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LONG-TIME TAILS OF THE HEAT-CONDUCTIVITY TIME CORRELATION FUNCTIONS FOR A MAGNETIZED PLASMA: A KINETIC-THEORY APPROACH

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The long-time behaviour of the longitudinal and the transverse heat-conductivity time correlation functions for a magnetized one-component plasma is studied by means of kinetic theory. To that end these correlation functions, which are defined as the inverse Laplace transforms of the dynamic heat conductivity coefficients, are expressed in terms of matrix elements of the kernel that appears in the kinetic equation for the phase-space density time correlation function. The explicit expression for the collision kernel in the disconnected approximation is used to write the dominant contributions to the heat-conductivity time correlation functions for large t as integrals over products of reduced time correlation functions for the particle density, the momentum density and the kinetic-energy density. By substituting the asymptotic expressions for the latter the decay of the long-time tails of the heat-conductivity time correlation functions is found to be proportional to $t^{-1/2}$. Hence, the results obtained by this kinetic approach corroborate the $t^{-1/2}$-tails of the Green–Kubo integrands for the heat conductivities that have been derived previously with the help of mode-coupling theory.

1. Introduction

Some twenty years ago Alder and Wainwright [1] found, by means of molecular dynamics calculations, that the velocity autocorrelation function for a fluid of neutral particles decays like $t^{-3/2}$ for large t. The algebraic decay of the long-time tail of this time correlation function can be explained with the use of mode-coupling theory [2, 3].

Recently mode-coupling theory has been used to evaluate the long-time behaviour of the Green–Kubo integrands for the self-diffusion, the viscosity and the heat conductivity coefficients of an unmagnetized [4] and a magnetized [5, 6] one-component plasma. For the unmagnetized plasma, the Green–Kubo integrand for the heat conductivity was found [4] to decay like $t^{-1/2}\cos(\omega_{\mathrm{p}}t+\Theta)$ with ω_{p} the plasma frequency. A consequence of this slow decay is that the dynamic heat conductivity coefficient $\lambda(\omega)$ for the unmagnetized plasma is divergent for the frequency $\omega = \omega_p$. In the case of a magnetized plasma the Green–Kubo integrands for the longitudinal and the transverse heat conductivity coefficients behave as $t^{-1/2}$ for long times [6]. This long-time behaviour leads to a divergency in the static heat conductivity coefficients.

An alternative method to study static and dynamic transport coefficients is furnished by kinetic theory. Starting from a formal kinetic equation for the phase-space density time correlation function one may write the transport coefficients as integrals of "kinetic transport-coefficient time correlation functions". These time correlation functions (t.c.f.'s) are given in terms of elements of a frequency matrix. The latter consists of a direct and an indirect part, which are both derived from the collision kernel of the kinetic equation by means of a projection operator method. It may be argued [7]–[10] that the long-time behaviour of the kinetic transport-coefficient t.c.f. is governed by the collision kernel in the "disconnected approximation". Upon adopting this point of view one can derive the long-time properties of the kinetic transport-coefficient t.c.f.'s from those of a set of reduced t.c.f.'s which describe the correlations between the particle density, the momentum density and the kinetic-energy density. The results can then be compared to those valid for the corresponding Green–Kubo integrands. Although the integrals of the kinetic transport-coefficient t.c.f.'s and of the associated Green–Kubo integrands should lead to the same static transport coefficients, the time-dependent functions themselves do not necessarily coincide. In particular, the long-time tails of the time-dependent functions may be different. However, if a divergency in a static transport coefficient is found as a consequence of a slowly decaying Green–Kubo integrand, one expects a similar divergency to arise from the tail of the kinetic transport-coefficient time correlation function.

The kinetic approach to determine long-time tails for time correlation functions of plasmas, as sketched above, has been employed some time ago [8, 10] to study the long-time behaviour of the kinetic t.c.f. for the shear viscosity of an unmagnetized plasma. More recently the same method has been used to calculate the long-time behaviour of the kinetic t.c.f. for the heat conductivity of a plasma, again for the unmagnetized case [11]. Although in the latter paper only contributions to the long-time tail that stem from the direct part of the frequency matrix have been considered, the result is in agreement with the findings from mode-coupling theory, as given above, apart from a term which does not contribute to the value of the static heat conductivity coefficient [11, 12].

In the present paper we will use the kinetic method to study the long-time behaviour of the kinetic time correlation function for the heat conductivity of a magnetized plasma. In doing so we will be able to assess the reality of the divergency in the static heat conductivity coefficient that we have encountered in our earlier work [6]. In the course of our treatment we shall also take the opportunity to extend the calculations of ref. [11] on the unmagnetized plasma, by including the contributions of the indirect part of the frequency matrix.

As a model we adopt the classical one-component plasma, which consists of charged particles immersed in an inert, neutralizing background. The interaction between the particles and with the background is purely electrostatic. The external magnetic field is assumed to be static and uniform in space.

The paper is organized as follows. After a review of the kinetic-theory method in section 2, the direct and the indirect parts of the frequency matrix element, which determines the kinetic t.c.f. for the heat conductivity, are written in terms of reduced t.c.f.'s in section 3 and 4, respectively. In section 5 the long-time behaviour of the kinetic heat-conductivity t.c.f. for an unmagnetized plasma is evaluated, with the inclusion of possible contributions from the indirect part of the frequency matrix. Finally, in section 6 we study the long-time behaviour of the kinetic t.c.f. for the longitudinal and the transverse heat conductivity coefficients of a magnetized plasma. We conclude with a discussion of our findings.

2. Frequency matrix

With the help of projection-operator techniques one can derive a kinetic equation for the Fourier–Laplace transform of the phase-space density time correlation function $C(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z)$ with $\boldsymbol{k}$ the wave vector, $\boldsymbol{p}$ and $\boldsymbol{p}'$ momentum variables and z the Laplace frequency variable. The kinetic equation can be written formally as [13]

$$[z - \Sigma(\boldsymbol{k}, z)]C(\boldsymbol{k}, z) = \tilde{C}(\boldsymbol{k}) , \tag{2.1}$$

where the momentum variables have been suppressed. The static correlation function $\tilde{C}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}')$ is given by $\lim_{z\to\infty} zC(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}', z)$. The kernel in momentum space $\Sigma(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z)$ is the sum of a free-streaming term,

$$\varphi^0(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = \frac{\boldsymbol{k}\cdot\boldsymbol{p}}{m}\,\delta(\boldsymbol{p}-\boldsymbol{p}') , \tag{2.2}$$

for particles with mass m, a Lorentz-force term,

$$\varphi^{\mathrm{L}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = -\mathrm{i}\omega_B(\boldsymbol{p}\wedge\hat{\boldsymbol{B}})\cdot\boldsymbol{\nabla}_p\delta(\boldsymbol{p}-\boldsymbol{p}') , \tag{2.3}$$

with $\omega_B = eB/mc$ the Larmor frequency in the magnetic field with strength B (e

is the charge of the particles) and $\hat{\boldsymbol{B}}$ a unit vector in the direction of the magnetic field, and a memory kernel φ, which is determined by the interparticle interaction. The latter is the sum of a static kernel,

$$\varphi^{s}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}') = -\frac{\boldsymbol{k} \cdot \boldsymbol{p}}{m} n f_0(p) c(k) , \tag{2.4}$$

with $c(k)$ the direct correlation function and $f_0(p)$ the normalized Maxwell–Boltzmann distribution, and a collision kernel φ^{c}, which will be discussed below.

The frequency matrix $\Omega_{\mu\nu}$, which determines the hydrodynamic modes [13], is given by

$$\Omega_{\mu\nu}(\boldsymbol{k}, z) = \langle \mu | \Sigma | \nu \rangle + \langle \mu | \Sigma \bar{Q} \frac{1}{z - \bar{Q}\Sigma\bar{Q}} \bar{Q}\Sigma | \nu \rangle . \tag{2.5}$$

Here we used the notation

$$\langle \mu | A | \nu \rangle = \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{p}'\, \psi_\mu(\boldsymbol{p}) A(\boldsymbol{p}, \boldsymbol{p}') \psi_\nu(\boldsymbol{p}') f_0(p') . \tag{2.6}$$

The functions $\psi_\mu(\boldsymbol{p})$, with $\mu = 0, \ldots, 4$, are polynomials of the momentum variable:

$$\begin{aligned} &\psi_0(\boldsymbol{p}) = 1, \qquad \psi_i(\boldsymbol{p}) = \frac{p_i}{(m k_\mathrm{B} T)^{1/2}} \quad (i = 1, 2, 3) , \\ &\psi_4(\boldsymbol{p}) = \frac{1}{\sqrt{6}} \left(\frac{p^2}{m k_\mathrm{B} T} - 3 \right) , \end{aligned} \tag{2.7}$$

with T the temperature. They obey the orthonormality condition

$$\int \mathrm{d}\boldsymbol{p}\, \psi_\mu(\boldsymbol{p}) \psi_\nu(\boldsymbol{p}) f_0(p) = \delta_{\mu\nu} . \tag{2.8}$$

The projector $\bar{Q}$ is the complement of the projector $\bar{P}$, which projects a momentum-dependent function onto the space spanned by the states $|\mu\rangle$, with $\mu = 0, \ldots, 4$.

The hydrodynamic mode frequencies are determined by the zeros of the matrix $z\delta_{\mu\nu} - \Omega_{\mu\nu}(\boldsymbol{k}, z)$. For small wavenumber k the determinant of this matrix is found to factorize. The heat mode is given by the solution of $z - z_T(z) = 0$ with

$$z_T(z) = \left(1 - \frac{3k_\mathrm{B}}{2c_V}\right) z + \frac{3k_\mathrm{B}}{2c_V} \Omega_{44}(\boldsymbol{k}, z) , \tag{2.9}$$

where c_V is the specific heat. The solution is, for small wavenumber k:

$$z_T = -\frac{\mathrm{i}}{nc_V} k^2(\lambda_\parallel \hat{k}_\parallel^2 + \lambda_\perp \hat{k}_\perp^2) , \tag{2.10}$$

where $\hat{k}_\parallel = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}}$ is the component of the unit vector $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$ in the direction of the magnetic field and $\hat{\boldsymbol{k}}_\perp = \hat{\boldsymbol{k}} - \hat{k}_\parallel \hat{\boldsymbol{B}}$ the transverse part of $\hat{\boldsymbol{k}}$. The longitudinal and the transverse heat conductivity coefficients, $\lambda_\parallel$ and $\lambda_\perp$, are obtained by taking the limit $z \to \mathrm{i}0$ of the expression

$$\lambda_\parallel(z)\hat{k}_\parallel^2 + \lambda_\perp(z)\hat{k}_\perp^2 = \mathrm{i}\,\frac{3}{2}\, nk_\mathrm{B} \lim_{\boldsymbol{k}\to\boldsymbol{0}} \frac{1}{k^2}\,[\Omega_{44}(\boldsymbol{k}, z) - \Omega_{44}(\boldsymbol{0}, z)] \,. \tag{2.11}$$

The frequency-dependent coefficients $\lambda_i(z)$, $i = \parallel, \perp$, are the Laplace transforms of time-dependent expressions. We write

$$\lambda_i(z) = \frac{1}{k_\mathrm{B} T^2} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}zt} F_i(t) \qquad (i = \parallel, \perp) , \tag{2.12}$$

which, in the limit $z \to \mathrm{i}0$, resemble the Green–Kubo formulae for the heat conductivity coefficients. We call the functions $F_i(t)$ the "kinetic heat-conductivity time correlation functions". Our aim is to study the long-time behaviour of $F_i(t)$. This will be done with the use of a special expression for the collision kernel φ^c. In the remainder of this section we will derive this expression.

