On the Validity of Magnetohydrodynamics for Ionic Mixtures

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The collective mode spectrum of the magnetized ionic mixture consisting of s components is studied by starting from the microscopic balance equations and the fluctuation formulas for the microscopic densities. Apart from a heat mode and $s-1$ diffusion modes with frequencies of second order in the wavenumber, four modes with complex finite frequencies for vanishing wavenumber are found. If the mixture consist of particles with equal ratios of charge and mass, these four modes become similar to the gyroplasmon modes of the magnetized one-component plasma of which the frequencies are real in lowest order in the wavenumber. Green–Kubo relations are derived for the transport coefficients which appear in the frequencies of the heat mode and the diffusion modes. The long-time behavior of the integrands of the Green–Kubo expressions is evaluated with the help of mode-coupling theory. The static transport coefficients are found to be finite unless the mixture consists of species with equal ratios of charge and mass. It is concluded that the presence of species with different charge–mass ratios is essential for the validity of magnetohydrodynamics for an ionic mixture.

1. INTRODUCTION

In recent years a large number of papers have been published in which the dynamic properties of classical Coulomb systems were studied.[1–3] The system which is investigated most frequently is the one-component plasma, which consists of identical charged particles immersed in a neutralizing background. With the help of kinetic theory, the collective mode spectra of

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the one-component plasma[4] and the one-component plasma in a uniform magnetic field[5] have been studied. The results of these studies have been corroborated by a different method to derive the mode spectra, which starts from the microscopic balance equations for the particle density, the momentum density, and the energy density.[6-8]

The damping of the collective modes is determined by the transport properties of the plasma. These transport properties can be calculated from Green–Kubo-type expressions that contain integrals over time correlation functions. To evaluate the long-time behavior of the Green–Kubo integrands for the transport coefficients, mode-coupling theory has been used, both for the unmagnetized[6] and for the magnetized one-component plasma.[9,10] In this way interesting results have been found for the Green–Kubo integrand of the heat conductivity. In the case of the unmagnetized one-component plasma the integrand decays, for long times, like $t^{-1/2} \cos(\omega_p + \Theta)$, with ω_p the plasma frequency. This long-time tail leads to a divergence in the dynamic heat conductivity coefficient $\lambda(\omega)$ for the frequency $\omega = \omega_p$. For the magnetized one-component plasma the Green–Kubo integrands for the longitudinal and the transverse heat conductivity coefficients both decay like $t^{-1/2}$. Consequently, the static heat conductivity coefficients are divergent. The transport coefficients have also been studied by means of kinetic theory. The divergences of the dynamic heat conductivity for the unmagnetized plasma and of the static heat conductivities for the magnetized plasma, found with the help of mode-coupling theory, have been confirmed by this method.[11,12] From the · divergence of the static heat conductivity for the one-component plasma in a magnetic field it can be concluded that magnetohydrodynamics is not well defined for this system.

In order to determine whether the divergence of the heat conductivity is an artefact of the adopted model, i.e., the magnetized one-component plasma, we will investigate in the following the properties of a more general model system, viz. an ionic mixture with several particle species in a magnetic field. Our aim is to establish formal expressions of Green–Kubo type for the dynamic heat conductivity and diffusion coefficients of a magnetized ionic mixture, and to study the static limit of these expressions.

The model adopted in this paper is the classical multicomponent ionic mixture. It consists of s species of particles with charges e_σ and masses m_σ $(\sigma = 1,..., s)$. The particles obey the classical equations of motion and move in an inert, neutralizing background. The interaction between the particles and with the background is purely electrostatic. The external magnetic field is static and uniform in space.

After reviewing the balance equations for the magnetized ionic mixture in Section 2, we derive the dispersion relation, which yields the collective

mode frequencies, and the modes in lowest order in the wavenumber k in Section 3. The zeroth-order dispersion relation is also investigated for an ionic mixture consisting of species with equal charge–mass ratios and for the unmagnetized ionic mixture. In Section 4 the frequencies of the heat mode and the diffusion modes are determined in second order in k. Green–Kubo-type expressions are derived for the transport coefficients which appear in these frequencies. Finally, Section 5 is devoted to the evaluation of the long-time behavior of the integrands of the dynamic coefficients occurring in the Green–Kubo expressions.

2. BALANCE EQUATIONS

In this section the balance equations for the microscopic partial particle density, the total momentum density, and the energy density for a multicomponent ionic mixture in a uniform magnetic field are given. These balance equations will be used to derive the collective modes of the system in the next section.

The balance equation for the Fourier transform of the partial particle density $n_\sigma(\mathbf{k})$ of the species with label $\sigma = 1,..., s$,

$$n_\sigma(\mathbf{k}) = \sum_\alpha \exp(-i\mathbf{k} \cdot \mathbf{r}_{\sigma\alpha}) \tag{2.1}$$

with $\mathbf{k}$ the wave vector and $\mathbf{r}_{\sigma\alpha}$ the position of particle α of component σ, reads

$$Ln_\sigma(\mathbf{k}) = -\frac{\mathbf{k}}{m_\sigma} \cdot \mathbf{g}_\sigma(\mathbf{k}) \tag{2.2}$$

The Liouville operator in phase space L determines the time derivative $\dot{F}$ of an arbitrary function F through $\dot{F} = iLF$.

The partial momentum density of species σ in Fourier space is given by

$$\mathbf{g}_\sigma(\mathbf{k}) = \sum_\alpha \mathbf{p}_{\sigma\alpha} \exp(-i\mathbf{k} \cdot \mathbf{r}_{\sigma\alpha}) \tag{2.3}$$

with $\mathbf{p}_{\sigma\alpha}$ the momentum of particle α of component σ. The balance equation for the total momentum density $\mathbf{g}(\mathbf{k}) = \sum_\sigma \mathbf{g}_\sigma(\mathbf{k})$ can be written as

$$L\mathbf{g}(\mathbf{k}) = -\mathbf{k} \cdot \tau - q_v \frac{\mathbf{k}}{k^2} q_v(\mathbf{k}) - i \sum_\sigma \omega_{B\sigma} \mathbf{g}_\sigma(\mathbf{k}) \wedge \hat{\mathbf{B}} \tag{2.4}$$

It contains a pressure tensor τ, which is finite in the limit of small

wavenumber k. The explicit expression for τ is not needed here. Further-more, the momentum balance equation contains a term which is propor-tional to the electric field generated by the charge density fluctuation $q_v(\mathbf{k}) = \sum_\sigma e_\sigma n_\sigma(\mathbf{k})$. Here $q_v = \sum_\sigma e_\sigma n_\sigma$ is the (equilibrium) charge density, with n_σ the (equilibrium) particle density of species σ. The last term in (2.4) accounts for the Lorentz force due to the uniform magnetic field, which points in the direction of the unit vector $\hat{\mathbf{B}}$. The cyclotron frequency for a particle of component σ in a magnetic field with strength B is given by $\omega_{B\sigma} = e_\sigma B / m_\sigma c$. For the unmagnetized ionic mixture this term is absent. We remark that only the total momentum density satisfies a simple balance equation.

The energy density consists of a kinetic and a potential part:

$$\varepsilon(\mathbf{k}) = \varepsilon^{\mathrm{kin}}(\mathbf{k}) + \varepsilon^{\mathrm{pot}}(\mathbf{k}) \tag{2.5}$$

with

$$\varepsilon^{\mathrm{kin}}(\mathbf{k}) = \sum_{\sigma\alpha} \frac{p_{\sigma\alpha}^2}{2m_\sigma} \exp(-i\mathbf{k}\cdot\mathbf{r}_{\sigma\alpha}) \tag{2.6}$$

$$\varepsilon^{\mathrm{pot}}(\mathbf{k}) = -\frac{1}{2V} \sum_{\mathbf{q}(\neq 0,\neq\mathbf{k})} \sideset{}{'}\sum_{\sigma_1\alpha_1,\sigma_2\alpha_2} e_{\sigma_1} e_{\sigma_2} \frac{\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})}{q^2(\mathbf{k}-\mathbf{q})^2}$$
$$\times \exp[i\mathbf{q}\cdot(\mathbf{r}_{\sigma_1\alpha_1} - \mathbf{r}_{\sigma_2\alpha_2}) - i\mathbf{k}\cdot\mathbf{r}_{\sigma_1\alpha_1}] \tag{2.7}$$

The prime on the summation symbol denotes the restriction $\sigma_i\alpha_i \neq \sigma_j\alpha_j$ $(i \neq j)$. The balance equation for the energy density reads

$$L\varepsilon(\mathbf{k}) = -\mathbf{k}\cdot\mathbf{j}_\varepsilon(\mathbf{k}) \tag{2.8}$$

The energy-current density $\mathbf{j}_\varepsilon(\mathbf{k})$ is the sum of a kinetic and a potential part:

$$\mathbf{j}_\varepsilon(\mathbf{k}) = \mathbf{j}_\varepsilon^{\mathrm{kin}}(\mathbf{k}) + \mathbf{j}_\varepsilon^{\mathrm{pot}}(\mathbf{k}) \tag{2.9}$$

with

$$\mathbf{k}\cdot\mathbf{j}_\varepsilon^{\mathrm{kin}}(\mathbf{k}) = \mathbf{k}\cdot\sum_{\sigma\alpha} \frac{\mathbf{p}_{\sigma\alpha}}{m_\sigma} \frac{p_{\sigma\alpha}^2}{2m_\sigma} \exp(-i\mathbf{k}\cdot\mathbf{r}_{\sigma\alpha}) \tag{2.10}$$

$$\mathbf{k}\cdot\mathbf{j}_\varepsilon^{\mathrm{pot}}(\mathbf{k}) = \frac{\mathbf{k}}{V}\cdot \sideset{}{'}\sum_{\sigma_1\alpha_1,\sigma_2\alpha_2} \frac{e_{\sigma_1} e_{\sigma_2}}{m_{\sigma_1}} \frac{\mathbf{p}_{\sigma_1\alpha_1}}{k^2} \exp(-i\mathbf{k}\cdot\mathbf{r}_{\sigma_2\alpha_2})$$
$$+ \frac{1}{V} \sum_{\mathbf{q}(\neq 0,\neq\mathbf{k})} \sideset{}{'}\sum_{\sigma_1\alpha_1,\sigma_2\alpha_2} \frac{e_{\sigma_1} e_{\sigma_2}}{m_{\sigma_1}} \left[\mathbf{q} - \frac{\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2} \right] \cdot \frac{\mathbf{p}_{\sigma_1\alpha_1}}{q^2}$$
$$\times \exp[i\mathbf{q}\cdot(\mathbf{r}_{\sigma_1\alpha_1} - \mathbf{r}_{\sigma_2\alpha_2}) - i\mathbf{k}\cdot\mathbf{r}_{\sigma_1\alpha_1}] \tag{2.11}$$

The balance equations can be used to derive the fluctuation formulas for the densities and the currents. The fluctuation formulas which are needed in the derivation of the collective mode spectrum of the (magnetized) multicomponent ionic mixture are given in Appendix A.

