiating the Bethe-Salpeter equation (7) with respect to the coupling constant λ that characterizes the strength of the potential V. When the result is multiplied by the adjoint $\bar{\chi}_P$ and (8) is used, one arrives at an alternative expression for the left-hand side of (13); as a result the normalization condition may be cast into the form [9]

$$-(2\pi)^3 i\int d^4q\,\mathrm{Tr}[\bar{\chi}_P(q)(\not{q}+\tfrac{1}{2}\not{P}-M)\chi_P(q)(\not{q}-\tfrac{1}{2}\not{P}-M)]=\lambda\,\mathrm{d}(M_B^2)/\mathrm{d}\lambda\,. \tag{14}$$

([8]) S. MANDELSTAM: *Proc. Roy. Soc.*, A **233**, 248 (1955); R. E. CUTKOSKY and M. LEON: *Phys. Rev.*, **135**, B 1445 (1964); D. LURIÉ, A. J. MACFARLANE and Y. TAKAHASHI: *Phys. Rev.*, **140**, B 1091 (1965).
([9]) N. NAKANISHI: *Phys. Rev.*, **138**, B 1182 (1965).

In the following only those bound-state functions will be considered for which the integral at the left-hand side has a finite value.

Bound states of two fermions may be described by a wave function

$$(15)\qquad \hat{\chi}_{P\alpha\beta}(q) = (2\pi)^{-2}\int d^4x \exp[iq\cdot x]\langle 0|T[\psi_\alpha(\tfrac{1}{2}x)\,\psi_{\beta'}(-\tfrac{1}{2}x)\,C^{-1}_{\beta'\beta}]|P, M_B\rangle\,,$$

for which an equation of the same form as (7) may be derived, with V replaced by $\hat{V}$, defined as

$$(16)\qquad \hat{V}_{\alpha\beta\alpha'\beta'}(p, q) = C^{-1}_{\beta\gamma}\,V_{\alpha\gamma'\alpha'\gamma}(p, q)\,C_{\gamma'\beta'}\,.$$

The adjoint wave function

$$(17)\qquad \hat{\bar{\chi}}_{P\alpha\beta}(q) = (2\pi)^{-2}\int d^4x \exp[-iq\cdot x]\langle P, M_B|T[\bar{\psi}_\beta(\tfrac{1}{2}x)\,\bar{\psi}_{\alpha'}(-\tfrac{1}{2}x)\,C_{\alpha'\alpha}]|0\rangle$$

satisfies an equation of the form (8). For a spinless composite state the wave function $\hat{\chi}_P$ fulfils the condition (9); for a state with parity $-\eta_P$ it transforms as in (10), while exchange symmetry leads to an invariance relation for $\hat{\chi}_P$ of the form (11) with $\eta_C = 1$. Normalized wave functions $\hat{\chi}_P$ and $\hat{\bar{\chi}}_P$ obey the relations (13) and (14).

In the following the Bethe-Salpeter equation will be studied in the ladder approximation; for V the potential will be inserted that corresponds to the exchange of massless particles

$$(18)\qquad V_{\alpha\beta\alpha'\beta'}(p, q) = V[(p-q)^2]\sum_i \lambda_i \Gamma^i_{\alpha\alpha'}\Gamma^i_{\beta'\beta}\,,$$

where $V[(p-q)^2]$ stands for $(2\pi)^{-2}(p-q)^{-2}$; the coupling constants λ_i measure the strength of the interactions characterized by the matrices $\Gamma^S = 1$, $\Gamma^V = \gamma^\mu$, $\Gamma^T = \sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$, $\Gamma^A = \gamma^\mu\gamma_5$, $\Gamma^P = i\gamma_5$ with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (for unitary γ^μ the matrices Γ^i satisfy $\gamma^0\Gamma^{i\dagger}\gamma^0 = \Gamma^i$ for all i). From (16) it then follows that the potential function $\hat{V}$ has the same form, with coupling constants $\hat{\lambda}_i$ equal to $\eta\lambda_i$, where $\eta = +1$ for $i = S, A, P$ and $\eta = -1$ for $i = V, T$.

3. – Equations for bound states with negligible mass.

When the coupling of the constituent fermions is strong the masses of the lowest bound states will be small compared to the masses of the constituents. Wave functions for such states satisfy an equation that is obtained from the Bethe-Salpeter equation by neglecting the mass M_B of the bound state with momentum vector $M_B\hat{P}^\mu$ (where $\hat{P}^\mu$ is a timelike unit vector). Normalizable solutions of this reduced bound-state equation will exist only for well-poised combinations of the coupling constants λ_i.

Upon expansion of the wave function (that still depends on $\hat{P}^\mu$ in general) in a complete set of Dirac matrices according to

$$\chi_{\hat{P}}(q) = \chi_S + \chi_V^\mu \gamma_\mu + \chi_T^{\mu\nu} \sigma_{\mu\nu} + \chi_A^\mu \gamma_\mu \gamma_5 + \chi_P i\gamma_5 \tag{19}$$

the Bethe-Salpeter equation (7) with potential function (18) may be written, for bound states with negligible mass, as a set of five integral equations:

$$(M^2 + q^2)\chi_S - 2Mq\cdot\chi_V = -i\Lambda_S \theta \chi_S \,, \tag{20}$$

$$2(-M\chi_S + q\cdot\chi_V) q^\mu + (M^2 - q^2)\chi_V^\mu = -i\Lambda_V \theta \chi_V^\mu \,, \tag{21}$$

$$(M^2 + q^2)\chi_T^{\mu\nu} + 2q_\lambda (q^\nu \chi_T^{\lambda\mu} - q^\mu \chi_T^{\lambda\nu}) + M\varepsilon^{\mu\nu\varkappa\lambda} q_\varkappa \chi_{A\lambda} = -i\Lambda_T \theta \chi_T^{\mu\nu} \,, \tag{22}$$

$$2M\varepsilon^{\mu\varkappa\lambda\nu} q_\varkappa \chi_{T\lambda\nu} - 2q\cdot\chi_A q^\mu + (M^2 + q^2)\chi_A^\mu = -i\Lambda_A \theta \chi_A^\mu \,, \tag{23}$$

$$(M^2 - q^2)\chi_P = -i\Lambda_P \theta \chi_P \,. \tag{24}$$

The integral operator θ occurring here is defined by

$$\theta f(\hat{P}, q) = \int d^4 q' \, V[(q - q')^2] f(\hat{P}, q') \,. \tag{25}$$

Furthermore the Λ_i stand for linear combinations of the coupling constants, given by

$$\begin{pmatrix} \Lambda_S \\ \Lambda_V \\ \Lambda_T \\ \Lambda_A \\ \Lambda_P \end{pmatrix} = \begin{pmatrix} 1 & 4 & 12 & -4 & -1 \\ 1 & -2 & 0 & -2 & 1 \\ 1 & 0 & -4 & 0 & -1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & -4 & 12 & 4 & -1 \end{pmatrix} \begin{pmatrix} \lambda_S \\ \lambda_V \\ \lambda_T \\ \lambda_A \\ \lambda_P \end{pmatrix} \tag{26}$$

for fermion-antifermion bound states; for fermion-fermion states the second and third columns have an additional minus sign. (In the following only fermion-antifermion states will be considered; the translation of the results to fermion-fermion states is trivial.) As is well known the Bethe-Salpeter equation splits (in the present case of negligible bound-state mass) into three groups of coupled equations, corresponding to the sectors (S-V), (T-A) and P.

For a fermion-antifermion bound state with vanishing total angular momentum J and positive parity η_P the relations (9) and (10) imply that the structure functions χ_A and χ_P vanish, while χ_S, χ_V and χ_T for such 0^+-states have the form

$$\chi_S = \chi_S(q^2, q\cdot\hat{P}) \,, \tag{27}$$

$$\chi_V^\mu = M^{-1} q^\mu \chi_{V1}(q^2, q\cdot\hat{P}) + \hat{P}^\mu \chi_{V2}(q^2, q\cdot\hat{P}) \,, \tag{28}$$

$$\chi_T^{\mu\nu} = M^{-1}(q^\mu \hat{P}^\nu - q^\nu \hat{P}^\mu)\chi_{T1}(q^2, q\cdot\hat{P}) \,. \tag{29}$$

For states with positive (negative) charge parity η_c the structure functions χ_S, χ_{V1}, χ_{T1} are even (odd) and χ_{V2} odd (even) in $q\cdot\hat{P}$, as follows from (11).

Fermion-antifermion 0^- bound states are characterized by vanishing χ_S and χ_V, while the other structure functions are given as

$$\chi_T^{\mu\nu} = M^{-1}\varepsilon^{\mu\nu\varkappa\lambda} q_\varkappa \hat{P}_\lambda \chi_{T2}(q^2, q\cdot\hat{P}) , \tag{30}$$

$$\chi_A^\mu = M^{-1}q^\mu \chi_{A1}(q^2, q\cdot\hat{P}) + \hat{P}^\mu \chi_{A2}(q^2, q\cdot\hat{P}) , \tag{31}$$

$$\chi_P = \chi_P(q^2, q\cdot\hat{P}) . \tag{32}$$

Positive (negative) charge parity implies even (odd) character of χ_{T2}, χ_{A2}, χ_P and odd (even) character of χ_{A1} with respect to $q\cdot\hat{P}$.

