Physica 144A (1987) 513–529
North-Holland, Amsterdam

SMALL-WAVENUMBER DIVERGENCY IN THE MODE-COUPLING AMPLITUDES FOR THE HEAT-CURRENT TIME CORRELATION FUNCTION OF A ONE-COMPONENT PLASMA IN A MAGNETIC FIELD

A.J. SCHOOLDERMAN and L.G. SUTTORP

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Received 22 January 1987

The long-time behaviour of the heat-current time correlation function of a one-component plasma in a magnetic field is studied with the use of mode-coupling theory. The coupling of two gyro-plasmon modes gives rise to contributions with amplitudes that are divergent for small wavenumber. These imply a slowly decaying long-time tail $\sim t^{-1/2}$ of the Green–Kubo integrand. As such a behaviour would lead to a divergency in the static heat conductivity, it is concluded that mode-coupling theory with diverging coupling amplitudes is not a reliable starting point for the study of the long-time behaviour of Green–Kubo integrands.

1. Introduction

For a fluid of neutral particles the equilibrium current–current time correlation functions, which determine the integrands of the Green–Kubo integral expressions for the transport coefficients, decay like $t^{-3/2}$ for large t. This well-known property of the correlation functions has been analyzed with the use of mode-coupling theory[1–3]. In this way it has been established that the current–current time correlation functions decay slowly owing to a coupling of the currents to products of two collective modes. In particular, the heat current is coupled to a product of two sound modes and to a product of a viscous and a thermal mode. Similar statements can be made for the current–current correlation functions that occur in the Green–Kubo expressions for the shear and the bulk viscosity and for the self-diffusion coefficient.

The mode spectrum of a plasma is rather different from that of a neutral gas as a consequence of the long-range character of the Coulomb interaction. In fact, for a classical one-component plasma, consisting of a set of charged particles that move in a neutralizing background, the sound modes are replaced by plasmon modes, which oscillate at the plasma frequency ω_p. The plasma

0378-4371/87/$03.50

collective modes are basic ingredients in a calculation of the long-time tails of the current–current correlation functions by means of the mode-coupling method[4]). As expected, the plasmon modes give rise to oscillating terms in the correlation functions. A peculiar feature, however, is the occurrence of an oscillating term of the form $t^{-1/2}\cos(\omega_p t + \theta)$ in the heat-current correlation function. The slow decay $\propto t^{-1/2}$ of this term is a consequence of the anomalous effectiveness of the coupling of the heat current to a particular product of modes, viz. a viscous mode and a plasmon mode. In fact, the amplitude that describes the coupling of these modes diverges linearly for vanishingly small wavenumbers. No such anomalous coupling amplitudes are encountered in the mode-coupling expressions for the Green–Kubo integrands of the viscosities and for the velocity autocorrelation function.

The slow decay of the Green–Kubo integrand for the heat conductivity does not lead to a divergence in the static heat conductivity, since the oscillating factor furnishes an effective damping of the integrand. Nevertheless, one may wonder whether a mode-coupling expression that contains a diverging coupling amplitude is a reliable starting point for a discussion of the long-time tails of the Green–Kubo integrands. To investigate this problem it is useful to study examples of systems, where similar slowly decaying tails show up.

Recently we have studied the collective modes of a one-component plasma in a magnetic field[5]). Subsequently we have employed these modes to investigate, by means of mode-coupling theory, the long-time behaviour of the velocity autocorrelation function for a magnetized plasma[6]). It turned out that in this case the mode-coupling amplitudes remain finite for small wave-number. Moreover, it was found that, owing to the anisotropy of the system, interference effects providing an additional damping of the asymptotic time behaviour occur. In fact, the asymptotic form of the velocity autocorrelation function contains (oscillating) terms proportional to t^{-2}, $t^{-5/2}$ and t^{-3}. Hence the velocity autocorrelation function for a magnetized plasma decays faster than that of a neutral fluid or an unmagnetized plasma.

In the present paper we shall study the long-time tail of the Green–Kubo integrand for the heat conductivity of a magnetized plasma. In particular it is our aim to determine whether anomalous mode-coupling amplitudes occur in this case and whether these lead to slowly decaying terms in the Green–Kubo integrand.

2. Basic formulae

The process of heat conduction through a one-component plasma that is situated in a uniform static magnetic field depends on the orientation of the heat current with respect to the magnetic field. As a consequence one may define a longitudinal and a transverse heat conductivity coefficient, $\lambda_{\parallel}$ and $\lambda_{\perp}$, respectively. The Green–Kubo formulae for these conductivities have the form[5])

$$\lambda_i = \frac{1}{k_B T^2} \int_0^\infty \mathrm{d}t\, F_i(t)\,, \quad i = \parallel, \perp\,, \tag{2.1}$$

with T the temperature and k_B Boltzmann's constant. The heat-current time correlation functions are defined by writing

$$\begin{aligned} F(\hat{\boldsymbol{k}}, t) &= \lim_{\boldsymbol{k}\to \boldsymbol{0}} \frac{1}{k^2} \frac{1}{V} \langle [Q\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k})]^* \,\mathrm{e}^{\mathrm{i}Lt} Q\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k})\rangle \\ &= \hat{k}_{\parallel}^2 F_{\parallel}(t) + \hat{k}_{\perp}^2 F_{\perp}(t)\,. \end{aligned} \tag{2.2}$$

Here $\hat{\boldsymbol{k}}$ is a unit vector in the direction of the wave vector $\boldsymbol{k}$; its components parallel with and perpendicular to the magnetic field are denoted by $\hat{k}_{\parallel}$ and $\hat{k}_{\perp}$, respectively. Furthermore V is the volume of the system. The Liouville operator L inside the brackets indicating a canonical average acts on the projected heat current given by

$$Q\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k}) = \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k}) - \frac{h}{m}\,\boldsymbol{k}\cdot \boldsymbol{g}(\boldsymbol{k})\,. \tag{2.3}$$

The heat current $\boldsymbol{j}_\varepsilon$ consists of a kinetic and a potential part[5,7]),

$$\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k}) = \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) + \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\,, \tag{2.4}$$

with

$$\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) = \sum_\alpha \frac{\boldsymbol{k}\cdot \boldsymbol{p}_\alpha}{m} \frac{p_\alpha^2}{2m}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot \boldsymbol{r}_\alpha}\,, \tag{2.5}$$

$$\begin{aligned} \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) &= \frac{e^2}{k^2 V} \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{\boldsymbol{k}\cdot \boldsymbol{p}_\alpha}{m}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot \boldsymbol{r}_\beta} \\ &\quad - \frac{e^2}{V} \sum_{\boldsymbol{k}'(\neq 0, \neq \boldsymbol{k})} \left[\frac{\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')}{k'^2(\boldsymbol{k}-\boldsymbol{k}')^2} - \frac{\boldsymbol{k}'}{k'^2}\right] \cdot \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{\boldsymbol{p}_\alpha}{m}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot \boldsymbol{r}_\alpha + \mathrm{i}\boldsymbol{k}'\cdot \boldsymbol{r}_{\alpha\beta}}\,. \end{aligned} \tag{2.6}$$