3. COLLECTIVE MODES

In the following we will study the long-living modes of a magnetized multicomponent ionic mixture. These so-called collective modes are characterized by frequencies of which the imaginary part is either negative and close to zero or vanishing in the long-wavelength limit. An ionic mixture consisting of s components will have $s+4$ collective modes, since the number of collective modes equals the number of conserved microscopic densities. The collective modes, denoted by $a_i(\mathbf{k})$, are particular independent linear combinations of the conserved microscopic densities: the partial particle densities $n_\sigma(\mathbf{k})$, the total momentum density $\mathbf{g}(\mathbf{k})$, and the energy density $\varepsilon(\mathbf{k})$. The time evolution of the collective mode $a_i(\mathbf{k})$ is given by

$$(z+L)\, a_i(\mathbf{k}, z) = a_i(\mathbf{k}) \tag{3.1}$$

with $a_i(\mathbf{k}, z)$ the Laplace transform of $a_i(\mathbf{k}, t)$

$$a_i(\mathbf{k}, z) = -i \int_0^\infty dt\, e^{izt} a_i(\mathbf{k}, t), \qquad \text{Im } z > 0 \tag{3.2}$$

A method to derive the collective modes from the balance equations and the fluctuation formulas, by using projection operator techniques, is given in a recent paper.[8] With the help of this method the frequencies z_i of the collective modes $a_i(\mathbf{k})$ are found as the eigenfrequencies of the frequency matrix Ω for small wavenumber. The elements of this matrix are

$$\Omega_{ij}(\mathbf{k}, z) = \Omega_{ij}^{(1)}(\mathbf{k}, z) + \Omega_{ij}^{(2)}(\mathbf{k}, z) \tag{3.3}$$

with the direct and the indirect parts given by

$$\Omega_{ij}^{(1)}(\mathbf{k}, z) = -\frac{1}{V} \langle \bar{a}_i^*(\mathbf{k})\, L a_j(\mathbf{k}) \rangle \tag{3.4}$$

$$\Omega_{ij}^{(2)}(\mathbf{k}, z) = \frac{1}{V} \langle \bar{a}_i^*(\mathbf{k})\, LQ \frac{1}{z+QLQ} QL a_j(\mathbf{k}) \rangle \tag{3.5}$$

The angle brackets denote an equilibrium ensemble average. The adjoints $\bar{a}_i(\mathbf{k})$ are defined by

$$\frac{1}{V} \langle \bar{a}_i^*(\mathbf{k}) \, a_j(\mathbf{k}) \rangle = \delta_{ij} \qquad (3.6)$$

The projection operator Q is the complement of a projection operator P which projects an arbitrary function in phase space onto a space spanned by the $s + 4$ conserved microscopic densities. As a basis set of independent linear combinations of the conserved microscopic densities we choose the total charge density $\beta^{1/2} k^{-1} q_v(\mathbf{k})$, the total momentum density $(\beta/m_v)^{1/2} \, \mathbf{g}(\mathbf{k})$, the partial particle densities $n_\sigma(\mathbf{k})$ with $\sigma = 2,..., s$, and the energy density $\varepsilon(\mathbf{k})$:

$$a_i(\mathbf{k}) \in \left\{ \frac{\beta^{1/2}}{k} \, q_v(\mathbf{k}), \left(\frac{\beta}{m_v}\right)^{1/2} \mathbf{g}(\mathbf{k}), n_2(\mathbf{k}),..., n_s(\mathbf{k}), \varepsilon(\mathbf{k}) \right\}, \qquad i = 0,..., s + 3$$

$$(3.7)$$

The charge density is divided by the wavenumber since the fluctuation formulas involving the charge density are proportional to k^2, as a consequence of the strong suppression of charge fluctuations by the Coulomb interaction. For convenience we have multiplied $k^{-1} q_v(\mathbf{k})$ and $\mathbf{g}(\mathbf{k})$ by normalization factors. Here, β is $1/k_B T$, with k_B Boltzmann's constant and T the temperature. Furthermore, $m_v = \sum_\sigma m_\sigma n_\sigma$ is the (equilibrium) total mass density.

With the help of the balance equations of the previous section and the fluctuation formulas given in Appendix A, one can determine the elements of the frequency matrix with respect to the basis set (3.7). The adjoints of the basis set (3.7) in lowest order in k follow from (3.6). The only matrix elements of the direct part $\Omega_{ij}^{(1)}$ which differ from zero for vanishing wavenumber are

$$\Omega_{i0}^{(1)}(\mathbf{k}, z) = \Omega_{0i}^{(1)}(\mathbf{k}, z) = \omega_p \hat{k}_i, \qquad i = 1, 2, 3 \qquad (3.8)$$

$$\Omega_{ij}^{(1)}(\mathbf{k}, z) = -i\omega_B \varepsilon_{ijk} \hat{B}_k, \qquad i, j = 1, 2, 3 \qquad (3.9)$$

with $\hat{k} = \mathbf{k}/k$ a unit vector in the direction of the wave vector. The plasma frequency is given by $\omega_p = q_v m_v^{-1/2}$. Furthermore, the collective cyclotron frequency is $\omega_B = q_v B/m_v c$.

In contrast with the indirect part of the frequency matrix for the (unmagnetized and magnetized) one-component plasma,[8] the matrix $\Omega^{(2)}$

for a multicomponent ionic mixture contains elements which are of order k^0, since one has in leading order in k

$$QL \frac{1}{k} q_v(\mathbf{k}) = -\hat{\mathbf{k}} \cdot \left[\sum_\sigma \frac{e_\sigma}{m_\sigma} \mathbf{g}_\sigma(\mathbf{k}) - \frac{q_v}{m_v} \mathbf{g}(\mathbf{k}) \right] \tag{3.10}$$

$$QL\mathbf{g}(\mathbf{k}) = -i \left[\sum_\sigma \omega_{B\sigma} \mathbf{g}_\sigma(\mathbf{k}) - \omega_B \mathbf{g}(\mathbf{k}) \right] \wedge \hat{\mathbf{B}} \tag{3.11}$$

From (3.10) and (3.11) we conclude that the matrix elements $\Omega_{ij}^{(2)}$ with $i, j = 0, 1, 2$ are of order k^0, if we choose the direction of the magnetic field parallel to the positive z axis. With the help of (3.11) one finds that the matrix elements $\Omega_{ij}^{(2)}$ with $i, j = 1, 2$ are independent of the wave vector $\mathbf{k}$, at least in lowest order in k:

$$\Omega_{ij}^{(2)}(\mathbf{k}, z, \mathbf{B}) = a(z) \, \varepsilon_{ijk} \hat{B}_k \tag{3.12}$$

where the dependence of the matrix element on $\mathbf{B}$ has been made explicit. The coefficient a is a function of the frequency z and the magnetic field strength B. The dependence on the latter is suppressed in (3.12). By using the hermitian character of the Liouville operator it can be shown that the frequency-dependent coefficient $a(z)$ satisfies the relation

$$a(z) = -[a(-z^*)]^* \tag{3.13}$$

for Im $z > 0$. By inspection of (3.10) and (3.11) it follows that one can write for the matrix elements $\Omega_{i0}^{(2)}$ and $\Omega_{0i}^{(2)}$, with $i = 1, 2, 3$, in zeroth order in k,

$$\Omega_{i0}^{(2)}(\mathbf{k}, z, \mathbf{B}) = b(z) \, \hat{k}_i + b'(z)(\hat{\mathbf{k}} \wedge \hat{\mathbf{B}})_i + b''(z) \, \hat{B}_i \hat{\mathbf{k}} \cdot \hat{\mathbf{B}} \tag{3.14}$$

$$\Omega_{0i}^{(2)}(\mathbf{k}, z, \mathbf{B}) = b(z) \, \hat{k}_i - b'(z)(\hat{\mathbf{k}} \wedge \hat{\mathbf{B}})_i + b''(z) \, \hat{B}_i \hat{\mathbf{k}} \cdot \hat{\mathbf{B}} \tag{3.15}$$

The coefficient $b(z)$ obeys the relation

$$b(z) = [b(-z^*)]^* \tag{3.16}$$

with similar relations for $b'(z)$ and $b''(z)$. Since $QLg_z(\mathbf{k}) = 0$ in zeroth order in the wavenumber, the matrix elements $\Omega_{30}^{(2)}$ and $\Omega_{03}^{(2)}$ vanish in order k^0, so that one ends up with

$$\Omega_{i0}^{(2)}(\mathbf{k}, z, \mathbf{B}) = b(z) \, \hat{k}_i + b'(z)(\hat{\mathbf{k}} \wedge \hat{\mathbf{B}})_i \tag{3.17}$$

$$\Omega_{0i}^{(2)}(\mathbf{k}, z, \mathbf{B}) = b(z) \, \hat{k}_i - b'(z)(\hat{\mathbf{k}} \wedge \hat{\mathbf{B}})_i \tag{3.18}$$

for $i = 1, 2$. Finally, due to (3.10), the matrix element $\Omega^{(2)}_{00}$ can be written in lowest order in k as

$$\Omega^{(2)}_{00}(\mathbf{k}, z, \mathbf{B}) = c(z) \tag{3.19}$$

with

$$c(z) = -[c(-z^*)]^* \tag{3.20}$$

The eigenvalue equation for the frequency matrix can now be evaluated in order k^0. One finds

$$
\begin{aligned}
z^s [[z^2 - zc(z) - \omega_p^2 \hat{k}_{\|}^2]\{z^2 + [-i\omega_B + a(z)]^2\} \\
- z(z\{[\omega_p + b(z)]^2 - [b'(z)]^2\} \\
- 2[-i\omega_B + a(z)][\omega_p + b(z)] \, b'(z)) \, \hat{k}_\perp^2] = 0
\end{aligned}
\tag{3.21}
$$

where $\hat{k}_{\|} = \mathbf{k} \cdot \hat{\mathbf{B}}$ and $\hat{k}_\perp^2 = 1 - \hat{k}_{\|}^2$. For a magnetized ionic mixture consisting of particle species with equal charge–mass ratio, i.e., with constant e_σ / m_σ for all σ, all matrix elements $\Omega^{(2)}_{ij}$ vanish in order k^0, as can be seen by inspection of (3.10) and (3.11). We call this a "well-poised" ionic mixture. For such a system the eigenvalue equation for the frequency matrix becomes

$$z^s [z^4 - (\omega_p^2 + \omega_B^2) z^2 + \omega_p^2 \, \omega_B^2 \hat{k}_{\|}^2] = 0 \tag{3.22}$$

This equation yields s modes with zero frequency for vanishing wavenumber and four "gyroplasmon" modes with real frequencies. Hence, the mode spectrum of this system is, apart from extra zero-frequency modes, the same as the mode spectrum for the magnetized one-component plasma.[8] The dispersion relation (3.21) has s zero-frequency solutions and furthermore a set of solutions which generally have a real and an imaginary part. Due to the four real solutions of (3.22), we expect that (3.21) has four solutions which lie just below the real axis and possibly other solutions with a large negative imaginary part. Since we are only interested in the long-living collective modes of the magnetized multicomponent ionic mixture, we will concentrate on the four solutions with small imaginary part. They depend on the orientation of the wave vector: $z_i = z_i(\hat{k}_{\|}) = z_i'(\hat{k}_{\|}) + i z_i''(\hat{k}_{\|})$ with z_i' and z_i'' real. With (3.13), (3.16), and (3.20) one can easily prove that, when $z_i = z_i' + i z_i''$ is a solution of (3.21), $z_i = -z_i' + i z_i''$ is also a solution. Furthermore, the imaginary parts of all solutions are even in $\hat{k}_{\|}$, whereas the real parts may be even or odd in $\hat{k}_{\|}$. By requiring that the four solutions of (3.21) should evolve continuously from the four gyroplasmon mode fre-

quencies of the well-poised ionic mixture, we can write the four generalized gyroplasmon mode frequencies as

$$z_{\lambda\rho} = \rho z'_\lambda + i z''_\lambda, \qquad \lambda = \pm 1, \qquad \rho = \pm 1 \tag{3.23}$$