After performing a Wick ([10]) rotation in the rest frame (where $\hat{P}^\mu = (1, 0)$) and introducing a Euclidean metric, with $q^4 = -iq^0$, $\hat{P}^4 = -i\hat{P}^0 = -i$, the integral equations (20)-(24) may be transformed to differential equations by using the identity $\Box_q q^{-2} = -(2\pi)^2\delta^{(4)}(q)$. For 0^+ fermion-antifermion states one then obtains with the help of (27)-(29)

$$\Box_q f_S = \Lambda_S \chi_S , \tag{33}$$

$$\Box_q(M^{-1}q^\mu f_{V1} + \hat{P}^\mu f_{V2}) = \Lambda_V(M^{-1}q^\mu \chi_{V1} + \hat{P}^\mu \chi_{V2}) , \tag{34}$$

$$\Box_q[M^{-1}(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)f_{T1}] = \Lambda_T M^{-1}(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)\chi_{T1} , \tag{35}$$

where the symbols f_i stand for the following combinations of structure functions:

$$f_S = (M^2 - q^2)\chi_S + 2q^2\chi_{V1} + 2Mq\cdot\hat{P}\chi_{V2} , \tag{36}$$

$$f_{V1} = -2M^2\chi_S + (M^2 - q^2)\chi_{V1} - 2Mq\cdot\hat{P}\chi_{V2} , \tag{37}$$

$$f_{V2} = (M^2 + q^2)\chi_{V2} , \tag{38}$$

$$f_{T1} = (M^2 + q^2)\chi_{T1} . \tag{39}$$

Likewise the equations for 0^- fermion-antifermion states get the form

$$\Box_q[M^{-1}(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)f_{T2}] = \Lambda_T M^{-1}(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)\chi_{T2} , \tag{40}$$

$$\Box_q(M^{-1}q^\mu f_{A1} + \hat{P}^\mu f_{A2}) = \Lambda_A(M^{-1}q^\mu \chi_{A1} + \hat{P}^\mu \chi_{A2}) , \tag{41}$$

$$\Box_q f_P = \Lambda_P \chi_P , \tag{42}$$

([10]) G. C. Wick: *Phys. Rev.*, **96**, 1124 (1954).

with the abbreviations

$$f_{T2} = (M^2 - q^2)\chi_{T2} + M^2\chi_{A2}\,, \tag{43}$$

$$f_{A1} = 4Mq\cdot\hat{P}\chi_{T2} + (M^2 + q^2)\chi_{A1} + 2Mq\cdot\hat{P}\chi_{A2}\,, \tag{44}$$

$$f_{A2} = -4q^2\chi_{T2} + (M^2 - q^2)\chi_{A2}\,, \tag{45}$$

$$f_P = (M^2 + q^2)\chi_P\,. \tag{46}$$

Spectral analysis of the rest frame wave function $\chi_{\hat{P}}$ and its adjoint $\bar{\chi}_{\hat{P}}$ shows that after Wick rotation these are related as

$$\bar{\chi}_{\hat{P}}(q^0, \boldsymbol{q}) = \gamma^0\chi_{\hat{P}}(q^{0*}, \boldsymbol{q})^\dagger\gamma^0\,. \tag{47}$$

The structure functions $\bar{\chi}_i$, that determine the adjoint wave function $\bar{\chi}_{\hat{P}}$ according to expressions of the form (19) with (27)-(32), are hence connected to those for $\chi_{\hat{P}}$:

$$\bar{\chi}_i(q\cdot\hat{P}, q^2) = \chi_i^*(-q\cdot\hat{P}, q^2)\,. \tag{48}$$

By using the charge parity properties of the structure functions the Wick-rotated normalization condition (14) may then be brought into the form

$$\begin{aligned} 4(2\pi)^3\eta_C\int \mathrm{d}^4q\{&(M^2 - q^2)|\chi_S|^2 - M^{-2}q^2(M^2 - q^2)|\chi_{V1}|^2 - \\ &-[M^2 + q^2 + 2(q\cdot\hat{P})^2]|\chi_{V2}|^2 - 4M^{-2}(M^2 + q^2)[q^2 + (q\cdot\hat{P})^2]|\chi_{T1}|^2 + \\ &+ 4q^2\,\mathrm{Re}\,(\chi_S\chi_{V1}^*) - 4iMq\cdot\hat{P}\,\mathrm{Im}\,(\chi_S\chi_{V2}^*) + \\ &+ 2iM^{-1}q\cdot\hat{P}(M^2 - q^2)\,\mathrm{Im}\,(\chi_{V1}\chi_{V2}^*)\} = \lambda_i\,\mathrm{d}(M_B^2)/\mathrm{d}\lambda_i \end{aligned} \tag{49}$$

for 0^+ bound states, and

$$\begin{aligned} 4(2\pi)^3\eta_C\int \mathrm{d}^4q\{&4M^{-2}(M^2 - q^2)[q^2 + (q\cdot\hat{P})^2]|\chi_{T2}|^2 - M^{-2}q^2(M^2 + q^2)|\chi_{A1}|^2 - \\ &-[M^2 - q^2 - 2(q\cdot\hat{P})^2]|\chi_{A2}|^2 - (M^2 + q^2)|\chi_P|^2 + 8[q^2 + (q\cdot\hat{P})^2]\,\mathrm{Re}\,(\chi_{T2}\chi_{A2}^*) + \\ &+ 2iM^{-1}q\cdot\hat{P}(M^2 + q^2)\,\mathrm{Im}\,(\chi_{A1}\chi_{A2}^*)\} = \lambda_i\,\mathrm{d}(M_B^2)/\mathrm{d}\lambda_i \end{aligned} \tag{50}$$

for 0^--states. The derivatives at the right-hand sides of (49) and (50) should be taken at fixed ratios λ_j/λ_k (for $j, k = S, \dots, P$) and are hence independent of i. The integrals will be finite only if certain conditions on the asymptotic behaviour of the structure functions for small and large q are fulfilled.

The normalization conditions (49) and (50) may be written in an alternative form by using eqs. (33)-(35) and (40)-(42). In fact, the structure functions χ_i

can be eliminated in favour of the functions f_i defined in (36)-(39) and (43)-(46), with the result (cf. [11])

$$(51)\quad 4(2\pi)^3\eta_C\int d^4q\{\Lambda_S^{-1}f_S^*(\Box_q f_S)-\Lambda_V^{-1}(M^{-1}q^\mu f_{V1}+\hat{P}^\mu f_{V2})^*\cdot \Box_q(M^{-1}q^\mu f_{V1}+\hat{P}^\mu f_{V2})-2\Lambda_T^{-1}M^{-2}(q^\mu\hat{P}^\nu-q^\nu\hat{P}^\mu)^*f_{T1}^*\cdot \Box_q[(q^\mu\hat{P}^\nu-q^\nu\hat{P}^\mu)f_{T1}]\}=\lambda_i\,d(M_B^2)/d\lambda_i$$

for 0^+-states and

$$(52)\quad 4(2\pi)^3\eta_C\int d^4q\{2\Lambda_T^{-1}M^{-2}(q^\mu\hat{P}^\nu-q^\nu\hat{P}^\mu)^*f_{T2}^*\cdot \Box_q[(q^\mu\hat{P}^\nu-q^\nu\hat{P}^\mu)f_{T2}]-\Lambda_A^{-1}(M^{-1}q^\mu f_{A1}+\hat{P}^\mu f_{A2})^*\cdot \Box_q(M^{-1}q^\mu f_{A1}+\hat{P}^\mu f_{A2})-\Lambda_P^{-1}f_P^*\Box_q f_P\}=\lambda_i\,d(M_B^2)/d\lambda_i$$

for 0^--states. For suitably chosen values of the coupling constants these forms of the normalization condition may be used to determine the sign of $\lambda_i\,d(M_B^2)/d\lambda_i$ without evaluating the integrals explicitly. In fact, upon partial integration the integrands in (51) and (52) are linear combinations of positive-definite expressions with coefficients $\pm\Lambda_i^{-1}$. If the coupling constants are such that the integrands as a whole have a definite sign the sign of $\lambda_i\,d(M_B^2)/d\lambda_i$ follows immediately, as will be shown in the next Section for a particular case.

In principle, the normalization condition (13) could be used as well to determine bounds on the asymptotic behaviour of the structure functions χ_i; however, since in the limit of vanishing small bound-state mass M_B that condition will contain the derivatives of the structure functions with respect to M_B, it is ess convenient and will not be considered further.

4. – Symmetric solutions for 0^+ fermion-antifermion states.

The equations for spinless fermion-antifermion bound states with positive parity permit solutions that are symmetric in the four-dimensional space characterized by q^μ. In fact, the partial differential equations (33)-(35) reduce to ordinary differential equations, if the structure functions χ_S, χ_{V1}, χ_{T1} are independent of $q\cdot\hat{P}$ and if χ_{V2} vanishes identically; in view of (11) and (19) this means a limitation to states with positive charge parity η_C. After the in-

(11) M. CIAFALONI: *Nuovo Cimento*, **51** A, 1090 (1967).

troduction of the variable $x = -q^2/M^2$, eqs. (33)-(35) get the form

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x}\right)[(1+x)\chi_S - 2x\chi_{V1}] = -\frac{1}{4}\Lambda_S\chi_S\,, \tag{53}$$

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 3\frac{\mathrm{d}}{\mathrm{d}x}\right)[-2\chi_S + (1+x)\chi_{V1}] = -\frac{1}{4}\Lambda_V\chi_{V1}\,, \tag{54}$$

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 3\frac{\mathrm{d}}{\mathrm{d}x}\right)[(1-x)\chi_{T1}] = -\frac{1}{4}\Lambda_T\chi_{T1}\,. \tag{55}$$

The equation for the structure function χ_{T1} is not coupled to those for χ_S and χ_{V1}; it is closely connected to the Goldstein equation ([4]). Since χ_{T1} has to yield a finite contribution to the integral (49) that determines the derivative of the bound-state mass with respect to the coupling constants, it should fulfil the requirements

$$\lim_{x\to 0} x^3|\chi_{T1}|^2 = 0\,, \tag{56}$$

$$\lim_{x\to -\infty} x^4|\chi_{T1}|^2 = 0\,. \tag{57}$$

The indicial equation of (55) for $x=0$ shows that its solutions vary like x^{-2} (possibly with an extra factor $\log(-x)$) or like x^0 for small x; the condition (56) then selects the solutions that are regular in $x=0$. Equation (55) is in fact a hypergeometric equation with parameters $2\pm w$ and 3, where w stands for $(1+\frac{1}{4}\Lambda_T)^{\frac{1}{2}}$, so that the regular solution is

$$\chi_{T1} = C\,{}_2F_1(2+w, 2-w, 3; x) \tag{58}$$

with normalization constant C. After analytic continuation of this solution its asymptotic behaviour for large negative x is found ([12]) to be given by $(-x)^{w-2}$ if $w>0$ and by $(-x)^{-2}\log(-x)$ if $w=0$. Hence the second condition (57) cannot be satisfied: a term with structure function χ_{T1} does not contribute to the bound-state wave functions considered here.