Here $\boldsymbol{r}_\alpha$, $\boldsymbol{p}_\alpha$ are the position and momentum of particle α; furthermore e and m are the charge and the mass of the particles. The momentum density $\boldsymbol{g}(\boldsymbol{k})$ in (2.3) is given by

$$g(\boldsymbol{k}) = \sum_\alpha \boldsymbol{p}_\alpha \, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} . \tag{2.7}$$

Finally h is the enthalpy per particle, which is related to the specific heat c_V and the isothermal compressibility κ_T of the plasma,

$$h = -k_B T + \tfrac{1}{3} c_V T + 3/(n\kappa_T) . \tag{2.8}$$

According to mode-coupling theory the long-time behaviour of the heat-current time correlation function $F(\hat{\boldsymbol{k}}, t)$ is dominated by contributions originating from the coupling of the projected heat current to the product of two collective modes,

$$F(\hat{\boldsymbol{k}}, t) \simeq \lim_{\boldsymbol{k}\to\boldsymbol{0}} \frac{1}{k^2 V} \frac{1}{2} \sum_{i,j} \sum_{\boldsymbol{q}} |A_{ij}(\boldsymbol{k}, \boldsymbol{q})|^2 \exp\{-i[z_i(\boldsymbol{q}) + z_j(\boldsymbol{k}-\boldsymbol{q})]t\} . \tag{2.9}$$

The summations are extended over the collective modes (with labels i, j) and over all values of the wave vector $\boldsymbol{q}$ (and $\boldsymbol{k}-\boldsymbol{q}$) of these modes. The frequencies of the modes are denoted by $z_i(\boldsymbol{q})$. The mode-coupling amplitudes A_{ij} are given by

$$A_{ij}(\boldsymbol{k}, \boldsymbol{q}) = \frac{1}{V} \langle [Q\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k})]^* a_i(\boldsymbol{q}) a_j(\boldsymbol{k}-\boldsymbol{q}) \rangle , \tag{2.10}$$

with $a_i(\boldsymbol{q})$ the collective modes.

The set of collective modes of a magnetized one-component plasma consists of a heat mode and of four gyro-plasmon modes[5]). The frequencies of these modes are

$$z_T(\boldsymbol{k}) = \frac{-ik^2}{nc_V} (\hat{k}_\parallel^2 \lambda_\parallel + \hat{k}_\perp^2 \lambda_\perp) , \tag{2.11}$$

$$z_{\lambda,\rho}(\boldsymbol{k}) = \rho w_\lambda(\hat{\boldsymbol{k}}) - ik^2 D_{\lambda,\rho}(\hat{\boldsymbol{k}}) , \tag{2.12}$$

with $\lambda = \pm 1$, $\rho = \pm 1$. The frequencies in zeroth order of $\boldsymbol{k}$ are given by

$$w_\lambda(\hat{\boldsymbol{k}}) = \tfrac{1}{2}(\omega_p^2 + \omega_B^2 + 2\omega_p\omega_B\hat{k}_\parallel)^{1/2} + \tfrac{1}{2}\lambda(\omega_p^2 + \omega_B^2 - 2\omega_p\omega_B\hat{k}_\parallel)^{1/2} , \tag{2.13}$$

with ω_p the plasma frequency and ω_B the Larmor frequency. They are solutions of the equation

$$w_\lambda^4(\hat{\boldsymbol{k}}) - (\omega_p^2 + \omega_B^2) w_\lambda^2(\hat{\boldsymbol{k}}) + \omega_p^2 \omega_B^2 \hat{k}_\parallel^2 = 0 . \tag{2.14}$$

The coefficients $D_{\lambda,\rho}(\hat{k})$ of damping and dispersion, which satisfy the relation

$$D^*_{\lambda,\rho}(\hat{k}) = D_{\lambda,-\rho}(\hat{k}) , \tag{2.15}$$

are functions of the dynamical viscosities that have been given elsewhere[5]).

The modes $a_{\mathrm{T}}(\boldsymbol{k})$ and $a_{\lambda\rho}(\boldsymbol{k})$ are linear combinations of the conserved quantities of the system. These are the particle density

$$n(\boldsymbol{k}) = \sum_\alpha \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} , \tag{2.16}$$

the momentum density $\boldsymbol{g}(\boldsymbol{k})$ given in (2.7) and the energy density

$$\begin{aligned}\varepsilon(\boldsymbol{k}) &= \varepsilon^{\mathrm{kin}}(\boldsymbol{k}) + \varepsilon^{\mathrm{pot}}(\boldsymbol{k}) \\ &= \sum_\alpha \frac{p_\alpha^2}{2m}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} - \frac{1}{2V}\sum_{\boldsymbol{k}'(\neq\boldsymbol{0},\neq\boldsymbol{k})} \frac{e^2\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')}{k'^2(\boldsymbol{k}-\boldsymbol{k}')^2} \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha+\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}_{\alpha\beta}} .\end{aligned} \tag{2.17}$$

The heat mode is

$$a_{\mathrm{T}}(\boldsymbol{k}) = C_{\mathrm{T}}[\varepsilon(\boldsymbol{k}) - hn(\boldsymbol{k})] , \tag{2.18}$$

with the normalization constant

$$C_{\mathrm{T}} = \frac{1}{(nk_{\mathrm{B}}c_V)^{1/2}T} . \tag{2.19}$$

The gyro-plasmon modes are given by

$$\begin{aligned}a_{\lambda\rho}(\boldsymbol{k}) = C_\lambda(\hat{\boldsymbol{k}})\Big[&\frac{k_{\mathrm{D}}}{k}\, n(\boldsymbol{k}) + \frac{1}{k_{\mathrm{B}}Tc_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_{\mathrm{B}})\,\frac{k}{k_{\mathrm{D}}}\,\varepsilon(\boldsymbol{k}) \\ &+ \frac{1}{(mk_{\mathrm{B}}T)^{1/2}}\,\boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{k}})\cdot\boldsymbol{g}(\boldsymbol{k})\Big] ,\end{aligned} \tag{2.20}$$

with k_{D} the Debye wave vector and $C_\lambda(\hat{\boldsymbol{k}})$ the normalization constant,

$$C_\lambda(\hat{\boldsymbol{k}}) = \frac{\omega_{\mathrm{p}}}{(2n)^{1/2}}\left[\frac{w_\lambda^2 - \omega_B^2\hat{k}_\parallel^2}{w_\lambda^2(\omega_{\mathrm{p}}^2 + \omega_B^2) - 2\omega_{\mathrm{p}}^2\omega_B^2\hat{k}_\parallel^2}\right]^{1/2} , \tag{2.21}$$

with $w_\lambda = w_\lambda(\hat{\boldsymbol{k}})$. Furthermore the vectors $\boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{k}})$ are

$$v_{\lambda\rho}(\hat{k}) = \frac{\rho w_\lambda \omega_p}{w_\lambda^2 - \omega_B^2}\, \hat{k}_\perp + \frac{\rho\omega_p}{w_\lambda}\, \hat{k}_\parallel - \frac{i\omega_p\omega_B}{w_\lambda^2 - \omega_B^2}\, \hat{k} \wedge \hat{B} , \tag{2.22}$$

with $\hat{B}$ a unit vector in the direction of the magnetic field, $\hat{k}_\parallel = \hat{k}_\parallel \hat{B}$ and $\hat{k}_\perp = \hat{k} - \hat{k}_\parallel$.