with z'_λ and z''_λ real. They satisfy the relations

$$z^*_{\lambda\rho} = -z_{\lambda,-\rho} \tag{3.24}$$

$$z'_\lambda(\hat{k}_{\|}) = \lambda z'_\lambda(-\hat{k}_{\|}) \tag{3.25}$$

$$z''_\lambda(\hat{k}_{\|}) = z''_\lambda(-\hat{k}_{\|}) \tag{3.26}$$

The modes $a_i(\mathbf{k})$ are the eigenvectors of the frequency matrix Ω that correspond to the mode frequencies z_i. The four generalized gyroplasmon modes of the magnetized multicomponent ionic mixture are, in lowest order in k,

$$a_{\lambda\rho}(\mathbf{k}) = \frac{\beta^{1/2}}{k} q_v(\mathbf{k}) + \left(\frac{\beta}{m_v}\right)^{1/2} \mathbf{v}_{\lambda\rho}(\mathbf{k}) \cdot \mathbf{g}(\mathbf{k}) \tag{3.27}$$

with the vector $\mathbf{v}_{\lambda\rho}(\mathbf{k})$ given by

$$
\begin{aligned}
\mathbf{v}_{\lambda\rho}(\mathbf{k}) = {} & \frac{\omega_p}{z_{\lambda\rho}} \hat{\mathbf{k}}_{\|} + \frac{1}{z^2_{\lambda\rho} + [-i\omega_B + a(z_{\lambda\rho})]^2} \\
& \times (\{z_{\lambda\rho}[\omega_p + b(z_{\lambda\rho})] - [-i\omega_B + a(z_{\lambda\rho})]\, b'(z_{\lambda\rho})\}\, \hat{\mathbf{k}}_{\perp} \\
& + \{z_{\lambda\rho}b'(z_{\lambda\rho}) + [-i\omega_B + a(z_{\lambda\rho})][\omega_p + b(z_{\lambda\rho})]\}\, \hat{\mathbf{k}} \wedge \hat{\mathbf{B}})
\end{aligned} \tag{3.28}
$$

where $\hat{\mathbf{k}}_{\perp} = \hat{\mathbf{k}} - \hat{k}_{\|}\hat{\mathbf{B}}$. The adjoints $\bar{a}_{\lambda\rho}(\mathbf{k})$ have the form

$$\bar{a}_{\lambda\rho}(\mathbf{k}) = N_{\lambda\rho}\left[\frac{\beta^{1/2}}{k} q_v(\mathbf{k}) + \left(\frac{\beta}{m_v}\right)^{1/2} \bar{\mathbf{v}}_{\lambda\rho}(\mathbf{k}) \cdot \mathbf{g}(\mathbf{k})\right] \tag{3.29}$$

with

$$\bar{\mathbf{v}}_{\lambda\rho}(\mathbf{k}) = f_{\|,\lambda\rho}\hat{\mathbf{k}}_{\|} + f_{\perp,\lambda\rho}\hat{\mathbf{k}}_{\perp} + f_{t,\lambda\rho}\hat{\mathbf{k}} \wedge \hat{\mathbf{B}} \tag{3.30}$$

The normalization constant $N_{\lambda\rho}$ and the coefficients $f_{i,\lambda\rho}$, with $i = \|$, $\perp$, t, can be determined from the orthonormality relation (3.6). We remark that, contrary to the case of the (magnetized) one-component plasma, the modes $a_{\lambda\rho}(\mathbf{k})$ differ from their adjoints $\bar{a}_{\lambda\rho}(\mathbf{k})$, even in lowest order in the wavenumber. The s modes $a_i(\mathbf{k})$ ($i = 1,..., s$) with zero frequency in order k^0 are degenerate. They (and their adjoints) are linear combinations of the partial particle densities $n_\sigma(\mathbf{k})$ ($\sigma \neq 1$) and the energy density $\varepsilon(\mathbf{k})$.

Finally, we discuss the mode spectrum for the unmagnetized multi-component ionic mixture. The dispersion relation for this system has been studied before by means of related microscopic methods.[13-15] For the unmagnetized multicomponent ionic mixture the coefficients $a(z)$, $b(z)$, and $b'(z)$ vanish because $QLg(\mathbf{k})$ is of order k^1 for zero magnetic field. The dispersion relation for this system is

$$z^{s+2}[z^2 - zc(z) - \omega_p^2] = 0 \tag{3.31}$$

The two zero-frequency solutions which show up here in addition to those occurring in (3.21) correspond to the two transverse viscous modes:

$$a_{\eta_1}(\mathbf{k}) = \bar{a}_{\eta_1}(\mathbf{k}) = \left(\frac{\beta}{m_v}\right)^{1/2} \hat{\mathbf{k}} \wedge \mathbf{g}(\mathbf{k}) \tag{3.32}$$

$$a_{\eta_2}(\mathbf{k}) = \bar{a}_{\eta_2}(\mathbf{k}) = \left(\frac{\beta}{m_v}\right)^{1/2} \hat{\mathbf{k}} \wedge [\hat{\mathbf{k}} \wedge \mathbf{g}(\mathbf{k})] \tag{3.33}$$

Apart from the $s+2$ zero-frequency modes, there are two modes with a finite complex frequency $z_\rho (\rho = \pm 1)$ for vanishing wavenumber. With (3.20) it can be shown that, when $z_\rho = \rho z' + iz''$ is a solution of (3.31), $z_{-\rho} = -\rho z' + iz''$ is also a solution. The corresponding modes are

$$a_\rho(\mathbf{k}) = \frac{\beta^{1/2}}{k} q_v(\mathbf{k}) + \left(\frac{\beta}{m_v}\right)^{1/2} \frac{\omega_p}{z_\rho} \hat{\mathbf{k}} \cdot \mathbf{g}(\mathbf{k}) \tag{3.34}$$

with the adjoints

$$\bar{a}_\rho(\mathbf{k}) = \frac{z_{-\rho}}{z_{-\rho} - z_\rho} \left[\frac{\beta^{1/2}}{k} q_v(\mathbf{k}) + \left(\frac{\beta}{m_v}\right)^{1/2} \frac{z_\rho}{\omega_p} \hat{\mathbf{k}} \cdot \mathbf{g}(\mathbf{k}) \right] \tag{3.35}$$

For the well-poised unmagnetized ionic mixture the coefficient $c(z)$ vanishes and one finds two oscillating plasmon modes with frequencies $z = \pm\omega_p$.

An alternative method to derive the mode spectrum is based on a formal kinetic equation for the one-particle time correlation function in phase space. In this way the mode spectrum has been derived for the unmagnetized and the magnetized one-component plasma.[4,5] The kinetic method can also be used to study the dispersion relation for the ionic mixture. For the unmagnetized binary ionic mixture this has been done by Baus.[16] With the help of projection operator techniques one derives a formal kinetic equation for the one-particle time correlation function which is the equilibrium ensemble average of the product of the initial phase-space density of species σ and the phase-space density of species σ' at time t. The kinetic equation contains a kernel Σ which consists of a free-

streaming term, a mean-field term, and a collision term. A matrix $G_{\mu\nu}$ is formed by matrix elements of Σ with respect to a "hydrodynamic" subspace in momentum space. This matrix satisfies an equation which contains a frequency matrix $\Omega_{\mu\nu}$. The collective mode frequencies are found as the eigenvalues of the frequency matrix.

For the one-component plasma the hydrodynamic subspace consists of the five functions in momentum space corresponding to the particle density, the momentum density, and the kinetic energy density. To derive the dispersion relation for the binary ionic mixture, a hydrodynamic subspace has been used which consists of the ten functions corresponding to the particle density, the momentum density, and the kinetic energy density for each species separately.[16] The dispersion relation obtained by using this hydrodynamic subspace reads, in lowest order in the wavenumber,

$$z(z^2 - \Omega_p^2) + iv(z)(z^2 - \omega_p^2) = 0 \tag{3.36}$$

where factors which yield the four zero-frequency modes, i.e., the heat mode, the diffusion mode, and the two viscous modes, and relaxation modes have been left out. Here, the "mean-field" frequency Ω_p is given by $\Omega_p^2 = \sum_\sigma n_\sigma e_\sigma^2 / m_\sigma$ and $v(z)$ is a frequency-dependent coefficient. For the well-poised binary ionic mixture one has $\Omega_p = \omega_p$ and (3.36) becomes

$$[z + iv(z)](z^2 - \omega_p^2) = 0 \tag{3.37}$$

This dispersion relation yields the two plasmon modes and a relaxation mode. Since the solutions of (3.36) and (3.37) should be connected continuously, the dispersion relation (3.36) yields two generalized plasmon modes and a relaxation mode. The appearance of a relaxation mode in (3.36) is due to the fact that the hydrodynamic subspace contains the partial momentum densities, for which no simple balance equations are available. It should be noted that it is also possible to derive a dispersion relation for the binary ionic mixture by using a hydrodynamic subspace which consists of only the partial particle densities, the total momentum density, and the kinetic energy density. In this way one obtains a dispersion relation which has the same form as (3.31).

The dispersion relation for the unmagnetized binary ionic mixture has also been studied by starting from the linearized hydrodynamic equations.[17–19] The appearance of two generalized plasmon modes in the mode spectrum is confirmed by this method.

The modes and the mode frequencies of the magnetized and the unmagnetized multicomponent ionic mixture have now been derived in lowest order in the wavenumber. In the following the frequencies of the heat mode and the diffusion modes will be studied in second order in k.

4. FREQUENCIES OF THE HEAT AND DIFFUSION MODES

The heat mode and the diffusion modes of a magnetized ionic mixture have zero frequency for vanishing wavenumber. In this section our aim is to derive the frequencies of the heat mode and the diffusion modes up to second order in the wavenumber k. We will find expressions of Green–Kubo type for the transport coefficients which appear in the frequencies.

In order to calculate the frequencies of the heat mode and the diffusion modes in order k^2, one needs a basis set in terms of which the heat mode and the diffusion modes in first order in k can be built up. We start by writing this basis set in order k^1 in the general form

$$a_j(\mathbf{k}) = c_j^{(\varepsilon)}\varepsilon(\mathbf{k}) + c_j^{(q)} \frac{q_v(\mathbf{k})}{k} + \mathbf{c}_j^{(g)} \cdot \mathbf{g}(\mathbf{k}) + \sum_{\sigma'(\neq 1)} c_j^{(\sigma')} n_{\sigma'}(\mathbf{k}) \qquad (4.1)$$

with $j = \varepsilon, \sigma$ ($\sigma \neq 1$). The coefficients $c_\varepsilon^{(\varepsilon)}$ and $c_\sigma^{(\sigma')}$ are of order k^0, since $c_\varepsilon^{(\varepsilon)} = 1$ and $c_\sigma^{(\sigma')} = \delta_{\sigma,\sigma'}$. The other coefficients are of first order in k. By requiring that the matrix elements

$$-\frac{1}{V} \langle b_i^*(\mathbf{k}) \, La_j(\mathbf{k}) \rangle$$

$$+ \frac{1}{V} \left\langle b_i^*(\mathbf{k}) \, LQ \frac{1}{z + QLQ} QLa_j(\mathbf{k}) \right\rangle, \qquad j = \varepsilon, \sigma \, (\sigma \neq 1) \quad (4.2)$$

with $b_i(\mathbf{k})$ chosen from the set (3.7), are of second order in the wavenumber for $z \to i0$, the coefficients can be determined. One arrives at

$$a_\varepsilon(\mathbf{k}) = \varepsilon(\mathbf{k}) - \frac{h_v}{q_v} q_v(\mathbf{k}) + \mathbf{c}_\varepsilon^{(g)} \cdot \mathbf{g}(\mathbf{k}) \equiv a_\varepsilon^{(0)}(\mathbf{k}) + \mathbf{c}_\varepsilon^{(g)} \cdot \mathbf{g}(\mathbf{k}) \qquad (4.3)$$

$$a_\sigma(\mathbf{k}) = n_\sigma(\mathbf{k}) - \frac{n_\sigma}{q_v} q_v(\mathbf{k}) + \mathbf{c}_\sigma^{(g)} \cdot \mathbf{g}(\mathbf{k}) \equiv a_\sigma^{(0)}(\mathbf{k}) + \mathbf{c}_\sigma^{(g)} \cdot \mathbf{g}(\mathbf{k}) \qquad (4.4)$$