The set of two coupled equations (53), (54) for the structure functions χ_S and χ_{V1} in the $(S\text{-}V)$-sector may be solved by considering their Euler transform ([13]). To that end the integral representations

$$\chi_S(x) = \int^{\mathscr{C}} \mathrm{d}t\,(t-x)^{-n}F(t)\,, \tag{59}$$

$$\chi_{V1}(x) = \int^{\mathscr{C}} \mathrm{d}t\,(t-x)^{-n}G(t)\,, \tag{60}$$

([12]) A. Erdélyi: *Higher Transcendental Functions*, I, II (New York, N.Y., 1953).
([13]) E. L. Ince: *Ordinary Differential Equations* (New York, N.Y., 1956).

with real parameter n, are to be inserted in the differential equations; the integrations are performed along a closed contour $\mathscr{C}$, which will be specified later (after eq. (73)). Upon substitution of (59) and (60) into (53) and (54) one gets differential equations for F and G:

$$t(1+t)\frac{\mathrm{d}^2F}{\mathrm{d}t^2}-(n-3)(2t+1)\frac{\mathrm{d}F}{\mathrm{d}t}+(n-2)(n-3)F- \\ -2t^2\frac{\mathrm{d}^2G}{\mathrm{d}t^2}+4(n-3)t\frac{\mathrm{d}G}{\mathrm{d}t}-2(n-2)(n-3)G=-\frac{1}{4}\Lambda_s F\,, \tag{61}$$

$$-2t\frac{\mathrm{d}^2F}{\mathrm{d}t^2}+2(n-4)\frac{\mathrm{d}F}{\mathrm{d}t}+t(1+t)\frac{\mathrm{d}^2G}{\mathrm{d}t^2}- \\ -[t(2n-7)+(n-4)]\frac{\mathrm{d}G}{\mathrm{d}t}+(n-2)(n-4)G=-\frac{1}{4}\Lambda_v G\,. \tag{62}$$

The indicial equations for $t=0$ and $t=1$ suggest the transformations

$$F(t)=t^{n-2}(1-t)^q F_0(t)\,, \tag{63}$$

$$G(t)=t^{n-3}(1-t)^q G_0(t)\,, \tag{64}$$

with a real parameter q chosen such that F_0 and G_0 are regular and not both zero in $t=1$. Inserting these expressions in (61), (62) and introducing the variable $u=1-t$, one arrives at a set of coupled equations for $F_0(u)$ and $G_0(u)$:

$$(u-1)(u-2)D_2F_0+[2(q+1)u-6q-n-1]D_1F_0+q(q+1)F_0+ \\ +2(u-1)D_2G_0+4qD_1G_0+u^{-1}qH=-\tfrac{1}{4}\Lambda_s F_0\,, \tag{65}$$

$$-2(u-1)^2D_2F_0-2[(2q+n)u-4q-n]D_1F_0-2(q+1)(q+n-2)F_0+ \\ +(u-1)(u-2)D_2G_0+[(2q+1)u-6q-n+1]D_1G_0+ \\ +(q+1)(q-1)G_0+u^{-1}q[(q+n-2)(F_0-G_0)-H]=-\tfrac{1}{4}\Lambda_v G_0\,, \tag{66}$$

where D_i stands for the differential operator $\mathrm{d}^i/\mathrm{d}u^i$; furthermore the abbreviation

$$H=4D_1(F_0-G_0)+2u^{-1}(1-u)(q-1)(F_0-G_0)-(n+q)F_0 \tag{67}$$

has been introduced. When series expansions of the form

$$F_0(u)=\sum_{p=0}^{\infty}a_p u^p\,, \tag{68}$$

$$G_0(u)=\sum_{p=0}^{\infty}b_p u^p \tag{69}$$

are substituted in (65) and (66), coupled recursion relations for a_p and b_p result:

$$(70)\quad 2(p+q+2)(p+q+1)a_{p+2}-(p+q+1)(3p+3q+n+1)a_{p+1}+ \\ +[(p+q)(p+q+1)+\tfrac{1}{4}\Lambda_S]a_p-2(p+q+2)(p+q+1)b_{p+2}+ \\ +2(p+q)(p+q+1)b_{p+1}=0\,,$$

$$(71)\quad -2(p+q+2)(p+q+1)a_{p+2}+2(p+q+1)(2p+2q+n)a_{p+1}- \\ -2(p+q+1)(p+q+n-2)a_p+2(p+q+2)(p+q+1)b_{p+2}- \\ -(p+q+1)(3p+3q+n-1)b_{p+1}+[(p+q+1)(p+q-1)+\tfrac{1}{4}\Lambda_V]b_p=0\,.$$

For special combinations of the coupling constants Λ_S and Λ_V these recursion relations will be fulfilled by sets of coefficients a_p, b_p that vanish outside a finite range of p. The structure functions χ_S and χ_{V1} then follow from (59) and (60) with (63) and (64) as

$$(72)\qquad \chi_S(x)=\sum_{p=0}^{r}a_p\int^{\mathscr{C}}\mathrm{d}t\,(t-x)^{-n}\,t^{n-2}(1-t)^{p+q}\,,$$

$$(73)\qquad \chi_{V1}(x)=\sum_{p=0}^{r}b_p\int^{\mathscr{C}}\mathrm{d}t\,(t-x)^{-n}\,t^{n-3}(1-t)^{p+q}$$

with a_r and/or b_r different from zero.

If q is not an integer a Jordan-Pocchammer–type contour ([13]) for $\mathscr{C}$ may now be adopted. In particular, an integration path will be chosen that starts in a real point t_0 with $0<t_0<1$ and consists of a loop around $t=1$ in the positive direction, followed in succession by a positive loop around 0 and x, a negative loop around 1 and a negative loop around 0 and x that ends at t_0. Giving the functions in the integrand their principal values for $t=t_0$ one finds then from Taylor expansion around $x=0$ the following expressions for χ_S and χ_{V1}:

$$(74)\qquad \chi_S(x)=\sum_{p=0}^{r}\hat{a}_p\,{}_2F_1(n,1-p-q,2;x)\,,$$

$$(75)\qquad \chi_{V1}(x)=\sum_{p=0}^{r}\hat{b}_p\,{}_2F_1(n,2-p-q,3;x)\,.$$

The coefficients $\hat{a}_p$ and $\hat{b}_p$ are given by

$$(76)\qquad \hat{a}_p=-(p+q)a_p\,,$$

$$(77)\qquad \hat{b}_p=\tfrac{1}{2}(p+q)(p+q-1)b_p\,,$$

where a common factor $2\pi i(\exp[2\pi i q]-1)$ could be suppressed, since the differential equations (53), (54) are homogeneous.

When q is an integer the above method to obtain χ_S and χ_{V1} breaks down, since the factor $\exp[2\pi i q]-1$ then vanishes. However the derivative of χ_S and χ_{V1} with respect to q will supply a nonvanishing solution in these cases ([13]); omitting common factors one arrives again at the expressions (74), (75).

The normalization condition (49) sets a bound to the asymptotic behaviour of χ_S and χ_{V1} for $x\to 0$ and $x\to-\infty$; in fact, the following requirements have to be satisfied:

$$\lim_{x\to 0} x^2[|\chi_S|^2+x|\chi_{V1}|^2-4x\,\mathrm{Re}\,(\chi_S\chi_{V1}^*)]=0\,, \tag{78}$$

$$\lim_{x\to-\infty} x^3[|\chi_S|^2+x|\chi_{V1}|^2-4\;\;\mathrm{Re}\,(\chi_S\chi_{V1}^*)]=0\,. \tag{79}$$

By choosing the contour $\mathscr{C}$ in the way described above a regular behaviour of χ_S and χ_{V1} at $x=0$ is guaranteed, so that the first condition is fulfilled. (If a different contour is employed, alternative solutions for χ_S and χ_{V1} may be found which behave, for small x, as x^{-1} and x^{-2}, respectively, and must hence be rejected.) The asymptotic behaviour of χ_S and χ_{V1}, as given by (74), (75), for large negative x is consistent with (79), if the parameters n, q and r obey the inequalities

$$n>2\,,\qquad r+q<-\tfrac{1}{2} \tag{80}$$

for $a_r\neq 0$ (if $a_r=0$ the second condition may be weakened to $r+q<0$).

In order to solve the recurrence relations (70) and (71) it is convenient to introduce combinations of a_p and b_p, *viz.*

$$c_p=a_p-b_p\,, \tag{81}$$

$$d_p=(p+q+n)a_p-2(p+q+1)c_{p+1}+(p+q-n+2)c_p\,. \tag{82}$$

Then (70) and (71) may be cast into the form

$$\begin{aligned}(p+q+1)[(p+q-n+2)(p+q-n+3)-2(n-1)(n-3)+\tfrac{1}{2}\Lambda_S]c_{p+1}-\\ -(p+q-n+2)[(p+q)(p+q+1)+\tfrac{1}{4}\Lambda_S]c_p-\\ -(p+q+n)d_{p+1}+[(p+q)(p+q+1)+\tfrac{1}{4}\Lambda_S]d_p=0\,,\end{aligned} \tag{83}$$

$$\begin{aligned}2(p+q+1)[(n-1)(n-3)+\tfrac{1}{4}\Lambda_V](c_{p+1}-c_p)+\\ +(p+q+1)(p+q+n)d_{p+1}-[(p+q+1)(p+q+2n-3)-\tfrac{1}{4}\Lambda_V]d_p=0\,.\end{aligned} \tag{84}$$

The parameter n, which has remained arbitrary, will be fixed now by writing

$$(n-1)(n-3)+\tfrac{1}{4}\Lambda_V=0\,, \tag{85}$$

so that the second recursion relation becomes independent of c_p. The coefficients d_p may be determined then iteratively. By substituting these into (83) a two-term relation for c_p is found which may be solved easily.