3. The mode-coupling amplitudes A_{ij}

The mode-coupling amplitudes follow from (2.10) by substituting (2.3), (2.18) and (2.20). To proceed further we need the three-factor fluctuation formulae that are the canonical ensemble averages of the product of a current (either $k \cdot j_\varepsilon$ or $k \cdot g$) and two densities (chosen from n, g and ε). Eight of the twelve possible products vanish upon averaging, since they contain an odd number of momenta. The remaining four fluctuation formulae are considered in the appendix. Here we give the results for small but non-vanishing wave vectors k, q and $l = k - q$,

$$\frac{1}{V} \langle [k \cdot g(k)]^* g(q) n(l) \rangle = nmk_B T \frac{l^2}{k_D^2}\, k , \tag{3.1}$$

$$\frac{1}{V} \langle [k \cdot g(k)]^* g(q) \varepsilon(l) \rangle = nm(k_B T)^2 k , \tag{3.2}$$

$$\frac{1}{V} \langle [k \cdot j_\varepsilon(k)]^* g(q) n(l) \rangle = n(k_B T)^2 (k - k \cdot \hat{q}\hat{q}) , \tag{3.3}$$

$$\frac{1}{V} \langle [k \cdot j_\varepsilon(k)]^* g(q) \varepsilon(l) \rangle = n(k_B T)^3 \left(\frac{5}{3} \frac{c_V}{k_B} + \frac{3}{nk_B T\kappa_T} - \frac{1}{2} \right) k + n(k_B T)^3 \left(\frac{3}{nk_B T\kappa_T} - \frac{3}{2} \right) (k - k \cdot \hat{q}\hat{q}) . \tag{3.4}$$

With the use of these formulae we shall derive expressions for the amplitudes A_{TT}, $A_{\lambda\rho,T}$ and $A_{\lambda\rho,\lambda'\rho'}$. The first of these, which describes the coupling of the heat current to the product of two heat modes, vanishes,

$$A_{TT}(k, q) = 0 , \tag{3.5}$$

because it is the average of a function odd in the momenta.

The amplitude $A_{\lambda\rho,T}$, which contains the product of a gyro-plasmon and a heat mode, is found from (3.1)–(3.4) as

$$A_{\lambda\rho,\mathrm{T}}(\boldsymbol{k},\boldsymbol{q}) = \frac{n(k_{\mathrm{B}}T)^{5/2}}{m^{1/2}} C_\lambda(\hat{\boldsymbol{q}}) C_{\mathrm{T}} \Big[\boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{q}}) \cdot \boldsymbol{k} \Big(\frac{4}{3} \frac{c_V}{k_{\mathrm{B}}} + \frac{1}{2} \Big) - \boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{q}}) \cdot (\boldsymbol{k} - \boldsymbol{k}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \Big(\frac{1}{3} \frac{c_V}{k_{\mathrm{B}}} + \frac{1}{2} \Big) \Big] , \tag{3.6}$$

where (2.8) has been used. Insertion of the expression (2.22) for $\boldsymbol{v}_{\lambda\rho}$ gives

$$A_{\lambda\rho,\mathrm{T}}(\boldsymbol{k},\boldsymbol{q}) = \frac{n(k_{\mathrm{B}}T)^{5/2}}{m^{1/2}} C_\lambda(\hat{\boldsymbol{q}}) C_{\mathrm{T}} \times \Big\{ \frac{c_V \omega_{\mathrm{p}}}{3k_{\mathrm{B}}(w_\lambda^2 - \omega_B^2)} \Big[\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{q}}_\perp \frac{\rho(4w_\lambda^2 - \omega_B^2 \hat{q}_\parallel^2)}{w_\lambda} + \boldsymbol{k}_\parallel \cdot \hat{\boldsymbol{q}}_\parallel \frac{\rho(4w_\lambda^2 - 3\omega_B^2 - \omega_B^2 \hat{q}_\parallel^2)}{w_\lambda} - \boldsymbol{k}_\perp \cdot (\hat{\boldsymbol{q}}_\perp \wedge \hat{\boldsymbol{B}}) 3\mathrm{i}\omega_B \Big] + \frac{1}{2} \boldsymbol{k}\cdot\hat{\boldsymbol{q}}\rho \frac{w_\lambda}{\omega_{\mathrm{p}}} \Big\} , \tag{3.7}$$

with $w_\lambda = w_\lambda(\hat{\boldsymbol{q}})$ satisfying a relation of the form (2.14).

In a similar way we may evaluate the mode-coupling amplitude $A_{\lambda\rho,\lambda'\rho'}$ describing the coupling to a product of two gyro-plasmon modes. On a par with (3.6) we get

$$A_{\lambda\rho,\lambda'\rho'}(\boldsymbol{k},\boldsymbol{q}) = \frac{n(k_{\mathrm{B}}T)^{3/2}}{m^{1/2}} C_\lambda(\hat{\boldsymbol{q}}) C_{\lambda'}(\hat{\boldsymbol{l}}) \times \Big[\frac{k_{\mathrm{D}}}{l} \boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{q}}) \cdot (\boldsymbol{k} - \boldsymbol{k}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) + \frac{k_{\mathrm{D}}}{q} \boldsymbol{v}_{\lambda'\rho'}(\hat{\boldsymbol{l}}) \cdot (\boldsymbol{k} - \boldsymbol{k}\cdot\hat{\boldsymbol{l}}\hat{\boldsymbol{l}}) \Big] , \tag{3.8}$$

where only the leading terms for small $\boldsymbol{q}$ and $\boldsymbol{l}$ have been retained, since only these will be needed in the following. Substituting (2.22) we arrive at

$$A_{\lambda\rho,\lambda'\rho'}(\boldsymbol{k},\boldsymbol{q}) = \frac{n(k_{\mathrm{B}}T)^{3/2}}{m^{1/2}} C_\lambda(\hat{\boldsymbol{q}}) C_{\lambda'}(\hat{\boldsymbol{q}}) \omega_{\mathrm{p}} \omega_B \frac{k_{\mathrm{D}}}{q} \times \Big\{ (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{q}}_\perp \hat{q}_\parallel^2 - \boldsymbol{k}_\parallel \cdot \hat{\boldsymbol{q}}_\parallel \hat{q}_\perp^2) \omega_B \times \Big[\frac{\rho}{w_\lambda(w_\lambda^2 - \omega_B^2)} - \frac{\lambda'\rho'}{w_{\lambda'}(w_{\lambda'}^2 - \omega_B^2)} \Big] - \mathrm{i}\boldsymbol{k}_\perp \cdot (\hat{\boldsymbol{q}}_\perp \wedge \hat{\boldsymbol{B}}) \Big(\frac{1}{w_\lambda^2 - \omega_B^2} - \frac{1}{w_{\lambda'}^2 - \omega_B^2} \Big) \Big\} . \tag{3.9}$$

Here terms of higher than the first order in $\boldsymbol{k}$ have been omitted, since they drop out of the mode-coupling integral (2.9).