Here h_v is the enthalpy per unit of volume[20]:

$$h_v = u_v + p$$

$$= 3q_v \frac{\partial p}{\partial q_v} - \beta \frac{\partial p}{\partial \beta} - \frac{3q_v}{2\beta} \frac{\partial n}{\partial q_v} \qquad (4.5)$$

where we used the equation of state $p = u_v/3 + n/(2\beta)$. The combinations

$a_\varepsilon^{(0)}(\mathbf{k})$ and $a_\sigma^{(0)}(\mathbf{k})$ are introduced for convenience. The vectors $\mathbf{c}_j^{(g)} = c_{j\|}^{(g)}\hat{\mathbf{B}} + \mathbf{c}_{j\perp}^{(g)}$ with $j = \varepsilon$ or $\sigma\ (\neq 1)$ are given by

$$c_{j\|}^{(g)} = \frac{k}{q_v \hat{k}_\|} \left\{ -c_j + \frac{\hat{k}_\perp^2}{-i\omega_B + a} \left[(b + \omega_p) b_j' + b' b_j \right] \right\} \tag{4.6}$$

$$\mathbf{c}_{j\perp}^{(g)} = -\frac{\omega_p k}{q_v(-i\omega_B + a)} \left[b_j' \hat{\mathbf{k}}_\perp - b_j \hat{\mathbf{k}} \wedge \hat{\mathbf{B}} \right] \tag{4.7}$$

where the frequency-dependent coefficients a, b, and b', as given by (3.12) and (3.17), are to be taken at $z = i0$. The coefficients b_j, b_j', and c_j are defined by

$$b_j(z) \mathbf{k}_\perp + b_j'(z) \mathbf{k} \wedge \hat{\mathbf{B}} = \frac{\beta}{m_v^{1/2}} \frac{1}{V} \left\langle \mathbf{g}_\perp^*(\mathbf{k}) LQ \frac{1}{z + QLQ} QLa_j^{(0)}(\mathbf{k}) \right\rangle \tag{4.8}$$

$$c_j(z) k^2 = \beta \frac{1}{V} \left\langle q_v^*(\mathbf{k}) LQ \frac{1}{z + QLQ} QLa_j^{(0)}(\mathbf{k}) \right\rangle \tag{4.9}$$

in first order of the wave vector. In (4.6) and (4.7) the values of these coefficients at $z = i0$ should be inserted.

In the previous section the generalized gyroplasmon modes were given in lowest order in k. Here we need these modes in first order in the wavenumber. Their general form is

$$a_{\lambda\rho}(\mathbf{k}) = c_{\lambda\rho}^{(\varepsilon)} \varepsilon(\mathbf{k}) + \beta^{1/2} \frac{q_v(\mathbf{k})}{k} + \left(\frac{\beta}{m_v} \right)^{1/2} \mathbf{v}_{\lambda\rho}(\mathbf{k}) \cdot \mathbf{g}(\mathbf{k}) + \sum_{\sigma(\neq 1)} c_{\lambda\rho}^{(\sigma)} n_\sigma(\mathbf{k}) \tag{4.10}$$

where the coefficients $c_{\lambda\rho}^{(\varepsilon)}$ and $c_{\lambda\rho}^{(\sigma)}$ are of order k^1. By requiring

$$-\frac{1}{V} \left\langle b_i^*(\mathbf{k}) La_{\lambda\rho}(\mathbf{k}) \right\rangle + \frac{1}{V} \left\langle b_i^*(\mathbf{k}) LQ \frac{1}{z_{\lambda\rho} + QLQ} QLa_{\lambda\rho}(\mathbf{k}) \right\rangle$$

$$= z_{\lambda\rho} \frac{1}{V} \left\langle b_i^*(\mathbf{k}) a_{\lambda\rho}(\mathbf{k}) \right\rangle + O(k^2) \tag{4.11}$$

with $b_i(\mathbf{k})$ again chosen from the set (3.7), the coefficients $c_{\lambda\rho}^{(\varepsilon)}$ and $c_{\lambda\rho}^{(\sigma)}$ can be evaluated. The explicit expressions are not needed here.

With the help of the orthonormality relations (3.6), the adjoints of the basis set (4.3) and (4.4) for the heat mode and the diffusion modes can be determined. Up to first order in k, these adjoints read

$$\bar{a}_i(\mathbf{k}) = (\mathsf{M}^{-1})_{i\varepsilon} \varepsilon(\mathbf{k}) + \bar{c}_i^{(q)} \frac{q_v(\mathbf{k})}{k} + \bar{\mathbf{c}}_i^{(g)} \cdot \mathbf{g}(\mathbf{k}) + \sum_{\sigma'(\neq 1)} (\mathsf{M}^{-1})_{i\sigma'} n_{\sigma'}(\mathbf{k}) \tag{4.12}$$

with $i = \varepsilon, \sigma$ $(\sigma \neq 1)$. The coefficients $\bar{c}_i^{(q)}$ and $\bar{\mathbf{c}}_i^{(g)}$ are of order k. The rows and the columns of the matrix M^{-1} are labeled by ε and σ (with $\sigma \neq 1$) in that order. It is the inverse of the matrix M:

$$\mathsf{M} = \begin{pmatrix} -\partial u_v/\partial\beta & -\partial n_{\sigma'}/\partial\beta \\ -\partial n_\sigma/\partial\beta & \partial n_\sigma/\partial\beta\tilde{\mu}_{\sigma'} \end{pmatrix} \tag{4.13}$$

Here the partial derivatives are defined in terms of the independent set $\beta, q_v, \beta\tilde{\mu}_\sigma$ $(\sigma \neq 1)$, as introduced in ref. 20. In writing a partial derivative with respect to a variable of this set, the variables that are meant to remain constant are suppressed. The requirement of orthogonality of the generalized gyroplasmon modes (4.10) with the adjoints (4.12), up to first order in the wavenumber, yields four relations from which expressions for the coefficients $\bar{c}_i^{(q)}$ and $\bar{\mathbf{c}}_i^{(g)}$ can be obtained. However, we will see that these coefficients do not appear in the frequencies of the heat mode and the diffusion modes.

Having determined the basis set for the heat mode and the diffusion modes in first order in the wavenumber and the associated adjoint basis set, we can evaluate the frequencies of the heat mode and the diffusion modes in order k^2. These frequencies follow from the eigenvalues of the matrix $\mathscr{M}$, the elements of which are

$$\mathscr{M}_{ij}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{k^2} \left[-\frac{1}{V} \langle \bar{a}_i^*(\mathbf{k}) \, La_j(\mathbf{k}) \rangle \right.$$

$$\left. + \frac{1}{V} \langle \bar{a}_i^*(\mathbf{k}) \, LQ \frac{1}{z + QLQ} QLa_j(\mathbf{k}) \rangle \right], \quad i, j \in \{\varepsilon, \sigma \, (\sigma \neq 1)\} \tag{4.14}$$

for $z \to i0$. By inserting the basis set for the heat mode and the diffusion modes and the corresponding adjoints in order k^1 as given by (4.3), (4.4), and (4.12), one easily checks that the expression between square brackets is indeed of order k^2. With the help of the explicit expressions for the vectors $\mathbf{c}_j^{(g)}$ $(j = \varepsilon, \sigma)$, as given in (4.6) and (4.7), the coefficients $\bar{c}_i^{(q)}$ and $\bar{\mathbf{c}}_i^{(g)}$ can be eliminated from the matrix elements (4.14). One arrives at

$$\mathscr{M}_{ij}(\hat{\mathbf{k}}, z) = \sum_n (\mathsf{M}^{-1})_{in} \left[\lim_{\mathbf{k} \to 0} \frac{1}{k^2} \frac{1}{V} \langle [a_n^{(0)}(\mathbf{k})]^* \, LQ \frac{1}{z + QLQ} QLa_j^{(0)}(\mathbf{k}) \rangle \right.$$

$$\left. - \frac{\hat{k}_\perp^2}{\beta(-i\omega_B + a)} (b_n b_j' + b_n' b_j) \right] \tag{4.15}$$

for $z \to i0$. The summation runs over ε, σ $(\sigma \neq 1)$. The first term within the

square brackets has the form that one could have expected on account of (4.3)–(4.4). In addition, however, a second term appears. It contains the coefficients a, b_j, and b_j', which depend on z, as before. The eigenvectors of $\mathscr{M}$ for $z \to i0$ are the heat mode and the diffusion modes. They are linear combinations of (4.3) and (4.4).

For the unmagnetized ionic mixture the coefficients a, b_j, and b_j' vanish, so that the additional term in (4.15) disappears. In this case the elements of the matrix $\mathscr{M}$ read

$$\mathscr{M}_{ij}(\hat{\mathbf{k}}, z) = \sum_n (\mathbf{M}^{-1})_{in} \lim_{\mathbf{k} \to 0} \frac{1}{k^2} \frac{1}{V} \left\langle [a_n^{(0)}(\mathbf{k})]^* LQ \frac{1}{z + QLQ} QLa_j^{(0)}(\mathbf{k}) \right\rangle \tag{4.16}$$

again for $z \to i0$. From rotation invariance it follows that $\mathscr{M}_{ij}$ is actually independent of $\hat{\mathbf{k}}$ in the absence of a magnetic field. When the expressions for $a_i^{(0)}(\mathbf{k})$ given by (4.3) and (4.4) are inserted, one observes that the elements of the matrix $\mathscr{M}$ for the unmagnetized ionic mixture are combinations of the functions

$$\tilde{F}_{\alpha\beta}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{k^2} \frac{1}{V} \left\langle [Q\mathbf{k} \cdot \mathbf{j}_\alpha(\mathbf{k})]^* \frac{1}{z + QLQ} Q\mathbf{k} \cdot \mathbf{j}_\beta(\mathbf{k}) \right\rangle \tag{4.17}$$

where $\mathbf{j}_\alpha$ and $\mathbf{j}_\beta$ are the energy-current density $\mathbf{j}_\varepsilon$ or the partial momentum density $\mathbf{g}_\sigma$. In Appendix B it is shown that these functions are finite for $z \to i0$ if the functions

$$F_{\alpha\beta}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{k^2} \frac{1}{V} \left\langle [Q\mathbf{k} \cdot \mathbf{j}_\alpha(\mathbf{k})]^* \frac{1}{z + L} Q\mathbf{k} \cdot \mathbf{j}_\beta(\mathbf{k}) \right\rangle \tag{4.18}$$

are finite for $z \to i0$. If this condition is satisfied, the elements of the matrix $\mathscr{M}$ obey the limit relation

$$\lim_{z \to i0} \mathscr{M}_{ij}(\hat{\mathbf{k}}, z)$$

$$= \sum_n (\mathbf{M}^{-1})_{in} \lim_{z \to i0} \lim_{\mathbf{k} \to 0} \frac{1}{k^2} \frac{1}{V} \left\langle [a_n^{(0)}(\mathbf{k})]^* LQ \frac{1}{z + L} QLa_j^{(0)}(\mathbf{k}) \right\rangle \tag{4.19}$$

For the magnetized ionic mixture one checks by inspection of the definitions of the coefficients a, b_j, and b_j' given by (3.12) and (4.8) that the elements (4.15) of the matrix $\mathscr{M}$ depend not only on $\tilde{F}_{\alpha\beta}(\hat{\mathbf{k}}, z)$, but also on the functions

$$\tilde{F}_{\sigma i, \sigma' j}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{V} \left\langle [Q g_{\sigma i}(\mathbf{k})]^* \frac{1}{z + QLQ} Q g_{\sigma' j}(\mathbf{k}) \right\rangle \qquad (4.20)$$

$$\tilde{F}_{\sigma i, \varepsilon}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{k} \frac{1}{V} \left\langle [Q g_{\sigma i}(\mathbf{k})]^* \frac{1}{z + QLQ} Q \mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k}) \right\rangle \qquad (4.21)$$

As shown in Appendix B, these functions are finite for $z \to i0$, if, apart from (4.18), also the functions

$$F_{\sigma i, \sigma' j}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{V} \left\langle [Q g_{\sigma i}(\mathbf{k})]^* \frac{1}{z + L} Q g_{\sigma' j}(\mathbf{k}) \right\rangle \qquad (4.22)$$

$$F_{\sigma i, \varepsilon}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{k} \frac{1}{V} \left\langle [Q g_{\sigma i}(\mathbf{k})]^* \frac{1}{z + L} Q \mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k}) \right\rangle \qquad (4.23)$$

are finite for $z \to i0$. If that is the case, one can prove that in the limit $z \to i0$ the matrix $\mathcal{M}_{ij}$ is again given by (4.19). No additional terms like those occurring in (4.15) appear in the limit relation (4.19).