The collision kernel φ^c can be written as the sum of a connected part and a disconnected part [9, 10, 14]. The latter describes collision processes in which the interaction of the colliding particles with the remainder of the system is accounted for by a pair of independent one-particle propagators. In the case of an unmagnetized one-component plasma an explicit expression for the disconnected collision kernel has been derived. It can be regarded as a generalisation of the Balescu–Guernsey–Lenard collision kernel to finite wavenumber and finite frequency and without restrictions on the strength of the coupling of the plasma. For a plasma in a magnetic field the disconnected collision kernel has the same form as for the unmagnetized plasma. However, the time-dependent phase-space density correlation function C appearing in φ^c now depends on the magnetic field. We assume that the disconnected part of the collision kernel determines the long-time behaviour of the time correlation functions and we will replace φ^c by its disconnected part from now on. The time-dependent collision kernel then gets the form

$$
\begin{aligned}
&\varphi^{c}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', t) f_0(p') = \\
&- k_B T \int \frac{d\boldsymbol{q}}{(2\pi)^3}\, c(q) v(q) \boldsymbol{q} \cdot \boldsymbol{\nabla}_p \boldsymbol{q} \cdot \boldsymbol{\nabla}_{p'} [C(\boldsymbol{k} - \boldsymbol{q}, \boldsymbol{p}, \boldsymbol{p}', t) S^{nn}(\boldsymbol{q}, t)] \\
&- \frac{k_B T}{2n} \int \frac{d\boldsymbol{q}}{(2\pi)^3}\, c(q) v(\boldsymbol{k} - \boldsymbol{q}) \\
&\times [\boldsymbol{q} \cdot \boldsymbol{\nabla}_p (\boldsymbol{k} - \boldsymbol{q}) \cdot \boldsymbol{\nabla}_{p'} C(\boldsymbol{k} - \boldsymbol{q}, \boldsymbol{p}, \bar{\boldsymbol{p}}'', t) C(\boldsymbol{q}, \bar{\boldsymbol{p}}''', \boldsymbol{p}', t) \\
&+ (\boldsymbol{k} - \boldsymbol{q}) \cdot \boldsymbol{\nabla}_p \boldsymbol{q} \cdot \boldsymbol{\nabla}_{p'} C(\boldsymbol{q}, \boldsymbol{p}, \bar{\boldsymbol{p}}'', t) C(\boldsymbol{k} - \boldsymbol{q}, \bar{\boldsymbol{p}}''', \boldsymbol{p}', t)] \\
&- \frac{k_B T}{2} \int \frac{d\boldsymbol{q}}{(2\pi)^3}\, v(q) a(\boldsymbol{q}, \boldsymbol{k} - \boldsymbol{q}) \\
&\times [\boldsymbol{k} \cdot \boldsymbol{\nabla}_p \boldsymbol{q} \cdot \boldsymbol{\nabla}_{p'} f_0(p) S^{nn}(\boldsymbol{q}, t) C(\boldsymbol{k} - \boldsymbol{q}, \bar{\boldsymbol{p}}'', \boldsymbol{p}', t) \\
&+ \boldsymbol{q} \cdot \boldsymbol{\nabla}_p \boldsymbol{k} \cdot \boldsymbol{\nabla}_{p'} f_0(p') C(\boldsymbol{k} - \boldsymbol{q}, \boldsymbol{p}, \bar{\boldsymbol{p}}'', t) S^{nn}(\boldsymbol{q}, t)] \,. \qquad (2.13)
\end{aligned}
$$

The Fourier transform of the Coulomb potential is denoted by $v(k)$. The function $a(\boldsymbol{k}, \boldsymbol{q})$ depends on the two- and three-particle equilibrium correlation functions; its explicit expression is not needed here. We used the notation

$$
C(\boldsymbol{q}, \boldsymbol{p}, \bar{\boldsymbol{p}}', t) = \int d\boldsymbol{p}'\; C(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{p}', t) \,. \qquad (2.14)
$$

The dynamic structure factor $S^{nn}(\boldsymbol{k}, t)$ is defined by

$$
n S^{nn}(\boldsymbol{k}, t) = C(\boldsymbol{k}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{p}}', t) \,. \qquad (2.15)
$$

The collision kernel (2.13) can be expanded in a complete set of functions in momentum space $\{\psi_\alpha(\boldsymbol{p})\}$ (and $\{\psi_\alpha(\boldsymbol{p}')\}$). We choose the orthonormal set starting with $\{\psi_\mu(\boldsymbol{p})\}$ with $\mu = 0, \ldots, 4$. When we use

$$
\delta(\boldsymbol{p} - \boldsymbol{p}') = \sum_\alpha \psi_\alpha(\boldsymbol{p}) \psi_\alpha(\boldsymbol{p}') f_0(p') \qquad (2.16)
$$

and perform partial integrations with respect to the momentum variables, factors $\boldsymbol{\nabla}_p \psi_\alpha(\boldsymbol{p})$ appear in the expression for the collision kernel. If we write

$$
\boldsymbol{\nabla}_p \psi_\alpha(\boldsymbol{p}) = \sum_\beta c_\beta^{(\alpha)} \psi_\beta(\boldsymbol{p}) \,, \qquad (2.17)
$$

we get on account of the orthonormality of the set $\{\psi_\alpha(\boldsymbol{p})\}$:

$$
-\boldsymbol{\nabla}_p [\psi_\alpha(\boldsymbol{p}) f_0(p)] = \sum_\beta c_\alpha^{(\beta)} \psi_\beta(\boldsymbol{p}) \,. \qquad (2.18)
$$

With the help of this relation we obtain from (2.13) the expanded form for the collision kernel

$$\varphi^{c}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', t) f_0(p') = \sum_{\alpha\beta} \nabla_{p}[\psi_\alpha(\boldsymbol{p}) f_0(p)]$$
$$\cdot \left\{ -k_{\rm B}T \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) v(q) \boldsymbol{q}\boldsymbol{q} \bar{G}_{\alpha\beta}(\boldsymbol{k}-\boldsymbol{q}, t) S^{nn}(\boldsymbol{q}, t) \right.$$
$$- \frac{k_{\rm B}T}{2n} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) v(\boldsymbol{k}-\boldsymbol{q}) [\boldsymbol{q}(\boldsymbol{k}-\boldsymbol{q}) \bar{G}_{\alpha 0}(\boldsymbol{k}-\boldsymbol{q}, t) \bar{G}_{0\beta}(\boldsymbol{q}, t)$$
$$+ (\boldsymbol{k}-\boldsymbol{q})\boldsymbol{q} \bar{G}_{\alpha 0}(\boldsymbol{q}, t) \bar{G}_{0\beta}(\boldsymbol{k}-\boldsymbol{q}, t)]$$
$$- \frac{k_{\rm B}T}{2n} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, v(q) a(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q}) [\boldsymbol{k}\boldsymbol{q} \delta_{\alpha 0} S^{nn}(\boldsymbol{q}, t) \bar{G}_{0\beta}(\boldsymbol{k}-\boldsymbol{q}, t)$$
$$\left. + \boldsymbol{q}\boldsymbol{k} \bar{G}_{\alpha 0}(\boldsymbol{k}-\boldsymbol{q}, t) S^{nn}(\boldsymbol{q}, t) \delta_{\beta 0}] \right\}$$
$$\cdot \nabla_{p'}[\psi_\beta(\boldsymbol{p}') f_0(p')] , \tag{2.19}$$

where we employed the abbreviations

$$\bar{G}_{\alpha\beta}(\boldsymbol{q}, t) = \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{p}'\, \psi_\alpha(\boldsymbol{p}) C(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{p}', t) \psi_\beta(\boldsymbol{p}') . \tag{2.20}$$

These functions are, for α, $\beta = 0, \ldots, 4$, linear combinations of the time correlation functions (t.c.f.'s) for the particle density, the momentum density and the kinetic-energy density. The latter can be written in terms of reduced t.c.f.'s the long-time behaviour of which is governed by the hydrodynamic modes [15]. As has been done before [10] we will write φ^{c} as the sum of a singular part $\varphi^{c}_{\rm S}$ and a regular part $\varphi^{c}_{\rm R}$, where $\varphi^{c}_{\rm S}$ follows from (2.19) by restricting the sums over α and β to the hydrodynamic states: α, $\beta = 0, \ldots, 4$. Hence only $\varphi^{c}_{\rm S}$ contains t.c.f.'s with long-time tails, whereas $\varphi^{c}_{\rm R}$ does not depend on such slowly decaying functions. The kernel Σ is likewise written as the sum of a singular part $\Sigma_{\rm S}$ and a regular part $\Sigma_{\rm R}$, namely as $\Sigma = \Sigma_{\rm S} + \Sigma_{\rm R}$ with $\Sigma_{\rm S} = \varphi^{c}_{\rm S}$. The long-time tails of the kinetic heat-conductivity t.c.f.'s F_i, defined by (2.12), are determined by the behaviour of the matrix element $\Omega_{44}(\boldsymbol{k}, z)$ for z in the vicinity of the real axis. In particular, a slowly decaying tail proportional to $t^{-1/2}$ is associated to a singular behaviour of $\Omega_{44}(\boldsymbol{k}, z)$ near $z = \mathrm{i}0$. Likewise, a tail proportional to $t^{-1/2}\cos(\omega t + \Theta)$ corresponds to a singular behaviour of $\Omega_{44}(\boldsymbol{k}, z)$ near $z = \pm\omega + \mathrm{i}0$. Such a singular behaviour can arise only from the singular part $\varphi^{c}_{\rm S}$ of the collision kernel, since only that part contains functions with long-time tails. Hence matrix elements of $\varphi^{c}_{\rm S}$ will govern the singular behaviour of $\Omega_{44}(\boldsymbol{k}, z)$ for z near the real axis.

The matrix element $\Omega_{44}(\boldsymbol{k}, z)$ consists of a direct part and an indirect part (cf. (2.5)):

$$\Omega_{44}(\boldsymbol{k}, z) = \Omega_{44}^{\mathrm{dir}}(\boldsymbol{k}, z) + \Omega_{44}^{\mathrm{indir}}(\boldsymbol{k}, z) , \tag{2.21}$$

with

$$\Omega_{44}^{\mathrm{dir}}(\boldsymbol{k}, z) = \langle 4|\Sigma|4\rangle , \tag{2.22}$$

$$\Omega_{44}^{\mathrm{indir}}(\boldsymbol{k}, z) = \langle 4|\Sigma\bar{Q} \frac{1}{z - \bar{Q}\Sigma\bar{Q}} \bar{Q}\Sigma|4\rangle . \tag{2.23}$$

Correspondingly, it follows from (2.11) and (2.12) that the kinetic heat-conductivity t.c.f.'s F_i can be written as a sum of a term F_i^{dir} originating from the direct part of the frequency matrix element Ω_{44} and a term F_i^{indir} stemming from the indirect part:

$$F_i(t) = F_i^{\mathrm{dir}}(t) + F_i^{\mathrm{indir}}(t) . \tag{2.24}$$

In the following section we will study the contributions to the kinetic heat-conductivity t.c.f.'s F_i which result from the direct part of $\Omega_{44}(\boldsymbol{k}, z)$. The analysis of the indirect part will follow in section 4.

