Normalization and Perturbation Theory for Tightly Bound States of the Spinor Bethe-Salpeter Equation.

L. G. SUTTORP

Institute of Theoretical Physics, University of Amsterdam - Amsterdam

(ricevuto il 19 Febbraio 1976)

Summary. — The normalization integrals for the tightly-bound-state solutions of the spinor Bethe-Salpeter equation that have been derived recently are evaluated. Ghost states are found to appear when the continuous parameters characterizing the type of fermion-boson interaction reach a critical value. Perturbation-theoretical methods are used to determine, for a pair of fermions with slightly different masses, the strength of the couplings that give rise to bound states with small values of the rest frame energy.

1. – Introduction.

The properties of bound states of particles with massless-boson exchange may be found by studying the Bethe-Salpeter equation. For constituent particles with spin $\frac{1}{2}$ this equation may be written, in the ladder approximation, as a set of coupled partial differential equations; due to the rather complex structure of the latter no general solutions for the bound-state wave function χ and the coupling constant λ have been obtained as yet. In the special case of strong binding, however, with binding energies equal to the sum of the constituent masses, a series of exact solutions has been derived recently (¹); the corresponding coupling constants were found to depend continuously on the parameters that characterize the type of fermion-boson interaction.

Solutions of the Bethe-Salpeter equation represent bound-state wave functions only if they satisfy a normalization condition. As is well known (²), the

(¹) L. G. SUTTORP: *Nuovo Cimento*, **29** A, 225 (1975).
(²) N. NAKANISHI: *Suppl. Prog. Theor. Phys.*, **43**, 1 (1969).

integral occurring in that condition is not necessarily positive; if it is negative, the corresponding solution cannot be normalized by adjusting a multiplicative constant and is therefore said to describe a « ghost » state. One of the purposes of the present paper is to determine the character of the tightly-bound-state solutions by explicit evaluation of the normalization integrals.

The solutions with strong binding are to be considered as the limits $\chi^{(0)}$ and $\lambda^{(0)}$ approached by the wave function χ and the coupling constant λ if the rest frame energy of the bound state $(2\varepsilon_B)$ tends to zero. The properties of physical bound states, for which ε_B is necessarily different from zero, may be obtained with the use of perturbation theory (3). Both χ and λ are then assumed to be analytical functions of ε_B and the Bethe-Salpeter equation is solved in successive orders of ε_B, with $\chi^{(0)}$ and $\lambda^{(0)}$ as the zeroth-order approximations. The dependence of χ and λ on the difference (2Δ) of the constituent masses may be determined in a similar way. In the following these perturbation-theoretical methods will be applied to the strong-binding solutions found previously.

In sect. **2** the coupled partial differential equations that follow from the Bethe-Salpeter equation for a fermion-antifermion pair with massless-boson exchange will be established. For suitably chosen combinations of the fermion-boson interactions these coupled sets of equations reduce to single partial differential equations that will be derived in sect. **3**. The general form of the normalization condition for the bound-state wave function is given in sect. **4**, while the perturbation integrals that represent the first-order corrections to the coupling constant λ for small values of Δ and ε_B are obtained in sect. **5**. The explicit evaluation of both the normalization and the perturbation integrals for the strong-binding solutions forms the subject of sect. **6** and **7**. In sect. **8**, finally, it is shown for a special case how higher-order perturbation theory may be used to derive more accurate approximations for the coupling constant.

2. – The spinor Bethe-Salpeter equation.

The wave function

$$\chi(q) = (2\pi)^{-2}\int d^4x \exp[iq\cdot x]\langle 0|T[\psi^a(\tfrac{1}{2}x)\,\psi^b(-\tfrac{1}{2}x)]|B\rangle \tag{1}$$

for bound states B of a fermion a and an antifermion $\bar{\text{b}}$ exchanging massless bosons satisfies a spinor Bethe-Salpeter equation that reads in the ladder

(3) R. F. Keam: *Journ. Math. Phys.*, **12**, 515 (1971); B. J. Brennan and R. F. Keam: *Prog. Theor. Phys.*, **49**, 1679 (1973); cf. for the boson Bethe-Salpeter equation H. J. W. Müller: *Nucl. Phys.*, **5** B, 277 (1968); F. Tanaka and N. Nakanishi: *Prog. Theor. Phys.*, **53**, 797 (1975).

approximation

$$[\not{q} + (1+\Delta)\varepsilon_B \hat{\not{P}} - (1+\Delta)]\,\chi(q)[\not{q} - (1-\Delta)\varepsilon_B\hat{\not{P}} - (1-\Delta)] = = -i(2\pi)^{-2}\sum_{i=1}^{5}\lambda_i\int \mathrm{d}^4q'\,(q-q')^{-2}\Gamma^i\,\chi(q')\,\Gamma^i\,. \tag{2}$$

Here $2\varepsilon_B$ is the mass of the bound state and 2Δ the mass difference of the fermion and antifermion (with half the sum of the masses of the constituents as the unit of mass). The vector $\hat{P}^\mu$ is a timelike unit vector parallel to the momentum vector of the bound state, so that it has the form (1, 0) in the rest frame of the latter. The coupling constants λ_i measure the strength of the interactions characterized by the matrices

$$\Gamma^S = 1\,, \qquad \Gamma^V = \gamma^\mu\,, \qquad \Gamma^T = \sigma^{\mu\nu} = \tfrac{1}{2}i[\gamma^\mu, \gamma^\nu]\,, \qquad \Gamma^A = \gamma^\mu\gamma_5\,,$$
$$\Gamma^P = i\gamma_5 \qquad \text{with} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,.$$

The wave function $\chi(q)$ may be expanded in a complete set of Dirac matrices. In fact, for 0^+ and 0^- bound states (with parity ± 1 and spin 0) one may write

$$\chi(q) = \chi_S + \not{q}\chi_{V1} + \hat{\not{P}}\chi_{V2} + 2q^\mu\hat{P}^\nu\sigma_{\mu\nu}\chi_{T1}\,, \tag{3}$$

$$\chi(q) = i\gamma_5\chi_P + \not{q}\gamma_5\chi_{A1} + \hat{\not{P}}\gamma_5\chi_{A2} + \varepsilon^{\varkappa\lambda\mu\nu}q_\varkappa\hat{P}_\lambda\sigma_{\mu\nu}\chi_{T2}\,, \tag{4}$$

respectively, where the structure functions χ_i depend on q^2 and $q\cdot\hat{P}$. When these expressions are inserted into the Bethe-Salpeter equation, a set of coupled integral equations is obtained for the structure functions; they may be transformed into coupled differential equations by performing a Wick rotation, introducing a Euclidean metric (with $q^4 = -iq^0$, $\hat{P}^4 = -i$) and employing the identity $\Box_q q^{-2} = -(2\pi)^2\delta^{(4)}(q)$ in the Euclidean q^μ-space.

For 0^+-states the structure functions are found to satisfy the equations

$$\Box_q f_S = \Lambda_S\chi_S\,, \tag{5}$$

$$\Box_q(q^\mu f_{V1} + \hat{P}^\mu f_{V2}) = \Lambda_V(q^\mu\chi_{V1} + \hat{P}^\mu\chi_{V2})\,, \tag{6}$$

$$\Box_q[(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)f_{T1}] = \Lambda_T(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu)\chi_{T1}\,. \tag{7}$$

Here the symbols f_i are abbreviations for the following expressions:

$$f_S = [(1-\Delta^2)(1-\varepsilon_B^2) - q^2 - 2\Delta\varepsilon_B q\cdot\hat{P}]\,\chi_S + 2q^2\chi_{V1} + + 2q\cdot\hat{P}\chi_{V2} + 4i\varepsilon_B[q^2 + (q\cdot\hat{P})^2]\,\chi_{T1}\,, \tag{8}$$

$$f_{V1} = [(1-\Delta^2)(1+\varepsilon_B^2) - q^2]\,\chi_{V1} - 2\chi_S - 2(q\cdot\hat{P} - \Delta\varepsilon_B)\,\chi_{V2} + + 4i[(1-\Delta^2)\,\varepsilon_B + \Delta q\cdot\hat{P}]\,\chi_{T1}\,, \tag{9}$$

$$f_{V2} = [(1-\Delta^2)(1-\varepsilon_B^2) + q^2]\,\chi_{V2} + 2\varepsilon_B[(1-\Delta^2)\,\varepsilon_B\, q\cdot\hat{P} - \Delta q^2]\,\chi_{V1} + \\ + 4i[(1-\Delta^2)\,\varepsilon_B\, q\cdot\hat{P} - \Delta q^2]\,\chi_{T1}\,, \tag{10}$$

$$f_{T1} = [(1-\Delta^2)(1+\varepsilon_B^2) + q^2 + 2\Delta\varepsilon_B\, q\cdot\hat{P}]\,\chi_{T1} + i\varepsilon_B\,\chi_S - \\ - i(1-\Delta^2)\,\varepsilon_B\,\chi_{V1} - i\Delta\,\chi_{V2}\,. \tag{11}$$

The effective coupling constants Λ_i are linear combinations ([1]) of the constants λ_i that occurred in (2).