The mode-coupling amplitudes (3.7) and (3.9) behave in a qualitatively different way for small wavenumbers q. Whereas $A_{\lambda\rho,\mathrm{T}}$ remains finite for $q \rightarrow 0$, the amplitudes $A_{\lambda\rho,\lambda'\rho'}$ diverge in the limit $q \rightarrow 0$. The coupling of two gyro-plasmon modes to the heat current is thus strongly enhanced, if the modes become soft. A similar anomalously behaving coupling amplitude has been found for the heat-current correlation function of an unmagnetized plasma[4]). In that case it leads to a slowly decaying oscillating term $\propto t^{-1/2} \cos(\omega_{\mathrm{p}} t + \theta)$ in the correlation function. In the next section we will consider the consequences of the occurrence of anomalous coupling amplitudes for the magnetized plasma by evaluating the mode-coupling integral (2.9).

4. Evaluation of the mode-coupling integral

The integral (2.9), which gives, according to mode-coupling theory, the long-time behaviour of the heat-current time correlation function, may be evaluated by substituting the coupling amplitudes (3.5), (3.7), (3.9) and the mode frequencies (2.11), (2.12). For large V the discrete summation over $\boldsymbol{q}$ can be replaced by an integration. As the mode-coupling amplitudes are linear in $\boldsymbol{k}$ the limit $\boldsymbol{k} \rightarrow \boldsymbol{0}$ in (2.9) is trivially carried out. Furthermore, since in this limit the frequencies depend on $\hat{\boldsymbol{q}}$ through $\hat{q}_{\|}$ only, the integral over the azimuthal angle, in a spherical coordinate system with polar axis in the direction of the magnetic field, is performed easily. In this way an expression of the general form (2.2), with a longitudinal and a transverse heat-current correlation function, is obtained.

The integral over the wavenumber $|\boldsymbol{q}|$ is of the form $\int \mathrm{d}q\, q^n \exp(-q^2 D t)$ (with integer n), and is hence proportional to $t^{-(n+1)/2}$. As a consequence the contributions with the lowest n dominate for large t. These originate from the coupling amplitudes (3.9) for which n equals 0. The anomalous coupling amplitudes for the gyro-plasmon modes thus determine, at least in the mode-coupling picture, the dominating terms, proportional to $t^{-1/2}$, in the tail of the heat-current correlation function. In the following we shall concentrate on these slowly decaying contributions.

In general the terms $\propto t^{-1/2}$ that follow from the mode-coupling integral (2.9) are accompanied by an oscillating factor, since the sum of the gyro-plasmon mode frequencies occurring in the exponent differs from zero in the long-wavelength limit. In fact, this sum is given by

$$z_{\lambda\rho}(\boldsymbol{q}) + z_{\lambda'\rho'}(-\boldsymbol{q}) = \rho w_{\lambda}(\hat{\boldsymbol{q}}) + \lambda'\rho' w_{\lambda'}(\hat{\boldsymbol{q}}) - \mathrm{i}q^2[D_{\lambda,\rho}(\hat{\boldsymbol{q}}) + D_{\lambda',\lambda'\rho'}(\hat{\boldsymbol{q}})] , \tag{4.1}$$

up to second order in the wavenumber. Here we used the relations

$$\rho w_\lambda(-\hat{q}) = \lambda\rho w_\lambda(\hat{q}) , \tag{4.2}$$

$$D_{\lambda,\rho}(-\hat{q}) = D_{\lambda,\lambda\rho}(\hat{q}) . \tag{4.3}$$

For the particular choice $\lambda' = \lambda$, $\rho' = -\lambda\rho$, however, the zeroth-order term at the right-hand side of (4.1) vanishes, so that the corresponding contributions to the mode-coupling integral (2.9) do not contain an oscillating factor. The same statement even holds for the whole set of contributions with $\lambda' = \lambda$ and arbitrary ρ, ρ', since the modes with $\lambda' = \lambda$, $\rho' = \lambda\rho$ do not couple to the heat current, according to (3.9). The longitudinal and the transverse parts $F_i^{\lambda\lambda}$ (with $i = \|, \perp$) of these contributions $F^{\lambda\lambda}(\hat{k}, t)$ to $F(\hat{k}, t)$ thus have the form

$$F_i^{\lambda\lambda}(t) \simeq \frac{C_i^{\lambda\lambda}}{t^{1/2}} . \tag{4.4}$$

The coefficients are found as

$$C_i^{\lambda\lambda} = \frac{n^2(k_B T)^3}{\pi^{3/2} m} k_D^2 \omega_p^2 \omega_B^4 \int_0^1 d\hat{q}_\| \frac{C_\lambda^4(\hat{q})}{w_\lambda^2(w_\lambda^2 - \omega_B^2)^2} \frac{\Phi_i(\hat{q})}{[2 \operatorname{Re} D_{\lambda,1}(\hat{q})]^{1/2}} , \tag{4.5}$$

where use has been made of (2.15) and where we introduce the abbreviations

$$\Phi_\|(\hat{q}) = \hat{q}_\|^2 \hat{q}_\perp^4 , \tag{4.6}$$

$$\Phi_\perp(\hat{q}) = \tfrac{1}{2} \hat{q}_\|^4 \hat{q}_\perp^2 . \tag{4.7}$$

Substituting the normalization constant (2.21) and the explicit expressions[5]) for $D_{\lambda,l}$ in terms of dynamical viscosity coefficients we obtain from (4.5)

$$C_i^{\lambda\lambda} = \frac{(nm)^{1/2}(k_B T)^2}{4\pi^{3/2}} \omega_p^2 I_i^{\lambda\lambda}(b) , \tag{4.8}$$

with the dimensionless integral

$$I_i^{\lambda\lambda}(b) = b^4 \int_0^1 dx \frac{\Phi_i(x)}{\bar{w}_\lambda(b^2 + 1 - 2\bar{w}_\lambda^2)^2} \left[\frac{b^2 + 1 - 2\bar{w}_\lambda^2}{D_\lambda(x)}\right]^{1/2} . \tag{4.9}$$

Here $b = \omega_B/\omega_p$ is the strength of the magnetic field in dimensionless units, while $\bar{w}_\lambda = w_\lambda/\omega_p$ follows from (2.13) as

$$\bar{w}_\lambda = \tfrac{1}{2}(b^2 + 1 + 2bx)^{1/2} + \tfrac{1}{2}\lambda(b^2 + 1 - 2bx)^{1/2} . \tag{4.10}$$