3. Direct part

In this section we determine the contributions to the long-time tail of the kinetic heat-conductivity t.c.f.'s F_i that originate from the direct part of the frequency matrix element Ω_{44} as given by (2.22). The direct part can be written as the sum of a matrix element of the regular kernel Σ_{R} and a matrix element of the singular collision kernel $\varphi_{\mathrm{S}}^{\mathrm{c}}$:

$$\Omega_{44}^{\mathrm{dir}}(\boldsymbol{k}, z) = \langle 4|\Sigma_{\mathrm{R}}(\boldsymbol{k}, z)|4\rangle + \langle 4|\varphi_{\mathrm{S}}^{\mathrm{c}}(\boldsymbol{k}, z)|4\rangle . \tag{3.1}$$

Since the kernel Σ_{R} is regular for small values of Im z, the singular behaviour of $\Omega_{44}^{\mathrm{dir}}$ will be determined by the second term at the right-hand side, which we shall call the singular part $\Omega_{44,\mathrm{S}}^{\mathrm{dir}}$:

$$\Omega_{44,\mathrm{S}}^{\mathrm{dir}}(\boldsymbol{k}, z) = \langle 4|\varphi_{\mathrm{S}}^{\mathrm{c}}(\boldsymbol{k}, z)|4\rangle . \tag{3.2}$$

The behaviour of $\Omega^{\mathrm{dir}}_{44,\mathrm{S}}$ for small Im z can be studied with the help of the explicit expression for $\varphi^{\mathrm{c}}_{\mathrm{S}}$ as given by (2.19) in which the sums over α and β are running now from 0 to 4. The terms of the singular collision kernel $\varphi^{\mathrm{c}}_{\mathrm{S}}$ containing the three-particle function $a(\boldsymbol{k}, \boldsymbol{q})$ drop out from $\Omega^{\mathrm{dir}}_{44,\mathrm{S}}$ since

$$\int \mathrm{d}\boldsymbol{p}\; \psi_\gamma(\boldsymbol{p}) \nabla_{\boldsymbol{p}} [\psi_0(\boldsymbol{p}) f_0(p)] = 0 \tag{3.3}$$

for $\gamma \neq 1, 2, 3$. As a consequence we arrive at the expression

$$\Omega^{\mathrm{dir}}_{44,\mathrm{S}}(\boldsymbol{k}, t) = -\frac{2}{3nm} \sum_{i,j} \mathscr{I}_{ijij}(\boldsymbol{k}, t)\,, \tag{3.4}$$

with

$$\begin{aligned} \mathscr{I}_{ij\alpha\beta}(\boldsymbol{k}, t) = & \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) v(q) q_i q_j \bar{G}_{\alpha\beta}(\boldsymbol{k}-\boldsymbol{q}, t) n S^{nn}(\boldsymbol{q}, t) \\ & + \frac{1}{2} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) v(\boldsymbol{k}-\boldsymbol{q}) \{ q_i (\boldsymbol{k}-\boldsymbol{q})_j \bar{G}_{\alpha 0}(\boldsymbol{k}-\boldsymbol{q}, t) \bar{G}_{0\beta}(\boldsymbol{q}, t) \\ & + (\boldsymbol{k}-\boldsymbol{q})_i q_j \bar{G}_{\alpha 0}(\boldsymbol{q}, t) \bar{G}_{0\beta}(\boldsymbol{k}-\boldsymbol{q}, t) \}\,. \end{aligned} \tag{3.5}$$

Summations over Latin indices are running from 1 to 3. We remark that the same result (3.4) is found, if the complete collision kernel φ^{c}, as given by (2.19), is used instead of its singular part $\varphi^{\mathrm{c}}_{\mathrm{S}}$. The functions $\bar{G}_{\alpha\beta}$ can be expressed in the reduced t.c.f.'s defined in ref. [15]. We have

$$(mk_{\mathrm{B}}T)^{1/2} \bar{G}_{0i}(\boldsymbol{q}, t) = q_{\parallel i} S^{ng}_{\parallel}(\boldsymbol{q}, t) + q_{\perp i} S^{ng}_{\perp}(\boldsymbol{q}, t) + (\boldsymbol{q} \wedge \hat{\boldsymbol{B}})_i S^{ng}_{\mathrm{t}}(\boldsymbol{q}, t)\,, \tag{3.6}$$

where $\boldsymbol{q}_{\parallel} = \boldsymbol{q} \cdot \hat{\boldsymbol{B}}\, \hat{\boldsymbol{B}}$ and $\boldsymbol{q}_{\perp} = \boldsymbol{q} - \boldsymbol{q}_{\parallel}$, and

$$\begin{aligned} mk_{\mathrm{B}}T \bar{G}_{ij}(\boldsymbol{q}, t) = & \, \delta_{ij} S^{gg}_1(\boldsymbol{q}, t) + \hat{q}_{\parallel i} \hat{q}_{\parallel j} S^{gg}_2(\boldsymbol{q}, t) \\ & + (\hat{q}_{\parallel i} \hat{q}_{\perp j} + \hat{q}_{\perp i} \hat{q}_{\parallel j}) S^{gg}_3(\boldsymbol{q}, t) + \hat{q}_{\perp i} \hat{q}_{\perp j} S^{gg}_4(\boldsymbol{q}, t) \\ & + \epsilon_{ijm} \hat{q}_{\perp m} \hat{q}_{\parallel} S^{gg}_5(\boldsymbol{q}, t) + \epsilon_{ijm} \hat{B}_m S^{gg}_6(\boldsymbol{q}, t)\,, \end{aligned} \tag{3.7}$$

with $\hat{\boldsymbol{q}}_{\parallel} = \hat{q}_{\parallel} \hat{\boldsymbol{B}}$. An expression for $\bar{G}_{i0}$ analogous to (3.6) can be derived as well; it differs from (3.6) by a minus sign in front of the last term.

Insertion of (3.6) and (3.7) in (3.4) with (3.5) yields

$$\Omega^{\rm dir}_{44,\rm S}(\boldsymbol{k},t) =$$
$$-\frac{2}{3m^2k_{\rm B}T}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,\frac{c(q)v(q)}{(\boldsymbol{k}-\boldsymbol{q})^2}\,S^{nn}(\boldsymbol{q},t)$$
$$\times[q^2(\boldsymbol{k}-\boldsymbol{q})^2S_1^{gg}(\boldsymbol{k}-\boldsymbol{q},t)+(k_\|q_\|-q_\|^2)^2S_2^{gg}(\boldsymbol{k}-\boldsymbol{q},t)$$
$$+2(k_\|q_\|-q_\|^2)(\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp-q_\perp^2)S_3^{gg}(\boldsymbol{k}-\boldsymbol{q},t)$$
$$+(\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp-q_\perp^2)^2S_4^{gg}(\boldsymbol{k}-\boldsymbol{q},t)]$$
$$-\frac{2}{3nm^2k_{\rm B}T}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(\boldsymbol{k}-\boldsymbol{q})$$
$$\times\{[(k_\|q_\|-q_\|^2)S_\|^{ng}(\boldsymbol{k}-\boldsymbol{q},t)+(\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp-q_\perp^2)S_\perp^{ng}(\boldsymbol{k}-\boldsymbol{q},t)]$$
$$\times[(k_\|q_\|-q_\|^2)S_\|^{ng}(\boldsymbol{q},t)+(\boldsymbol{k}_\perp\cdot\boldsymbol{q}_\perp-q_\perp^2)S_\perp^{ng}(\boldsymbol{q},t)]$$
$$+[(\boldsymbol{q}\wedge\boldsymbol{k})\cdot\hat{\boldsymbol{B}}]^2S_{\rm t}^{ng}(\boldsymbol{q},t)S_{\rm t}^{ng}(\boldsymbol{k}-\boldsymbol{q},t)\}\,. \tag{3.8}$$

We expand the right-hand side around $\boldsymbol{k}=\boldsymbol{0}$, up to order k^2. Upon introducing a cylindrical coordinate system for $\boldsymbol{q}$, with the polar axis in the direction of the magnetic field, we perform the integration over the azimuthal angle. The terms of order k vanish and one ends up with an expression of second order in k for $\Omega^{\rm dir}_{44,\rm S}(\boldsymbol{k},z)-\Omega^{\rm dir}_{44,\rm S}(\boldsymbol{0},z)$. This expression can be rewritten by using relations between the reduced t.c.f.'s, for example [15]

$$\mathrm{i}nm\,\frac{\partial}{\partial t}\,S^{nn}(\boldsymbol{q},t)=q_\|^2S_\|^{ng}(\boldsymbol{q},t)+q_\perp^2\,S_\perp^{ng}(\boldsymbol{q},t)\,. \tag{3.9}$$

Other relations express the time derivative of S_i^{ng} ($i=\|,\perp,\mathrm{t}$) in combinations of S_i^{gg} ($i=1,\ldots,6$).

The contributions to the long-time tail of $F_i^{\rm dir}(t)$ which arise from the singular part of $\Omega^{\rm dir}_{44}$ can be written down now. For large t we find

$$F_i^{\rm dir}(t)\simeq-\frac{k_{\rm B}T}{m^2}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)\Big(\mathrm{i}nm\,\frac{\partial S^{nn}\mathscr{F}_i}{\partial t}+nS^{nn}\mathscr{G}_i+\mathscr{H}_i\Big)\,, \tag{3.10}$$

with

$$\mathscr{F}_\|=\frac{1}{2}\,q_\|^2\,\frac{\partial^2S_\|^{ng}}{\partial q_\|^2}+\frac{1}{2}\,q_\perp^2\,\frac{\partial^2S_\perp^{ng}}{\partial q_\|^2}-g_\|S_+^{ng}\,, \tag{3.11}$$

$$\mathscr{F}_\perp=\frac{1}{4}\,q_\|^2\,\frac{1}{q_\perp}\,\frac{\partial}{\partial q_\perp}\Big(q_\perp\,\frac{\partial S_\|^{ng}}{\partial q_\perp}\Big)+\frac{1}{4}\,q_\perp\,\frac{\partial}{\partial q_\perp}\Big(q_\perp\,\frac{\partial S_\perp^{ng}}{\partial q_\perp}\Big)-g_\perp S_+^{ng}\,. \tag{3.12}$$

Here we introduced the abbreviations $g_{\|} = 1 - \hat{q}_{\|}^2$, $g_{\perp} = 1 - \frac{1}{2}\hat{q}_{\perp}^2$ and $S_{+}^{ng} = \hat{q}_{\|}^2 S_{\|}^{ng} + \hat{q}_{\perp}^2 S_{\perp}^{ng}$, $S_{-}^{ng} = S_{\|}^{ng} - S_{\perp}^{ng}$. Furthermore, we used

$$\mathscr{G}_{\|} = g_{\|}[S_1^{gg} + \hat{q}_{\|}^4(S_2^{gg} - S_3^{gg})] - \hat{q}_{\perp}^6(S_3^{gg} - S_4^{gg}) , \tag{3.13}$$

$$\mathscr{G}_{\perp} = g_{\perp}[S_1^{gg} + \hat{q}_{\|}^4(S_2^{gg} - S_3^{gg})] + \tfrac{1}{2}\hat{q}_{\|}^4\hat{q}_{\perp}^2(S_3^{gg} - S_4^{gg}) , \tag{3.14}$$

$$\mathscr{H}_{\|} = q_{\|}^2 q_{\|} \frac{\partial(S_{-}^{ng} S_{+}^{ng})}{\partial q_{\|}} - q_{\|}^2\hat{q}_{\perp}^4(S_{-}^{ng})^2 - 2q_{\|}^2\hat{q}_{\perp}^2 S_{-}^{ng} S_{+}^{ng} , \tag{3.15}$$

$$\mathscr{H}_{\perp} = -\frac{1}{2} q_{\|}^2 q_{\perp} \frac{\partial(S_{-}^{ng} S_{+}^{ng})}{\partial q_{\perp}} - \tfrac{1}{2}q_{\perp}^2\hat{q}_{\|}^4(S_{-}^{ng})^2 + q_{\|}^2\hat{q}_{\perp}^2 S_{-}^{ng} S_{+}^{ng} + \tfrac{1}{2}q_{\perp}^2(S_{t}^{ng})^2 . \tag{3.16}$$

The reduced t.c.f.'s are functions of the independent variables $q_{\|}$ and $q_{\perp}$.