5. LONG-TIME TAILS

In the previous section the frequencies of the heat mode and the diffusion modes for the unmagnetized and the magnetized ionic mixture have been derived in second order in the wavenumber. We concluded that the transport coefficients which appear in these frequencies are finite if the functions $F_{\alpha\beta}(\hat{\mathbf{k}}, z)$, $F_{\sigma i, \sigma' j}(\hat{\mathbf{k}}, z)$, and $F_{\sigma i, \varepsilon}(\hat{\mathbf{k}}, z)$ are finite for $z \to i0$. In this section we will determine the long-time behavior of the inverse Laplace transforms of these functions by means of mode-coupling theory. From the long-time behavior one can determine whether these functions are indeed finite for $z \to i0$.

For the unmagnetized ionic mixture we determine the long-time behavior of the time correlation functions

$$F_{\alpha\beta}(\hat{\mathbf{k}}, t) = \lim_{\mathbf{k} \to 0} \frac{1}{k^2} \frac{1}{V} \left\langle [Q \mathbf{k} \cdot \mathbf{j}_\alpha(\mathbf{k})]^* e^{iLt} Q \mathbf{k} \cdot \mathbf{j}_\beta(\mathbf{k}) \right\rangle \qquad (5.1)$$

where $\mathbf{j}_\alpha$ and $\mathbf{j}_\beta$ are the energy-current density $\mathbf{j}_\varepsilon$ or the partial momentum density $\mathbf{g}_\sigma$. The projected energy-current density and the projected partial momentum density are

$$Q \mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k}) = \mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k}) - \frac{h_v}{m_v} \mathbf{k} \cdot \mathbf{g}(\mathbf{k}) \qquad (5.2)$$

$$Q \mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k}) = \mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k}) - \frac{n_\sigma m_\sigma}{m_v} \mathbf{k} \cdot \mathbf{g}(\mathbf{k}) \qquad (5.3)$$

According to mode-coupling theory the long-time behavior of a time correlation function $F_{\alpha\beta}(\hat{\mathbf{k}}, t)$ is dominated by contributions which stem from the coupling of the projected current $Q\mathbf{k} \cdot \mathbf{j}_\alpha$ (and $Q\mathbf{k} \cdot \mathbf{j}_\beta$) with the product of two collective modes $a_i(\mathbf{k})$ and $a_j(\mathbf{k})$ [and their adjoints $\bar{a}_i(\mathbf{k})$ and $\bar{a}_j(\mathbf{k})$]. The mode-coupling expression which gives $F_{\alpha\beta}(\hat{\mathbf{k}}, t)$ for long times reads

$$F_{\alpha\beta}(\hat{\mathbf{k}}, t) \simeq \lim_{\mathbf{k} \to \mathbf{0}} \frac{1}{k^2 V} \frac{1}{2} \sum_{i, j} \sum_{\mathbf{q}} A_{ij}^\alpha(\mathbf{k}, \mathbf{q}) [\bar{A}_{ij}^\beta(\mathbf{k}, \mathbf{q})]^*$$
$$\times \exp\{-i[z_i(\mathbf{q}) + z_j(\mathbf{k} - \mathbf{q})] t\} \qquad (5.4)$$

where the summations are extended over all collective modes and over all values of the wave vector $\mathbf{q}$ of these modes. The mode-coupling amplitudes $A_{ij}^\alpha(\mathbf{k}, \mathbf{q})$ and $\bar{A}_{ij}^\alpha(\mathbf{k}, \mathbf{q})$ are given by

$$A_{ij}^\alpha(\mathbf{k}, \mathbf{q}) = \frac{1}{V} \langle [Q\mathbf{k} \cdot \mathbf{j}_\alpha(\mathbf{k})]^* \, a_i(\mathbf{q}) \, a_j(\mathbf{k} - \mathbf{q}) \rangle \qquad (5.5)$$

$$\bar{A}_{ij}^\alpha(\mathbf{k}, \mathbf{q}) = \frac{1}{V} \langle [Q\mathbf{k} \cdot \mathbf{j}_\alpha(\mathbf{k})]^* \, \bar{a}_i(\mathbf{q}) \, \bar{a}_j(\mathbf{k} - \mathbf{q}) \rangle \qquad (5.6)$$

Since the unmagnetized multicomponent ionic mixture is isotropic, the time correlation functions (5.1) for this system will not depend on the orientation of the wave vector $\mathbf{k}$, so that we may write them as $F_{\alpha\beta}(t)$. To determine the contributions of the different possible couplings of two modes a_i and a_j, three-factor fluctuation formulas are needed in leading order in the wavenumber. These three-factor fluctuation formulas represent the equilibrium ensemble averages of the product of a current, i.e., $\mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k})$ or $\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})$, with the momentum density $\mathbf{g}(\mathbf{k})$ and one of the densities $k^{-1}q_v(\mathbf{k})$, $n_{\sigma'}(\mathbf{k})$, $\varepsilon(\mathbf{k})$. The derivation of these fluctuation formulas for the multicomponent ionic mixture, which is analogous to that for the one-component plasma,[10] can be found in Appendix C. Here we give the results for small but nonvanishing wave vectors $\mathbf{k}$, $\mathbf{q}$, and $l = \mathbf{k} - \mathbf{q}$:

$$\frac{1}{V} \langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \frac{1}{l} q_v(l) \rangle$$

$$= l \frac{m_\sigma}{\beta^2} \frac{\partial n_\sigma}{\partial q_v} \mathbf{k} \qquad (5.7)$$

$$\frac{1}{V} \langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, n_{\sigma'}(l) \rangle$$

$$= \frac{m_\sigma}{\beta} \frac{D n_\sigma}{D \beta \tilde{\mu}_{\sigma'}} \mathbf{k} \qquad (5.8)$$

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \varepsilon(\mathbf{l}) \right\rangle$$

$$= -\frac{m_\sigma}{\beta^2} \left[\beta \frac{\partial n_\sigma}{\partial \beta} - n_\sigma \right] \mathbf{k} \tag{5.9}$$

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \frac{1}{l} \, q_v(\mathbf{l}) \right\rangle$$

$$= \frac{q_v}{l\beta^2} (\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}), \tag{5.10}$$

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, n_\sigma(\mathbf{l}) \right\rangle$$

$$= -\frac{1}{\beta^2} \left(q_v \frac{\partial n_\sigma}{\partial q_v} + \beta \frac{\partial n_\sigma}{\partial \beta} - n_\sigma \right) \mathbf{k} + \frac{q_v}{\beta^2} \frac{\partial n_\sigma}{\partial q_v} (\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}) \tag{5.11}$$

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \varepsilon(\mathbf{l}) \right\rangle$$

$$= \frac{1}{2\beta^3} \left(6q_v \beta \frac{\partial p}{\partial q_v} - 10\beta^2 \frac{\partial p}{\partial \beta} - 3q_v \frac{\partial n}{\partial q_v} + 3\beta \frac{\partial n}{\partial \beta} - 3n \right) \mathbf{k}$$

$$+ \frac{3q_v}{2\beta^3} \left(2\beta \frac{\partial p}{\partial q_v} - \frac{\partial n}{\partial q_v} \right) (\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}) \tag{5.12}$$

where the partial derivatives are defined in terms of the independent set β, q_v, $\beta\tilde{\mu}_\sigma$ ($\sigma \neq 1$). The operator $D/D\beta\tilde{\mu}_\sigma$ is defined in (A.9).

Slowly decaying contributions to the mode-coupling expression for the time correlation function $F_{\alpha\beta}(t)$ arise if the damping coefficients of both modes i and j vanish for small wavenumber. Hence, the generalized plasmon modes can be excluded from the sum over the modes, since they decay exponentially fast, even in the long-wavelength limit. Turning to the other modes, one easily checks that a pair of viscous modes does not couple to the current densities $\mathbf{j}_\alpha$. Furthermore, the mode-coupling amplitude for two modes both chosen from the set of diffusion modes and the heat mode is at least of first order in the wavenumbers. Finally, if one of the modes is a viscous mode and the other the heat mode or a diffusion mode, the mode-coupling amplitude is of zeroth order in the wavenumbers. The latter coupling leads to a contribution to (5.4) with the slowest decay.

This follows by writing the summation over the wave vector $\mathbf{q}$ in the mode-coupling expression (5.4) as an integration and using the identity

$$\int_0^\infty dq\, q^{2n} \exp(-q^2 Dt) = \sqrt{\pi}\, \frac{(2n)!}{n!}\, (4Dt)^{-n-1/2} \tag{5.13}$$

In this way one finds that the long-time behavior of the time correlation functions $F_{\alpha\beta}(t)$ for the unmagnetized ionic mixture is governed by a long-time tail which decays like $t^{-3/2}$. Hence, the transport coefficients occurring in the frequencies of the heat mode and the diffusion modes are all finite for an unmagnetized ionic mixture.

The reasoning presented here is not conclusive for all types of unmagnetized ionic mixtures. In fact, if the mixture is well poised, the plasmon modes are not damped for small wavenumbers, so that these modes can play a role in the long-time behavior. The mode-coupling amplitudes for the coupling of the energy-current density with a plasmon mode and a viscous mode contain a term of order q^{-1}, which arises from (5.10). As a consequence, the long-time behavior of the energy-current autocorrelation function $F_{\varepsilon\varepsilon}(t)$ is given by

$$F_{\varepsilon\varepsilon}(t) \simeq \frac{\omega_p^2}{3\beta^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{q^2} \sum_\rho \exp\{ -i[z_\rho(\mathbf{q}) + z_\eta(\mathbf{q})]\, t\} \tag{5.14}$$

Inserting the mode frequencies $z_\rho(\mathbf{q})$ and $z_\eta(\mathbf{q})$ of the two plasmon modes and the viscous modes, respectively, up to second order in the wavenumber and employing (5.13), one ends up with an expression of the form $t^{-1/2} \cos(\omega_p t + \Theta)$, with Θ a phase factor. Hence, a slowly decaying tail of the same form as found previously for the (unmagnetized) one-component plasma[6] is obtained: in this respect the well-poised ionic mixture closely resembles the corresponding one-component system. It should be noted that the other time correlation functions $F_{\alpha\beta}(t)$ for the well-poised mixture have oscillating long-time tails proportional to t^{-1} or $t^{-3/2}$. Since the tails are oscillating, the Laplace transforms of all functions $F_{\alpha\beta}(t)$, with the inclusion of $F_{\varepsilon\varepsilon}(t)$, are finite as $z \to i0$, so that the transport coefficients occurring in the heat mode and the diffusion modes are finite even for an (unmagnetized) well-poised ionic mixture.