Furthermore, $\Phi_i(x)$ is obtained from (4.6) and (4.7) by putting $q_\parallel = x$ and $q_\perp = (1 - x^2)^{1/2}$. Finally the denominator $D_\lambda(x)$ is

$$D_\lambda(x) = f_0 + f_1\bar{w}_\lambda + f_2\bar{w}_\lambda^2 + f_3\bar{w}_\lambda^3 , \tag{4.11}$$

with the coefficients

$$f_0 = b^2x^4(3\eta_1' - 4\eta_2' + \eta_3' + 6\zeta') + b^2x^2(-\tfrac{5}{3}\eta_1' + 4\eta_2' + \eta_3' + \eta_V' - 2\zeta') , \tag{4.12}$$

$$f_1 = bx^4(-2\eta_4'' - 4\eta_5'') + bx^2(2\eta_4'' + 2\eta_5'') , \tag{4.13}$$

$$\begin{aligned} f_2 = {} & x^4(-2\eta_1' - 2\eta_2' + 4\eta_3') \\ & + x^2[(4\eta_2' - 4\eta_3' - 6\zeta') + b^2(-\tfrac{5}{3}\eta_1' + 4\eta_2' - 2\eta_3' + \eta_V' - 2\zeta')] \\ & + [(\tfrac{2}{3}\eta_1' - 2\eta_2' - \eta_V' + 2\zeta') + b^2(\tfrac{5}{3}\eta_1' - 4\eta_2' - \eta_V' + 2\zeta')] , \end{aligned} \tag{4.14}$$

$$f_3 = bx^2(2\eta_4'' + 2\eta_5'') + b(-2\eta_4'') . \tag{4.15}$$

Here $\eta_i' = \mathrm{Re}\,\eta_i$ $(i = 1, 2, 3, V)$, $\zeta' = \mathrm{Re}\,\zeta$ and $\eta_i'' = \mathrm{Im}\,\eta_i$ $(i = 4, 5)$ are the dynamical viscosity coefficients at the frequency $z = \omega_p\bar{w}_\lambda$.

As (4.4) shows the mode-coupling expression for the time correlation functions $F_i(t)$ in the asymptotic regime contains terms $F_i^{\lambda\lambda}(t)$ that are proportional to $t^{-1/2}$ for large t. These slowly decaying terms can not be compensated by the contributions arising from the coupling of different products of modes. In fact, an explicit evaluation of the contributions $F_i^{\lambda,-\lambda}(t)$ and $F_i^{\lambda,\mathrm{T}}(t)$, which originate from the coupling of the heat current to the product of two gyro-plasmon modes with opposite λ and from the coupling to the product of a gyro-plasmon and a thermal mode, respectively, yields expressions of the form

$$F_i^{\lambda,-\lambda}(t) \simeq c_1\cos[(\omega_p + \omega_B)t + \theta_1]t^{-5/2} + c_2\cos[|\omega_p - \omega_B|t + \theta_2]t^{-5/2} , \tag{4.16}$$

$$F_i^{\lambda,\mathrm{T}}(t) \simeq c_3\cos(\omega_p t + \theta_3)t^{-5/2} , \tag{4.17}$$

for large t, so that these contributions decay much faster.

The coefficients $C_i^{\lambda\lambda}$ in (4.4), as given by (4.8) with (4.9), are in general different from zero. However, if the magnetic field is turned off the coefficients vanish. In that case the dominant terms in the mode-coupling expression for the heat-current correlation function are proportional to $t^{-1/2}\cos(\omega_p t + \theta)$ and $t^{3/2}$ as discussed in ref. 4. For weak magnetic field ($b \ll 1$) one may prove

$$I_i^{\lambda\lambda}(b) = \begin{cases} \dfrac{b^4}{(\frac{4}{3}\eta' + \eta_V')^{1/2}}\, C_i^1 & (\lambda = 1)\,, \quad (4.18)\\[2ex] \dfrac{b^2}{(2\eta)^{1/2}}\, C_i^{-1} & (\lambda = -1)\,, \quad (4.19)\end{cases}$$

with $C_\parallel^1 = \frac{8}{105}$, $C_\perp^1 = \frac{1}{35}$, $C_\parallel^{-1} = \frac{8}{15}$ and $C_\perp^{-1} = \frac{2}{15}$. Here we used the approximations $\eta_i \simeq \eta = \eta' + i\eta''$ ($i = 1, 2, 3$), $\eta_i \simeq 0$ ($i = 4, 5$), $\zeta \simeq 0$, valid for weak fields; in (4.18) the real parts of the dynamical viscosities for $z = \omega_p$ occur, whereas in (4.19) the static real viscosity coefficient appears.

In order to evaluate the mode-coupling integral (2.9), with the amplitudes (3.9) inserted, we have so far taken the limit $\boldsymbol{k} \to \boldsymbol{0}$ before carrying out the integration over $\boldsymbol{q}$. As a check on the validity of this procedure we have studied the mode-coupling integral for finite $\boldsymbol{k}$ with the help of numerical methods. In particular, we have investigated the behaviour of the Laplace transform of the integral in (2.9) for small but finite $\boldsymbol{k}$ and z, with z the Laplace variable associated with t. To simplify the calculations we neglected the frequency dependence of the dynamical viscosity coefficients contained in the damping and the dispersion coefficients of $z_{\lambda\rho}$. In this way we have found that in the limit $\boldsymbol{k} \to \boldsymbol{0}$ and for small finite z the integral is proportional to $z^{-1/2}$. This is consistent with the behaviour proportional to $t^{-1/2}$ obtained above, so that the interchange of the limit and the integration is indeed harmless.

The occurrence of a slowly decaying tail in the mode-coupling expression for the Green–Kubo integrands is a peculiarity of the heat conductivity. We have checked explicitly that the Green–Kubo integrands of the (seven) anisotropic viscosity coefficients of a one-component plasma in a magnetic field decay as $t^{-3/2}$ or even faster for large t. In a previous paper[6]) we have shown that the velocity autocorrelation function is for large t proportional to t^{-2}, $t^{-5/2}$ or even t^{-3}.

In conclusion we have found that the small-wavenumber divergency in the amplitudes for the coupling of the heat current to the product of two gyro-plasmon modes implies the presence of a term proportional to $t^{-1/2}$ in the mode-coupling expression for the long-time tail of the heat-current time correlation function. This slowly decaying term is not modulated by an oscillating factor. Hence a divergent result is found, if the heat-current time correlation function, with this long-time tail, is integrated over the time. Since this integral represents the static heat conductivity up to a factor, it would mean that in a magnetized one-component plasma the static heat conductivity is non-existent, at least for an infinite system. However, it should be remarked that it is not at all obvious that a mode-coupling expression with amplitudes diverging for small wavenumbers makes sense in the first place. It may be argued that the very concept of coupling of large-scale collective modes loses its meaning under these circumstances. It would be interesting to pursue this problem by starting from kinetic theory. However, the offending terms in the mode-coupling amplitudes stem from the potential part of the heat current $\boldsymbol{j}_\varepsilon$ in (3.3); the purely kinetic part of $\boldsymbol{j}_\varepsilon$ gives a finite contribution to the mode-coupling amplitude in the limit of small wavenumbers.