The simplest solution of (84), with (85) inserted, is obtained by putting d_p identically equal to zero. An obvious way to fulfil (83) is found then by taking $c_p \equiv 0$ as well. A nontrivial solution for a_p and b_p follows if $p+q+n=0$, $a_p \neq 0$, for some p; adjusting q such that $q+n=0$, $a_0 \neq 0$, one obtains $a_0 = b_0 = C_n$, $a_p = b_p = 0$ $(p \neq 0)$, with arbitrary constant C_n, so that (74) and (75) become

$$\chi_S(x) = C_n n\, {}_2F_1(n, n+1, 2; x)\,, \tag{86}$$

$$\chi_{V1}(x) = \tfrac{1}{2} C_n n(n+1)\, {}_2F_1(n, n+2, 3; x)\,. \tag{87}$$

The solution $a_0 = b_0 = C_n$ satisfies (70), with $q = -n$, if

$$n(n-1) + \tfrac{1}{4}\Lambda_S = 0\,, \tag{88}$$

while (71) is obeyed as it stands. The normalization condition restricts, according to (80), the parameter n to values larger than 2.

When a (finite) number of coefficients c_p is different from zero, the parameter q may be chosen such that $c_1 \neq 0$, $c_r \neq 0$, $c_p = 0$ for $p \leqslant 0$, $p \geqslant r+1$ (with $r \geqslant 1$). The recursion relation (83) for $p=0$ is satisfied by $d_p \equiv 0$, $c_0 = 0$, $c_1 \neq 0$ if one has

$$(q-n+2)(q-n+3) - 2(n-1)(n-3) + \tfrac{1}{2}\Lambda_S = 0 \tag{89}$$

or if $q+1=0$. Likewise, (83) for $p=r$ is fulfilled by $d_p \equiv 0$, $c_r \neq 0$, $c_{r+1} = 0$ if Λ_S obeys moreover the relation

$$(r+q)(r+q+1) + \tfrac{1}{4}\Lambda_S = 0 \tag{90}$$

or if $r+q-n+2=0$. If $q+1 \neq 0$ and $r+q-n+2 \neq 0$, the parameters n and q must hence be related according to the quadratic equation

$$(q+n)^2 - 3(q+n) + 4qr + 2r(r+1) = 0\,, \tag{91}$$

which follows by elimination of Λ_S from (89) and (90). This equation determines a set of parabolas in the (n, q)-plane (Fig. 1). The normalization integral in (49) is finite if the parameters n and q satisfy the inequalities given by (80) (for $a_r \neq 0$, as will be the case). The parabolas are thus each cut into two distinct branches: an upper branch, given by

$$q = -n - 2r + \tfrac{3}{2} \pm \sqrt{4nr + 2r^2 - 8r + \tfrac{9}{4}}\,, \tag{92}$$

with the upper sign and bounded by $n = r + 2 + (2r^2 + \frac{9}{4})^{\frac{1}{2}}$, $q = -r - \frac{1}{2}$; and a lower branch, following from (92) with the lower sign and starting from $n = 2$, $q = -2r - \frac{1}{2} - (2r^2 + \frac{9}{4})^{\frac{1}{2}}$. In other words, the allowed branches of the

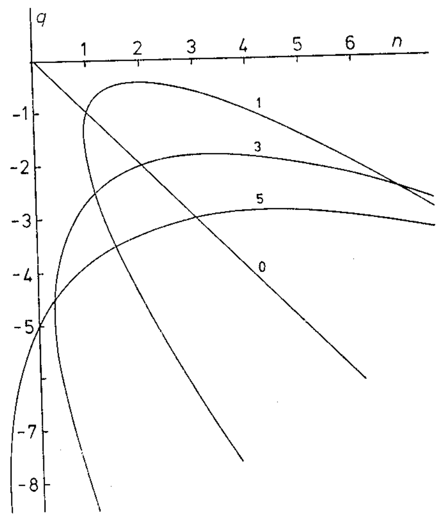

Fig. 1. – Curves for the parameters n and q in the cases $r = 0, 1, 3, 5$.

parabolas are situated in the domain defined by

$$n - q > 2r + \tfrac{5}{2} + \sqrt{2r^2 + \tfrac{9}{4}}\,. \tag{93}$$

If r tends to zero the lower branch of the parabola (92) degenerates to the straight line given by $q + n = 0$ and the domain (93) to $n - q > 4$ or $n > 2$; the relation (90) then reduces to (88).

The solution for the coefficients c_p is obtained from (83), with $d_p \equiv 0$, which may be written as

$$p(p + q + 1)(p + 2q - 2n + 5)c_{p+1} = \\ = (p - r)(p + q - n + 2)(p + r + 2q + 1)c_p\,, \tag{94}$$

if use is made of (89)-(90). Introducing the Pocchammer symbol $(a)_p = a(a+1)...$ $...(a + p - 1)$ for $p \geqslant 1$, $(a)_0 = 1$, one finds then for $0 \leqslant p \leqslant r - 1$

$$c_{p+1} = (-)^p \binom{r-1}{p} \frac{(2q + r + 2)_p (q - n + 3)_p}{(q + 2)_p (2q - 2n + 6)_p} C_{r,n,q}\,, \tag{95}$$

with arbitrary constant $C_{r,n,q}$. Only those combinations of parameters r, n and q are to be considered here for which none of the factors in numerator and denominator vanish, *i.e.* $2q+r+2$, $q-n+3$, $2q-2n+6$ and $q+2$ are taken to be different from $0, -1, \ldots, -(r-2)$ for $r \geqslant 2$. (If any of these factors vanishes for some p, with $0 \leqslant p \leqslant r-1$, the solution would reduce in fact to one with a smaller value of r.) From the definitions (81) and (82) the coefficients a_p and b_p are now found for $0 \leqslant p \leqslant r$ (with $r \geqslant 1$) as

$$(96) \qquad a_p = (-)^{p+1} \frac{(q+1)(q+n-3+p)}{(2q+r+1)r} \binom{r}{p} \frac{(2q+r+1)_p (q-n+3)_p}{(q+1)_p (2q-2n+6)_p} C_{r,n,q} ,$$

$$(97) \qquad b_p = a_p - c_p .$$

When the restrictions on (r, n, q) given above are taken into account, vanishing denominators in a_p may occur only if $2q+r+1=0$ or if $2q-2n+r+5=0$ (for $p=r$); according to (91) these relations are equivalent to $(q+n)\cdot(q+n-3)=0$ or $(q+n+r)(q+n+r-3)=0$, respectively. For $q+n-3=0$ or $q+n+r-3=0$ (with $p=r$) the numerator in (96) vanishes as well and the limiting value of the expression along the parabolas defined by (91) is understood; the cases $q+n=0$ and $q+n+r=0$ may be omitted since they lead to solutions contained already in (86), (87). For allowed values of (r, n, q) the coefficient a_r turns out to be different from zero.

By choosing instead of (89) the alternative $q+1=0$ additional solutions may be found, which are however not normalizable since (80) cannot be satisfied. The use of the alternative $r+q-n+2=0$ instead of (90) does not yield normalizable solutions either.

Up to now only solutions with $d_p \equiv 0$ have been considered. The relation (84) with (85) also admits solutions d_p with $d_p \equiv 0$ for $p \leqslant l-1$ and $d_l \neq 0$ if i) $q=-l$ or ii) $q=-n-l+1$ with $n \neq 1$. Defining $m \geqslant l$ by writing $d_m \neq 0$, $d_p \equiv 0$ for $p > m$ one finds from the recurrence relation (84) for $p=m$ the equality $\frac{1}{4}\Lambda_Y = (m+q+1)(m+q+2n-3)$; this is consistent with (85) if $n=-m-q$ or $n=-m-q+2$. Since the latter conditions imply $n \leqslant -l-q+2$, the case i) will lead to nonnormalizable solutions with $n \leqslant 2$. In case ii) one finds $m=l+1$ so that only d_l and d_{l+1} are different from zero. These are spurious solutions however: although (84) for $p=l-1$ is satisfied for $d_{l-1}=0$, $d_l \neq 0$, the original recurrence relation (71) for $p=l-1$ may be employed to derive $d_l=0$ from $d_{l-1}=0$. (Indeed (84) has been obtained from (71) by multiplying by $p+q+n$ and introducing subsequently c_p and d_p.)

To conclude, normalizable solutions of the coupled equations (53) and (54) have been found in the form of linear combinations of hypergeometric functions given by (74) and (75). These functions contain parameters n, q and integer r that satisfy the parabola eq. (91) for $r \geqslant 1$ (or its degenerate case $n+q=0$ for $r=0$) and obey the inequality (93). The coefficients in front of the hypergeometric functions are given by (76), (77) with a_p, b_p defined in (96), (97) with (95) for $r \geqslant 1$, and a_0, b_0 equal to a constant for $r=0$.

The coupling constant combinations Λ_S and Λ_V follow from the relations (85) and (90) which may be written as

$$1-\Lambda_S = 4(r+q+\tfrac{1}{2})^2\,, \tag{98}$$

$$4-\Lambda_V = 4(n-2)^2\,, \tag{99}$$

so that the inequalities $\Lambda_S \leqslant 1$ and $\Lambda_V \leqslant 4$ follow immediately. If r vanishes, Λ_S and Λ_V satisfy the quadratic equation

$$(\Lambda_S-\Lambda_V)^2 + 12(2\Lambda_S+\Lambda_V) = 0\,, \tag{100}$$

since then q equals $-n$. For $r \geqslant 1$ the values of Λ_S and Λ_V are determined by an equation of fourth degree, corresponding, for each r, to a pair of curves in the $(1-\Lambda_S,\ 4-\Lambda_V)$-plane (see Fig. 2). These curves transform into each other if $1-\Lambda_S$ and $4-\Lambda_V$ are interchanged, as follows from the symmetry of (98), (99) and (91) under an interchange of $n-2$ and $-r-q-\frac{1}{2}$.