For 0^--states the equations that determine the structure functions are

$$\Box_q[(q^\mu \hat{P}^\nu - q^\nu \hat{P}^\mu)\, f_{T2}] = \Lambda_T (q^\mu \hat{P}^\nu - q^\nu \hat{P}^\mu)\,\chi_{T2}\,, \tag{12}$$

$$\Box_q[q^\mu f_{A1} + \hat{P}^\mu f_{A2}] = \Lambda_A (q^\mu \chi_{A1} + \hat{P}^\mu \chi_{A2})\,, \tag{13}$$

$$\Box_q f_P = \Lambda_P\, \chi_P\,, \tag{14}$$

with f_i given as

$$f_{T2} = [(1-\Delta^2)(1-\varepsilon_B^2) - q^2 - 2\Delta\varepsilon_B\, q\cdot\hat{P}]\,\chi_{T2} + \chi_{A2} - i\varepsilon_B\,\chi_P\,, \tag{15}$$

$$f_{A1} = [(1-\Delta^2)(1-\varepsilon_B^2) + q^2]\,\chi_{A1} + 4q\cdot\hat{P}\chi_{T2} + 2(q\cdot\hat{P} - \Delta\varepsilon_B)\,\chi_{A2} + 2i\Delta\chi_P\,, \tag{16}$$

$$f_{A2} = [(1-\Delta^2)(1+\varepsilon_B^2) - q^2]\,\chi_{A2} - 4q^2\,\chi_{T2} - 2\varepsilon_B[(1-\Delta^2)\,\varepsilon_B\, q\cdot\hat{P} - \Delta q^2]\,\chi_{A1} - \\ - 2i(1-\Delta^2)\,\varepsilon_B\,\chi_P\,, \tag{17}$$

$$f_P = [(1-\Delta^2)(1+\varepsilon_B^2) + q^2 + 2\Delta\varepsilon_B\, q\cdot\hat{P}]\,\chi_P - 4i\varepsilon_B[q^2 + (q\cdot\hat{P})^2]\,\chi_{T2} - \\ - 2i[(1-\Delta^2)\,\varepsilon_B\, q\cdot\hat{P} - \Delta q^2]\,\chi_{A1} + 2i[(1-\Delta^2)\,\varepsilon_B + \Delta q\cdot\hat{P}]\,\chi_{A2}\,. \tag{18}$$

When the bound-state rest frame energy $2\varepsilon_B$ and the mass difference 2Δ of the constituents tend to zero the eqs. (5)-(7) with (8)-(11) and (12)-(14) with (15)-(18) reduce to those given in ref. ([1]). In that paper the 0^+ equations have been shown to possess solutions that are symmetric in q^μ-space. The structure functions χ_S and χ_{V1} turned out to be linear combinations of hypergeometric functions depending on $x = -q^2$:

$$\chi_S(x) = \sum_{p=0}^{r} \hat{a}_p^{(r)}\, {}_2F_1(n,\, 1-p-q,\, 2;\, x)\,, \tag{19}$$

$$\chi_{V1}(x) = \sum_{p=0}^{r} \hat{b}_p^{(r)}\, {}_2F_1(n,\, 2-p-q,\, 3;\, x)\,, \tag{20}$$

while χ_{V2} and χ_{T1} were found to vanish. The solutions are labelled by a non-negative integer r and a real parameter n; furthermore q is given as

$$q = q_\pm \equiv -n - 2r + \tfrac{3}{2} \pm (4nr + 2r^2 - 8r + \tfrac{9}{4})^{\frac{1}{2}} \tag{21}$$

for $r \geqslant 1$ (both choices for the sign are possible), while $q = -n$ for $r = 0$. The coefficients $\hat{a}_p^{(r)}$ and $\hat{b}_p^{(r)}$ in (19)-(20) are in general complicated functions of n, q and r; they contain an arbitrary multiplicative constant $C_{n,q}^{(r)}$ that has to be determined by a normalization condition (see sect. 4). For $r = 0$ these coefficients read

$$\hat{a}_0^{(0)} = nC_n^{(0)}, \qquad \hat{b}_0^{(0)} = \tfrac{1}{2}n(n+1)\,C_n^{(0)}. \tag{22}$$

The solutions (19)-(20) satisfy the Bethe-Salpeter equations (5)-(7) with $\varepsilon_B = \Delta = 0$ if the coupling constants Λ_S, Λ_V are taken to be

$$\Lambda_S = -\,4(q+r)(q+r+1)\,, \tag{23}$$

$$\Lambda_V = -\,4(n-1)(n-3)\,. \tag{24}$$

The coupled differential equations given above simplify considerably if the interactions are such that one or more of the coupling constants Λ_i vanish; these cases will be considered in the next section.

3. – Bethe-Salpeter equations for interactions with $\Lambda_i \neq 0$, $\Lambda_j = 0$.

When a coupling constant Λ_j vanishes the corresponding differential equation in the coupled set (5)-(7) or (12)-(14) reduces to an algebraic equation, that may be used to eliminate one or more structure functions from the remaining equations. If two of the three coupling constants that occur in each set are zero only one differential equation (with scalar, vector or tensor character) will be left.

A single scalar differential equation is found for 0^+-states with interactions characterized by $\Lambda_S \neq 0$, $\Lambda_V = \Lambda_T = 0$. It follows when the algebraic relations $f_{V1} = f_{V2} = f_{T1} = 0$ (with f_i given by (9)-(11)) are used to eliminate χ_{V1}, χ_{V2} and χ_{T1} from (5) with (8); the result is

$$\Box_q[(N/D_K)\,\chi_S] = \Lambda_S\,\chi_S \tag{25}$$

with numerator and denominator functions

$$N = \prod_{\mu=\pm1}[(1+\mu\Delta)^2(1-\varepsilon_B^2) + q^2 + 2\mu(1+\mu\Delta)\,\varepsilon_B\, q\cdot\hat{P}]\,, \tag{26}$$

$$D_K = (1-\Delta^2)(1-\varepsilon_B^2) - q^2 - 2\Delta\varepsilon_B\, q\cdot\hat{P}\,. \tag{27}$$

The single partial differential equation (25), which reduces for $\Delta = \varepsilon_B = 0$

to an equation studied earlier [1,4], may be called the generalized Kummer equation.

For 0^--states a single scalar differential equation is obtained if $\Lambda_P \neq 0$, $\Lambda_A = \Lambda_T = 0$; it has the form

$$\Box_q[(N/D_G)\,\chi_P] = \Lambda_P\,\chi_P \tag{28}$$

with the same numerator function as in (25) and a denominator given by

$$D_G = (1-\Delta^2)(1+\varepsilon_B^2) + q^2 + 2\Delta\varepsilon_B\, q\cdot\hat{P}\,. \tag{29}$$

Equation (28) is a generalization for $\Delta \neq 0$, $\varepsilon_B \neq 0$ of the equation of GOLDSTEIN [1,5].

Vectorial equations arise for 0^+-states with $\Lambda_V \neq 0$, $\Lambda_S = \Lambda_T = 0$ and for 0^--states with $\Lambda_A \neq 0$, $\Lambda_P = \Lambda_T = 0$. However, these vector equations still contain a pair of structure functions. Equations for a single function are obtained if bound states with infinitesimally small mass that are described by structure functions symmetric in q^μ-space are considered. In the case of 0^+-states with $\Lambda_V \neq 0$ a vectorial form of the generalized Kummer equation (25) is then found for χ_{V1}, while for 0^--states with $\Lambda_A \neq 0$ the function χ_{A1} has to satisfy a vectorial generalized Goldstein equation analogous to (28).

Finally, tensorial equations are encountered for 0^+-states with $\Lambda_T \neq 0$, $\Lambda_S = \Lambda_V = 0$ and for 0^--states with $\Lambda_T \neq 0$, $\Lambda_P = \Lambda_A = 0$. In the former case an equation of Goldstein type shows up for χ_{T1} and in the latter case a generalized Kummer equation for χ_{T2}.

All interactions with $\Lambda_i \neq 0$, $\Lambda_j = 0$ $(j \neq i)$ are thus found to lead to scalar, vector and tensor forms of the generalized Kummer and Goldstein equations. For $\Delta = \varepsilon_B = 0$ these equations have completely different properties: whereas the Kummer equation has a discrete set of solutions, no such solutions exist for the Goldstein equation. The solutions χ_i for the three Kummer-type cases (0^+ with $\Lambda_S \neq 0$ or $\Lambda_V \neq 0$, 0^- with $\Lambda_T \neq 0$) have been found as [1]

$$\chi_i(t,\psi) = C^i_{r,l}\left(\frac{1-t}{2}\right)^{\frac{1}{2}(l-k)}\left(\frac{1+t}{2}\right)^{m-\frac{1}{2}(l-k)-1} tP_r^{(l+1,2m-l-5)}(t)\,C^{(p)}_{l-k}(\cos\psi)\,. \tag{30}$$

The arguments t and $\cos\psi$ of the Jacobi and Gegenbauer polynomials are defined by $t = (1-q^2)/(1+q^2)$ and $\cos\psi = iq\cdot\hat{P}/(q^2)^{\frac{1}{2}}$ The parameters k and p have the values (0, 1), (1, 1) and (1, 2) for the S, $V1$ and T cases, respectively.

---

(4) W. KUMMER: *Nuovo Cimento*, **31**, 219 (1964).
(5) J. S. GOLDSTEIN: *Phys. Rev.*, **91**, 1516 (1953).