Since the long-time behaviour of the reduced t.c.f.'s is known from kinetic theory [15] the long-time tails of $F_i^{\mathrm{dir}}(t)$ can be determined now. This calculation is postponed to section 6. First we will carry on with the study of the indirect part of the frequency matrix element Ω_{44}.

4. Indirect part

In this section we study the contributions of the indirect part of the frequency matrix element Ω_{44},

$$\Omega_{44}^{\mathrm{indir}}(\boldsymbol{k}, z) = \langle 4|\Sigma\bar{Q} \frac{1}{z - \bar{Q}\Sigma\bar{Q}} \bar{Q}\Sigma|4\rangle , \tag{4.1}$$

to the kinetic heat-conductivity time correlation functions $F_i(t)$. According to (2.11) and (2.12), we have to consider (4.1) in first and in second order in the wavenumber k. Using the parity invariance of the collision kernel

$$\varphi^{\mathrm{c}}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z, \boldsymbol{B}) = \varphi^{\mathrm{c}}(-\boldsymbol{k}, -\boldsymbol{p}, -\boldsymbol{p}', z, \boldsymbol{B}) \tag{4.2}$$

and, consequently, of the kernel Σ, one easily finds that (4.1) vanishes in order k^1, as expected. Thus we have to evaluate (4.1) in order k^2.

The projector $\bar{Q}$ projects onto the space spanned by the states $|\gamma\rangle$, with $\gamma \geq 5$, of the orthonormal set introduced in section 2. Hence we can write (4.1) as

$$\Omega_{44}^{\mathrm{indir}}(\boldsymbol{k}, z) = \sum_{\gamma,\delta \geq 5} \langle 4|\Sigma|\gamma\rangle\langle\gamma| \frac{1}{z - \bar{Q}\Sigma\bar{Q}} |\delta\rangle\langle\delta|\Sigma|4\rangle . \tag{4.3}$$

In order k^2 the two factors k can be distributed over the three matrix elements at the right-hand side in six distinct ways. From the parity invariance of the kernel Σ one finds that solely the part of the matrix element $\langle 4|\Sigma|\gamma\rangle$ that is even in k contributes, when $|\gamma\rangle$ is even in the momentum variable, and that its odd part in k yields a contribution only, when $|\gamma\rangle$ is odd in the momentum variable. A similar statement can be made for the matrix element $\langle\delta|\Sigma|4\rangle$. Likewise, the part of the matrix element $\langle\gamma|(z-\bar{Q}\Sigma\bar{Q})^{-1}|\delta\rangle$ that is even in k plays a role only when $\langle\gamma|$ and $|\delta\rangle$ are both even or both odd in the momentum variable; its odd part remains if one of the states is odd in the momentum variable and the other even. Further conditions on the intermediate states $|\gamma\rangle$ and $|\delta\rangle$ can be deduced from the rotational symmetry of the system. Since a magnetized plasma possesses cylinder symmetry around the magnetic-field axis, the matrix element $\langle 4|\Sigma|\gamma\rangle$ vanishes in order k^0 unless $|\gamma\rangle$ is a cylinder-symmetric state. Likewise, only cylinder-symmetric states $|\gamma\rangle$ contribute if $\langle 4|\Sigma|\gamma\rangle$ is chosen to be of order k^2, since in that case both the second and third factor in (4.3) are of order k^0. Of course analogous statements can be made for the state $|\delta\rangle$. In the special case of a vanishing magnetic field the system becomes isotropic. The states $|\gamma\rangle$ and $|\delta\rangle$ have to be spherical-symmetric then under the conditions specified above.

To assess whether $\Omega_{44}^{\rm indir}$ becomes singular near the real z-axis we have to study the behaviour of the three factors at the right-hand side of (4.3) for Im z tending to 0. To that end we shall need the precise form of Σ. However, there is one case for which a general statement can be made without using any detailed information about Σ. In fact, in order k^0 only the collision kernel $\varphi^{\rm c}$ contributes to the matrix element $\langle 4|\Sigma|\gamma\rangle$. From the microscopic conservation of energy it follows that

$$\lim_{z\to {\rm i}0} \frac{1}{z}\int {\rm d}\boldsymbol{p}\, \frac{p^2}{2m}\, \varphi^{\rm c}(\boldsymbol{k}=\boldsymbol{0}, \boldsymbol{p}, \boldsymbol{p}', z) \tag{4.4}$$

is finite [13]. Using moreover

$$\int {\rm d}\boldsymbol{p}\, \varphi^{\rm c}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}', z) = 0\,, \tag{4.5}$$

we thus find in order k^0

$$\langle 4|\Sigma|\gamma\rangle \propto z\,, \tag{4.6}$$

for z tending to i0. An analogous conclusion can be drawn for the matrix element $\langle\delta|\Sigma|4\rangle$ in order k^0. It should be noted that the behaviour of $\langle 4|\Sigma|\gamma\rangle$ near other points of the real z-axis cannot be obtained in such a straightforward manner.

Proceeding to a general discussion of (4.3) we split the kernel Σ in its regular and singular parts. The first matrix element then becomes

$$\langle 4|\Sigma|\gamma\rangle = \langle 4|\varphi_{\mathrm{S}}^{\mathrm{c}}|\gamma\rangle + \langle 4|\Sigma_{\mathrm{R}}|\gamma\rangle \,. \tag{4.7}$$

The third matrix element of (4.3) can be written analogously. As before we shall assume that the contribution of Σ_{R} remains finite for all z near the real axis. In order to discuss the second factor at the right-hand side of (4.3) we use the operator identity:

$$\frac{1}{z-\bar{Q}\Sigma\bar{Q}} = \Xi_{\mathrm{R}} + \frac{1}{z-\bar{Q}\Sigma\bar{Q}}\,\bar{Q}\varphi_{\mathrm{S}}^{\mathrm{c}}\bar{Q}\Xi_{\mathrm{R}}\,, \tag{4.8}$$

with the abbreviation

$$\Xi_{\mathrm{R}} = \frac{1}{z-\bar{Q}\Sigma_{\mathrm{R}}\bar{Q}}\,. \tag{4.9}$$

By iteration we can write now

$$\langle\gamma|\,\frac{1}{z-\bar{Q}\Sigma\bar{Q}}\,|\delta\rangle = \langle\gamma|\Xi_{\mathrm{R}} + \Xi_{\mathrm{R}}\bar{Q}\varphi_{\mathrm{S}}^{\mathrm{c}}\bar{Q}\Xi_{\mathrm{R}} + \cdots|\delta\rangle\,. \tag{4.10}$$

All factors containing Ξ_{R} are assumed to remain finite if z approaches the real axis. Hence the singular behaviour of (4.10) will be determined by that of $\bar{Q}\varphi_{\mathrm{S}}^{\mathrm{c}}\bar{Q}$. In this way we arrive at the conclusion that a singular behaviour of $\Omega_{44}^{\mathrm{indir}}$, as given by (4.3), near the real z-axis can arise only if any of the expressions $\langle 4|\varphi_{\mathrm{S}}^{\mathrm{c}}|\gamma\rangle$, $\bar{Q}\varphi_{\mathrm{S}}^{\mathrm{c}}\bar{Q}$ or $\langle\gamma|\varphi_{\mathrm{S}}^{\mathrm{c}}|4\rangle$ becomes infinite. We shall study these factors in succession now, starting with the last one. For each of them we shall consider separately the contributions of order k^0, k^1 and k^2.

From (2.19) it follows that the matrix element $\langle\gamma|\varphi_{\mathrm{S}}^{\mathrm{c}}|4\rangle$ contains a factor

$$\int \mathrm{d}\boldsymbol{p}\;\psi_\gamma(\boldsymbol{p})\boldsymbol{\nabla}_p[\psi_\alpha(\boldsymbol{p})f_0(p)] \tag{4.11}$$

with $\alpha = 1,\ldots,4$. For $\alpha = 1,2,3$ this factor is different from zero only if $\gamma = 5,\ldots,9$ with

$$\psi_5(\boldsymbol{p}) = \frac{p_x p_y}{mk_{\mathrm{B}}T}\,,\qquad \psi_6(\boldsymbol{p}) = \frac{p_x p_z}{mk_{\mathrm{B}}T}\,,\qquad \psi_7(\boldsymbol{p}) = \frac{p_y p_z}{mk_{\mathrm{B}}T}\,, \tag{4.12}$$

$$\psi_8(\boldsymbol{p}) = \frac{1}{2mk_{\mathrm{B}}T}\,(p_x^2 - p_y^2)\,,\qquad \psi_9(\boldsymbol{p}) = \frac{1}{2}\,\sqrt{3}\,\frac{1}{mk_{\mathrm{B}}T}\,(p_z^2 - \tfrac{1}{3}p^2)\,. \tag{4.13}$$

For $\alpha = 4$, the factor (4.11) differs from zero only if $\gamma = 10, 11, 12$ with

$$\psi_{i+9}(\boldsymbol{p}) = \frac{p_i}{\sqrt{10}(mk_{\mathrm{B}}T)^{1/2}} \left(\frac{p^2}{mk_{\mathrm{B}}T} - 5\right) \qquad (i = 1, 2, 3) . \tag{4.14}$$

Thus $\langle \gamma | \varphi^{\mathrm{c}}_{\mathrm{S}} | 4 \rangle$ is different from 0 only for $\gamma = 5, \ldots, 12$. Writing the matrix element as c_γ we have found

$$\bar{Q} \varphi^{\mathrm{c}}_{\mathrm{S}} | 4 \rangle = \sum_{\gamma=5}^{12} | \gamma \rangle c_\gamma . \tag{4.15}$$

Using the expression for $\varphi^{\mathrm{c}}_{\mathrm{S}}$ that follows from (2.19) we get the following explicit formula for c_γ:

$$c_\gamma(\boldsymbol{k}, t) = \left(\frac{2k_{\mathrm{B}}T}{3m}\right)^{1/2} \frac{1}{n} \sum_{i,j} \sum_{\alpha=1}^{4} \left[\int \mathrm{d}\boldsymbol{p}\, \psi_\gamma(\boldsymbol{p}) \frac{\partial}{\partial p_i} [\psi_\alpha(\boldsymbol{p}) f_0(p)]\right] \mathscr{I}_{ij\alpha j}(\boldsymbol{k}, t) , \tag{4.16}$$

where (3.5) has been employed. For $\langle 4 | \varphi^{\mathrm{c}}_{\mathrm{S}} \bar{Q}$ an expression analogous to (4.15) can be derived. It should be remarked that the terms containing the three-particle function $a(\boldsymbol{k}, \boldsymbol{q})$ do not contribute to (4.16) because of (3.3). Using parity and cylinder-symmetry arguments, as before, one finds that in order k^0 and k^2 we only need to discuss c_γ for $\gamma = 9$, whereas in order k^1 we shall have to consider the cases $\gamma = 10, 11, 12$.