We now turn to a discussion of the magnetized ionic mixture. In this case we have to consider the long-time behavior of three types of functions, $F_{\alpha\beta}(\hat{\mathbf{k}}, t)$, $F_{\sigma i, \sigma' j}(\hat{\mathbf{k}}, t)$, and $F_{\sigma i, \varepsilon}(\hat{\mathbf{k}}, t)$. As we have seen, the two generalized plasmon modes and the two viscous modes of the unmagnetized ionic mixture merge into a set of four generalized gyroplasmon modes when a magnetic field is switched on. If the mixture is

not well poised, these gyroplasmon modes are exponentially damped, so that they cannot contribute to slowly decaying tails. Only the heat mode and the diffusion modes may contribute to the tails. Since the basis functions for these modes and their adjoints, given in (4.1) and (4.12), both contain $q_v(\mathbf{k})/k$ and $\mathbf{g}(\mathbf{k})$ with coefficients that are of order k, it follows by inspection of (5.7)–(5.12) that the product of mode-coupling amplitudes connecting the projected currents (5.2)–(5.3) with a pair of modes chosen from the set of the heat mode and the diffusion modes is of second order in the wavenumbers. Hence, the long-time tail of $F_{\alpha\beta}(\hat{\mathbf{k}}, t)$ is proportional to $t^{-5/2}$. To discuss the long-time behavior of the other two functions, viz. $F_{\sigma i, \sigma' j}(\hat{\mathbf{k}}, t)$ and $F_{\sigma i, \varepsilon}(\hat{\mathbf{k}}, t)$, we need three-factor fluctuation formulas containing the full currents $\mathbf{g}_\sigma(\mathbf{k})$ instead of their components $\hat{\mathbf{k}} \cdot \mathbf{g}_\sigma(\mathbf{k})$ along the wave vector $\mathbf{k}$. These follow trivially from (5.7)–(5.9) by replacing at the right-hand sides the vectors $\mathbf{k}$ by the unit tensor U. Employing these three-factor formulas, we easily arrive at the conclusion that the tails of the functions $F_{\sigma i, \sigma' j}(\hat{\mathbf{k}}, t)$ and $F_{\sigma i, \varepsilon}(\hat{\mathbf{k}}, t)$ are likewise proportional to $t^{-5/2}$. Hence, all three types of functions yield convergent Laplace transforms in the limit $z \to i0$, so that the finiteness of the transport coefficients occurring in the heat and diffusion mode frequencies of a magnetized ionic mixture is guaranteed.

Once again well-poised ionic mixtures are exceptions to the general rule. The gyroplasmon modes are no longer damped in the long-wave limit, so that they may contribute to the long-time behavior. As a matter of fact, they give rise to a slowly decaying tail in the function $F_{\varepsilon\varepsilon}(\hat{\mathbf{k}}, t)$, as can be seen from the expression

$$
\begin{aligned}
F_{\varepsilon\varepsilon}(\hat{\mathbf{k}}, t) \simeq \frac{\omega_p^2}{2\beta^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{q^2} \sum_{\lambda\rho, \lambda'\rho'} N_{\lambda\rho} N_{\lambda'\rho'} \\
\times |\hat{\mathbf{k}} \cdot (\mathsf{U} - \hat{\mathbf{q}}\hat{\mathbf{q}}) \cdot [\mathbf{v}_{\lambda\rho}(\mathbf{q}) + \mathbf{v}_{\lambda'\rho'}(-\mathbf{q})]|^2 \\
\times \exp\{-i[z_{\lambda\rho}(\mathbf{q}) + z_{\lambda'\rho'}(-\mathbf{q})]\, t\}
\end{aligned}
\tag{5.15}
$$

Choosing $\lambda' = \lambda$ and $\rho' = -\lambda\rho$, the sum of the mode frequencies in the exponent vanishes in lowest order of the wavenumber. Using (5.13), as before, one arrives at an expression for the long-time tail that is proportional to $t^{-1/2}$, without an accompanying oscillating factor. The other functions have long-time tails that decay faster. Hence, the transport coefficients appearing in the frequencies of the heat mode and the diffusion modes for a well-poised magnetized ionic mixture are divergent. This conclusion generalizes that obtained for a magnetized one-component plasma.[10] For both cases magnetohydrodynamic theory loses its meaning, at least if dissipation effects are to be included in the theory.

APPENDIX A.   FLUCTUATION FORMULAS

In the main text we have used fluctuation formulas for the multicomponent ionic mixture that are valid in leading order in the wavenumber. In ref. 21 the following fluctuation formulas have been derived:

$$\frac{1}{V} \langle [q_v(\mathbf{k})]^* \, q_v(\mathbf{k}) \rangle = \beta^{-1} k^2 \tag{A.1}$$

$$\frac{1}{V} \langle [\mathbf{g}_\sigma(\mathbf{k})]^* \, \mathbf{g}_{\sigma'}(\mathbf{k}) \rangle = \beta^{-1} \delta_{\sigma\sigma'} n_\sigma m_\sigma \, \mathsf{U} \tag{A.2}$$

$$\frac{1}{V} \langle [n_\sigma(\mathbf{k})]^* \, n_{\sigma'}(\mathbf{k}) \rangle = \frac{Dn_\sigma}{D\beta\tilde{\mu}_{\sigma'}} \tag{A.3}$$

$$\frac{1}{V} \langle [\varepsilon(\mathbf{k})]^* \, \varepsilon(\mathbf{k}) \rangle = -\frac{\partial u_v}{\partial \beta} \tag{A.4}$$

$$\frac{1}{V} \langle [q_v(\mathbf{k})]^* \, n_\sigma(\mathbf{k}) \rangle = \frac{1}{\beta} \frac{\partial n_\sigma}{\partial q_v} k^2 \tag{A.5}$$

$$\frac{1}{V} \langle [q_v(\mathbf{k})]^* \, \varepsilon(\mathbf{k}) \rangle = \frac{3}{2\beta^2} \left(2\beta \frac{\partial p}{\partial q_v} - \frac{\partial n}{\partial q_v} \right) k^2 \tag{A.6}$$

$$\frac{1}{V} \langle [n_\sigma(\mathbf{k})]^* \, \varepsilon(\mathbf{k}) \rangle = -\frac{\partial n_\sigma}{\partial \beta} \tag{A.7}$$

$$\frac{1}{V} \langle [\varepsilon(\mathbf{k})]^* \, \hat{\mathbf{k}} \cdot \tau(\mathbf{k}) \rangle = -\frac{\partial p}{\partial \beta} \hat{\mathbf{k}} \tag{A.8}$$

Here U is the second-rank unit tensor. The operator $D/D\beta\tilde{\mu}_\sigma$ is defined by

$$\frac{D}{D\beta\tilde{\mu}_\sigma} = (1 - \delta_{\sigma,1}) \frac{\partial}{\partial \beta\tilde{\mu}_\sigma} - \delta_{\sigma,1} \sum_{\sigma'(\neq 1)} \frac{e_{\sigma'}}{e_1} \frac{\partial}{\partial \beta\tilde{\mu}_{\sigma'}} \tag{A.9}$$

The partial derivatives are defined in terms of the independent set β, q_v, $\beta\tilde{\mu}_\sigma$ $(\sigma \neq 1)$, as introduced in ref. 20. In writing a partial derivative with respect to a variable of this set, the variables that are meant to remain constant are suppressed. Furthermore, u_v is the internal energy per unit of volume and p the thermodynamic pressure.

APPENDIX B.   GREEN–KUBO RELATIONS

In this Appendix we will first determine the relation between the elements of the matrix $\mathcal{M}$ in order k^2 and the functions (4.18) for the

unmagnetized ionic mixture. Subsequently, we shall generalize the results for the magnetized mixture.

For the unmagnetized ionic mixture we wish to show that the matrix elements

$$\tilde{A}_{ij}(\mathbf{k}, z) = \frac{1}{V} \left\langle \left[a_i^{(0)}(\mathbf{k}) \right]^* LQ \frac{1}{z + QLQ} QL a_j^{(0)}(\mathbf{k}) \right\rangle \qquad \text{(B.1)}$$

in order k^2 are finite for $z \to i0$, if the functions

$$F_{\alpha\beta}(\hat{\mathbf{k}}, z) = \lim_{\mathbf{k} \to 0} \frac{1}{k^2 V} \left\langle \left[Q\mathbf{k} \cdot \mathbf{j}_\alpha(\mathbf{k}) \right]^* \frac{1}{z + L} Q\mathbf{k} \cdot \mathbf{j}_\beta(\mathbf{k}) \right\rangle \qquad \text{(B.2)}$$

are finite for $z \to i0$. With the help of the operator identity

$$\frac{1}{z + QL} = \frac{1}{z + L} + \frac{1}{z + L} PL \frac{1}{z + QL} \qquad \text{(B.3)}$$

we write (B.1) as

$$\tilde{A}_{ij}(\mathbf{k}, z) = A_{ij}(\mathbf{k}, z) + \sum_n \frac{1}{V} \left\langle \left[a_i^{(0)}(\mathbf{k}) \right]^* LQ \frac{1}{z + L} a_n(\mathbf{k}) \right\rangle$$

$$\times \frac{1}{V} \left\langle \left[\bar{a}_n(\mathbf{k}) \right]^* LQ \frac{1}{z + QLQ} QL a_j^{(0)}(\mathbf{k}) \right\rangle \qquad \text{(B.4)}$$

where the summation is over the collective modes $a_n(\mathbf{k})$ of the unmagnetized ionic mixture. Furthermore, we used the notation

$$A_{ij}(\mathbf{k}, z) = \frac{1}{V} \left\langle \left[a_i^{(0)}(\mathbf{k}) \right]^* LQ \frac{1}{z + L} QL a_j^{(0)}(\mathbf{k}) \right\rangle \qquad \text{(B.5)}$$

Since our aim is to analyse (B.1) in order k^2, only the generalized plasmon modes have to be included in the second term of the right-hand side of (B.4). This term reads, in second order in k,

$$\frac{1}{V} \left\langle \left[a_i^{(0)}(\mathbf{k}) \right]^* LQ \frac{1}{z + L} \frac{\beta^{1/2}}{k} q_v(\mathbf{k}) \right\rangle \tilde{A}_{q_v, j}(\mathbf{k}, z) \qquad \text{(B.6)}$$

where the subscript q_v indicates that $a_i^{(0)}(\mathbf{k})$ in the matrix element $\tilde{A}_{ij}$ is replaced by $\beta^{1/2} k^{-1} q_v(\mathbf{k})$. By using $Q(z + L)^{-1} = z^{-1} [Q - Q(z + L)^{-1} L]$, we rewrite the first factor of (B.6):

$$\frac{1}{V}\left\langle\, [a_i^{(0)}(\mathbf{k})]^* \, LQ \, \frac{1}{z+L} \frac{\beta^{1/2}}{k} \, q_v(\mathbf{k}) \,\right\rangle$$

$$= -\frac{1}{z} A_{i,q_v}(\mathbf{k}, z)$$

$$-\frac{1}{z}\sum_n \frac{1}{V}\left\langle\, [a_i^{(0)}(\mathbf{k})]^* \, LQ \, \frac{1}{z+L} \, a_n(\mathbf{k}) \,\right\rangle \frac{1}{V}\left\langle\, [\bar{a}_n(\mathbf{k})]^* \, L \frac{\beta^{1/2}}{k} \, q_v(\mathbf{k}) \,\right\rangle$$

$$\text{(B.7)}$$

When the second term at the right-hand side is evaluated in order k^1, one arrives at

$$\frac{1}{V}\left\langle\, [a_i^{(0)}(\mathbf{k})]^* \, LQ \, \frac{1}{z+L} \frac{\beta^{1/2}}{k} \, q_v(\mathbf{k}) \,\right\rangle = -\frac{z}{z^2-\omega_p^2} A_{i,q_v}(\mathbf{k}, z) \quad \text{(B.8)}$$