Furthermore r and l are nonnegative integers (with $l \geqslant k$ and $l \equiv 1$ for $i = V1$), while m is a real parameter given by

$$m = l + r + \tfrac{7}{2} + (2r^2 + 2rl + l^2 + 4r + 3l + \tfrac{9}{4})^{\frac{1}{2}}\,. \tag{31}$$

The constants $C^i_{r,l}$ will follow from the normalization of the bound-state wave function. The coupling constants Λ_i that correspond to the solutions (30) are

$$\Lambda_i = -\,4(m-2)(m-l-3)\,. \tag{32}$$

4. – Normalization.

The bound-state wave function χ satisfies a normalization condition which reads [1,6]

$$\begin{aligned}\mathcal{N} \equiv \tfrac{1}{8} i(2\pi)^3 \int \mathrm{d}^4 q \,\mathrm{Tr}\,\{\bar\chi[q + (1+\Delta)\,\varepsilon_B \hat{P} - (1+\Delta)]\,\chi(1-\Delta)\,\varepsilon_B^{-1}\hat{P} - \\ - \bar\chi(1+\Delta)\,\varepsilon_B^{-1}\hat{P}\chi[q - (1-\Delta)\,\varepsilon_B\hat{P} - (1-\Delta)]\} = 1\end{aligned} \tag{33}$$

(with $\bar\chi$ the adjoint wave function); it may be derived by considering the pole structure of the inverse Green function that describes the propagator of a fermion-antifermion pair.

After Wick rotation and introduction of the structure functions the normalization condition for $0^\pm$-states may be brought into the form

$$\mathcal{N} = I_\pm(\partial'/\partial\varepsilon_B^2) = 1\,. \tag{34}$$

For 0^+-states this condition contains the integral I_+ which is defined as

$$\begin{aligned}I_+(D) = (2\pi)^3\int \mathrm{d}^4 q\,\{\bar\chi_S D f_S - (q^\mu \bar\chi_{V1} + \hat{P}^\mu \bar\chi_{V2})\,D(q^\mu f_{V1} + \hat{P}^\mu f_{V2}) + \\ + 2(q^\mu \hat{P}^\nu - q^\nu \hat{P}^\mu)\,\bar\chi_{T1} D[(q^\mu\hat{P}^\nu - q^\nu\hat{P}^\mu) f_{T1}]\}\,;\end{aligned} \tag{35}$$

the operator $\partial'/\partial\varepsilon_B^2$ occurring in (34) differentiates only the factors multiplying the structure functions χ_i in f_i (given by (8)-(11)), not the structure functions themselves. For 0^--states the normalization integral in (34) has the general

(6) S. MANDELSTAM: *Proc. Roy. Soc.*, A **233**, 248 (1955); R. E. CUTKOSKY and M. LEON: *Phys. Rev.*, **135**, B 1445 (1964); D. LURIÉ, A. J. MACFARLANE and Y. TAKAHASHI: *Phys. Rev.*, **140**, B 1091 (1965).

form

$$I_-(D) = (2\pi)^3 \int d^4q \, \{-\bar{\chi}_P D f_P + (q^\mu \bar{\chi}_{A1} + \hat{P}^\mu \bar{\chi}_{A2}) D (q^\mu f_{A1} + \hat{P}^\mu f_{A2}) - - 2(q^\mu \hat{P}^\nu - q^\nu \hat{P}^\mu) \bar{\chi}_{T2} D[(q^\mu \hat{P}^\nu - q^\nu \hat{P}^\mu) f_{T2}]\} \tag{36}$$

with f_i given by (15)-(18).

The adjoint structure functions $\bar{\chi}_i$ that occur in (35), (36) are connected to the structure functions χ_i by the relation

$$\bar{\chi}_i(q\cdot\hat{P}, q^2) = \chi_i^*(-q\cdot\hat{P}, q^2)\,, \tag{37}$$

as follows from spectral analysis of the wave function χ and its adjoint $\bar{\chi}$. If this relation is used in (35) or (36) and the expressions for f_i are inserted, it is found that the normalization integrals are not necessarily positive: « ghost » states with negative norm may show up. In sects. **6** and **7** it will indeed be shown that some of the strong-binding solutions given in (19)-(20) and (30) have ghost state character.

5. – Perturbation theory.

The strong-binding solutions of the Bethe-Salpeter equation have been deduced by putting the parameters Δ and ε_B equal to zero. If the structure functions and the coupling constants are assumed to be analytical functions of Δ and ε_B, one may employ perturbation theory to derive the properties of bound states with Δ and ε_B different from zero. In fact, when power series expansions for χ_i and Λ_i are substituted into eqs. (5)-(7) or (12)-(14) and the coefficients of successive powers of Δ and ε_B are set equal to zero, inhomogeneous differential equations are found from which χ_i and Λ_i may be derived to any order of approximation, at least in principle.

The first-order contributions to the coupling constants Λ_i for nonvanishing Δ and ε_B may be found without solving a differential equation. The change of the coupling constant λ (defined by writting $\Lambda_i = c_i\lambda$ with constant c_i) that follows if the bound state is given a small rest frame energy $2\varepsilon_B$ may be evaluated by differentiating the spinor Bethe-Salpeter equation (2) with respect to ε_B, multiplying the result by the adjoint wave function $\bar{\chi}$ and integrating over q^μ; one gets in this way [2]

$$\lambda(\partial\lambda/\partial\varepsilon_B^2)^{-1} = -\tfrac{1}{4} i(2\pi)^3 \int d^4q \, \mathrm{Tr}\,\{\bar{\chi}[\not{q} + (1+\Delta)\varepsilon_B \hat{\not{P}} - (1+\Delta)] \cdot \cdot \chi[\not{q} - (1-\Delta)\varepsilon_B\hat{\not{P}} - (1-\Delta)]\}\,. \tag{38}$$

When the wave functions χ and $\bar{\chi}$ are expanded in terms of the structure functions one obtains, after Wick rotation, for $0^\pm$ states

$$\lambda(\partial\lambda/\partial\varepsilon_B^2)^{-1} = I_\pm(1) \tag{39}$$

with $I_\pm$ defined in (35) and (36).

The use of perturbation theory makes sense only if the integral on the right-hand side of (39) is finite. As a consequence conditions on the asymptotic behaviour of the structure functions have to be satisfied. Since these conditions have a form similar to that found from the requirement that the normalization integral (34) be convergent, the relation (39) is often called a normalization condition as well. This nomenclature (which has been followed in ref. [1]) will not be used here: the physical origins of (34) and (39) are in fact rather different.

The derivative of the coupling constant λ with respect to the mass difference 2Δ of the constituents may be obtained in a similar way, be differentiating the Bethe-Salpeter equation with respect to Δ. If use is made of (38) the result may be written as

$$\begin{aligned}(\partial\lambda/\partial\Delta)(\partial\lambda/\partial\varepsilon_B^2)^{-1} = -\tfrac{1}{4}i(2\pi)^3\!\int\! \mathrm{d}^4q\, \mathrm{Tr}\,\{\bar\chi(\varepsilon_B\hat{P}-1)\,\chi\,\cdot\\ \cdot[\hat{q}-(1-\Delta)\,\varepsilon_B\hat{P}-(1-\Delta)] + \bar\chi[\hat{q}+(1+\Delta)\,\varepsilon_B\hat{P}-(1+\Delta)]\,\chi(\varepsilon_B\hat{P}+1)\}\,,\end{aligned} \tag{40}$$

or, after Wick rotation, as

$$(\partial\lambda/\partial\Delta)(\partial\lambda/\partial\varepsilon_B^2)^{-1} = I_\pm(\partial'/\partial\Delta)\,; \tag{41}$$

the operator $\partial'/\partial\Delta$ is meant to differentiate only the factors appearing in f_i in front of the structure functions χ_i. Again, the requirement of convergence of the integral will lead to restrictions on the asymptotic behaviour of the structure functions.

An auxiliary identity that will be useful in evaluating the perturbation and normalization integrals is obtained if the Bethe-Salpeter equation (2) is acted upon by the operator $q^\mu\partial/\partial q^\mu$. Upon multiplication by $\bar\chi$ and integration over q^μ one finds then the relation

$$\begin{aligned}\int\! \mathrm{d}^4q\, \mathrm{Tr}\,\{\bar\chi[(1+\Delta)\,\varepsilon_B\hat{P}-(1+\Delta)]\,\chi[\hat{q}-(1-\Delta)\,\varepsilon_B\hat{P}-(1-\Delta)]\,+\\ +\,\bar\chi[\hat{q}+(1+\Delta)\,\varepsilon_B\hat{P}-(1+\Delta)]\,\chi[-(1-\Delta)\,\varepsilon_B\hat{P}-(1-\Delta)]\} = 0\,,\end{aligned} \tag{42}$$

which for $0^\pm$-states may be brought into the form

$$I_\pm(q^\mu\partial'/\partial q^\mu - 2) = 0\,; \tag{43}$$

the differentiations $\partial'/\partial q^\mu$ have to be carried out at constant χ_i and $\sqrt{q^2}\,\chi_j$ with $i = S,\ V2,\ A2,\ P$ and $j = V1,\ T1,\ T2,\ A1$ (so that the wave function given by (3) or (4) is not differentiated).

6. – Normalization and perturbation integrals for $\Lambda_i \neq 0$ solutions.