In zeroth order of the wavenumber c_9 is found to have the following form:

$$\frac{\sqrt{2}}{3} \frac{1}{nm} \sum_{i,j} (\delta_{i1} + \delta_{i2} - 2\delta_{i3}) \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) q_i q_j \\ \times \{ \bar{G}_{ij}(\boldsymbol{q}, t) n S^{nn}(\boldsymbol{q}, t) + \bar{G}_{i0}(\boldsymbol{q}, t) \bar{G}_{0j}(\boldsymbol{q}, t) \} . \tag{4.17}$$

Likewise, in order k^1 one finds for c_γ, with $\gamma = 10, 11, 12$:

$$-\frac{\sqrt{10}}{3} \frac{1}{nm} \sum_{i,j} \delta_{\gamma, i+9} I_{ij}(\boldsymbol{k}, t) \tag{4.18}$$

with

$$I_{ij}(\boldsymbol{k}, t) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) v(q) q_i q_j \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\boldsymbol{q}} [\bar{G}_{4j}(\boldsymbol{q}, t)] n S^{nn}(\boldsymbol{q}, t) \\ + \frac{1}{2} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\, c(q) \{ q_i \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\boldsymbol{q}} [q_j \bar{G}_{40}(\boldsymbol{q}, t) v(q)] \bar{G}_{0j}(\boldsymbol{q}, t) \\ - q_j \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\boldsymbol{q}} [q_i \bar{G}_{0j}(\boldsymbol{q}, t) v(q)] \bar{G}_{40}(\boldsymbol{q}, t) \} . \tag{4.19}$$

Finally, in second order in k the coefficient c_9 becomes

$$\frac{\sqrt{2}}{6}\,\frac{1}{nm}\sum_{i,j}(\delta_{i1}+\delta_{i2}-2\delta_{i3})J_{ij}(\boldsymbol{k},t) \tag{4.20}$$

with

$$\begin{aligned}J_{ij}(\boldsymbol{k},t) &= \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)q_i\,q_j\boldsymbol{kk}\colon \boldsymbol{\nabla}_{\boldsymbol{q}}\boldsymbol{\nabla}_{\boldsymbol{q}}[\bar{G}_{ij}(\boldsymbol{q},t)]nS^{nn}(\boldsymbol{q},t)\\ &+\frac{1}{2}\int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)\{q_i\boldsymbol{kk}\colon \boldsymbol{\nabla}_{\boldsymbol{q}}\boldsymbol{\nabla}_{\boldsymbol{q}}[q_j\bar{G}_{i0}(\boldsymbol{q},t)v(q)]\bar{G}_{0j}(\boldsymbol{q},t)\\ &+q_j\boldsymbol{kk}\colon \boldsymbol{\nabla}_{\boldsymbol{q}}\boldsymbol{\nabla}_{\boldsymbol{q}}[q_i\bar{G}_{0j}(\boldsymbol{q},t)v(q)]\bar{G}_{i0}(\boldsymbol{q},t)\}\,.\end{aligned} \tag{4.21}$$

We now turn to a consideration of the expression $\bar{Q}\varphi^{\mathrm{c}}_{\mathrm{S}}\bar{Q}$, which is contained in (4.10). In a way analogous to (4.15) we can write

$$\bar{Q}\varphi^{\mathrm{c}}_{\mathrm{S}}\bar{Q}=\sum_{\gamma,\delta=5}^{12}|\gamma\rangle c_{\gamma,\delta}\langle\delta| \tag{4.22}$$

with

$$\begin{aligned}c_{\gamma,\delta}(\boldsymbol{k},t) &= -\frac{k_{\mathrm{B}}T}{n}\sum_{i,j}\sum_{\alpha=1}^{4}\left[\int \mathrm{d}\boldsymbol{p}\,\psi_\gamma(\boldsymbol{p})\,\frac{\partial}{\partial p_i}\,[\psi_\alpha(\boldsymbol{p})f_0(p)]\right]\\ &\times\sum_{\beta=1}^{4}\left[\int \mathrm{d}\boldsymbol{p}'\,\psi_\delta(\boldsymbol{p}')\,\frac{\partial}{\partial p'_j}\,[\psi_\beta(\boldsymbol{p}')f_0(p')]\right]\mathscr{I}_{ij\alpha\beta}(\boldsymbol{k},t)\,.\end{aligned} \tag{4.23}$$

As before the terms containing the three-particle function $a(\boldsymbol{k},\boldsymbol{q})$ do not contribute. In order k^0 we have to analyse $c_{\gamma,\delta}$ for γ and δ chosen both from the set $\{5,\ldots,9\}$ or for γ and δ chosen from $\{10,11,12\}$. In first order in k the combinations $\gamma=9$, $\delta=10,11,12$ (and the reverse) must be studied. Finally, in second order $c_{\gamma,\delta}$ has to be considered only for $\gamma=\delta=9$, since we can invoke the cylinder symmetry of the remaining factors in (4.3), with (4.10) inserted, to dismiss any other combination of γ and δ.

We start by considering the coefficients $c_{\gamma,\delta}$ in order k^0. For $c_{\gamma,\gamma}$ with $\gamma=i+9$ we find in this order:

$$-\frac{5}{3}\,\frac{1}{nm}\int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)q_i^2\{\bar{G}_{44}(\boldsymbol{q},t)nS^{nn}(\boldsymbol{q},t)-\bar{G}_{40}(\boldsymbol{q},t)\bar{G}_{04}(\boldsymbol{q},t)\}\,. \tag{4.24}$$

It is an example of the twelve possible choices of γ, δ for which $c_{\gamma,\delta}$ in order k^0

is not equal to zero, viz. $\gamma = \delta = 5, \ldots, 12$ and $(\gamma, \delta) = (5, 8), (8, 5), (6, 7), (7, 6)$.

In order k^1 the following expression for $c_{\gamma,9}$ with $\gamma = 10, 11, 12$ is found from (4.23):

$$\frac{\sqrt{5}}{3} \frac{1}{nm} \sum_{i,j} \delta_{\gamma,i+9}(\delta_{j1} + \delta_{j2} - 2\delta_{j3}) I_{ij}(\boldsymbol{k}, t) , \tag{4.25}$$

where (4.19) has been used. An analogous expression can be derived for $c_{9,\gamma}$.

Finally, in second order in the wavenumber $c_{9,9}$ becomes

$$- \frac{1}{6nm} \sum_{i,j} (\delta_{i1} + \delta_{i2} - 2\delta_{i3})(\delta_{j1} + \delta_{j2} - 2\delta_{j3}) J_{ij}(\boldsymbol{k}, t) , \tag{4.26}$$

where we have employed (4.21).

With (3.6), (3.7) and

$$\bar{G}_{40}(\boldsymbol{q}, t) = \frac{1}{\sqrt{6}} \left[\frac{2}{k_B T} S^{\epsilon^{\mathrm{kin}} n}(\boldsymbol{q}, t) - 3nS^{nn}(\boldsymbol{q}, t) \right] , \tag{4.27}$$

$$(mk_B T)^{1/2} \bar{G}_{4i}(\boldsymbol{q}, t) = \frac{2}{\sqrt{6} k_B T} [q_{\parallel i} S_{\parallel}^{\epsilon^{\mathrm{kin}} g}(\boldsymbol{q}, t) + q_{\perp i} S_{\perp}^{\epsilon^{\mathrm{kin}} g}(\boldsymbol{q}, t) + (\boldsymbol{q} \wedge \hat{\boldsymbol{B}})_i S_{\mathrm{t}}^{\epsilon^{\mathrm{kin}} g}(\boldsymbol{q}, t)] - \frac{3}{\sqrt{6}} (mk_B T)^{1/2} \bar{G}_{0i}(\boldsymbol{q}, t) , \tag{4.28}$$

$$\bar{G}_{44}(\boldsymbol{q}, t) = \frac{2}{3(k_B T)^2} S^{\epsilon^{\mathrm{kin}} \epsilon^{\mathrm{kin}}}(\boldsymbol{q}, t) - \frac{2}{k_B T} S^{\epsilon^{\mathrm{kin}} n}(\boldsymbol{q}, t) + \frac{3}{2} nS^{nn}(\boldsymbol{q}, t) \tag{4.29}$$

the expressions (4.17), (4.19), (4.21) and (4.24) can be written in terms of reduced t.c.f.'s. After substitution of the asymptotic long-time expressions for these reduced t.c.f.'s the behaviour of $\Omega_{44}^{\mathrm{indir}}$ near the real z-axis and, hence, of the kinetic heat-conductivity time correlation functions $F_i(t)$ for large t can be determined. However, before doing so we will first study the singular behaviour of Ω_{44} for the unmagnetized plasma in section 5 and continue with the magnetized plasma in section 6.

5. Unmagnetized plasma

In this section we will determine the long-time tails of the kinetic heat-conductivity time correlation functions $F_i(t)$, defined by (2.12), for the case of the unmagnetized plasma. First we consider the direct part $F_i^{\mathrm{dir}}(t)$, given by (3.10) for the magnetized plasma. For vanishing magnetic field strength B (or Larmor frequency ω_B) the reduced t.c.f.'s get a simpler form. One has

$$S_{\parallel}^{ng}(\boldsymbol{q}, t) = S_{\perp}^{ng}(\boldsymbol{q}, t) = S_{+}^{ng}(\boldsymbol{q}, t) , \qquad S_{-}^{ng}(\boldsymbol{q}, t) = 0 , \qquad S_{\mathrm{t}}^{ng}(\boldsymbol{q}, t) = 0 ,$$
$$S_2^{gg}(\boldsymbol{q}, t) = S_3^{gg}(\boldsymbol{q}, t) = S_4^{gg}(\boldsymbol{q}, t) . \tag{5.1}$$

From (3.10)–(3.16) it follows that $F_{\parallel}^{\mathrm{dir}}(t) = F_{\perp}^{\mathrm{dir}}(t) \equiv F^{\mathrm{dir}}(t)$ becomes for large t:

$$F^{\mathrm{dir}}(t) \simeq \bar{F}(t) + \frac{\mathrm{d}}{\mathrm{d}t} \bar{\bar{F}}(t) , \tag{5.2}$$

with

$$\bar{F}(t) = -\frac{2nk_{\mathrm{B}}T}{3m^2} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3} c(q)v(q)S^{nn}S_1^{gg} , \tag{5.3}$$

$$\bar{\bar{F}}(t) = -\frac{n^2k_{\mathrm{B}}T}{6} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3} c(q)v(q)S^{nn} \frac{\partial}{\partial t} \left(-\frac{\partial^2 S^{nn}}{\partial q^2} + \frac{2}{q} \frac{\partial S^{nn}}{\partial q} + \frac{2}{q^2} S^{nn} \right) . \tag{5.4}$$