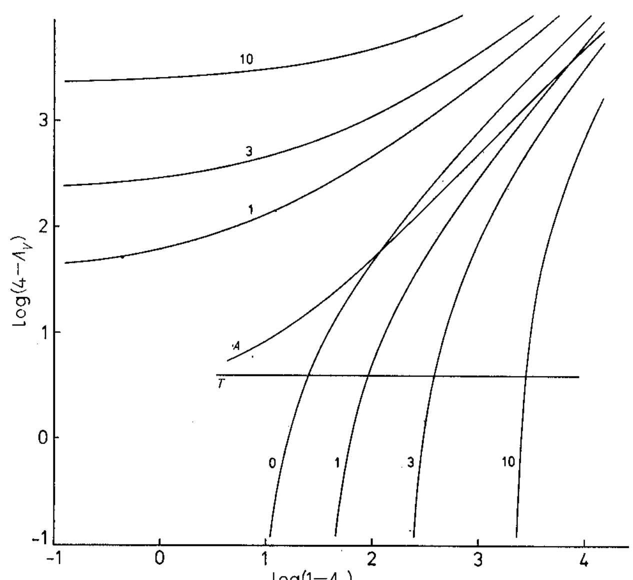

Fig. 2. – Graphs for the coupling constant combinations Λ_S and Λ_V with $r = 0, 1, 3, 10$. The curves T and A correspond to pure tensor and axial vector interactions.

For pure S, V, T, A or P interactions Λ_S and Λ_V satisfy additional constraints, *viz.* $\Lambda_S = \Lambda_V$ for S, $\Lambda_S = -2\Lambda_V$ for V, $\Lambda_V = 0$ for T, $\Lambda_S = 2\Lambda_V$ for A and $\Lambda_S = -\Lambda_V$ for P (see eq. (26)). Only for pure T and A interactions are these constraints compatible with the relations (91), (98), (99) and (80);

for these interactions a series of discrete solutions is thus obtained (see Fig. 2, where the pure T and A curves have been drawn as well).

For a pure T interaction (99) with (80) implies that n equals 3; then q and $\lambda_T = \frac{1}{12}\Lambda_S$ follow from (92), with the lower sign, and (98). In particular one finds for $r = 0$ and 1 the values $q = -3$, $\lambda_T = -2$ and $q = -6.37$, $\lambda_T = -7.83$, respectively. For a pure A interaction (98) and (99) with (91) lead to the fourth-degree equation

$$(101) \qquad n^4 - (28r + 8)n^3 - (10r^2 - 168r - 7)n^2 + \\ + (4r^3 + 40r^2 - 306r + 36)n + (r^4 - 8r^3 - 25r^2 + 164r - 36) = 0$$

for n (with $r = 0, 1, \ldots$). The solutions for $r = 0$ and 1 that are compatible with (80) are $n = 6$ and $n = 30.89$, respectively, so that one gets with (92) and (99) for q and $\lambda_A = -\frac{1}{2}\Lambda_V$ the values $q = -6$, $\lambda_A = 30$ and $q = -42.34$, $\lambda_A = 1667.55$, respectively. (The coupling constants λ_A cannot be found in general by solving the Diophantine equations proposed by NAKANISHI [14] as the latter lead to integer $n - q - 1$, denoted by N in his paper.)

Additional insight in the coupling constant spectrum is obtained by considering mixtures of S-P and V-A interactions. For S-P mixtures Λ_S and

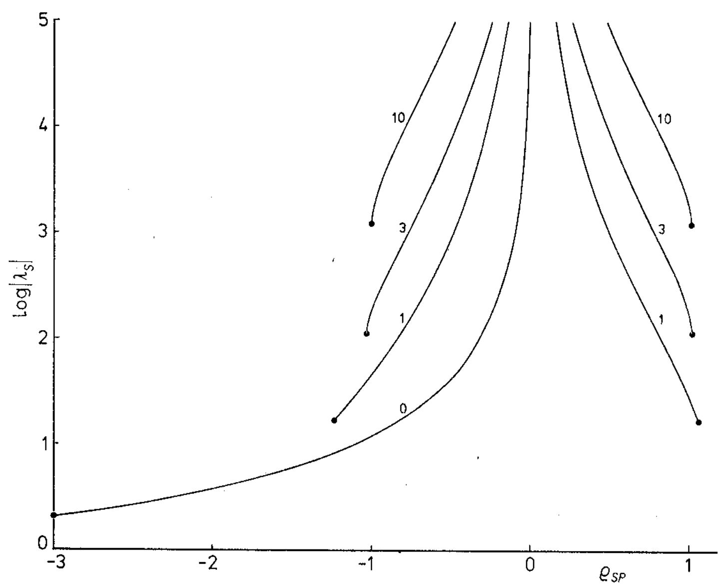

Fig. 3. – The scalar coupling constant λ_S as a function of the mixing parameter ϱ_{SP} for $(S + \varrho P)$-coupling, with $r = 0, 1, 3, 10$. (λ_S is negative for all ϱ_{SP}.)

[14] N. NAKANISHI: *Journ. Math. Phys.*, **12**, 1578 (1971).

Λ_V may be written as

(102) $$\Lambda_S = \lambda_S(1 - \varrho_{SP}), \qquad \Lambda_V = \lambda_S(1 + \varrho_{SP})$$

with $\varrho_{SP} = \lambda_P/\lambda_S$, as follows from (26). The spectrum of λ_S as a function of ϱ_{SP} is given in Fig. 3; for $r = 0$ in particular λ_S is determined by

(103) $$\lambda_S = 3(\varrho_{SP} - 3)/\varrho_{SP}^2 .$$

For V-A interactions one obtains similarly, with

(104) $$\Lambda_S = 4\lambda_V(1 - \varrho_{VA}), \qquad \Lambda_V = -2\lambda_V(1 + \varrho_{VA})$$

(where $\varrho_{VA} = \lambda_A/\lambda_V$), the spectrum presented in Fig. 4; the curve with $r = 0$ is found from

(105) $$\lambda_V = 6(5\varrho_{VA} - 3)/(\varrho_{VA} - 3)^2 .$$

An alternative form of the solutions (74), (75) is obtained by introducing the variable $z = x/(x - 1)$ instead of x, with the result

(106) $$\chi_S(z) = (1 - z)^n \sum_{p=0}^{r} \hat{a}_p\, {}_2F_1(n, p + q + 1, 2; z),$$

(107) $$\chi_{V1}(z) = (1 - z)^n \sum_{p=0}^{r} \hat{b}_p\, {}_2F_1(n, p + q + 1, 3; z).$$

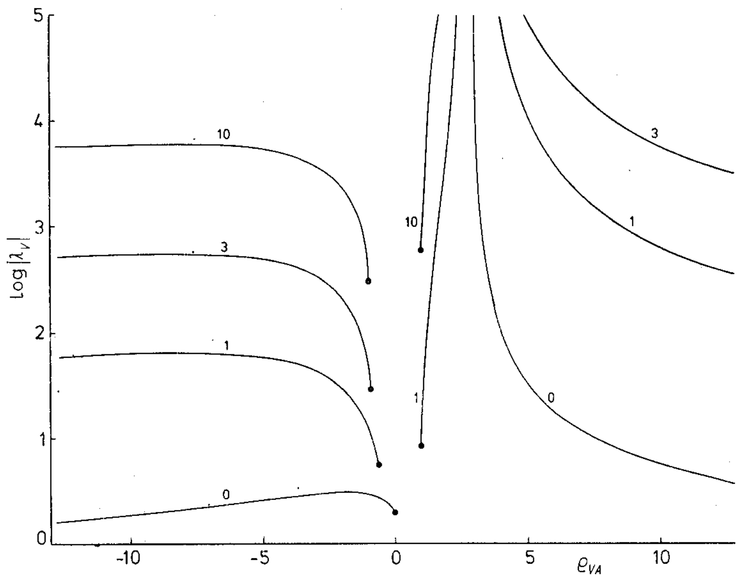

Fig. 4. – The vector coupling constant λ_V as a function of ϱ_{VA} for $(V + \varrho A)$-coupling, with $r = 0, 1, 3, 10$. (The sign of λ_V equals that of ϱ_{AA}.)

These expressions show that for negative integer q (with $q+r\leqslant -1$) the solutions simplify to polynomials in z, multiplied by a power of $1-z$. In fact the hypergeometric function may be written in that case as a Jacobi polynomial, since for negative integer a one has [12]

$$(108)\qquad {}_2F_1(a, b, c; z) = \frac{(-a)!}{(c)_{-a}} P^{(c-1, a+b-c)}_{-a}(1-2z)\,.$$

In a similar way one may show that for integer n (with $n\geqslant 3$) the solutions reduce to polynomials in z, again multiplied by a power of $1-z$. Some special cases of these polynomial-type solutions have been obtained earlier. BASTAI *et. al* [5] found a solution for pure vector coupling, with $\lambda_V=-2$, corresponding to $\Lambda_S=-8$, $\Lambda_V=4$, $n=2$, $q=-2$, $r=0$; it reads according to (106) and (107) (or (86) and (87))

$$(109)\qquad \chi_S(z) = 2C_2(1-z)^3\,,\qquad \chi_{V1}(z) = C_2(1-z)^2(3-2z)$$

(the normalization integral in (49) is logarithmically divergent in this case; indeed $n=2$ violates the conditions (80)). The solution presented by KEAM [7] and NAKANISHI [14] for pure axial vector coupling is recovered by putting $\lambda_A=30$ or $\Lambda_S=-120$, $\Lambda_V=-60$, $n=6$, $q=-6$, $r=0$; it has the form

$$(110)\qquad \chi_S(z) = 2C(1-z)^6\,{}_2F_1(6, -5, 2; z)\,,$$

$$(111)\qquad \chi_{V1}(z) = 7C(1-z)^6\,{}_2F_1(6, -5, 3; z)\,.$$

When one of the coupling constants Λ_S or Λ_V vanishes the coupled equations (53), (54) reduce to a single differential equation and an algebraic relation between χ_S and χ_{V1}. Since the former may be brought into the form of the hypergeometric equation the solutions are easily obtained; they have been discussed earlier by KUMMER [6]. In Appendix A their equivalence with the solutions obtained above (for vanishing Λ_S or Λ_V) is shown in detail.