The normalization integral (34) and the perturbation integrals (39), (41) that determine $\partial\lambda/\partial\varepsilon_B^2$ and $\partial\lambda/\partial\Delta$ will be evaluated now for the solutions corresponding to interactions with $\Lambda_i \neq 0$ $(i = S, V, T)$, as discussed in sect. 4. The structure functions χ_i for these cases have been given in (30), while their adjoints follow from (37) as $\bar{\chi}_i = (-)^{l-k}\chi_i^*$. Due to the vanishing of the coupling constants Λ_j $(j \neq i)$ the structure functions satisfy linear equations $f_j = 0$ (with f_j given in (8)-(11) or (15)-(18)); these may be used to express the integrals exclusively in terms of the structure functions χ_i.

For the scalar and tensor cases the normalization integral is found to have the general form

$$\mathscr{N} = (2\pi)^4 \int_{-1}^{1} \mathrm{d}t \int_0^\pi \mathrm{d}\psi \frac{(1-t)^{k+2}}{(1+t)^{k+3}t}(2\sin\psi)^{2p}\cos^2\psi\,\bar{\chi}_i\chi_i\,; \tag{44}$$

here the integrand has been simplified by using the auxiliary identity (43), which may be written as

$$\int_{-1}^{1} \mathrm{d}t \int_0^\pi \mathrm{d}\psi \frac{(1-t)^{k+1}}{(1+t)^{k+3}t^2}(1-2t)(2\sin\psi)^{2p}\bar{\chi}_i\chi_i = 0\,. \tag{45}$$

When (30) is inserted into (44) the scalar and tensor normalization integrals become a product of integrals over ψ and t:

$$\mathscr{N} = (-)^{l+p+1}(2\pi)^4|C^i_{r,l}|^2 I_\psi I_t\,. \tag{46}$$

The integral over ψ is of the general type (A.8) with $q = 1$; its explicit form follows from (A.9)-(A.10) as

$$I_\psi = \pi[(1-\tfrac{1}{2}\delta_{l,k})\,\delta_{p,1} + (l^2+2l-2)\,\delta_{p,2}]\,. \tag{47}$$

The integral over t may be written as a linear combination of three integrals of the type (A.1), which have been evaluated in (A.6) and (A.7):

$$I_t = \frac{(l+r+1)!\,\Gamma(2m-l+r-4)P_0(m,l)}{4r!\,\Gamma(2m+r-3)(2m+2r-1)(2m+2r-3)(2m+2r-5)} \tag{48}$$

with the polynomial $P_0(m,l) = 2m^2 + m(-2l-10) + 4l^2 + 12l + 15$. Both the integrals (47) and (48) may be shown to have positive signs for all $l \geqslant k$, $r \geqslant 0$ and m following from (31).

In the vector case the normalization integral is found as

$$(49)\qquad \mathcal{N} = (2\pi)^4 \int_{-1}^{1} dt \int_{0}^{\pi} d\psi \frac{(1-t)^2}{(1+t)^4 t} [2 + \cos^2\psi(t-1-t^{-1})](2\sin\psi)^2 \bar{\chi}_{V1}\chi_{V1}\,.$$

Upon substitution of (30) the righ-hand side leads to an expression analogous to (46), with (47) and (48); however, the integral I_t turns out to contain now instead of $P_0(m, l)$ the polynomial $P_1(m, 1) = -46m^2 + 276m - 335$, which implies a negative sign of the t-integral for all $r \geqslant 0$ and m determined by (31) with $l \equiv 1$.

In view of the signs of I_ψ and I_t one obtains a sign $(-)^{l+k}$ for the normalization integral $\mathcal{N}$ of the three types of solutions. Thus, only the scalar solutions with even l, the vector solutions (with $l \equiv 1$) and the tensor solutions with odd l can be properly normalized. The scalar odd l and the tensor even l solutions correspond to ghost states.

The perturbation integrals (39) and (41) that determine $\partial\lambda/\partial\varepsilon_B^2$ and $\partial\lambda/\partial\Delta$ may be evaluated in a similar fashion by writing them first as an integral over χ_i and $\bar{\chi}_i$. The integral (39) with (35) and (36) gets the form

$$(50)\qquad \lambda\,(\partial\lambda/\partial\varepsilon_B^2)^{-1} = (-)^{k+p+1}(2\pi)^4 \int_{-1}^{1} dt \int_{0}^{\pi} d\psi \frac{(1-t)^{k+1}}{(1+t)^{k+4} t} (2\sin\psi)^{2p} \bar{\chi}_i\chi_i\,,$$

while (41) becomes up to terms linear in Δ and ε_B

$$(51)\qquad (\partial\lambda/\partial\Delta)(\partial\lambda/\partial\varepsilon_B^2)^{-1} = 2\Delta(-)^{k+p+1}(2\pi)^4 \int_{-1}^{1} dt \int_{0}^{\pi} d\psi \frac{(1-t)^{k+2}}{(1+t)^{k+3} t} (2\sin\psi)^{2p} \bar{\chi}_i\chi_i\,;$$

here (45) has been used to simplify the integrands. The expression (51) shows that the mixed second derivative $\partial^2\lambda/\partial\Delta\,\partial\varepsilon_B$ vanishes for $\Delta = \varepsilon_B = 0$. When (30) is substituted, the integrals (50), (51) again factorize, as in (46); the ψ and t integrals may be evaluated with the help of the identities given in the appendix. If the normalization condition (46) with (47), (48) is employed to eliminate the constants $C^i_{r,l}$ one obtains for the perturbation integrals of the non-ghost solutions

$$(52)\qquad \lambda^{-1}\frac{\partial\lambda}{\partial\varepsilon_B^2} = \frac{P_{k-p+1}(m, l)(2m-l-5)}{2(2m-2l-2r-7)(2m+2r-1)(2m+2r-5)} \cdot \left[\left(1-\frac{1}{2}\delta_{l,k}\right)\delta_{p,1} + \frac{l^2+2l-2}{l(l+2)}\delta_{p,2}\right]$$

(with $(k, p) \equiv (0, 1)$, $(1, 1)$ and $(1, 2)$ for the S, V and T solutions, respectively) and

$$\lambda^{-1}\frac{\partial\lambda}{\partial\Delta^2} = \frac{P_0(m, l)(2m-l-5)}{(2m-2l-2r-7)(2m+2r-1)(2m+2r-5)}\,. \tag{53}$$

For the scalar and tensor nonghost solutions the derivative of $|\lambda|$ with respect to ε_B^2 is found to be positive, while it is negative for the vector solutions. For nonrelativistic bound-state problems a decrease of the absolute value of the coupling constant always corresponds to a decrease of the binding energy, so that $\partial|\lambda|/\partial\varepsilon_B^2$ is negative in that case. Likewise the boson Bethe-Salpeter equation that describes bound states of spin-0 particles leads to a negative sign for the derivative (7). The physical meaning of the solutions with anomalous signs that occur here is not clear.

The derivative $\partial|\lambda|/\partial\Delta^2$ given by (53) is positive for all values of l, r and m compatible with (31). Hence, if the constituent particles are given (at fixed ε_B) slightly different masses $(1 \pm \Delta)$, the absolute value $|\lambda|$ of the coupling constant increases. This may be compared with the result found for the boson Bethe-Salpeter equation; in that case the coupling constant turns out to decrease, when Δ acquires a nonvanishing value (7).

7. – Normalization and perturbation integrals for symmetric 0^+-solutions.

The Bethe-Salpeter equations for 0^+ bound states with $\Delta = \varepsilon_B = 0$ possess a series of solutions of which the general form has been given in (19) and (20) with (21)-(24). When these solutions are considered as zeroth-order approximations in a perturbation theory the integrals (39) and (41) may be used again to obtain the first-order corrections to the coupling constant. Before evaluating these perturbation corrections the normalization integral will be calculated.

The normalization integral given in (34) with (35) and (37) becomes upon insertion of the expressions (8)-(11) for f_i

$$\mathcal{N} = -(2\pi)^3\int \mathrm{d}^4q\,\{|\chi_S|^2 + [q^2 + 2(q\cdot\hat{P})^2]|\chi_{V1}|^2 + \\ + 4[q^2 + (q\cdot\hat{P})^2]\,\mathrm{Im}\,[(\chi_S - \chi_{V1})^*\chi'_{T1}]\}\,. \tag{54}$$

In the last term the derivative $\chi'_{T1} = \partial\chi_{T1}/\partial\varepsilon_B$ of the tensor structure function with respect to ε_B occurs; it is found as the limit for vanishing ε_B of $\varepsilon_B^{-1}\chi_{T1}$ and may be obtained as the solution of (7), with (11) inserted. This inhomoge-

(7) G. C. Wick: *Phys. Rev.*, **96**, 1124 (1954); R. E. Cutkosky: *Phys. Rev.*, **96**, 1135 (1954).