Acknowledgement

This investigation is part of the research programme of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)", which is financially supported by the "Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (Z.W.O.)".

Appendix

Three-factor fluctuation formulae

In this appendix we will derive the three-factor fluctuation formulae (3.1)–(3.4) for nonvanishing wave vectors $\boldsymbol{k}$, $\boldsymbol{q}$ and $\boldsymbol{l}$, by starting from the expressions for n, $\boldsymbol{g}$, ε and $\boldsymbol{j}_\varepsilon$ as given in (2.16), (2.7), (2.17) and (2.4), respectively. These fluctuation formulae have been discussed before in ref. 4. However, in that paper an approximation was involved to derive some of the results, which moreover contain several printing errors. In the following we shall show that no approximation is needed to derive the final formulae. Furthermore, the grand-canonical ensemble was employed in ref. 4. However, it is well known[8]) that the use of the grand-canonical ensemble for a one-component plasma may lead to divergent expressions. On the other hand, the canonical ensemble is particularly suited to describe the one-component plasma, since in this ensemble the inert neutralizing background keeps a constant density. We use the canonical ensemble throughout.

The first of the set of fluctuation formulae (3.1)–(3.4) follows directly by inserting (2.7) and (2.16) and performing the average over the momenta. Upon using the small-wave-vector expansion[9]) of the structure factor, viz. $1 + nh(\boldsymbol{k}) \simeq k^2/k_{\mathrm{D}}^2$, one recovers (3.1).

The energy density $\varepsilon(\boldsymbol{l})$ in the left-hand side of (3.2) is the sum of a kinetic and a potential term. The contribution of the former is readily evaluated, with the result

$$\frac{1}{V}\langle[\boldsymbol{k}\cdot\boldsymbol{g}(\boldsymbol{k})]^*\boldsymbol{g}(\boldsymbol{q})\varepsilon^{\mathrm{kin}}(\boldsymbol{l})\rangle = nm(k_{\mathrm{B}}T)^2\boldsymbol{k}\,, \tag{A.1}$$

up to first order in the wave vectors. When the momentum averages have been carried out, the contribution from the potential energy $\varepsilon^{\mathrm{pot}}$ reduces to a well-known two-factor fluctuation formula[10,11]):

$$\begin{aligned}\frac{1}{V}\langle[\boldsymbol{k}\cdot\boldsymbol{g}(\boldsymbol{k})]^*\boldsymbol{g}(\boldsymbol{q})\varepsilon^{\mathrm{pot}}(\boldsymbol{l})\rangle &= mk_{\mathrm{B}}T\boldsymbol{k}\,\frac{1}{V}\langle[n(\boldsymbol{l})]^*\varepsilon^{\mathrm{pot}}(\boldsymbol{l})\rangle\\ &= 3nm(k_{\mathrm{B}}T)^2\left(\frac{1}{nk_{\mathrm{B}}T\kappa_T}-1\right)\frac{l^2}{k_{\mathrm{D}}^2}\,\boldsymbol{k}\,,\end{aligned} \tag{A.2}$$

up to third order in the wave vectors.

To derive (3.3) we will consider separately the kinetic and the potential parts of the heat current $\boldsymbol{j}_\varepsilon(\boldsymbol{k})$. The former yields

$$\frac{1}{V}\langle[\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\mathrm{kin}}(\boldsymbol{k})]^*\boldsymbol{g}(\boldsymbol{q})n(\boldsymbol{l})\rangle = \frac{5}{2}\,n(k_{\mathrm{B}}T)^2\,\frac{l^2}{k_{\mathrm{D}}^2}\,\boldsymbol{k}\,, \tag{A.3}$$

in leading order of the wave vectors. The potential part gives upon averaging over the momenta

$$\begin{aligned}&\frac{1}{V}\langle[\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^*\boldsymbol{g}(\boldsymbol{q})n(\boldsymbol{l})\rangle\\ &\quad=\frac{e^2k_{\mathrm{B}}T}{V^2}\sum_{\boldsymbol{k}'(\neq\boldsymbol{0},\neq\boldsymbol{k})}\frac{1}{k'^2}\left[\boldsymbol{k}'-\frac{\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')}{(\boldsymbol{k}-\boldsymbol{k}')^2}\right]\\ &\quad\quad\times\left\langle\sum_{\substack{\alpha,\beta,\gamma\\ \alpha\neq\beta}}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}_{\alpha\beta}+\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{r}_{\alpha\gamma}}\right\rangle,\end{aligned} \tag{A.4}$$

where we omitted a term that vanishes in the thermodynamic limit. We now introduce correlation functions h_s, which for integer $s\geq 2$ are defined as

$$n^s h_s(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_{s-1}) = \frac{1}{V}\left\langle\sum_{\alpha_1,\ldots,\alpha_s}{}'\exp\left(\mathrm{i}\sum_{j=1}^{s-1}\boldsymbol{k}_j\cdot\boldsymbol{r}_{\alpha_1\alpha_{j+1}}\right)\right\rangle, \tag{A.5}$$

for $\boldsymbol{k}_j\neq\boldsymbol{0}$ $(j=1,\ldots,s-1)$ and $\Sigma_{j=1}^{s-1}\boldsymbol{k}_j\neq\boldsymbol{0}$. The prime at the summation symbol indicates that all particles $\alpha_1,\ldots,\alpha_s$ are to be different. In terms of these correlation functions the right-hand side of (A.4) becomes in the thermodynamic limit

$$\frac{e^2n^2k_BT}{l^2}(\boldsymbol{l}-\boldsymbol{l}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})[1+nh(\boldsymbol{l})]$$
$$+e^2n^2k_BT\int\frac{d\boldsymbol{k}'}{(2\pi)^3}\frac{1}{k'^2}\left[\boldsymbol{k}'-\frac{\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')}{(\boldsymbol{k}-\boldsymbol{k}')^2}\right]$$
$$\times[nh_3(-\boldsymbol{k}',\boldsymbol{l})+h(\boldsymbol{k}')+h(\boldsymbol{l}-\boldsymbol{k}')]\,, \tag{A.6}$$

where we replaced the summation by an integration. In the first term, which originates from the contribution $\boldsymbol{k}'=\boldsymbol{l}$ in (A.4), we may insert the small-wave-vector expression for $1+nh$, as before. In the second term we expand the integrand around $\boldsymbol{k}=\boldsymbol{0}$, for fixed $\boldsymbol{l}$, so that we get from (A.6)