The normalization constant $C_{r,n,q}$ which appears in the solutions is determined in principle (apart from a phase factor) by the condition (49). However, since only solutions with vanishing bound-state mass have been considered, the right-hand side of (49) is still unknown. The relation may be used however to obtain the sign of the derivative $\lambda_i\,\mathrm{d}(M_B^2)/\mathrm{d}\lambda_i$ evaluated at fixed ratio of the coupling constants. By using the alternative form (51) this sign may be found easily in the special case that Λ_S and Λ_V have opposite sign (or if one of them vanishes). In fact, if $0\leqslant\Lambda_S<1$ and $\Lambda_V<0$ (corresponding to $-r-1\leqslant q<-r-\frac{1}{2}$ with n determined by (92) with the upper sign, for $r\geqslant 1$) one obtains in this way a negative sign of the derivative $\lambda_i\,\mathrm{d}(M_B^2)/\mathrm{d}\lambda_i$. For $0\leqslant\Lambda_V<4$ and $\Lambda_S<0$ (so that $2<n\leqslant 3$, q given by (92) with the lower

sign, for $r \geqslant 0$) a positive derivative is found. If Λ_S and Λ_V are both negative the integral in (49) has to be evaluated explicitly to determine the sign of the derivative. Details of this calculation are given in Appendix C.

The solutions (74), (75) of the Bethe-Salpeter equation were obtained by starting from the Wick-rotated form of the equation, which entailed the use of imaginary q_0 (written as iq^4). Analytical continuation back to real values of q^0 is accomplished by replacing $x = -M^{-2}(\boldsymbol{q}^2 + q_4^2)$ in (74), (75) by $M^{-2}(q^{02} - \boldsymbol{q}^2 + i0)$. The solutions are then found to be singular for $q^{02} = \boldsymbol{q}^2 + M^2 - i0$.

All results in this Section apply to 0^+ fermion-antifermion bound states with wave function defined by (5). They are equally valid for 0^- fermion-fermion bound states with wave function (15), if λ_V and λ_T are reversed in sign.

5. – Symmetric solutions for 0^- fermion-antifermion states.

Fermion-antifermion bound states with negative parity and vanishing spin are described by eqs. (40)-(42). As for 0^+-states solutions of the structure functions χ_i may be considered that are symmetric in q^μ. Inspection of the equations leads to the conclusion that such solutions will be characterized by vanishing χ_{T2} and χ_{A2}, while χ_{A1} and χ_P should be functions of q^2 only. For positive charge parity states only χ_P will then be different from zero and for negative charge parity states only χ_{A1}. Introducing the variable $x = -q^2/M^2$ one gets from (41), (42) with (44)-(46) the set of uncoupled differential equations

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 3\frac{\mathrm{d}}{\mathrm{d}x}\right)[(1-x)\chi_{A1}] = -\frac{1}{4}\Lambda_A\chi_{A1}\,, \tag{112}$$

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x}\right)[(1-x)\chi_P] = -\frac{1}{4}\Lambda_P\chi_P\,. \tag{113}$$

Both these equations are of the type considered by GOLDSTEIN [4] and may be discussed in a way analogous to that of eq. (55). Since the structure functions should give finite contributions to the normalization integral in (50) the asymptotic behaviour for small and large x is restricted by the conditions

$$\lim_{x\to 0} x^{c_i}|\chi_i|^2 = 0\,, \tag{114}$$

$$\lim_{x\to -\infty} x^{c_i+1}|\chi_i|^2 = 0\,, \tag{115}$$

with $c_{A1} = 3$ and $c_P = 2$. The indicial equations for $x = 0$ imply that only solutions regular in the origin are to be considered; these have the form

$$\chi_i = C_i\,{}_2F_1[\tfrac{1}{2}(c_i+1) + w_i,\ \tfrac{1}{2}(c_i+1) - w_i,\ c_i;\ x] \tag{116}$$

with $w_i = \frac{1}{2}[(c_i - 1)^2 + \Lambda_i]^{\frac{1}{2}}$. Analytical continuation of these solutions so as to determine their asymptotic form for x going to $-\infty$ shows that χ_i behaves, for $w_i > 0$, like $(-x)^{w_i - \frac{1}{2}(c_i+1)}$, while for $w_i = 0$ an extra logarithmic factor $\log(-x)$ is present. In neither case the condition (115) is fulfilled: for 0^--states no symmetric solutions may be found. The situation here is thus completely different from that of the preceding Section where an infinite set of solutions for 0^+-states was obtained.

If the coupling constant Λ_A vanishes, eq. (41) is equivalent to a pair of algebraic relations for the structure functions χ_i. The differential equation (40) may be solved easily in that case as is shown in Appendix B; it leads to normalizable solutions for the structure functions that are asymmetric in the four-dimensional q^μ-space.

6. – Conclusion.

The spinor Bethe-Salpeter equation for tightly bound composite states of a pair of spin-$\frac{1}{2}$ fermions that exchange massless bosons has been shown to have a series of solutions symmetric in relative momentum space. For spinless bound states with positive parity the normalizable solutions, which are linear combinations of hypergeometric functions, lead to a discrete spectrum of coupling constants; these depend continuously on the parameters that determine the type of interaction at the vertices, as is shown for some cases in Fig. 3 and 4. For spinless bound states with negative parity on the contrary no normalizable symmetric solutions can be obtained. For special combinations of interactions nonsymmetric solutions may be found as well, as will be shown in the Appendices A and B.

Appendix A

Solutions for 0^+ fermion-antifermion states with $\Lambda_S = 0$ or $\Lambda_V = 0$.

The differential system (33)-(34) contains a homogeneous equation which is easily solved, if one of the coupling constants Λ_S or Λ_V vanishes. These cases will be treated in detail in this Appendix.

When Λ_V is zero it follows from (34) with (37), (38) that χ_{V2} vanishes while χ_S and χ_{V1} are related according to

$$\chi_{V1} = \frac{2M^2}{M^2 - q^2}\chi_S \tag{A.1}$$

(only the null solution of the homogeneous equation is to be considered, as is obvious from its integral-equation counterpart (21)). Substitution of (A.1) in (33) gives

$$\square_q\left[\frac{(M^2+q^2)^2}{M^2-q^2}\,\chi_s\right]=\Lambda_s\chi_s\,. \tag{A.2}$$

An equation of this form has been studied by KUMMER ([6]). The general solution for states without spin follows by writing χ_s in the form

$$\chi_s(x,\psi)=(1+x)(1-x)^{-m}(-x)^{\frac{1}{2}l}f(x)\,C_l^1(\cos\psi) \tag{A.3}$$

with $x=-q^2/M^2$ and C_l^1 a Gegenbauer polynomial ($l=0,1,\dots$) depending on $\cos\psi=iq\cdot\hat{P}/(q^2)^{\frac{1}{2}}$. Introducing the variable $z=x/(x-1)=q^2/(q^2+M^2)$ one finds from (A.2)

$$z(1-z)\frac{\mathrm{d}^2f}{\mathrm{d}z^2}+[l+2-2(m-1)z]\frac{\mathrm{d}f}{\mathrm{d}z}+ \\ +(m-2)\frac{(m-1)z-(l+2)}{1-z}f=\frac{1}{4}\Lambda_s\frac{1-2z}{1-z}f\,. \tag{A.4}$$

This equation is a hypergeometric equation if m is chosen such that

$$(m-2)(m-l-3)+\tfrac{1}{4}\Lambda_s=0\,. \tag{A.5}$$

The normalization requirements (78), (79), with (A.1) inserted, yield

$$\lim_{z\to 0} z^{\frac{1}{2}l+1}f(z)=0\,, \tag{A.6}$$

$$\lim_{z\to 1}(1-z)^{m-\frac{1}{2}l-\frac{5}{2}}f(z)=0\,, \tag{A.7}$$

from which it may be shown that only degenerate hypergeometric functions, with $a=-r$ ($r=0,1,2,\dots$), $b=2m+r-3$, $c=l+2$ and m given as

$$m=l+r+\tfrac{7}{2}+\sqrt{2r^2+2rl+l^2+4r+3l+\tfrac{9}{4}} \tag{A.8}$$

are acceptable as solutions of (A.4); in this way one obtains for the structure functions

$$\chi_s(z,\psi)=C_{r,l}(1-2z)z^{\frac{1}{2}l}(1-z)^{m-\frac{1}{2}l-1}{}_2F_1(-r,2m+r-3,l+2;z)\,C_l^1(\cos\psi)\,, \tag{A.9}$$

$$\chi_{V1}(z,\psi)=\frac{2(1-z)}{1-2z}\,\chi_s(z,\psi)\,. \tag{A.10}$$

For vanishing l these solutions are symmetric in the four-dimensional space of q^μ. These symmetric solutions may be shown to be equivalent to those found in Sect. 4 for $\Lambda_V=0$. In view of (85) with (80) the latter are obtained from (106), (107) by putting $n=3$, $r=0, 1, 2, \ldots$ and deriving q from (92) with the lower sign; they may be written as

$$\text{(A.11)} \quad \chi_S(z) = -\sum_{p=0}^{r} \tfrac{1}{2} a_p (p+q)[(1+p+q)-(p+q-1)(1-z)](1-z)^{1-p-q},$$

$$\text{(A.12)} \quad \chi_{V1}(z) = \sum_{p=0}^{r} \tfrac{1}{2} b_p (p+q)(p+q-1)(1-z)^{2-p-q}.$$

For $r=0$ the coefficients a_0 and b_0 are equal to a constant C_3, while for $r\geqslant 1$ these coefficients are given by the expressions (96), (97), *viz.*

$$\text{(A.13)} \quad a_p = -\frac{q(q+1)}{(2q+r+1)r}\,\frac{(-r)_p(2q+r+1)_p}{(2q)_p\,p!}\,C_{r,3,q},$$

$$\text{(A.14)} \quad b_p = -\frac{q^2(q-1)(q+1)}{(2q+r+1)r}\,\frac{(-r)_p(2q+r+1)_p}{(p+q)(p+q-1)(2q)_p\,p!}\,C_{r,3,q}.$$

For $r=0$, $q=-3$ the expression (A.11) gives immediately

$$\text{(A.15)} \quad \chi_S(z) = 3C_3(1-2z)(1-z)^4,$$

while division of (A.12) by (A.11) leads to the relation (A.10). This relation is found for $r\geqslant 1$ as well, if the expression (92) for q in terms of r is used; then (A.11), with (A.13) inserted, may be cast into the form

$$\text{(A.16)} \quad \chi_S(z) = (-)^{r+1} C_{r,3,q}\,\frac{q^2(q+1)(q-1)(r+1)!}{4r(2q)_r(2q+r+1)}\cdot$$
$$\cdot(1-2z)(1-z)^{-r-q+1}\,{}_2F_1(-r, -2q-r+1, 2; z).$$

The solutions (A.15) and (A.16) are indeed proportional to those given in (A.9) for $l=0$, as follows by identifying m with $-r-q+2$; then (90) coincides with (A.5).