neous differential equation for χ_{T1} is in general rather difficult to solve. Its solution is simple however if Λ_T vanishes; in that case χ'_{T1} is found as

$$\chi'_{T1} = -i(\chi_S - \chi_{V1})/(1+q^2)\,. \tag{55}$$

The normalization integral (54) may be expressed then completely in terms of the structure functions χ_S and χ_{V1}:

$$\mathcal{N} = \frac{1}{4}(2\pi)^5 \int\limits_{-\infty}^{0} \mathrm{d}x \left[x|\chi_S|^2 - \frac{1}{2}x^2|\chi_{V1}|^2 + \frac{3x^2}{1-x}|\chi_S - \chi_{V1}|^2 \right], \tag{56}$$

where use has been made of the fact that the structure functions depend only on $x = -q^2$. When the solutions (19) and (20) are substituted in the integrand the first two terms lead to integrals of the general form (A.11). The third term gives rise to an integral of the type (A.16), if the identity ([8])

$$(c-1)\,{}_2F_1(a, b, c-1; x) = (c-b-1)\,{}_2F_1(a, b, c; x) + b\,{}_2F_1(a, b+1, c; x) \tag{57}$$

is employed to transform χ_S (19) into a linear combination of hypergeometric functions with third parameter equal to 3 (just as χ_{V1}). The difference of structure functions $(\chi_S - \chi_{V1})$ is then found to have the general form

$$\chi_S - \chi_{V1} = \sum_{p=0}^{r} \hat{c}_p^{(r)}\,{}_2F_1(n, 1-p-q, 3; x)\,, \tag{58}$$

where the coefficients $\hat{c}_p^{(r)}$ are given by

$$\hat{c}_p^{(r)} = \tfrac{1}{2}(p+q+1)\,\hat{a}_p^{(r)} - \tfrac{1}{2}(p+q)\,\hat{a}_{p+1}^{(r)} - \hat{b}_{p+1}^{(r)}\,. \tag{59}$$

If (19), (20) and (58) are inserted into (56) the normalization integral becomes

$$\begin{aligned} \mathcal{N} = \tfrac{1}{4}(2\pi)^5 \sum_{p_1,p_2=0}^{r} [&- \hat{a}_{p_1}^{(r)}\hat{a}_{p_2}^{(r)} K(n; 1-p_1-q, 1-p_2-q; 2) - \\ &- \tfrac{1}{2} b_{p_1}^{(r)} b_{p_2}^{(r)} K(n; 2-p_1-q, 2-p_2-q; 3) + \\ &+ 3\hat{c}_{p_1}^{(r)}\hat{c}_{p_2}^{(r)} M(n; 1-p_1-q, 1-p_2-q; 3)\,. \end{aligned} \tag{60}$$

The K-integrals have been given explicitly in (A.14); the M-integrals however are linear combinations of generalized Saalschutzian hypergeometric func-

([8]) A. ERDÉLYI: *Higher Transcendental Functions*, Vol. 1 (New York, N. Y., 1953).

tions [8,9] (see (A.18)-(A.20)) that have to be evaluated numerically. The norm is finite if the parameters n, q and r fulfil the inequalities

$$n > \tfrac{3}{2}, \qquad r + q < 0 . \tag{61}$$

For the solutions with $r = 0$ the normalization integral is

$$\begin{aligned}\mathcal{N} = \frac{3}{16}(2\pi)^5 n^2(n-1)^2|C_n^{(0)}|^2\{M(n;\, n+1, n+1;\, 3) - \\ - 8\Gamma(2n-3)\Gamma(2n-2)\Gamma(2n-1)\Gamma(2n)/[\Gamma(n)^2\Gamma(n+1)^2\Gamma(4n-2)]\} .\end{aligned} \tag{62}$$

For integer $n \geqslant 3$ the inequality (A.23) may be employed to show that $\mathcal{N}$ is positive, while for $n = 2$ it turns out to be negative. For noninteger n the sign of $\mathcal{N}$ can be determined only by numerical evaluation of the M-integral with the help of (A.21). It is found that the expression between the brackets in (62) vanishes for $n = n_0$ with $n_0 = 2.77$, so that the normalization constant $C_n^{(0)}$ has then to be taken infinitely large in order to satisfy the normalization condition $\mathcal{N} = 1$. If n is smaller than n_0 the normalization integral is negative: the solutions with $n < n_0$ correspond to ghost states; for $n > n_0$ no ghost states show up. Hence it turns out that a transition from a nonghost to a ghost state may occur if the parameter n changes continuously. Since n determines the ratio Λ_S/Λ_V of the scalar and vector effective coupling constants (see (21), (23) and (24)), nonghost states go over into ghost states if the parameters characterizing the type of the fermion-boson interaction reach definite critical values. This « phase transition » from nonghost to ghost state solutions is a phenomenon that may occur only if the Bethe-Salpeter equation is equivalent to a set of coupled equations. In the $\Lambda_i \neq 0$ cases discussed in the preceding section the ghost state solutions could be distinguished by a discrete parameter (l) from the nonghost state solutions.

The numerical results for the normalization integrals of the solutions with $r \geqslant 1$ show similar features: for $1.5 < n < n_0$ and $q = q_-$ (see eq. (21)) the solutions correspond to ghost states, whereas the solutions with $n > n_0$, $q = q_-$ and $n > r + \frac{3}{2} + \frac{1}{2}(8r^2 - 8r + 9)^{\frac{1}{2}}$, $q = q_+$ have a positive norm; the critical value n_0 is found as $n_0 = 2.79$, $n_0 = 2.74$ and $n_0 = 2.68$ for $r = 1$, 2 and 3, respectively.

The perturbation integral (39) that gives $\partial\lambda/\partial\varepsilon_B^2$ reads in the present case of nonvanishing χ_S and χ_{V1}

$$\lambda(\partial\lambda/\partial\varepsilon_B^2)^{-1} = -(2\pi)^3\int \mathrm{d}^4q(1+q^2)[|\chi_S|^2 - q^2|\chi_{V1}|^2] , \tag{63}$$

---

(9) A. M. Mathai and R. K. Saxena: *Generalized Hypergeometric Functions with Applications in Statistical and Physical Sciences* (Berlin, 1973).

where the auxiliary identity

$$\int \mathrm{d}^4q\,[|\chi_S|^2 - q^2|\chi_{V1}|^2 + 2q^2\,\mathrm{Re}\,(\chi_S\,\chi_{V1}^*)] = 0 \tag{64}$$

that follows from (43) has been employed. Upon introduction of the variable $x = -q^2$ and insertion of (19) and (20) one encounters integrals of products of hypergeometric functions of the type (A.12) for which explicit expressions have been given in (A.15):

$$\lambda(\partial\lambda/\partial\varepsilon_B^2)^{-1} = -\tfrac{1}{4}(2\pi)^5 \sum_{p_1,p_2=0}^{r} [\hat{a}_{p_1}^{(r)}\hat{a}_{p_2}^{(r)} L(n;\, 1-p_1-q,\, 1-p_2-q;\, 2) - \\ - \hat{b}_{p_1}^{(r)}\hat{b}_{p_2}^{(r)} L(n;\, 2-p_1-q,\, 2-p_2-q;\, 3)]\,. \tag{65}$$

The integrals at the right-hand side are convergent only if the parameters n, q and r are such that

$$n > 2\,, \qquad r + q < -\tfrac{1}{2}\,. \tag{66}$$

These conditions on the parameters are stronger than those given in (61): the convergence of the perturbation integral for $\lambda(\partial\lambda/\partial\varepsilon_B^2)^{-1}$ implies the convergence of the normalization integral, while the opposite is not true. Since the solutions at $\varepsilon_B = 0$ are meaningful only if they are considered as zeroth-order approximations to the physical solutions with $\varepsilon_B \neq 0$, the conditions (66) have to be fulfilled. (In fact, these conditions were imposed in ref. [1].)

The perturbation integral (41) that determines the change of the coupling constant λ if the constituents acquire different masses $(1 \pm \Delta)$ reads for the symmetric 0^+ solutions

$$(\partial\lambda/\partial\Delta)(\partial\lambda/\partial\varepsilon_B^2)^{-1} = -2\Delta(2\pi)^3\int \mathrm{d}^4q\,[|\chi_S|^2 - q^2|\chi_{V1}|^2] \tag{67}$$

up to terms linear in Δ and ε_B. The mixed derivative $\partial^2\lambda/\partial\Delta\,\partial\varepsilon_B$ is thus again found to vanish. The right-hand side becomes, upon insertion of the solutions (19) and (20), a linear combination of K-integrals that have been defined in (A.11) and evaluated in (A.14):

$$(\partial\lambda/\partial\Delta^2)(\partial\lambda/\partial\varepsilon_B^2)^{-1} = -\tfrac{1}{4}(2\pi)^5 \sum_{p_1,p_2=0}^{r} [\hat{a}_{p_1}^{(r)}\hat{a}_{p_2}^{(r)} K(n;\, 1-p_1-q,\, 1-p_2-q;\, 2) - \\ - \hat{b}_{p_1}^{(r)}\hat{b}_{p_2}^{(r)} K(n;\, 2-p_1-q,\, 2-p_2-q;\, 3)\,. \tag{68}$$

The convergence of the integrals is ensured if the inequalities (66) are satisfied.

For the solutions with $r = 0$ one gets from (65) with (22) and (A.15)

$$\lambda\left(\frac{\partial\lambda}{\partial\varepsilon_B^2}\right)^{-1} = \frac{1}{4}(2\pi)^5\,|C_n^{(0)}|^2\,\frac{3\Gamma(2n-1)^3\Gamma(2n-4)}{\Gamma(n)^3\Gamma(n-1)\Gamma(4n-3)}\,, \tag{69}$$

while the quotient of (68) and (65) gives then

$$\lambda^{-1}\frac{\partial\lambda}{\partial\Delta^2}=\frac{2(n-2)(2n-1)}{(n-1)(4n-3)}\,. \tag{70}$$

The constant $C_n^{(0)}$ may be eliminated from (69) by making use of the normalization condition $\mathcal{N}=1$, with $\mathcal{N}$ given by (62).