To determine the long-time tail of $F^{\mathrm{dir}}(t)$ we need the long-time behaviour of the reduced t.c.f.'s $S^{nn}(\boldsymbol{q}, t)$ and $S_1^{gg}(\boldsymbol{q}, t)$ for the unmagnetized plasma. From kinetic theory [10, 15] it is found that the long-time behaviour of $S^{nn}(\boldsymbol{q}, t)$ is governed by the plasmon modes with frequency z_ρ ($\rho = \pm 1$) given by

$$z_\rho = \rho\omega_{\mathrm{p}}(1 + \tfrac{1}{2}q^2\gamma) - \mathrm{i}\tfrac{1}{2}q^2\Gamma . \tag{5.5}$$

The dynamic coefficients $\gamma(z)$ and $\Gamma(z)$ are defined by

$$\gamma(z) = \frac{1}{nm\kappa_T\omega_{\mathrm{p}}^2} + \frac{k_{\mathrm{B}}T}{mz} \frac{\bar{b}^2(z)}{z + \mathrm{i}c(z)} , \tag{5.6}$$

$$\Gamma(z) = \frac{1}{nm} [\tfrac{4}{3}\eta(z) + \eta_V(z)] , \tag{5.7}$$

where κ_T is the isothermal compressibility, $\eta(z)$ the dynamic shear viscosity coefficient and $\eta_V(z)$ the dynamic bulk viscosity coefficient. Furthermore, the dynamic coefficients $\bar{b}(z)$ and $c(z)$ are defined by writing for small wavenumber q, up to first order in q,

$$\langle 3|\varphi^0 + \varphi^c|4\rangle + \langle 3|\Sigma\bar{Q}\,\frac{1}{z-\bar{Q}\Sigma\bar{Q}}\,\bar{Q}\Sigma|4\rangle = \left(\frac{k_B T}{m}\right)^{1/2} q\bar{b}(z)\,, \tag{5.8}$$

$$\langle 4|\varphi^c|4\rangle + \langle 4|\,\Sigma\bar{Q}\,\frac{1}{z-\bar{Q}\Sigma\bar{Q}}\,\bar{Q}\Sigma|4\rangle = -\mathrm{i}c(z)\,, \tag{5.9}$$

where symmetry properties of the formal collision kernel φ^c have been used and the wave vector $\boldsymbol{q}$ has been chosen along the z-axis. These dynamic coefficients are to be evaluated at the finite frequency $z = \rho\omega_p$. For vanishingly small frequency $\gamma(z)$ satisfies the relation

$$\lim_{z\to \mathrm{i}0} \gamma(z) = \frac{c_s^2}{\omega_p^2} \tag{5.10}$$

with c_s the velocity of sound. The long-time behaviour of $S_1^{gg}(\boldsymbol{q}, t)$ is governed by the shear modes with frequency z_η given by

$$z_\eta = -\mathrm{i}q^2\,\frac{\eta}{nm} \tag{5.11}$$

with η the static shear viscosity coefficient. For long times and in leading order in the wavenumber we have

$$S^{nn}(\boldsymbol{q}, t) = \frac{1}{2}\,\frac{q^2}{k_D^2} \sum_{\rho=\pm 1} \exp\{-\mathrm{i}\rho\omega_p(1+\tfrac{1}{2}q^2\gamma)t - \tfrac{1}{2}q^2\Gamma t\}\,, \tag{5.12}$$

$$S_1^{gg}(\boldsymbol{q}, t) = nmk_B T\,\exp\{-q^2\nu t\}\,, \tag{5.13}$$

where the kinematic viscosity $\nu = \eta/nm$ and the Debye wavenumber $k_D = (ne^2/k_B T)^{1/2}$ have been introduced.

The long-time tail of $\bar{F}(t)$, given by (5.3), can be evaluated by substituting (5.12), (5.13) and employing the identity

$$\int_0^\infty \mathrm{d}q\; q^{2n}\,\mathrm{e}^{-\mathrm{i}\omega t - q^2 Dt} = \sqrt{\pi}\,\frac{(2n)!}{n!}\,\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{(4Dt)^{n+1/2}}\,. \tag{5.14}$$

One finds

$$\bar{F}(t) \simeq \frac{(k_B T\omega_p)^2}{6\pi^{3/2}}\,\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}\omega_p t}}{(\nu + \frac{1}{2}\mathrm{i}\omega_p\gamma + \frac{1}{2}\Gamma)^{1/2}}\,\frac{1}{t^{1/2}}\,, \tag{5.15}$$

where it has been used that $c(q) = -e^2/(k_B T q^2)$ for small q. This long-time tail of $F^{\mathrm{dir}}(t)$, of the form $t^{-1/2}\cos(\omega_p t + \Theta)$, was found some years ago with the use of the same method [11]. It can be interpreted as brought about by the coupling of a plasmon mode and a shear mode. If one uses mode-coupling theory to obtain the long-time tails of the Green–Kubo integrand of the heat-conductivity coefficient, the same long-time tail is found [4].

To determine the contribution of the time-derivative term to the long-time tail of $F^{\mathrm{dir}}(t)$ we evaluate $\bar{\bar{F}}(t)$ given by (5.4). The integrand at the right-hand side of (5.4) contains terms with factors t and t^2 that arise when we take the derivative with respect to the wavenumber of the exponential functions contained in S^{nn}. On account of (5.14) terms of order $q^{-2}t^0$, q^0t^1 and q^2t^2 in the integrand lead to long-time tails of $\bar{\bar{F}}(t)$ proportional to $t^{-1/2}$ multiplied by an oscillating factor. One finds

$$\bar{\bar{F}}(t) \simeq -\frac{19(k_B T)^2\omega_p}{192\pi^{3/2}} \operatorname{Re} \frac{\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\omega_p t}}{(\mathrm{i}\omega_p\gamma + \Gamma)^{1/2}} \frac{1}{t^{1/2}} . \tag{5.16}$$

This means that the time-derivative term in (5.2) has a long-time tail which decays as $t^{-1/2}\cos(2\omega_p t + \Theta')$. This result, which can be interpreted as being a consequence of the coupling of two plasmon modes, agrees with that obtained before with the help of kinetic theory [11, 12]. It is not found if mode-coupling theory is employed to determine the long-time tails of the Green–Kubo integrand of the heat conductivity [4]. However, since it is a pure time-derivative term, it does not contribute to the static heat conductivity coefficient defined by

$$\lambda = \frac{1}{k_B T^2} \int_0^\infty \mathrm{d}t\, F(t) , \tag{5.17}$$

since $\bar{\bar{F}}(t=0) = \bar{\bar{F}}(t \to \infty) = 0$. We conclude that the long-time tail of $F^{\mathrm{dir}}(t)$ is the same as the long-time tail of the Green–Kubo heat-conductivity integrand, apart from a term which does not contribute to the static heat conductivity coefficient.

For the indirect part we have to consider (4.3) in order k^2. Its singular behaviour near the real z-axis is determined by the expressions $\langle 4|\varphi_S^c|\gamma\rangle$, $\bar{Q}\varphi_S^c\bar{Q}$ and $\langle\gamma|\varphi_S^c|4\rangle$ and hence by the coefficients c_γ and $c_{\gamma,\delta}$ defined in the previous section. Owing to the isotropy of the unmagnetized plasma c_9 in order k^0 vanishes, while in second order of the wavenumber c_9 cannot play a role either. Furthermore, if we choose the wave vector along the z-axis, c_γ in order k^1 differs from 0 only for $\gamma = 12$; it is given by (4.18). By a similar reasoning it follows that for the evaluation of $\bar{Q}\varphi_S^c\bar{Q}$ only $c_{12,12}$ in order k^0 is needed. It is given by (4.24).

Apart from (5.12) and (5.13), explicit expressions for the other reduced t.c.f.'s for the unmagnetized plasma, which appear in the coefficients c_γ and $c_{\gamma,\delta}$, can be derived from kinetic theory. One finds, for long times and in leading order of the wavenumber,

$$S_{\mathrm{t}}^{ng}(\boldsymbol{q}, t) = \frac{1}{2} n k_{\mathrm{B}} T \sum_{\rho=\pm 1} \frac{\rho}{\omega_{\mathrm{p}}} \mathrm{e}^{-\mathrm{i} z_\rho t} , \tag{5.18}$$

$$S^{\epsilon^{\mathrm{kin}} n}(\boldsymbol{q}, t) = \frac{3}{2} n k_{\mathrm{B}} T \frac{q^2}{k_{\mathrm{D}}^2} \left\{ - \frac{1}{n c_V} \left(\frac{\partial \bar{P}}{\partial T} \right)_n \mathrm{e}^{-\mathrm{i} z_T t} + \frac{1}{2} \sum_{\rho=\pm 1} \mathrm{e}^{-\mathrm{i} z_\rho t} \right.$$
$$\left. + \frac{1}{2} \left(\frac{2}{3} \right)^{1/2} \sum_{\rho=\pm 1} \frac{\rho \omega_{\mathrm{p}} \bar{b}(\rho \omega_{\mathrm{p}})}{\rho \omega_{\mathrm{p}} + \mathrm{i} c(\rho \omega_{\mathrm{p}})} \mathrm{e}^{-\mathrm{i} z_\rho t} \right\} , \tag{5.19}$$

$$S_{\parallel}^{\epsilon^{\mathrm{kin}} g}(\boldsymbol{q}, t) = S_{\perp}^{\epsilon^{\mathrm{kin}} g}(\boldsymbol{q}, t)$$
$$= \frac{3}{4} n (k_{\mathrm{B}} T)^2 \left\{ \sum_{\rho=\pm 1} \frac{\rho}{\omega_{\mathrm{p}}} \mathrm{e}^{-\mathrm{i} z_\rho t} \right.$$
$$\left. + \left(\frac{2}{3} \right)^{1/2} \sum_{\rho=\pm 1} \frac{\bar{b}(\rho \omega_{\mathrm{p}})}{\rho \omega_{\mathrm{p}} + \mathrm{i} c(\rho \omega_{\mathrm{p}})} \mathrm{e}^{-\mathrm{i} z_\rho t} \right\} , \tag{5.20}$$

$$S_{\mathrm{t}}^{\epsilon^{\mathrm{kin}} g}(\boldsymbol{q}, t) = 0 , \tag{5.21}$$

$$S^{\epsilon^{\mathrm{kin}} \epsilon^{\mathrm{kin}}}(\boldsymbol{q}, t) = \frac{9}{4} n (k_{\mathrm{B}} T)^2 \frac{k_{\mathrm{B}}}{c_V} \mathrm{e}^{-\mathrm{i} z_T t} , \tag{5.22}$$

where z_T is the frequency of the heat mode,

$$z_T = -\mathrm{i} q^2 \frac{\lambda}{n c_V} , \tag{5.23}$$

with λ the heat conductivity coefficient. In the derivation of (5.19) and (5.20) we have used the "pole approximation" [15].

By substituting the asymptotic expressions for the reduced t.c.f.'s in the integral at the right-hand side of (4.24) (with (4.27) and (4.29) inserted), which gives $c_{12,12}$ in order k^0, one finds that the integrand is of order q^0. Upon evaluating the integrand in (4.18), which yields c_{12} in order k^1, one encounters derivatives with respect to the wavenumber, which may act on the exponential function of a reduced time correlation function. If these derivatives are carried out, an extra factor t shows up in the integrand, as before. In fact, the integrand is found to contain, for small wavenumber, terms of order q^0 and $q^2 t$. If (5.14) is employed now we find that both $c_{12,12}$ in order k^0 and c_{12} in order k^1

possess long-time tails of the form $\sim t^{-3/2}$ or $t^{-3/2}\cos(\omega t+\Theta'')$, with $\omega=\omega_{\mathrm{p}}$ or $2\omega_{\mathrm{p}}$. Such tails do not lead to infinities in $\Omega_{44}^{\mathrm{indir}}$ as z approaches the real axis. We conclude that the long-time behaviour of the kinetic heat-conductivity t.c.f. $F(t)$ of an unmagnetized plasma is dominated by the contributions from the direct part of the frequency matrix element Ω_{44}, which have been given in (5.15) and (5.16).