For vanishing Λ_S eq. (33) simplifies to an algebraic equation for χ_S, χ_{V1} and χ_{V2}. Solutions symmetric in four-dimensional q^μ-space will follow now by putting χ_{V2} equal to zero and substituting in (34) the expression

$$\text{(A.17)} \quad \chi_S = -\frac{2q^2}{M^2-q^2}\,\chi_{V1}.$$

The result is

$$\text{(A.18)} \quad \Box_q\left[q^\mu\,\frac{(M^2+q^2)^2}{M^2-q^2}\,\chi_{V1}\right] = \Lambda_V q^\mu \chi_{V1},$$

where χ_{V1} depends on q^2 only. This equation is closely connected to (A.2) and may be solved in a similar way. In fact, writing χ_{V1} as $(1+x)(1-x)^{-m}f$ and introducing $z=x/(x-1)$, one obtains an equation of the form (A.4) with $l=1$. Since the normalization requirements (78), (79), with (A.17) inserted, lead to relations like (A.6), (A.7) (with $l=1$), the normalizable symmetric solutions for χ_s and χ_{V1} have the form (cf. (A.9))

$$\chi_s(z) = -\frac{2z}{1-2z}\chi_{V1}(z)\,, \tag{A.19}$$

$$\chi_{V1}(z) = C_r(1-2z)(1-z)^{m-1}{}_2F_1(-r, 2m+r-3, 3; z) \tag{A.20}$$

with $r = 0, 1, 2, \ldots$ and m following from (A.8) for $l = 1$. The coupling constant Λ_V is given by (cf. (A.5))

$$(m-2)(m-4) + \tfrac{1}{4}\Lambda_V = 0\,. \tag{A.21}$$

The solutions given in (106), (107) for $\Lambda_S = 0$ (and hence $q = -r-1$, $n = r + \frac{5}{2} + (2r^2 + 2r + \frac{9}{4})^{\frac{1}{2}}$, with $r = 1, 2, \ldots$) are proportional to those given here. In fact, since the coefficients a_p and b_p that follow from (96), (97) are

$$a_p = \frac{n-r+p-4}{p-r-1}\,\frac{(-r)_p(-r-n+2)_p}{(-2r-2n+4)_p p!}\,C_{r,n,-r-1}\,, \tag{A.22}$$

$$b_p = \frac{2(n-3)(p-r)}{(-r-n+1)(p-r-1)}\,\frac{(-r)_p(-r-n+1)_p}{(-2r-2n+4)_p p!}\,C_{r,n,-r-1}\,, \tag{A.23}$$

the expressions (106), (107) for χ_s and χ_{V1} may be written as

$$\chi_s(z) = -(1-z)^n\sum_{s=0}^{r}\left[\sum_{p=0}^{r-s}\frac{(p-r-1)(-r+s)_p}{(-r)_p}\,a_p\right]\frac{(-r)_s(n)_s}{(2)_s s!}\,z^s\,, \tag{A.24}$$

$$\chi_{V1}(z) = (1-z)^n\sum_{s=0}^{r}\left[\sum_{p=0}^{r-s}\frac{(p-r-1)(p-r-2)(-r+s)_p}{2(-r)_p}\,b_p\right]\frac{(-r)_s(n)_s}{(3)_s s!}\,z^s\,, \tag{A.25}$$

where the identity $(-r)_p(-r+p)_s = (-r)_s(-r+s)_p$ has been used. The sums over p may be performed by using Vandermonde's [15] theorem in the form

$$\sum_{p=0}^{r-s}\frac{(-r+s)_p(t)_p}{(u)_p p!} = \frac{(u-t)_{r-s}}{(u)_{r-s}} \tag{A.26}$$

with s and $r-s$ nonnegative integers. As a result the functions χ_s and χ_{V1} given in (A.24), (A.25) are found to satisfy the relation (A.19), while χ_{V1} may be cast into the form

$$\chi_{V1}(z) = -C_{r,n,-r-1}\frac{r(n-3)(n)_{r-1}}{2(2n+r-2)_{r-1}}\cdot (1-2z)(1-z)^n{}_2F_1(2n+r-2, -r+1, 3; z)\,. \tag{A.27}$$

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[15] A. M. Mathai and R. K. Saxena: *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences* (Berlin, 1973).

When r and n in this formula are replaced by $r+1$ and $m-1$, respectively, the expression (A.20) is recovered apart from a proportionality constant; the eigenvalue equation (85) coincides in that case with (A.21).

Appendix B

Solutions for 0^--states with $\Lambda_A = 0$ or $\Lambda_T = 0$.

For vanishing Λ_A eq. (41) implies that f_{A1} and f_{A2} are zero. From (44) and (45) it then follows that χ_{A1}, χ_{A2} and χ_{T2} are related as

$$\chi_{A1} = -\frac{4Mq\cdot\hat{P}}{M^2-q^2}\chi_{T2}\,, \tag{B.1}$$

$$\chi_{A2} = \frac{4q^2}{M^2-q^2}\chi_{T2}\,. \tag{B.2}$$

Substitution of these expressions in (40) with (43) gives

$$\Box_q\left[(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)\frac{(M^2+q^2)^2}{M^2-q^2}\chi_{T2}\right] = \Lambda_T(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)\chi_{T2}\,. \tag{B.3}$$

The form of this equation is similar to that of (A.2); it may be solved by writing χ_{T2} as

$$\chi_{T2}(x,\psi) = (1+x)(1-x)^{-m}(-x)^{\frac{1}{2}(l-1)}f(x)\,C^2_{l-1}(\cos\psi) \tag{B.4}$$

with $x=-q^2/M^2$, $\cos\psi = iq\cdot\hat{P}/(q^2)^{\frac{1}{2}}$ and $l\geqslant 1$. After introduction of the variable $z=x/(x-1)$ an equation like (A.4) is obtained (with Λ_S replaced by Λ_T). Since the normalization requirement (50), with (B.1), (B.2) inserted, yields once more conditions of the form (A.6), (A.7), the solutions are (cf. (A.9), (A.10))

$$\chi_{T2}(z,\psi) = \tag{B.5}$$
$$= C'_{r,l}(1-2z)\,z^{\frac{1}{2}(l-1)}(1-z)^{m-\frac{1}{2}l-\frac{1}{2}}\,{}_2F_1(-r, 2m+r-3, l+2; z)\,C^2_{l-1}(\cos\psi)\,,$$

$$\chi_{A1}(z,\psi) = 4i\frac{\sqrt{z(1-z)}}{1-2z}\cos\psi\,\chi_{T2}(z,\psi)\,, \tag{B.6}$$

$$\chi_{A2}(z,\psi) = \frac{4z}{1-2z}\chi_{T2}(z,\psi)\,, \tag{B.7}$$

where r is a nonnegative integer and m is given by (A.8); the eigenvalue Λ_T is obtained from a relation of the form (A.5).

When Λ_T is zero, it follows from (40) that f_{T2} given by (43) vanishes, so that one has

$$\chi_{T2} = -\frac{M^2}{M^2 - q^2}\chi_{A2}\,. \tag{B.8}$$

Equation (41) will lead in general to a set of coupled equations for χ_{A1} and χ_{A2}. A single equation results if symmetric structure functions are considered; these have been studied already in Sect. **5**.

Appendix C

Evaluation of normalization integrals.

The expression (49) for the derivative $\lambda_i\, \mathrm{d}(M_B^2)/\mathrm{d}\lambda_i$ of the square of the bound-state mass with respect to the coupling constants may be evaluated for the symmetric solutions given in (74), (75) by writing first the hypergeometric function as a Meyer function ([15]):

$$ {}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(-x)^{-m} G_{22}^{12}\left(-x\left|\begin{matrix}1+m-a,\ 1+m-b\\ m;\ 1+m-c\end{matrix}\right.\right), \tag{C.1}$$

and using then Meyer's integral formula

$$\int_0^\infty \mathrm{d}y\, G_{22}^{12}\left(y\left|\begin{matrix}a_1,\ a_2\\ b_1;\ b_2\end{matrix}\right.\right) G_{22}^{12}\left(y\left|\begin{matrix}c_1,\ c_2\\ d_1;\ d_2\end{matrix}\right.\right) = G_{44}^{33}\left(1\left|\begin{matrix}c_1, & c_2, & -b_1; & -b_2\\ d_1, & -a_1, & -a_2; & d_2\end{matrix}\right.\right), \tag{C.2}$$

valid if $b_1 + d_1 > -1$, $\max(a_1, a_2) + \max(c_1, c_2) < 1$. As a result one obtains for the integral over a product of hypergeometric functions

$$\int_{-\infty}^0 \mathrm{d}x\,(-x)^{m_1+m_2-2}\,{}_2F_1(a_1, b_1, c_1; x)\,{}_2F_1(a_2, b_2, c_2; x) = \tag{C.3}$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)\Gamma(b_2)} G_{44}^{33}\left(1\left|\begin{matrix}m_1-a_1, & m_1-b_1, & 1-m_2; & c_2-m_2\\ a_2-m_2, & b_2-m_2, & m_1-1; & -c_1+m_1\end{matrix}\right.\right)$$

for $m_1 + m_2 > 1$, $\min(a_1, b_1) + \min(a_2, b_2) > m_1 + m_2 - 1$. In view of (49) and

(74), (75) integrals of the following type have to be evaluated:

$$\text{(C.4)}\qquad I(a_1, b_1; a_2, b_2; c) = \int\limits_{-\infty}^{0} \mathrm{d}x\, (-x)^{c-1}(1+x)\, {}_2F_1(a_1, b_1, c; x)\, {}_2F_1(a_2, b_2, c; x)\,,$$

$$\text{(C.5)}\qquad J(a_1, b_1; a_2, b_2; c) = \int\limits_{-\infty}^{0} \mathrm{d}x\, (-x)^{c}\, {}_2F_1(a_1, b_1, c; x)\, {}_2F_1(a_2, b_2, c+1; x)\,,$$

where $x = -q^2/M^2$.