$$n(k_BT)^2(\boldsymbol{l}-\boldsymbol{l}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})$$
$$+e^2n^2k_BT\mathcal{P}\int\frac{d\boldsymbol{k}'}{(2\pi)^3}\frac{1}{k'^2}(\boldsymbol{k}-\boldsymbol{k}\cdot\hat{\boldsymbol{k}}'\hat{\boldsymbol{k}}')$$
$$\times[nh_3(-\boldsymbol{k}',\boldsymbol{l})+h(\boldsymbol{k}')+h(\boldsymbol{l}-\boldsymbol{k}')]\,, \tag{A.7}$$

where $\boldsymbol{k}'=\boldsymbol{0}$ is excluded from the principal value of the integral. The combination of correlation functions that occurs between the square brackets vanishes for $\boldsymbol{l}=\boldsymbol{0}$, since the three-particle correlation function satisfies a perfect-screening relation[11,12]), which reads in Fourier space

$$nh_3(\boldsymbol{k},\boldsymbol{0})+2h(\boldsymbol{k})=0\,, \tag{A.8}$$

for arbitrary $\boldsymbol{k}$. Hence the second term in (A.7) is at least of second order in the wave vectors $\boldsymbol{k}$ and $\boldsymbol{l}$, so that it may be neglected with respect to the first term in (A.7). Consequently we obtain, by elimination of $\boldsymbol{l}$ in favour of $\boldsymbol{k}$,

$$\frac{1}{V}\langle[\boldsymbol{k}\cdot\boldsymbol{j}_F^{\mathrm{pot}}(\boldsymbol{k})]^*g(\boldsymbol{q})n(\boldsymbol{l})\rangle=n(k_BT)^2(\boldsymbol{k}-\boldsymbol{k}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})\,, \tag{A.9}$$

up to leading order in the wave vectors. Comparing with the kinetic contribution (A.3) we conclude that the potential contribution dominates. Thus (3.3) is proved.

It should be noted that in the course of the derivation the (exact) property of perfect screening was enough to proceed to the suppression of the second term in (A.7). In ref. 4 the so-called convolution approximation[13]) has been invoked to derive the required three-factor fluctuation formulae. This approximation becomes exact in the limit of small wave vectors. However, it has to be used inside an integral, so that one of the wave vectors in h_3 is not necessarily small.

Fortunately the perfect screening property, which is contained in the convolution approximation, is the only feature that is essential in the derivation.

Finally we turn to the fourth fluctuation formula of section 3. Both $\boldsymbol{j}_\varepsilon$ and ε are the sum of a kinetic and a potential part, so that the fluctuation expression falls apart into four terms. Of these the purely kinetic contribution is trivially obtained,

$$\frac{1}{V}\,\langle [\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{kin}}(\boldsymbol{k})]^* \boldsymbol{g}(\boldsymbol{q})\,\varepsilon^{\mathrm{kin}}(\boldsymbol{l})\rangle = 5n(k_{\mathrm{B}}T)^3\boldsymbol{k}\,, \tag{A.10}$$

up to terms of leading order in the wave vectors.

The mixed potential–kinetic contribution to (3.4) is similar to (A.4) once the momentum variables have been averaged. Upon introducing correlation functions according to (A.5) we get, on a par with (A.6),

$$\begin{aligned}\frac{1}{V}\,\langle \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* \boldsymbol{g}(\boldsymbol{q})\,\varepsilon^{\mathrm{kin}}(\boldsymbol{l})\rangle &= \frac{3e^2n^2(k_{\mathrm{B}}T)^2}{2l^2}\,(\boldsymbol{l}-\boldsymbol{l}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})[1+nh(\boldsymbol{l})]\\ &+ e^2n^2(k_{\mathrm{B}}T)^2\int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\,\frac{1}{k'^2}\left[\boldsymbol{k}' - \frac{\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')}{(\boldsymbol{k}-\boldsymbol{k}')^2}\right]\\ &\times [\tfrac{3}{2}nh_3(-\boldsymbol{k}',\boldsymbol{l}) + \tfrac{5}{2}h(\boldsymbol{k}') + \tfrac{3}{2}h(\boldsymbol{l}-\boldsymbol{k}')]\,.\end{aligned} \tag{A.11}$$

The first term is evaluated by expansion of $1 + nh$ for small $\boldsymbol{l}$, as before. In the second term we expand the first factor in square brackets in powers of $\boldsymbol{k}$, while the perfect-screening relation (A.8) is used to eliminate h_3, for small $\boldsymbol{l}$. The leading terms of (A.11) are then found as

$$\begin{aligned}\frac{1}{V}\,\langle [\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\mathrm{pot}}(\boldsymbol{k})]^* \boldsymbol{g}(\boldsymbol{q})\,\varepsilon^{\mathrm{kin}}(\boldsymbol{l})\rangle &= \tfrac{3}{2}n(k_{\mathrm{B}}T)^3(\boldsymbol{k}-\boldsymbol{k}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})\\ &+ e^2n^2(k_{\mathrm{B}}T)^2\mathscr{P}\int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\,\frac{1}{k'^2}\,(\boldsymbol{k}-\boldsymbol{k}\cdot\hat{\boldsymbol{k}}'\hat{\boldsymbol{k}}')h(\boldsymbol{k}')\,,\end{aligned} \tag{A.12}$$

where the principal value symbol indicates the exclusion of $\boldsymbol{k}' = \boldsymbol{0}$ from the integration. The integral is equal to $\frac{4}{3}\boldsymbol{k}u^{\mathrm{pot}}/ne^2$, with u^{pot} the potential part of the internal energy per particle; it may be expressed in κ_T and c_V,

$$\frac{u^{\mathrm{pot}}}{k_{\mathrm{B}}T} = \frac{c_V}{4k_{\mathrm{B}}} + \frac{9}{4nk_{\mathrm{B}}T\kappa_T} - \frac{21}{8}\,. \tag{A.13}$$

The mixed kinetic–potential contribution to (3.4) may be calculated in the same way as (A.2); the result is

$$\frac{1}{V}\,\langle [\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\rm kin}(\boldsymbol{k})]^* \boldsymbol{g}(\boldsymbol{q})\varepsilon^{\rm pot}(\boldsymbol{l})\rangle = \frac{15}{2}\, n(k_{\rm B}T)^3\Big(\frac{1}{nk_{\rm B}T\kappa_T}-1\Big)\,\frac{l^2}{k_{\rm D}^2}\,\boldsymbol{k}\,, \qquad \text{(A.14)}$$

in leading (third) order in the wave vectors.