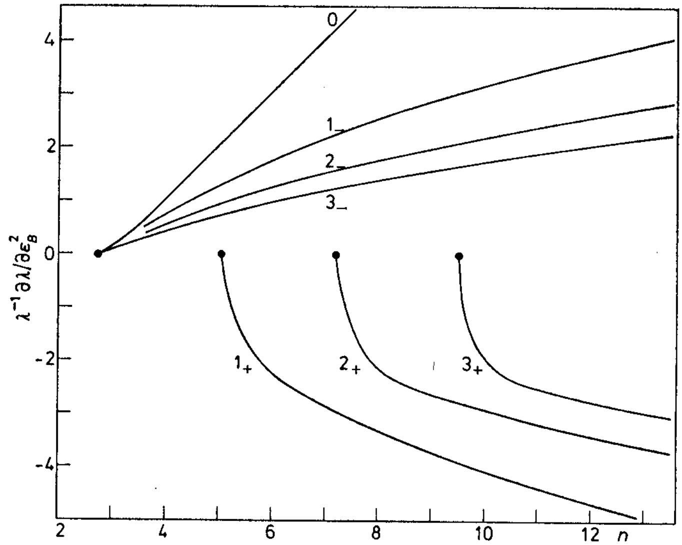

Fig. 1. – The derivative $\lambda^{-1}\partial\lambda/\partial\varepsilon_B^2$ for the symmetric 0^+ solutions with $r=0, 1, 2, 3$; $q=q_\pm$.

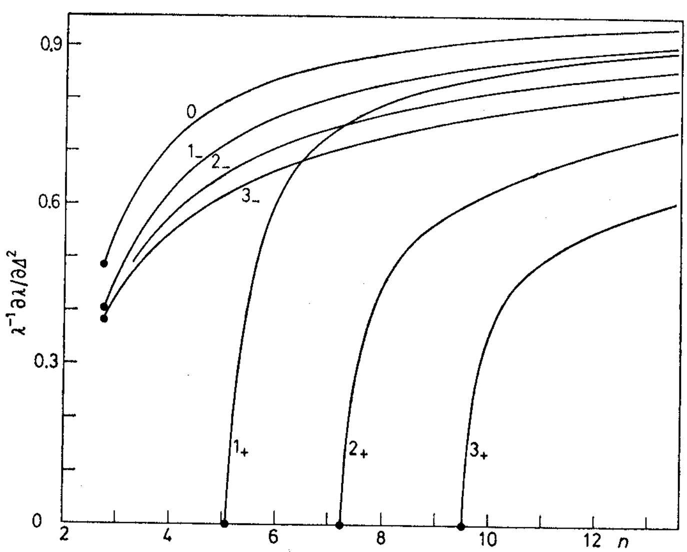

Fig. 2. – The derivative $\lambda^{-1}\partial\lambda/\partial\Delta^2$ for the solutions with $r=0, 1, 2, 3$; $q=q_\pm$.

For higher r the general expressions (60), (65) and (68) may be combined in a similar way. One obtains then, with the help of the numerical results for the normalization integral $\mathcal{N}$, the curves for $\lambda^{-1}\partial\lambda/\partial\varepsilon_B^2$ and $\lambda^{-1}\partial\lambda/\partial\Delta^2$ as a function of n that are given in fig. 1 and 2. (The parameter n is limited to values larger than n_0 for nonghost states.)

The curves for $\lambda^{-1}\partial\lambda/\partial\varepsilon_B^2$ show that the absolute value $|\lambda|$ of the coupling constant increases with increasing ε_B^2 for the $r=0$ and the $r\geqslant 1$, $q=q_-$ solutions, while the «normal» negative sign of $\partial|\lambda|/\partial\varepsilon_B^2$ shows up for the $r\geqslant 1$ $q=q_+$ solutions. The derivative $\partial|\lambda|/\partial\Delta^2$ on the other hand is found to be positive definite for all symmetric 0^+ solutions. These results are consistent with those obtained in the preceding section for the $\Lambda_i\neq 0$ cases; indeed the numerical values of the derivatives given here coincide for $n=3$, $q=q_-$ and $n=r+\frac{5}{2}+(2r^2+2r+\frac{9}{4})^{\frac{1}{2}}$, $q=q_+$ with those found from (52) and (53) for $\Lambda_S\neq 0$ and $\Lambda_V\neq 0$ solutions, respectively.

8. – Higher-order perturbation theory for $\Lambda_S\neq 0$ solutions with $\Delta\neq 0$.

In the preceding sections perturbation integrals have been evaluated for the solutions with $\Delta=\varepsilon_B=0$. These integrals yield the corrections to the coupling constant λ in lowest-order perturbation theory. The higher-order corrections to λ may be found by solving the Bethe-Salpeter equation successively in increasing order of Δ and ε_B. In the present section this procedure will be followed to obtain the coupling constant in higher-order perturbation theory for 0^+-states bound by interactions with $\Lambda_S\neq 0$.

The Bethe-Salpeter equation for $\Lambda_S\neq 0$ interactions is given by the generalized Kummer equation (25). The zeroth-order solutions $\chi_S^{(0)}$, $\Lambda_S^{(0)}$ of that equation (corresponding to $\Delta=\varepsilon_B=0$) have been written in (30) and (32). When ε_B is different from zero, the partial differential equation (25) cannot be solved by the method of separation of variables as was used for the zeroth-order equation. For $\varepsilon_B=0$, $\Delta\neq 0$ however such a separation is still possible; the following discussion will be limited to that case.

The generalized Kummer equation (25) with $\varepsilon_B=0$ may be solved by writing the structure function χ_S in the general form

$$\chi_S=t(1-t)^{\frac{1}{2}l}(1+t)^{m-\frac{1}{2}l-1}f(t)\,C_l^1(\cos\psi)\,, \tag{71}$$

where the variables $t=(1-\Delta^2-q^2)/(1-\Delta^2+q^2)$ and $\cos\psi=iq\cdot\hat{P}/(q^2)^{\frac{1}{2}}$ have been introduced. The function χ_S is a solution if m is chosen such that

$$(m-2)(m-l-3)+\tfrac{1}{4}\Lambda_S=0\,, \tag{72}$$

and if $f(t)$ satisfies the ordinary differential equation

$$(1-t^2)(1-t^2\Delta^2)f''+2\{-(m-1)t+m-l-3+ \\ +\Delta^2 t[(m+1)t^2-(m-l-3)t-2]\}f'+ \\ +\{(m-2)(m-2l-5)+\Delta^2[m(m+1)t^2-2m(m-l-3)t-2]\}f=0\,. \tag{73}$$

For $\Delta=0$ the solutions $f^{(0)}$ that correspond to (30) are Jacobi polynomials $P_r^{(l+1,2m-l-5)}$ with m given by (31); the relation (72) then reduces to (32). For $\Delta\neq 0$ one may insert power series for both $f(t)$ and m:

$$f(t)=f^{(0)}(t)+\Delta^2 f^{(1)}(t)+\Delta^4 f^{(2)}(t)+\dots\,, \tag{74}$$

$$m=m^{(0)}+\Delta^2 m^{(1)}+\Delta^4 m^{(2)}+\dots\,. \tag{75}$$

If $r=0$, the zeroth-order solutions are $f^{(0)}(t)=1$ and $m^{(0)}=2l+5$. Equation (73) reads then in first order in Δ^2

$$(1-t^2)f^{(1)''}-2(l+2)(2t-1)f^{(1)'}+ \\ +[2(l+3)(2l+5)t^2-2(l+2)(2l+5)t-2+m^{(1)}(2l+3)]=0\,. \tag{76}$$

Substituting the Jacobi polynomial expansion

$$f^{(1)}(t)=\sum_{p=0}^{\infty}c_p^{(1)}P_p^{(l+1,3l+5)}(t)\,, \tag{77}$$

using Jacobi's differential equation, multiplying the result by $(1-t)^{l+1}\cdot(1+t)^{3l+5}P_i^{(l+1,3l+5)}(t)$ and integrating over t, one arrives at the relation

$$c_p^{(1)}p(p+4l+7)I_{p,p}^{(l+1,3l+5;0)}=8(l+3)(2l+5)I_{p,0}^{(l+1,3l+5;2)}- \\ -4(2l+5)(3l+8)I_{p,0}^{(l+1,3l+5;1)}+[8(l+2)(l+3)+m^{(1)}(2l+3)]I_{p,0}^{(l+1,3l+5;0)} \tag{78}$$

for the coefficients $c_p^{(1)}$ and $m^{(1)}$; here the notation (A.1) for integrals over Jacobi polynomials has been introduced. The case $p=0$ yields an equation for $m^{(1)}$ from which one may derive with the help of (A.6) and (A.7)

$$m^{(1)}=(l+2)(2l+3)/(4l+9)\,. \tag{79}$$

The coefficients $c_1^{(1)}$ and $c_2^{(1)}$ are found as

$$c_1^{(1)}=1/[4(l+2)]\,,\qquad c_2^{(1)}=4(l+3)/(4l+9)^2\,. \tag{80}$$

The coefficients $c_p^{(1)}$ with $p \geqslant 3$ vanish as may be proved with the help of (A.5), while $c_0^{(1)}$ is left undetermined; it may be put equal to zero since a nonvanishing $c_0^{(1)}$ would lead to a contribution proportional to $f^{(0)}$ in $f^{(1)}$ and hence to an effective factor $1+\Delta^2 c_0^{(1)}$ in f.