The integral in (C.4) may be written in the form

$$\text{(C.6)}\qquad \frac{\Gamma(c)^2}{\Gamma(a_1)\,\Gamma(b_1)\,\Gamma(a_2)\,\Gamma(b_2)} \sum_{p=0}^{1} (-)^p\, G^{33}_{44}\left(1 \left| \begin{matrix} c+p-a_1,\ c+p-b_1,\ 0;\ c-1 \\ a_2-1,\ b_2-1,\ c+p-1;\ p \end{matrix} \right.\right),$$

as follows by putting $m_1 = c+p$ and $m_2 = 1$ in (C.3). For $p = 0, 1$ one may use now the relations [15]

$$\text{(C.7)}\qquad G^{33}_{44}\left(1 \left| \begin{matrix} a_1,\ b_1,\ c;\ d \\ a_2,\ b_2,\ d;\ c \end{matrix} \right.\right) = G^{22}_{22}\left(1 \left| \begin{matrix} a_1,\ b_1 \\ a_2,\ b_2 \end{matrix} \right.\right)$$

and

$$\text{(C.8)}\qquad G^{33}_{44}\left(1 \left| \begin{matrix} a_1,\ b_1,\ c;\ d \\ a_2,\ b_2,\ d+1;\ c+1 \end{matrix} \right.\right) =$$

$$= G^{22}_{22}\left(1 \left| \begin{matrix} a_1-1,\ b_1 \\ a_2+1,\ b_2 \end{matrix} \right.\right) + (c-a_2)\, G^{22}_{22}\left(1 \left| \begin{matrix} a_1-1,\ b_1 \\ a_2,\ b_2 \end{matrix} \right.\right) -$$

$$- (d+1-a_1)\, G^{22}_{22}\left(1 \left| \begin{matrix} a_1,\ b_1 \\ a_2+1,\ b_2 \end{matrix} \right.\right) - (c-a_2)(d+1-a_1)\, G^{22}_{22}\left(1 \left| \begin{matrix} a_1,\ b_1 \\ a_2,\ b_2 \end{matrix} \right.\right),$$

respectively. The Meyer G^{22}_{22}-functions for unit argument occurring here are known explicitly:

$$\text{(C.9)}\qquad G^{22}_{22}\left(1 \left| \begin{matrix} 1-a_1,\ 1-b_1 \\ a_2,\ b_2 \end{matrix} \right.\right) = \frac{\Gamma(a_1+a_2)\,\Gamma(a_1+b_2)\,\Gamma(b_1+a_2)\,\Gamma(b_1+b_2)}{\Gamma(a_1+a_2+b_1+b_2)}\,.$$

Upon substitution of these expressions in (C.6) the integral (C.4), with a_1 and

a_2 equal to a common value a, is found to be

(C.10) $$I(a, b_1; a, b_2; c) =$$
$$= \frac{\Gamma(c)^2\Gamma(2a-c-1)\Gamma(b_1+b_2-c-1)\Gamma(a+b_1-c-1)\Gamma(a+b_2-c-1)}{4\Gamma(a)^2\Gamma(b_1)\Gamma(b_2)\Gamma(2a+b_1+b_2-2c)} \cdot$$
$$\cdot\{-(s_a+s_b-2c-2)[s_a s_b + c(c-3)(s_a+s_b)-2c^3+4c^2+2c]+$$
$$+(s_a-c-1)(s_b-c-1)(s_a+s_b-2c-2)^2+\Delta^2(s_a-s_b+1)(s_a-c-1)\},$$

where the abbreviations $s_a=2a$, $s_b=b_1+b_2$ and $\Delta=b_1-b_2$ have been introduced.

The integral in (C.5) may be evaluated by using (C.3) and the identity

(C.11) $$G^{33}_{44}\left(1\,\middle|\,\begin{matrix}a_1,\ b_1,\ c;\ d\\ a_2,\ b_2,\ d;\ c+1\end{matrix}\right) = -G^{22}_{22}\left(1\,\middle|\,\begin{matrix}a_1,\ b_1\\ a_2+1,\ b_2\end{matrix}\right) + (a_2-c)\,G^{22}_{22}\left(1\,\middle|\,\begin{matrix}a_1,\ b_1\\ a_2,\ b_2\end{matrix}\right).$$

With the help of (C.9) the following expression for the J-integral (with $a_1=a_2=a$) is found:

(C.12) $$J(a, b_1; a, b_2; c) =$$
$$= \frac{\Gamma(c)\Gamma(c+1)\Gamma(2a-c-1)\Gamma(b_1+b_2-c-1)\Gamma(a+b_1-c-1)\Gamma(a+b_2-c-1)}{2\Gamma(a)^2\Gamma(b_1)\Gamma(b_2)\Gamma(2a+b_1+b_2-2c-1)} \cdot$$
$$\cdot[(s_a+s_b-2c-2)(c-1)-\Delta(s_a-c-1)]$$

with the same abbreviations as in (C.10).

The derivative $\lambda_i\,d(M_B^2)/d\lambda_i$ for the solutions (74), (75) may be expressed now as a double sum over the I and J integrals evaluated above:

(C.13) $$\lambda_i\,d(M_B^2)/d\lambda_i = (2\pi)^5 M^6 \sum_{p_1,p_2=0}^{r} [\hat{a}_{p_1}\hat{a}_{p_2} I(n, 1-p_1-q; n, 1-p_2-q; 2) -$$
$$-\hat{b}_{p_1}\hat{b}_{p_2} I(n, 2-p_1-q; n, 2-p_2-q; 3) +$$
$$+4\hat{a}_{p_1}\hat{b}_{p_2} J(n, 1-p_1-q; n, 2-p_2-q; 2)].$$

For $r=0$ (so that $q=-n$, $p_1=p_2=0$) one finds in particular

(C.14) $$I(n, n+c-1; n, n+c-1; c) =$$
$$= \frac{4\Gamma(c)^2\Gamma(2n-c-1)\Gamma(2n+c-3)\Gamma(2n-2)^2}{\Gamma(n-1)^2\Gamma(n+c-1)^2\Gamma(4n-2)}(4n^2-qn-2c^2+4c+3)$$

with $c = 2, 3$, and

$$J(n, n+1; n, n+2; 2) = \frac{6\Gamma(2n-3)\Gamma(2n-2)\Gamma(2n-1)\Gamma(2n)}{\Gamma(n-1)\Gamma(n)\Gamma(n+1)\Gamma(n+2)\Gamma(4n-2)}, \tag{C.15}$$

so that (C.13) becomes then

$$\lambda_i \, \mathrm{d}(M_B^2)/\mathrm{d}\lambda_i = (2\pi)^5 M^6 |C_n|^2 \frac{3(n-1)\Gamma(2n-1)^3\Gamma(2n-4)}{\Gamma(4n-3)\Gamma(n)^4}, \tag{C.16}$$

since $\hat{a}_0 = nC_n$, $\hat{b}_0 = \frac{1}{2}n(n+1)\,C_n$ in this case. For normalizable solutions, with $n > 2$, the sign of the right-hand side is positive; for $2 < n \leqslant 3$ this could be proved without evaluating the integrals, as shown in Sect. **4**.

For the special solutions derived in Appendices A and B the normalization integrals may be evaluated independently ([16]) by expressing, with the help of (108), the hypergeometric functions in terms of Jacobi polynomials. Writing then $\mathrm{d}^4 q = 2\pi M^4 z(1-z)^{-3} \sin^2\psi \, \mathrm{d}\psi \, \mathrm{d}z$ and employing the recursion and orthogonality relations of the Gegenbauer and Jacobi polynomials, one finds for the solutions (A.9), (A.10) with $\Lambda_V = 0$

$$\lambda_i \frac{\mathrm{d}(M_B^2)}{\mathrm{d}\lambda_i} = (-)^l |C_{r,l}|^2 I_{r,l,m} \tag{C.17}$$

with the abbreviation

$$I_{r,l,m} = (2\pi)^5 M^6 \binom{r+l+1}{r}^{-2} \cdot \frac{(r+l+1)!\,\Gamma(2m-l+r-4)(2m-2l-2r-7)}{r!\,\Gamma(2m+r-3)(2m+2r-3)(2m-l-5)}. \tag{C.18}$$

Since $I_{r,l,m}$ is positive, as may be shown by using (A.8), the sign of the derivative $\lambda_i \, \mathrm{d}(M_B^2)/\mathrm{d}\lambda_i$ is determined by that of $(-)^l$. For the $\Lambda_S = 0$ solutions given in (A.19), (A.20) one obtains analogously

$$\lambda_i \frac{\mathrm{d}(M_B^2)}{\mathrm{d}\lambda_i} = -\,|C_r|^2 I_{r,1,m}, \tag{C.19}$$

which is negative for all r. Finally for the $\Lambda_A = 0$ solutions (B.5)-(B.7) the normalization integral is found to be

$$\lambda_i \frac{\mathrm{d}(M_B^2)}{\mathrm{d}\lambda_i} = (-)^{l+1} |C'_{r,l}|^2 l(l+2)\, I_{r,l,m} \tag{C.20}$$

with the sign determined by that of $(-)^{l+1}$.

([16]) K. SETO: *Prog. Theor. Phys.*, **42**, 1394 (1969).

● RIASSUNTO (*)

Si ottengono soluzioni esatte per l'equazione spinoriale di Bethe-Salpeter che descrive stati strettamente legati di fermioni di spin $\frac{1}{2}$ con scambio di bosoni senza massa. Le costanti di accoppiamento corrispondenti formano uno spettro discreto che dipende in maniera continua dai parametri che caratterizzano il tipo d'interazione fermione-bosone.

(*) *Traduzione a cura della Redazione.*

Точные решения уравнения Бете-Салпетера для сильно связанных состояний.

Резюме (*). — Получаются точные решения для спинорного уравнения Бете-Салпетера, которое описывает сильно связанные состояния фермионов со спином $\frac{1}{2}$ с обменом безмассовым бозоном. Соответствующие константы связи образуют дискретный спектр, который непрерывно зависит от параметров, характеризующих тип фермион-бозонного взаимодействия.

(*) ***Переведено редакцией.***