With the help of this relation we find for (B.4) in second order in $\mathbf{k}$

$$\tilde{A}_{ij}(\mathbf{k}, z) = A_{ij}(\mathbf{k}, z) - \frac{z}{z^2-\omega_p^2} A_{i,q_v}(\mathbf{k}, z) \, \tilde{A}_{q_v,j}(\mathbf{k}, z) \quad \text{(B.9)}$$

By taking, for $a_i^{(0)}(\mathbf{k})$, the combination $\sum_{\sigma(\neq 1)} (e_\sigma - e_1 m_\sigma/m_1) a_\sigma^{(0)}(\mathbf{k})$ one can prove from (B.9)

$$\tilde{A}_{q_v,j}(\mathbf{k}, z) = A_{q_v,j}(\mathbf{k}, z)\left[1 + \frac{z}{z^2-\omega_p^2} A_{q_v,q_v}(\mathbf{k}, z)\right]^{-1} \quad \text{(B.10)}$$

in second order in k. Upon inserting this relation in (B.9), we arrive at

$$\tilde{A}_{ij}(\mathbf{k}, z) = A_{ij}(\mathbf{k}, z)$$

$$-\frac{z}{z^2-\omega_p^2} A_{i,q_v}(\mathbf{k}, z) A_{q_v,j}(\mathbf{k}, z)\left[1 + \frac{z}{z^2-\omega_p^2} A_{q_v,q_v}(\mathbf{k}, z)\right]^{-1}$$

$$\text{(B.11)}$$

valid in order k^2. From this relation we conclude that the matrix elements $\tilde{A}_{ij}$ are finite for $z \to i0$ if all A_{ij} in order k^2 are finite as $z \to i0$ or, alternatively, if the functions $F_{\alpha\beta}$ given by (B.2) are finite. In the limit $z \to i0$ the matrix elements $\tilde{A}_{ij}$ and A_{ij} coincide.

For the magnetized ionic mixture the second term on the right-hand side of (B.4) yields, apart from (B.6), a second term, since $Lg_\perp(\mathbf{k})$ contains a contribution of order k^0. In second order in the wavenumber one finds

$$\tilde{A}_{ij}(\mathbf{k}, z) = A_{ij}(\mathbf{k}, z)$$

$$+ \frac{1}{V} \left\langle [a_i^{(0)}(\mathbf{k})]^* LQ \frac{1}{z+L} \frac{\beta^{1/2}}{k} q_v(\mathbf{k}) \right\rangle \tilde{A}_{q_v, j}(\mathbf{k}, z)$$

$$+ \frac{1}{V} \left\langle [a_i^{(0)}(\mathbf{k})]^* LQ \frac{1}{z+L} \left(\frac{\beta}{m_v}\right)^{1/2} \mathbf{g}_\perp(\mathbf{k}) \right\rangle \cdot \tilde{\mathbf{A}}_{g_\perp, j}(\mathbf{k}, z) \qquad (B.12)$$

The subscript $g_\perp$ indicates that $a_i^{(0)}(\mathbf{k})$ in $\tilde{A}_{ij}(\mathbf{k}, z)$ is replaced by $(\beta/m_v)^{1/2} \mathbf{g}_\perp(\mathbf{k})$. The evaluation of the first factors of the second and the third terms on the right-hand side of (B.12) proceeds in a way analogous to the calculation of the first factor of expression (B.6). One arrives at the relations

$$\frac{1}{V} \left\langle [a_i^{(0)}(\mathbf{k})]^* LQ \frac{1}{z+L} \frac{\beta^{1/2}}{k} q_v(\mathbf{k}) \right\rangle$$

$$= -\frac{z}{z^4 - z^2(\omega_p^2 + \omega_B^2) + \omega_p^2 \omega_B^2 \hat{k}_\parallel^2} [(z^2 - \omega_B^2) A_{i, q_v}(\mathbf{k}, z)$$

$$+ \omega_p (z\hat{\mathbf{k}}_\perp - i\omega_B \hat{\mathbf{k}} \wedge \hat{\mathbf{B}}) \cdot \mathbf{A}_{i, g_\perp}(\mathbf{k}, z)] \qquad (B.13)$$

$$\frac{1}{V} \left\langle [a_i^{(0)}(\mathbf{k})]^* LQ \frac{1}{z+L} \left(\frac{\beta}{m_v}\right)^{1/2} \mathbf{g}_\perp(\mathbf{k}) \right\rangle$$

$$= -\frac{z\omega_p}{(z^2 - \omega_B^2)[z^4 - z^2(\omega_p^2 + \omega_B^2) + \omega_p^2 \omega_B^2 \hat{k}_\parallel^2]} [(z^2 - \omega_B^2) A_{i, q_v}(\mathbf{k}, z)$$

$$+ \omega_p (z\hat{\mathbf{k}}_\perp - i\omega_B \hat{\mathbf{k}} \wedge \hat{\mathbf{B}}) \cdot \mathbf{A}_{i, g_\perp}(\mathbf{k}, z)](z\hat{\mathbf{k}}_\perp + i\omega_B \hat{\mathbf{k}} \wedge \hat{\mathbf{B}})$$

$$- \frac{1}{z^2 - \omega_B^2} [z\mathbf{A}_{i, g_\perp}(\mathbf{k}, z) + i\omega_B \mathbf{A}_{i, g_\perp}(\mathbf{k}, z) \wedge \hat{\mathbf{B}}] \qquad (B.14)$$

valid in first order in k.

By inserting (B.13) and (B.14) in (B.12), one obtains the equivalent of (B.9) for the magnetized ionic mixture. Analogous relations, valid in order k^0 or k^1, can be derived by replacing $a_i^{(0)}(\mathbf{k})$ and/or $a_j^{(0)}(\mathbf{k})$ in (B.12) by $k^{-1} q_v(\mathbf{k})$ or $\mathbf{g}_\perp(\mathbf{k})$. With the help of these relations one can show that the contribution of order k^2 of $\tilde{A}_{ij}(\mathbf{k}, z)$ is finite for $z \to i0$ if the k^2 contribution of $A_{ij}(\mathbf{k}, z)$, the k^1 contributions of $A_{i, q_v}(\mathbf{k}, z)$ and $\mathbf{A}_{i, g_\perp}(\mathbf{k}, z)$, and the k^0 contributions of $A_{q_v, q_v}(\mathbf{k}, z)$, $\mathbf{A}_{q_v, g_\perp}(\mathbf{k}, z)$, and $A_{g_\perp, g_\perp}(\mathbf{k}, z)$ are finite for $z \to i0$. This conclusion is equivalent to the requirement that the functions $F_{\alpha\beta}(\mathbf{k}, z)$, $F_{\sigma i, \sigma' j}(\mathbf{k}, z)$, and $F_{\sigma i, \varepsilon}(\mathbf{k}, z)$ given by (4.18), (4.22), and (4.23) are finite for $z \to i0$.

With the help of the relations derived in this Appendix, one can prove

$$\lim_{z \to i0} \lim_{\mathbf{k} \to 0} \tilde{A}_{g_\perp, g_\perp}(\mathbf{k}, z) = \lim_{z \to i0} \left(1 + \frac{ia}{\omega_B} \right) \lim_{\mathbf{k} \to 0} A_{g_\perp, g_\perp}(\mathbf{k}, z) \qquad \text{(B.15)}$$

$$\lim_{z \to i0} \lim_{\mathbf{k} \to 0} \frac{1}{k} \tilde{A}_{g_\perp, i}(\mathbf{k}, z) = \lim_{z \to i0} \left(1 + \frac{ia}{\omega_B} \right) \lim_{\mathbf{k} \to 0} \frac{1}{k} A_{g_\perp, i}(\mathbf{k}, z) \qquad \text{(B.16)}$$

and, finally,

$$\lim_{z \to i0} \left[\lim_{\mathbf{k} \to 0} \frac{1}{k^2} \tilde{A}_{ij}(\mathbf{k}, z) - \frac{\hat{k}_\perp^2}{\beta(-i\omega_B + a)} (b_i b_j' + b_i' b_j) \right]$$

$$= \lim_{z \to i0} \lim_{\mathbf{k} \to 0} \frac{1}{k^2} A_{ij}(\mathbf{k}, z) \qquad \text{(B.17)}$$

Using the last relation, one easily checks that the matrix elements $\mathscr{M}_{ij}(\hat{\mathbf{k}}, z)$ for the magnetized ionic mixture, given by (4.15), can be written as (4.19).

APPENDIX C. THREE-FACTOR FLUCTUATION FORMULAS

We derive the three-factor fluctuation formulas (5.7)–(5.12) for small but nonvanishing wave vectors $\mathbf{k}, \mathbf{q}$, and $l = \mathbf{k} - \mathbf{q}$ in this Appendix. The expressions for $q_v(\mathbf{k})$, $g_\sigma(\mathbf{k})$, $n_\sigma(\mathbf{k})$, $\varepsilon(\mathbf{k})$, and $j_\varepsilon(\mathbf{k})$ have been given in Section 2. The partial derivatives are defined in terms of the independent set β, q_v, $\beta\tilde{\mu}_\sigma$ ($\sigma \neq 1$), as in Appendix A. In the course of this Appendix frequent use is made of results of refs. 20 and 21.

To obtain the first three-factor fluctuation formula (5.7), the expression (2.3) for $g_\sigma(\mathbf{k})$ and $q_v(\mathbf{k}) = \sum_\sigma e_\sigma n_\sigma(\mathbf{k})$, with n_σ given by (2.1), are inserted. After performing the average over the momenta, one can write

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \frac{1}{l} q_v(l) \right\rangle = \frac{m_\sigma n_\sigma}{l\beta} \left[e_\sigma + \sum_{\sigma_1} n_{\sigma_1} e_{\sigma_1} h_{\sigma\sigma_1}^{(2)}(l) \right] \mathbf{k} \qquad \text{(C.1)}$$

The Ursell functions $h_{\sigma_1,\dots,\sigma_m}^{(m)}(\mathbf{k}_2,\dots,\mathbf{k}_m)$ are defined by

$$n_{\sigma_1} \cdots n_{\sigma_m} h_{\sigma_1,\dots,\sigma_m}^{(m)}(\mathbf{k}_2,\dots,\mathbf{k}_m)$$

$$= \frac{1}{V} \left\langle \sum_{\alpha_1,\dots,\alpha_m}' \exp[i\mathbf{k}_2 \cdot (\mathbf{r}_{\sigma_1\alpha_1} - \mathbf{r}_{\sigma_2\alpha_2}) + \cdots + i\mathbf{k}_m \cdot (\mathbf{r}_{\sigma_1\alpha_1} - \mathbf{r}_{\sigma_m\alpha_m})] \right\rangle$$

$$\text{(C.2)}$$

for $\mathbf{k}_i \neq 0$ ($i = 2,\dots,m$) and $\sum_{i=2}^m \mathbf{k}_i \neq 0$. The prime on the summation

symbol denotes the restriction $\sigma_i \alpha_i \neq \sigma_j \alpha_j$ $(i \neq j)$. The two-particle Ursell function $h^{(2)}_{\sigma_1 \sigma_2}$ may be expanded as

$$h^{(2)}_{\sigma_1\sigma_2}(\mathbf{k}) = a^{(0)}_{\sigma_1\sigma_2} + a^{(1)}_{\sigma_1\sigma_2} k^2 + a^{(2)}_{\sigma_1\sigma_2} k^4 + \cdots \tag{C.3}$$

With the help of

$$\sum_{\sigma_2} n_{\sigma_2} e_{\sigma_2} a^{(0)}_{\sigma_1\sigma_2} = -e_{\sigma_1} \tag{C.4}$$

$$n_{\sigma_1} \sum_{\sigma_2} n_{\sigma_2} e_{\sigma_2} a^{(1)}_{\sigma_1\sigma_2} = \beta^{-1} \frac{\partial n_{\sigma_1}}{\partial q_v} \tag{C.5}$$

one finds for (C.1) the result (5.7) in leading order in l.