The most difficult part of (3.4) is the purely potential contribution. After averaging over the momenta and introducing the correlation functions (A.5) up to fourth order we get

$$\begin{aligned}
\frac{1}{V}\,\langle [\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^{\rm pot}(\boldsymbol{k})]^* \boldsymbol{g}(\boldsymbol{q})\varepsilon^{\rm pot}(\boldsymbol{l})\rangle &= -\frac{1}{2}\, e^4 n^2 k_{\rm B}T \int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\,\frac{\mathrm{d}\boldsymbol{k}''}{(2\pi)^3} \\
&\times \Big[\boldsymbol{k}' - \frac{\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')}{(\boldsymbol{k}-\boldsymbol{k}')^2}\Big]\,\frac{\boldsymbol{k}''\cdot(\boldsymbol{l}-\boldsymbol{k}'')}{k'^2 k''^2(\boldsymbol{l}-\boldsymbol{k}'')^2} \\
&\times [n^2 h_4(-\boldsymbol{k}', \boldsymbol{l}-\boldsymbol{k}'', \boldsymbol{k}'') + nh_3(-\boldsymbol{k}', \boldsymbol{k}'') + nh_3(-\boldsymbol{k}', \boldsymbol{l}-\boldsymbol{k}'') \\
&+ nh_3(\boldsymbol{l}-\boldsymbol{k}'-\boldsymbol{k}'', \boldsymbol{k}'') + nh_3(-\boldsymbol{k}'+\boldsymbol{k}'', \boldsymbol{l}-\boldsymbol{k}'') \\
&+ h(-\boldsymbol{k}'+\boldsymbol{k}'') + h(\boldsymbol{l}-\boldsymbol{k}'-\boldsymbol{k}'')] \\
&- e^4 n^4 k_{\rm B}T\mathcal{P}\int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\Big[\boldsymbol{k}' - \frac{\boldsymbol{k}'\cdot(\boldsymbol{k}-\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')}{(\boldsymbol{k}-\boldsymbol{k}')^2}\Big]\,\frac{\boldsymbol{k}'\cdot(\boldsymbol{l}-\boldsymbol{k}')}{k'^4(\boldsymbol{l}-\boldsymbol{k}')^2} \\
&\times [1+nh(\boldsymbol{k}')][1+nh(\boldsymbol{l}-\boldsymbol{k}')] \\
&- \frac{1}{2}\,\frac{e^4 n^3 k_{\rm B}T}{l^2}\,(\boldsymbol{l}-\boldsymbol{l}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})\int \frac{\mathrm{d}\boldsymbol{k}''}{(2\pi)^3}\,\frac{\boldsymbol{k}''\cdot(\boldsymbol{l}-\boldsymbol{k}'')}{k''^2(\boldsymbol{l}-\boldsymbol{k}'')^2} \\
&\times [nh_3(-\boldsymbol{l}, \boldsymbol{l}-\boldsymbol{k}'') + h(\boldsymbol{k}'') + h(\boldsymbol{l}-\boldsymbol{k}'')]\,. \qquad \text{(A.15)}
\end{aligned}$$

As before the principal value symbol points to the exclusion of the origin from the integration. The last term at the right-hand side of (A.15) is proportional to a two-factor fluctuation expression which may be evaluated as in (A.2). In fact, one has

$$\begin{aligned}
\frac{1}{V}\,\langle [n(\boldsymbol{l})]^*\varepsilon^{\rm pot}(\boldsymbol{l})\rangle &= -\frac{1}{2}\, n^2 e^2\int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\,\frac{\boldsymbol{k}'\cdot(\boldsymbol{l}-\boldsymbol{k}')}{k'^2(\boldsymbol{l}-\boldsymbol{k}')^2} \\
&\times [nh_3(-\boldsymbol{l}, \boldsymbol{l}-\boldsymbol{k}') + h(\boldsymbol{k}') + h(\boldsymbol{l}-\boldsymbol{k}')]\,. \qquad \text{(A.16)}
\end{aligned}$$

Likewise, the first two terms may be related to another known two-factor fluctuation expression, since one may prove for small $\boldsymbol{k}$

$$\frac{1}{V}\langle[\varepsilon^{\text{pot}}(\boldsymbol{k})]^*\varepsilon^{\text{pot}}(\boldsymbol{k})\rangle = \frac{1}{4}n^2e^4\int\frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{k}''}{(2\pi)^3}\frac{1}{k'^2k''^2}$$
$$\times[n^2h_4(-\boldsymbol{k}',-\boldsymbol{k}'',\boldsymbol{k}'') + nh_3(-\boldsymbol{k}',\boldsymbol{k}'') + nh_3(-\boldsymbol{k}',-\boldsymbol{k}'')$$
$$+ nh_3(-\boldsymbol{k}'-\boldsymbol{k}'',\boldsymbol{k}'') + nh_3(-\boldsymbol{k}'+\boldsymbol{k}'',-\boldsymbol{k}'')$$
$$+ h(-\boldsymbol{k}'+\boldsymbol{k}'') + h(-\boldsymbol{k}'-\boldsymbol{k}'')]$$
$$+\frac{1}{2}n^2e^4\int\frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\frac{1}{k'^4}[1+nh(\boldsymbol{k}')]^2\,. \tag{A.17}$$

The left-hand side is equal to $nk_BT^2(c_V - \frac{3}{2}k_B)$, so that the evaluation of (A.15) is now complete. The result is

$$\frac{1}{V}\langle[\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\text{pot}}(\boldsymbol{k})]^*\boldsymbol{g}(\boldsymbol{q})\varepsilon^{\text{pot}}(\boldsymbol{l})\rangle$$
$$= \frac{4}{3}n(k_BT)^3\Big(\frac{c_V}{k_B}-\frac{3}{2}\Big)\boldsymbol{k} + 3n(k_BT)^3\Big(\frac{1}{nk_BT\kappa_T}-1\Big)(\boldsymbol{k}-\boldsymbol{k}\cdot\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})\,, \tag{A.18}$$

up to terms of first order in the wave vectors. Upon adding (A.10), (A.12) with (A.13) inserted, (A.14) and (A.18) we arrive at (3.4).

References

1) M.H. Ernst, E.H. Hauge and J.M.J. van Leeuwen, Phys. Lett. A **34** (1971) 419; Phys. Rev. A **4** (1971) 2055; J. Stat. Phys. **15** (1976) 7.
2) K. Kawasaki, Progr. Theor. Phys. **45** (1971) 1691, **46** (1971) 1299.
3) Y. Pomeau, Phys. Rev. A **5** (1972) 2569, **7** (1973) 1134; Phys. Lett. A **38** (1972) 245.
4) M.C. Marchetti and T.R. Kirkpatrick, Phys. Rev. A **32** (1985) 2981.
5) L.G. Suttorp and A.J. Schoolderman, Physica **141A** (1987) 1.
6) L.G. Suttorp and A.J. Schoolderman, Physica **143A** (1987) 494.
7) B. Bernu and P. Vieillefosse, Phys. Rev. A **18** (1978) 2345.
8) E.H. Lieb and H. Narnhofer, J. Stat. Phys. **12** (1975) 291, **14** (1976) 465.
9) F.H. Stillinger Jr and R. Lovett, J. Chem. Phys. **49** (1968) 1991.
10) L.G. Suttorp and J.S. Cohen, Physica **133A** (1985) 357.
11) P. Vieillefosse, J. Stat. Phys. **41** (1985) 1015.
12) L. Blum, C. Gruber, J.L. Lebowitz and P. Martin, Phys. Rev. Lett. **48** (1982) 1769.
13) T. O'Neil and N. Rostoker, Phys. Fluids **8** (1965) 1109.