6. Magnetized plasma

In this section we will determine the long-time tails of the kinetic longitudinal and transverse heat-conductivity time correlation functions $F_i(t)$, as defined by (2.12), for the magnetized plasma. First the direct part $F_i^{\mathrm{dir}}(t)$, given by (3.10), is considered. As in the analysis for the unmagnetized plasma in the previous section, we shall treat separately the time-derivative term in the integrand of (3.10) and the other two terms. The contribution of the latter is denoted by $\bar{F}_i(t)$:

$$\bar{F}_i(t)=-\frac{k_{\mathrm{B}}T}{m^2}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)(nS^{nn}\mathscr{G}_i+\mathscr{H}_i)\,. \tag{6.1}$$

The long-time behaviour of S^{nn} and of the reduced t.c.f.'s occurring in $\mathscr{G}_i$ and $\mathscr{H}_i$ is governed by the collective modes of the magnetized plasma, viz. the heat mode and the four so-called gyro-plasmon modes [15]. In fact, the latter yield the dominant contributions for small values of the wavenumber. Their frequencies $z_{\lambda\rho}$, with $\lambda=\pm1$, $\rho=\pm1$, are given by

$$z_{\lambda\rho}=\rho w_{\lambda}(\hat{\boldsymbol{q}})-\mathrm{i}q^2D_{\lambda\rho}(\hat{\boldsymbol{q}})\,, \tag{6.2}$$

up to second order in the wavenumber. The leading term contains the frequency w_λ which is defined as

$$w_\lambda=\tfrac{1}{2}(\omega_{\mathrm{p}}^2+\omega_B^2+2\omega_{\mathrm{p}}\omega_B\hat{q}_{\|})^{1/2}+\tfrac{1}{2}\lambda(\omega_{\mathrm{p}}^2+\omega_B^2-2\omega_{\mathrm{p}}\omega_B\hat{q}_{\|})^{1/2}\,. \tag{6.3}$$

The function $D_{\lambda\rho}$ depends on w_λ, ω_{p}, ω_B and on dynamic coefficients at frequency $z=\rho w_\lambda$.

In contrast to the mode frequencies of the unmagnetized plasma the gyroplasmon mode frequencies depend on the orientation of the wave vector $\boldsymbol{q}$, even in leading order of q. From (3.13)–(3.16) it follows that the integrand of (6.1) contains a sum of products of reduced t.c.f.'s of the form $S^{nn}S_i^{gg}$, with $i=1,\ldots,4$ or $S_j^{ng}S_k^{ng}$, with j and k chosen from $\|$, $\perp$ or t. All these reduced t.c.f.'s are, for large t and small q, linear combinations of exponentials $\exp(-\mathrm{i}z_{\lambda\rho}t)$, so that each term in the integrand can be said to arise from a specific "coupling" of modes $\lambda\rho$ and $\lambda'\rho'$. The product of exponentials associated to such a coupling has got an exponent of which the leading term for small q is $-\mathrm{i}(\rho w_\lambda + \rho' w_{\lambda'})t$. In general this expression depends on the orientation of the wave vector, so that the subsequent integration over $\boldsymbol{q}$ gives rise to interference effects [5]. However, if $\lambda = \lambda'$ and $\rho = -\rho'$ no interference effects occur in leading order of the wavenumber. The two cases, which we shall call the "interference coupling" and the "non-interference coupling", respectively, will be considered separately in the following.

The contribution of the non-interference coupling terms to the long-time tail of $\bar{F}_i(t)$ follows by inserting the asymptotic expressions for the reduced t.c.f.'s in (3.13)–(3.16). It is found then that $S^{nn}\mathscr{G}_i$ is proportional to q^2 for small q. With the help of (5.14) it follows immediately that this term yields a tail proportional to $t^{-1/2}$. The functions $\mathscr{H}_i$ contain a derivative of a product of two reduced t.c.f.'s, which is proportional to an exponential function with an exponent that starts with a term of order q^2t in the present case of non-interference coupling. Hence, differentiation of the exponential brings down a factor qt, so that $\mathscr{H}_i$ is the sum of two contributions, one proportional to q^2 and the other to q^4t. Upon substitution into (6.1) it is seen that both yield a tail proportional to $t^{-1/2}$. A detailed calculation yields the following result for the long-time tail of $\bar{F}_i(t)$ that arises from non-interference couplings:

$$\bar{F}_i(t) \simeq \sum_{\lambda=\pm 1} \frac{C_i^\lambda}{t^{1/2}} . \tag{6.4}$$

The proportionality factor C_i^λ is given by

$$C_i^\lambda = \frac{(k_\mathrm{B}T)^2}{4\pi^{3/2}}\, \omega_\mathrm{p}^4 \omega_B^4 \int_0^1 \mathrm{d}\hat{q}_\parallel \, \frac{1}{w_\lambda^2(2w_\lambda^2 - \omega_\mathrm{p}^2 - \omega_B^2)^2}\, \frac{\Phi_i(\hat{\boldsymbol{q}})}{[2\,\mathrm{Re}\, D_{\lambda,1}(\hat{\boldsymbol{q}})]^{1/2}} , \tag{6.5}$$

with

$$\Phi_\parallel(\hat{\boldsymbol{q}}) = \hat{q}_\parallel^2 \hat{q}_\perp^4 , \tag{6.6}$$

$$\Phi_\perp(\hat{\boldsymbol{q}}) = \tfrac{1}{2} \hat{q}_\parallel^4 \hat{q}_\perp^2 . \tag{6.7}$$

This result is in agreement with the long-time tail found from mode-coupling theory for the Green–Kubo integrand of the heat conductivity coefficients [6].

The long-time tails of $\bar{F}_i(t)$ that are brought about by the interference coupling of two gyro-plasmon modes are evaluated using the techniques from the appendix of ref. [5]. One finds that these couplings lead to long-time tails of the form $t^{-\nu}\cos(\omega t+\Theta)$ with $\nu \geq 1$ and $\omega = 2(\omega_p^2+\omega_B^2)^{1/2}$ or $\omega = |\omega_p \pm \omega_B|$. It should be noted that the heat mode plays no role in the analysis, because the long-time behaviour of the reduced t.c.f.'s in (3.10) is not dominated by the heat mode for small wavenumber.

We now turn to a discussion of the time-derivative term of $F_i(t)$ in (3.10). We shall write it as $\mathrm{d}\bar{\bar{F}}_i(t)/\mathrm{d}t$, with

$$\bar{\bar{F}}_i(t) = -\frac{\mathrm{i}nk_BT}{m}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)S^{nn}\mathscr{F}_i\,. \tag{6.8}$$

Analysis of the non-interference and the interference coupling contributions proceeds in the same way as above. It turns out that the non-interference couplings lead to a long-time tail in $\mathrm{d}\bar{\bar{F}}_i(t)/\mathrm{d}t$ which also decays with $t^{-1/2}$. It has the form

$$\frac{(k_BT)^2}{128\pi^{3/2}}\,\omega_p^4\omega_B^4\sum_{\lambda=\pm1}\int_0^1\mathrm{d}\hat{q}_\parallel\,\frac{(w_\lambda^2-\omega_B^2)[3(w_\lambda^2-\omega_B^2)-16(2w_\lambda^2-\omega_p^2-\omega_B^2)]}{w_\lambda^2(2w_\lambda^2-\omega_p^2-\omega_B^2)^4}\times\frac{\Phi_i(\hat{\boldsymbol{q}})}{[2\,\mathrm{Re}\,D_{\lambda,1}(\hat{\boldsymbol{q}})]^{1/2}}\,\frac{1}{t^{1/2}}\,. \tag{6.9}$$

The interference-coupling contributions to $\mathrm{d}\bar{\bar{F}}_i/\mathrm{d}t$ may give rise to long-time tails with an even slower decay. For the longitudinal case the dominating contribution to the long-time tail originates from the interference of two gyro-plasmon modes with $\lambda=\lambda'=1$ that propagate in a direction perpendicular to the magnetic field, with a zeroth-order frequency $(\omega_p^2+\omega_B^2)^{1/2}\equiv\omega_0$. The ensuing contribution to the long-time tail of $\mathrm{d}\bar{\bar{F}}_\parallel/\mathrm{d}t$ is found to be

$$\frac{(k_BT)^2}{32\pi}\,\frac{\omega_p^5\omega_B}{\omega_0^{7/2}}\,\mathrm{Re}\left[\frac{\mathrm{e}^{-2\mathrm{i}\omega_0 t+\frac{3}{4}\pi\mathrm{i}}}{[2D_{1,1}(\hat{\boldsymbol{q}}=\hat{\boldsymbol{q}}_\perp)]^{1/2}}\right]. \tag{6.10}$$

For the transverse case, on the other hand, the dominating contribution stems from the interference of two gyro-plasmon modes that travel parallel to the magnetic field, with a frequency ω_p in zeroth-order of the wavenumber. The contribution to the long-time tail of $\mathrm{d}\bar{\bar{F}}_\perp/\mathrm{d}t$ is

$$- \frac{(k_B T)^2}{16\pi^{3/2}} \omega_p^2 \operatorname{Re}\left[\frac{e^{-2i\omega_p t}}{[2D_{\pm 1,1}(\hat{q} = \hat{q}_\|)]^{1/2}} \right] \frac{1}{t^{1/2}} , \tag{6.11}$$

where the upper sign should be taken for $\omega_p > \omega_B$ and the lower sign for $\omega_p < \omega_B$.

As (6.10) and (6.11) show the interference coupling of two gyro-plasmon modes gives rise to a long-time tail in the time-derivative term $d\bar{\bar{F}}_i(t)/dt$ that decays slowly or even remains finite indefinitely. Unlike the corresponding contributions in $\bar{F}_i$ neither (6.10) nor (6.11) can be neglected with respect to (6.9). However, it should be noted that both of them oscillate with a finite frequency. Hence, they do not lead to a singular behaviour of the static heat conductivities. Instead, they give rise to infinities in the dynamic conductivities at a finite frequency, viz. $2\omega_0$ or $2\omega_p$. It is easily seen that interference coupling of gyro-plasmon modes will always lead to oscillating tails, and hence possibly to a singular behaviour of the dynamic conductivities at finite frequencies only. Since our main concern in the present paper is to analyse the behaviour of the conductivities near $z = i0$, we shall not consider (6.10) and (6.11) any further and we shall concentrate on the non-interference coupling contributions in the following.

As we have seen the non-interference coupling contributions in the direct part of $F_i(t)$ lead to non-oscillating tails proportional to $t^{-1/2}$ given by (6.4) and (6.9). The first of these agrees with the long-time tail of the Green–Kubo integrand for the heat conductivity coefficients, as found from mode-coupling theory, whereas the second has no counterpart in mode-coupling theory. Hence, the same situation as encountered for the unmagnetized plasma occurs here: both theories predict a tail of the same type, but the proportionality factors agree only if a time-derivative term is dismissed. For the unmagnetized plasma it could be argued that the latter does not contribute to the static heat conductivity. Here such a statement cannot be made; the time-derivative term does contribute to the static heat conductivity coefficients.