In an analogous fashion the second-order correction to m may be obtained; the result is

$$m^{(2)} = \frac{(l+2)(2l+3)}{4(4l+9)^3(4l+11)}(192l^3+1388l^2+3316l+2613)\,. \tag{81}$$

The relation (72) may be employed now to derive the first- and second-order corrections to the coupling constant Λ_S. Up to order Δ^4 one finds

$$\frac{1}{4}\Lambda_S = -(l+2)(2l+3)\left[1+\frac{3l+5}{4l+9}\Delta^2+\right.$$
$$\left.+\frac{704l^4+6212l^3+20304l^2+29111l+15441}{4(4l+9)^3(4l+11)}\Delta^4\right]. \tag{82}$$

In a recent paper ([10]) the generalized Kummer equation (25), with $\Delta \neq 0$, $\varepsilon_B = 0$, has been studied numerically; in fact, the coupling constant has been determined for the cases $r=0$, $1 \leqslant l \leqslant 5$ (and for some cases with $r>0$) in the range $0 \leqslant \Delta \leqslant 0.95$, with results presented in a form that is accurate to within 2%. The perturbation formula (82) for the coupling constant up to order Δ^4 confirms these numerical results in the range $0 \leqslant \Delta \leqslant 0.5$; for $\Delta \ll 0.5$ the accuracy of (82) is of course much better than 2%, whereas for $\Delta > 0.5$ more terms in the perturbation series should be included.

9. – Conclusion.

An explicit evaluation of the normalization integrals for the strong-binding solutions of the spinor Bethe-Salpeter equation has shown that some of these solutions have negative norm and correspond hence to ghost states. In particular, it has been found that ghost state solutions appear when the continuous parameters that characterize the interaction at the fermion-boson vertices approach certain critical values.

By employing a perturbation theory for the Bethe-Salpeter equation that treats the bound-state rest frame energy $2\varepsilon_B$ and the mass difference 2Δ of the fermion constituents as small parameters, the coupling constants have been determined for which bound states with small but nonvanishing Δ and ε_B may occur. The results that have been presented in fig. 1 and 2 show that only

---

([10]) B. J. Brennan: *Journ. Math. Phys.*, **16**, 2241 (1975).

part of the (nonghost) strong-binding solutions have a negative sign for $\partial|\lambda|/\partial\varepsilon_B^2$, which may be called the «normal» sign, since it is found in nonrelativistic bound-state problems and for the boson Bethe-Salpeter equation. The derivative $\partial|\lambda|/\partial\Delta^2$ turns out to be positive for all strong-binding solutions of the spinor Bethe-Salpeter equation.

APPENDIX

Integrals of Jacobi polynomials and hypergeometric functions.

The perturbation and normalization integrals for the $\Lambda_i \neq 0$ solutions and for the symmetric 0^+ solutions, as obtained in sect. **6** and **7**, contain in their integrands products of Jacobi polynomials and of hypergeometric functions, respectively. Integrals of that type will be considered in more detail in this appendix.

The general form of the integrals encountered for $\Lambda_i \neq 0$ solutions is

$$I_{m,n}^{(\alpha,\beta;\gamma)} = \frac{1}{2}\int_{-1}^{1} \mathrm{d}t \left(\frac{1-t}{2}\right)^{\alpha} \left(\frac{1+t}{2}\right)^{\beta+\gamma} P_m^{(\alpha,\beta)}(t) P_n^{(\alpha,\beta)}(t) \tag{A.1}$$

with $\alpha > -1$, $\beta+\gamma > -1$. This integral may be expressed in terms of a terminating Saalschutzian generalized hypergeometric function of unit argument (11). When a Bailey-transform relation (12) is employed, the result may be brought into the form

$$I_{m,n}^{(\alpha,\beta;\gamma)} = (-)^{m+n} \frac{\Gamma(\beta+\gamma+1,\, m-\gamma,\, n-\gamma,\, \alpha+m+n+1)}{m!\,n!\,\Gamma(-\gamma,\, -\gamma,\, \alpha+\beta+\gamma+m+n+2)} \cdot {}_4F_3\left(\begin{matrix}\beta+\gamma+1,\, -\alpha-\beta+\gamma-m-n,\, -m,\, -n \\ -\alpha-m-n,\, \gamma+1-m,\, \gamma+1-n\end{matrix}\,\middle|\, 1\right) \tag{A.2}$$

with $\Gamma(p_1, \dots, p_n) = \prod_{i=1}^{n} \Gamma(p_i)$.

If α and/or γ are integer, the right-hand side of (A.2) is understood to be defined by a limiting procedure. In particular, if $\gamma = -1$, the hypergeometric function for the case $m \leqslant n$ is found as

$$\lim_{\gamma\to-1} {}_4F_3(1) = \frac{\Gamma(\alpha+n+1,\, \beta+m+1,\, \alpha+\beta+m+n+1)}{m!\,\Gamma(\beta+1,\, \alpha+m+n+1,\, \alpha+\beta+n+1)}, \tag{A.3}$$

---

(11) A. ERDÉLYI: *Tables of Integral Transforms*, Vol. **2** (New York, N. Y., 1954), p. 288.

(12) L. J. SLATER: *Generalized Hypergeometric Functions* (Cambridge, 1966), p. 121.

where Saalschutz's theorem [9] for the sum of a terminating ${}_3F_2(1)$ has been used. Substituting this expression in (A.2) one gets for $m \geqslant n$

$$I^{(\alpha,\beta;-1)}_{m,n} = (-)^{m+n} \frac{\Gamma(\alpha+n+1, \beta+m+1)}{m!\beta\Gamma(\alpha+\beta+n+1)} . \tag{A.4}$$

When γ tends to a positive integer or zero a number of different cases (characterized by the relative magnitude of m, n and γ) have to be distinguished in evaluating (A.2). A more convenient form is obtained by reversing the order of terms in the terminating hypergeometric series and using once more the Bailey-transform relation; the result is

$$I^{(\alpha,\beta;\gamma)}_{m,n} = \frac{\Gamma(\beta+\gamma+1, \gamma+m+1, \gamma+n+1, \alpha+m+n+1)}{m!n!\Gamma(\gamma+m-n+1, \gamma-m+n+1, \alpha+\beta+\gamma+m+n+2)} \cdot {}_4F_3\left(\begin{matrix}\beta-\gamma, -\alpha-\beta-\gamma-m-n-1, -m, -n\\ -\alpha-m-n, -\gamma-m, -\gamma-n\end{matrix}\middle| 1\right) . \tag{A.5}$$

This expression shows that for γ approaching a nonnegative integer the integral (A.1) is different from zero only if $|n-m| \leqslant \gamma$.

In the case $\gamma = 0$ the hypergeometric function occurring in (A.5) reduces to a terminating ${}_3F_2(1)$, summable by Saalschutz's theorem. One may recover in this way the well-known normalization integral for Jacobi polynomials:

$$I^{(\alpha,\beta;0)}_{m,n} = \frac{\Gamma(\alpha+m+1, \beta+m+1)}{m!(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)} \delta_{m,n} . \tag{A.6}$$

For positive integer γ the asymmetric expression

$$I^{(\alpha,\beta;\gamma)}_{m,n} = \frac{\Gamma(\alpha+n+1, \beta+\gamma+m+1, \gamma+m+1, \alpha+\beta+2m+1)}{m!n!\Gamma(\alpha+\beta+\gamma+m+n+2, \gamma+m-n+1, \alpha+\beta+m+1)} \cdot {}_4F_3\left(\begin{matrix}-\beta-m, -\alpha-\beta-\gamma-n-m-1, -\gamma-m+n, -m\\ -\beta-\gamma-m, -\alpha-\beta-2m, -\gamma-m\end{matrix}\middle| 1\right) \tag{A.7}$$

(that may be obtained from (A.2) by employing similar auxiliary relations as used in deriving (A.5)) is best suited to evaluate I_{mn} for $m \leqslant n$, especially for small values of γ.