The study of the long-time tail of the direct part of $F_i(t)$ is finished now and we will proceed with the evaluation of the long-time behaviour of the indirect part of $F_i(t)$. The functions $c_\gamma(\boldsymbol{k}, t)$ and $c_{\gamma,\delta}(\boldsymbol{k}, t)$ which play a central role in the analysis of the indirect part have been given in section 4. The long-time behaviour of the reduced t.c.f.'s appearing in the expressions for c_γ and $c_{\gamma,\delta}$ will be substituted. Upon taking the Laplace transform the singular behaviour of (4.3) for $z \to i0$ can be determined.

In zeroth order in the wavenumber the behaviour of the first and the third factor in (4.3) near $z = i0$ follows directly from (4.6). To determine their behaviour in first and in second order of k we have to consider c_γ. We start with c_γ in order k^1, given by (4.18) with (4.19). When the derivative with respect to the wave vector $\boldsymbol{q}$ acts on the exponent of a reduced t.c.f., terms with a factor t appear in the integrand. Since only these terms can lead to a long-time tail decaying with $t^{-1/2}$, we analyse them separately. With the help of (3.6), (4.27) and (4.28) the non-interference coupling contributions to these terms are obtained as

$$\mathrm{i}t\,\frac{2}{3}\sqrt{\frac{5}{3}}\,\frac{1}{n(mk_{\mathrm{B}}T)^{3/2}}\sum_i \delta_{\gamma,i+9}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)q_i\sum_{\lambda\rho}\rho[\boldsymbol{k}\cdot\boldsymbol{\nabla}_q w_\lambda(\hat{\boldsymbol{q}})]$$
$$\times[n(q_\|^2 S^{\epsilon^{\mathrm{kin}}g}_{\|,\lambda\rho}+q_\perp^2 S^{\epsilon^{\mathrm{kin}}g}_{\perp,\lambda\rho})S^{nn}_{\lambda,-\rho}-(q_\|^2 S^{ng}_{\|,\lambda\rho}+q_\perp^2 S^{ng}_{\perp,\lambda\rho})S^{\epsilon^{\mathrm{kin}}n}_{\lambda,-\rho}]\,, \tag{6.12}$$

where we introduced the component $S_{\lambda\rho}$ of a reduced t.c.f. S by writing the part of S that is dominated by the gyro-plasmon modes $\lambda\rho$ as $\Sigma_{\lambda\rho} S_{\lambda\rho}$ in lowest-order of the wavenumber. By using (3.9) and

$$im\,\frac{\partial}{\partial t}\,S^{\epsilon^{\mathrm{kin}}n}(\boldsymbol{q},t)=q_\|^2 S_\|^{\epsilon^{\mathrm{kin}}g}(\boldsymbol{q},t)+q_\perp^2 S_\perp^{\epsilon^{\mathrm{kin}}g}(\boldsymbol{q},t)\,, \tag{6.13}$$

it is verified easily that the expression (6.12) vanishes. As a consequence the non-interference coupling contributions to c_γ with $\gamma=10,\ 11,\ 12$ in order k^1 decay like $t^{-3/2}$ or faster. Consequently, the Laplace transform $c_\gamma(\boldsymbol{k},z)$ in order k^1 is finite near $z=\mathrm{i}0$.

In order k^2 we have to study c_9 that is given by (4.20) with (4.21). After inserting (3.6) and (3.7) and writing the wave vector derivatives as derivatives with respect to $q_\|$ and $q_\perp$ we may evaluate the terms of c_9 which can lead to non-interference coupling contributions with tails proportional to $t^{1/2}$ and even $t^{3/2}$. These terms are

$$\frac{\sqrt{2}}{12nm^2}\sum_{i=\|,\perp}k_i^2\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\,c(q)v(q)q^2$$
$$\times\Big[-2\mathrm{i}nm\,\frac{\partial}{\partial t}\,\{S^{nn}(2\hat{q}_\|^2\mathcal{D}_i S_\|^{ng}-\hat{q}_\perp^2\mathcal{D}_i S_\perp^{ng})\}$$
$$+3q_\|^2\hat{q}_\perp^2(S_\perp^{ng}\mathcal{D}_i S_\|^{ng}-S_\|^{ng}\mathcal{D}_i S_\perp^{ng})$$
$$+\psi_i\,\frac{\partial S_\|^{ng}}{\partial q_i}\,[8\hat{q}_\|^2 S_\|^{ng}+(1-2\hat{q}_\|^2)S_\perp^{ng}]$$
$$+\psi_i\,\frac{\partial S_\perp^{ng}}{\partial q_i}\,[(1-2\hat{q}_\|^2)S_\|^{ng}+4\hat{q}_\perp^2 S_\perp^{ng}]+6n\psi_i q^{-2}S^{nn}\,\frac{\partial S_3^{gg}}{\partial q_i}\Big]\,, \tag{6.14}$$

where we used the abbreviations $\psi_\|=-2q_\|\hat{q}_\perp^2$ and $\psi_\perp=q_\perp\hat{q}_\|^2$. The operators

$\mathscr{D}_i$ are defined by:

$$\mathscr{D}_{\parallel} S = \frac{\partial^2}{\partial q_{\parallel}^2} S , \tag{6.15}$$

$$\mathscr{D}_{\perp} S = \frac{1}{2} \frac{1}{q_{\perp}} \frac{\partial}{\partial q_{\perp}} \left(q_{\perp} \frac{\partial}{\partial q_{\perp}} S \right) . \tag{6.16}$$

By substituting the explicit expressions for the reduced t.c.f.'s it is found that non-interference coupling of two gyro-plasmon modes does not give rise to $t^{3/2}$- and $t^{1/2}$-tails. We conclude that c_9 in second order in k decays with $t^{-1/2}$ and that the Laplace transform $c_9(\boldsymbol{k}, z)$ in order k^2 behaves as $z^{-1/2}$ for small z.

Finally, we will analyse the long-time behaviour of the coefficient $c_{\gamma,\delta}$ in order k^0, k^1 and k^2 for the choices of γ, δ given in section 4. In order k^0 we may prove from (4.24), by substituting the explicit expressions for the reduced t.c.f.'s, that the wave vector integrand of $c_{\gamma,\delta}$ is of order q^0. This leads to a $t^{-3/2}$ decay. Consequently, the Laplace transform $c_{\gamma,\delta}(\boldsymbol{k}, z)$ in order k^0 is finite for small z.

In order k^1 one verifies with the help of (4.25) that $c_{\gamma,9}$ and $c_{9,\delta}$ with $\gamma = 10, 11, 12$ and $\delta = 10, 11, 12$ decay for long times with $t^{-1/2}$. It follows that the Laplace transform $c_{9,\gamma}(\boldsymbol{k}, z)$ in order k^1 behaves as $z^{-1/2}$ for small z.

In second order in the wavenumber we have to consider $c_{9,9}$ given by (4.26). On a par with (6.14) we may deduce the terms which can lead to non-interference coupling contributions with tails that are proportional to $t^{3/2}$. These are

$$-\frac{1}{6nm^2 k_B T} \sum_{i=\parallel,\perp} k_i^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3} c(q) v(q) q^2$$
$$\times [nS^{nn}\{(1+3\hat{q}_{\parallel}^2)\mathscr{D}_i S_1^{gg} + 4\hat{q}_{\parallel}^4 \mathscr{D}_i S_2^{gg} - 4\hat{q}_{\parallel}^2 \hat{q}_{\perp}^2 \mathscr{D}_i S_3^{gg} + \hat{q}_{\perp}^4 \mathscr{D}_i S_4^{gg}\}$$
$$+ q^2(2\hat{q}_{\parallel}^2 S_{\parallel}^{ng} - \hat{q}_{\perp}^2 S_{\perp}^{ng})(2\hat{q}_{\parallel}^2 \mathscr{D}_i S_{\parallel}^{ng} - \hat{q}_{\perp}^2 \mathscr{D}_i S_{\perp}^{ng})] . \tag{6.17}$$

Upon inserting, as before, the explicit expressions for the reduced t.c.f.'s and focussing on the non-interference couplings one may check that the long-time tails proportional to $t^{3/2}$, which arise by a repeated differentiation of the exponential factors in the reduced t.c.f.'s, cancel. Hence, the long-time behaviour of (6.17) is governed by tails proportional to $t^{1/2}$. Consequently, the Laplace transform $c_{9,9}(\boldsymbol{k}, z)$ in order k^2 is proportional to $z^{-3/2}$ for small z.

Having established now the behaviour near $z = \mathrm{i}0$ of all relevant coefficients c_γ and $c_{\gamma,\delta}$ it is an easy matter to derive the z-dependence of (4.3) near the origin; it turns out that it stays finite as z tends to i0. Hence, the non-oscillating part of the asymptotic expression for the indirect part of $F_i(t)$ decays faster than t^{-1}, so that it can be neglected in comparison with the non-oscillating contribution that arises from the direct part.

7. Concluding remarks

The kinetic approach followed in this paper has led to explicit expressions for the long-time tails of the time correlation functions associated to the longitudinal and the transverse heat conductivities of a one-component plasma in a uniform magnetic field. Apart from oscillating terms these tails contain a monotonously decaying contribution that is proportional to $t^{-1/2}$. Qualitatively this result corroborates earlier findings for the long-time behaviour of the Green–Kubo integrands for the heat conductivities, which have been obtained by means of mode-coupling theory. Quantitatively the agreement is only partial: the asymptotic expression that follows from kinetic theory is the sum of two terms of which one is identical to that found from mode-coupling theory, whereas the other term, which is a total time derivative, has no counterpart in the mode-coupling expression. As has been noted already in the introduction this discrepancy need not come as a surprise, since the two time correlation functions of which the long-time tails are compared are closely related but differently defined functions.

The long-time tails proportional to $t^{-1/2}$ give rise to a divergency proportional to $z^{-1/2}$ in the dynamical heat conductivities if the frequency z approaches i0 from the upper half plane. Hence both the earlier mode-coupling treatment and the present kinetic approach lead to the conclusion that the static heat conductivity, which determines the damping of the collective heat mode, is no longer finite if a magnetic field is turned on. Such a conclusion implies that strictly speaking a magnetohydrodynamic description that incorporates dissipative effects is meaningless for a magnetized one-component plasma. Of course the inconsistency of a hydrodynamic description is not unprecedented: as is well known hydrodynamics cannot be used either for systems of neutral particles in a space with dimension less than 3. It might be argued that the magnetic field effectively reduces the spatial dimension, since it essentially forces the free-particle motion into spiral orbits. In our opinion this argument is too vague, however, to explain an infinite heat conductivity on its basis.

In arriving at the above conclusion a crucial assumption has been made, namely that the long-time behaviour of the relevant time correlation functions is indeed governed by the "disconnected" part of the kinetic kernel. This assumption is closely related to that usually made in mode-coupling theories, according to which the long-time behaviour of a Green–Kubo integrand is dominated by the coupling of pairs of modes. Furthermore, the discussion of the present paper has been confined to the special model of a one-component plasma with an inert neutralizing background. It remains to be investigated whether the main conclusion of our treatment is unaffected if the assumption on the preponderance of the disconnected part of the kinetic kernel is abandoned or if a more realistic model for a plasma is adopted.

In closing it may be remarked that it would be highly interesting if the present results concerning the slowly decaying tails of the heat-conductivity time correlation functions for a one-component magnetized plasma could be verified by means of molecular dynamics.

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