Integrals over squares of Gegenbauer polynomials of the form

$$J^{(p,q)}_n \equiv \int_0^\pi d\psi\, (2\sin\psi)^{2p}(\cos\psi)^{2q}[C^p_n(\cos\psi)]^2 \tag{A.8}$$

with $p \geqslant 1$ and q a nonnegative integer may be found from the integrals given above by using the relationship between Gegenbauer and Jacobi polynomials. Alternatively the orthogonality and recursion relations for the Gegenbauer polynomials may be employed. In either way one obtains for $q = 0, 1$

$$J_n^{(p,0)} = 2\pi \frac{\Gamma(n+2p)}{n!(n+p)\Gamma(p)^2}, \tag{A.9}$$

$$J_n^{(p,1)} = \left[\frac{p(2n+1)+(n+1)(n-1)}{2(n+p+1)(n+p-1)}(1-\delta_{n,0}\delta_{p,1}) + \frac{1}{4}\delta_{n,0}\delta_{p,1}\right] J_n^{(p,0)}. \tag{A.10}$$

The perturbation integrals for the symmetric 0^+ solutions are linear combinations of the following integrals:

$$K(a;\, b_1, b_2;\, c) \equiv \int_{-\infty}^{0} \mathrm{d}x\, (-x)^{c-1}\, {}_2F_1(a, b_1, c; x)\, {}_2F_1(a, b_2, c; x)\,, \tag{A.11}$$

$$L(a;\, b_1, b_2;\, c) \equiv \int_{-\infty}^{0} \mathrm{d}x\, (-x)^{c-1}(1-x)\, {}_2F_1(a, b_1, c; x)\, {}_2F_1(a, b_2, c; x) \tag{A.12}$$

with $c > 0$, $\min(a, b_1) + \min(a, b_2) > c$ or $c+1$, respectively (so that convergence is guaranteed). These integrals may be evaluated by employing the identity

$$\int_{-\infty}^{0} \mathrm{d}x\, (-x)^{m_1+m_2-2}\, {}_2F_1(a_1, b_1, c_1; x)\, {}_2F_1(a_2, b_2, c_2; x) = \tag{A.13}$$
$$= \frac{\Gamma(c_1, c_2)}{\Gamma(a_1, b_1, a_2, b_2)}\, G^{33}_{44}\left(1 \,\middle|\, \begin{matrix} m_1-a_1,\, m_1-b_1,\, 1-m_2;\, c_2-m_2 \\ a_2-m_2,\, b_2-m_2,\, m_1-1;\, -c_1+m_1 \end{matrix}\right),$$

that follows by writing the hypergeometric functions as Meijer functions and applying Meijer's integral formula ([9]). With the help of this formula and the recursion relations for the Meijer functions one obtains the following expressions for the integrals (A.11)-(A.12) (see appendix C of ref. ([1])):

$$K(a;\, b_1, b_2;\, c) = \frac{\Gamma(2a-c,\, b_1+b_2-c,\, a+b_1-c,\, a+b_2-c,\, c,\, c)}{\Gamma(a, a, b_1, b_2, 2a+b_1+b_2-2c)}, \tag{A.14}$$

$$L(a;\, b_1, b_2;\, c) = \tag{A.15}$$
$$= \frac{\Gamma(2a-c-1,\, b_1+b_2-c-1,\, a+b_1-c-1,\, a+b_2-c-1,\, c,\, c)}{4\Gamma(a, a, b_1, b_2, 2a+b_1+b_2-2c)}\cdot$$
$$\cdot\{(s_a+s_b-2c-2)[s_a s_b + c(c-3)(s_a+s_b) - 2c^3 + 4c^2 + 2c] +$$
$$+ (s_a-c-1)(s_b-c-1)(s_a+s_b-2c-2)^2 - \Delta^2(s_a+s_b-2c-1)(s_a-c-1)\},$$

with the abbreviations $s_a = 2a$, $s_b = b_1 + b_2$, $\Delta = b_1 - b_2$.

The normalization integral for the symmetric 0^+ solutions contains, apart from contributions of the form of (A.11), an integral of a different type, *viz.*

$$M(a;\, b_1, b_2;\, c) \equiv \int_{-\infty}^{0} \mathrm{d}x\, (-x)^{c-1}(1-x)^{-1}\, {}_2F_1(a, b_1, c; x)\, {}_2F_1(a, b_2, c; x) \tag{A.16}$$

with $c > 0$, $\min(a, b_1) + \min(a, b_2) > c - 1$. This integral is likewise related to a Meijer function of unit argument [13]:

$$M(a;\, b_1, b_2;\, c) = \frac{\Gamma(c)^2}{\Gamma(a, b_2, c-a, c-b_2)}\, G^{42}_{44}\left(1 \left| \begin{matrix} 1-a, 1-b_2;\, a+b_1-c+1, 1 \\ a-c+1, b_1-c+1, -a-b_2+c, 0 \end{matrix} \right.\right). \tag{A.17}$$

The Meijer function at the right-hand side may be written as a linear combination of two generalized hypergeometric functions; the M-integral becomes then

$$M(a;\, b_1, b_2;\, c) = \Gamma_1\, {}_4F_3\left(\begin{matrix} 2a-c+1, a+b_1-c+1, -b_2+c, a \\ 2a+b_1-c+1, a-b_2+1, a+1 \end{matrix} \middle| 1\right) + \Gamma_2\, {}_4F_3\left(\begin{matrix} a+b_2-c+1, b_1+b_2-c+1, -a+c, b_2 \\ a+b_1+b_2-c+1, -a+b_2+1, b_2+1 \end{matrix} \middle| 1\right) \tag{A.18}$$

with coefficients

$$\Gamma_1 = \frac{\Gamma(-a+b_2, 2a-c+1, a+b_1-c+1, c, c)}{\Gamma(2a+b_1-c+1, a+1, b_2, -a+c)}, \tag{A.19}$$

$$\Gamma_2 = \frac{\Gamma(a-b_2, a+b_2-c+1, b_1+b_2-c+1, c, c)}{\Gamma(a+b_1+b_2-c+1, b_2+1, a, -b_2+c)}. \tag{A.20}$$

General summation formulae for the Saalschutzian ${}_4F_3$-functions of unit argument occurring here have not been found in the literature, so that the M-integrals must be obtained by evaluating the ${}_4F_3$-functions numerically.

The expansion (A.18) for the M-integral breaks down if $a - b_2$ is an integer; this is indeed the case when the contribution $M(n;\, n+1, n+1;\, 3)$ to the normalization integral of symmetric 0^+ solutions with $r = 0$ is considered. The difficulty may be overcome by writing first $b_2 = n + 1 + \delta$ and taking the limit $\delta \to 0$ afterwards. In that way one gets

$$M(n;\, n+1, n+1;\, 3) = \frac{4}{\Gamma(n, n+1)} \lim_{\delta\to 0} \delta^{-1} \cdot \tag{A.21}$$
$$\cdot\left[-\frac{\Gamma(2n-2, 2n-1, n)}{\Gamma(-\delta, 3n-1, n+1, 3-n)}\, {}_4F_3\left(\begin{matrix} 2n-2, 2n-1, n, 2-n-\delta \\ -\delta, 3n-1, n+1 \end{matrix} \middle| 1\right) + \right.$$
$$+ \frac{\Gamma(2n-1+\delta, 2n+\delta, n+1+\delta)}{\Gamma(\delta+2, 3n+\delta, n+2+\delta, 2-n-\delta)} \cdot$$
$$\left. \cdot\, {}_4F_3\left(\begin{matrix} 2n-1+\delta, 2n+\delta, n+1+\delta, 3-n \\ \delta+2, 3n+\delta, n+2+\delta \end{matrix} \middle| 1\right)\right].$$

([13]) A. M. MATHAI and R. K. SAXENA: *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences* (Berlin, 1973), p. 37, 98.

For integer n ($\geqslant 3$) only the second term contributes and the limit $\delta \to 0$ may be performed, with the result

$$\text{(A.22)} \qquad M(n;\, n+1, n+1;\, 3) = \\ = 4(-)^{n+1} \frac{\Gamma(2n-1, 2n)}{(n-1)\Gamma(3n, n+2)}\, {}_4F_3 \left(\begin{matrix} 2n-1, 2n, n+1, 3-n \\ 2, 3n, n+2 \end{matrix} \middle| 1 \right).$$

By employing the relations between terminating Saalschutzian ${}_4F_3(1)$-functions one may prove that the right-hand side is positive and that it fulfils the inequality

$$\text{(A.23)} \qquad M(n;\, n+1, n+1;\, 3) \geqslant 4\, \frac{\Gamma(2n, 2n-2, 2n-2, 2n-3)}{\Gamma(n, n, n+1, n+1, 4n-3)}$$

for all integer $n \geqslant 3$. For noninteger n the limit in (A.21) has to be evaluated numerically.

● RIASSUNTO (*)

Si sono valutati gli integrali di normalizzazione per le soluzioni di stati strettamente legati dell'equazione spinoriale di Bethe-Salpeter che sono state dedotte recentemente. Si trova che compaiono stati fantasma quando i parametri continui che caratterizzano il tipo d'interazione fermione-bosone raggiungono un valore critico. Si usano metodi di teoria delle perturbazioni per determinare, per una coppia di fermioni con masse leggermente diverse, la forza degli accoppiamenti che danno origine a stati legati con piccoli valori dell'energia nel sistema di riferimento in quiete.

(*) *Traduzione a cura della Redazione.*

Нормировка и теория возмущений для плотно связанных состояний спинорного уравнения Бете-Салпетера.

Резюме (*). — Вычисляются нормировочные интегралы для решений спинорного уравнения Бете-Салпетера, которые описывают плотно связанные состояния. Обнаружено, что возникают состояния — « духи », когда непрерывные параметры, характеризующие тип фермион-бозонного взаимодействия, достигают критической величины. Используются методы теории возмущений для определения, для пары фермионов со слегка различными массами, величин связей, которые приводят к связанным состояниям с малыми значениями энергии в системе покоя.

(*) *Переведено редакцией.*