The derivation of (5.8) proceeds in a way similar to that of (5.7). After expanding the two-particle Ursell function, one finds in leading order in the wave vectors

$$\frac{1}{V} \langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \mathbf{g}(\mathbf{q}) n_{\sigma'}(l) \rangle = \frac{n_\sigma m_\sigma}{\beta} (\delta_{\sigma\sigma'} + n_{\sigma'} a^{(0)}_{\sigma\sigma'}) \mathbf{k} \tag{C.6}$$

With the help of

$$n_{\sigma_1} n_{\sigma_2} a^{(0)}_{\sigma_1\sigma_2} = \frac{D n_{\sigma_1}}{D \beta \tilde{\mu}_{\sigma_2}} - n_{\sigma_1} \delta_{\sigma_1\sigma_2} \tag{C.7}$$

we recover (5.8).

To derive (5.9), we separately consider the kinetic and the potential parts of the energy density. The contribution of the kinetic part of $\varepsilon(l)$ is found by averaging over the momenta and expanding the appearing two-particle Ursell function:

$$\frac{1}{V} \langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \mathbf{g}(\mathbf{q}) \varepsilon^{\mathrm{kin}}(l) \rangle = \frac{n_\sigma m_\sigma}{2\beta^2} \left(5 + 3 \sum_{\sigma_1} n_{\sigma_1} a^{(0)}_{\sigma\sigma_1} \right) \mathbf{k} \tag{C.8}$$

in leading order in the wave vectors. After performing the average over the momenta, the contribution of $\varepsilon^{\mathrm{pot}}(l)$ can be expressed in a two-factor fluctuation formula which has been evaluated elsewhere[21]:

$$\frac{1}{V} \langle [\mathbf{k} \cdot \mathbf{g}_\sigma(\mathbf{k})]^* \mathbf{g}(\mathbf{q}) \varepsilon^{\mathrm{pot}}(l) \rangle$$

$$= \frac{m_\sigma}{\beta} \frac{1}{V} \langle [n_\sigma(l)]^* \varepsilon^{\mathrm{pot}}(l) \rangle \mathbf{k}$$

$$= -3 \frac{n_\sigma m_\sigma}{\beta^2} \sum_{\sigma_1} n_{\sigma_1} (a^{(0)}_{\sigma\sigma_1} + \beta q_v e_{\sigma_1} a^{(1)}_{\sigma\sigma_1}) \mathbf{k} \tag{C.9}$$

With (C.5) and

$$n_{\sigma_1} \sum_{\sigma_2} n_{\sigma_2} a^{(0)}_{\sigma_1\sigma_2} = -2q_v \frac{\partial n_{\sigma_1}}{\partial q_v} + \frac{2}{3} \beta \frac{\partial n_{\sigma_1}}{\partial \beta} + n_{\sigma_1} \qquad (C.10)$$

one finds, by adding (C.8) and (C.9), the result (5.9).

Now we turn to the derivation of the three-factor fluctuation formulas containing the energy-current density. For the contribution of the kinetic part of $\mathbf{j}_\varepsilon(\mathbf{k})$ to (5.10) one finds, with the help of (C.4) and (C.5),

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}^{kin}_\varepsilon(\mathbf{k})]^* g(\mathbf{q}) \frac{1}{l} q_v(\mathbf{l}) \right\rangle = \frac{5}{2} \frac{l}{\beta^3} \frac{\partial n}{\partial q_v} \mathbf{k} \qquad (C.11)$$

in leading order in the wave vectors. The potential part of the energy-current density gives a contribution

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}^{pot}_\varepsilon(\mathbf{k})]^* g(\mathbf{q}) \frac{1}{l} q_v(\mathbf{l}) \right\rangle$$

$$= \frac{q_v}{l^2 \beta} \sum_{\sigma_1\sigma_2} n_{\sigma_1} e_{\sigma_1} e_{\sigma_2} [\delta_{\sigma_1\sigma_2} + n_{\sigma_2} h^{(2)}_{\sigma_1\sigma_2}(l)][\hat{\mathbf{I}} - \hat{\mathbf{I}} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}]$$

$$+ \frac{1}{l\beta} \frac{1}{V} \sum_{\mathbf{k}'(\neq\mathbf{0},\neq\mathbf{k})} \frac{1}{k'^2} \left[\mathbf{k}' - \frac{\mathbf{k}' \cdot (\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2} \right] \sum_{\sigma_1\sigma_2} n_{\sigma_1} n_{\sigma_2} e_{\sigma_1} e_{\sigma_2}$$

$$\times \left[e_{\sigma_1} h^{(2)}_{\sigma_1\sigma_2}(\mathbf{k}') + e_{\sigma_2} h^{(2)}_{\sigma_1\sigma_2}(\mathbf{l}-\mathbf{k}') + \sum_{\sigma_3} n_{\sigma_3} e_{\sigma_3} h^{(3)}_{\sigma_1\sigma_2\sigma_3}(-\mathbf{k}',\mathbf{l}) \right] \qquad (C.12)$$

where a term that vanishes in the thermodynamic limit has been omitted. To proceed, we expand $h^{(2)}_{\sigma_1\sigma_2}$ in the first term on the right-hand side and use (C.4) and

$$\sum_{\sigma_1\sigma_2} n_{\sigma_1} n_{\sigma_2} e_{\sigma_1} e_{\sigma_2} a^{(1)}_{\sigma_1\sigma_2} = \beta^{-1} \qquad (C.13)$$

which follows from (C.5). The sum over the wave vectors $\mathbf{k}'$ in the second term on the right-hand side of (C.12) is written as an integral and, after expanding the integrand for fixed l around $\mathbf{k} = \mathbf{0}$, one finds for the right-hand side of (C.12)

$$\frac{q_v}{\beta^2} [\hat{\mathbf{I}} - \hat{\mathbf{I}} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}] + \frac{1}{l\beta} \mathscr{P} \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{1}{k'^2} [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{k}}'\hat{\mathbf{k}}'] \sum_{\sigma_1\sigma_2} n_{\sigma_1} n_{\sigma_2} e_{\sigma_1} e_{\sigma_2}$$

$$\times \left[e_{\sigma_1} h^{(2)}_{\sigma_1\sigma_2}(\mathbf{k}') + e_{\sigma_2} h^{(2)}_{\sigma_1\sigma_2}(\mathbf{l}-\mathbf{k}') + \sum_{\sigma_3} n_{\sigma_3} e_{\sigma_3} h^{(3)}_{\sigma_1\sigma_2\sigma_3}(-\mathbf{k}',\mathbf{l}) \right] \qquad (C.14)$$

Here the principal value excludes $\mathbf{k}' = \mathbf{0}$ from the integral. From the perfect-screening relation for the three-particle Ursell function

$$(e_{\sigma_1} + e_{\sigma_2})\, h^{(2)}_{\sigma_1\sigma_2}(\mathbf{k}) + \sum_{\sigma_3} n_{\sigma_3} e_{\sigma_3} h^{(3)}_{\sigma_1\sigma_2\sigma_3}(\mathbf{k}, \mathbf{0}) = 0 \qquad (\text{C.15})$$

it follows that the integral in (C.14) vanishes for $l = 0$. Hence, this integral is of second order in the wave vectors $\mathbf{k}$ and l. In leading order one finds, by writing $\mathbf{k} - \mathbf{q}$ instead of l:

$$\frac{1}{V}\left\langle [\mathbf{k} \cdot \mathbf{j}_\varepsilon(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \frac{1}{l} q_v(l) \right\rangle = \frac{q_v}{l\beta^2}\, [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}] \qquad (\text{C.16})$$

so that (5.10) is proved.

The evaluation of the three-factor fluctuation formula (5.11) is similar to that of (5.10). The kinetic part of the energy-current density gives a contribution

$$\frac{1}{V}\langle [\mathbf{k} \cdot \mathbf{j}^{\text{kin}}_\varepsilon(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q})\, n_\sigma(l)\rangle = -\frac{5}{3\beta^2}\left[3q_v \frac{\partial n_\sigma}{\partial q_v} - \beta \frac{\partial n_\sigma}{\partial \beta} - 3n_\sigma \right]\mathbf{k} \qquad (\text{C.17})$$

When a term which vanishes in the thermodynamic limit is omitted, the contribution of the potential part of the energy-current density can be written as

$$\frac{1}{V}\langle [\mathbf{k} \cdot \mathbf{j}^{\text{pot}}_\varepsilon(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q})\, n_\sigma(l)\rangle$$

$$= \frac{q_v}{\beta^2}\frac{\partial n_\sigma}{\partial q_v}\, [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}]$$

$$+ \frac{1}{\beta}\mathscr{P} \int \frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{k'^2}\, [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{k}}'\hat{\mathbf{k}}']\sum_{\sigma_1} n_\sigma n_{\sigma_1} e_{\sigma_1}$$

$$\times \left[e_\sigma h^{(2)}_{\sigma\sigma_1}(\mathbf{k}') + e_\sigma h^{(2)}_{\sigma\sigma_1}(l - \mathbf{k}') + \sum_{\sigma_2} n_{\sigma_2} e_{\sigma_2} h^{(3)}_{\sigma_1\sigma_2\sigma}(-\mathbf{k}', l) \right] \qquad (\text{C.18})$$

For small l the dependence of the integrand on $\hat{\mathbf{k}}'$ gets simpler and, with the help of the relation

$$\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^2}\sum_{\sigma_1\sigma_2} n_{\sigma_1} e_{\sigma_1}[2\delta_{\sigma_2\sigma_3} e_{\sigma_3} h^{(2)}_{\sigma_1\sigma_3}(\mathbf{k}) + n_{\sigma_2} e_{\sigma_2} h^{(3)}_{\sigma_1\sigma_2\sigma_3}(\mathbf{k}, \mathbf{0})]$$

$$= -6 \sum_{\sigma_1} n_{\sigma_1}[\beta^{-1} a^{(0)}_{\sigma_1\sigma_3} + q_v e_{\sigma_1} a^{(1)}_{\sigma_1\sigma_3}] \qquad (\text{C.19})$$

and (C.5) and (C.10), one finds

$$
\frac{1}{V} \langle \, [\mathbf{k} \cdot \mathbf{j}_\varepsilon^{\text{pot}}(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, n_\sigma(l) \rangle = \frac{4}{3\beta^2} \left[3q_v \frac{\partial n_\sigma}{\partial q_v} - 2\beta \frac{\partial n_\sigma}{\partial \beta} - 3n_\sigma \right] \mathbf{k}
$$

$$
+ \frac{q_v}{\beta^2} \frac{\partial n_\sigma}{\partial q_v} \, [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}] \qquad \text{(C.20)}
$$

in leading order in the wave vectors. Addition of (C.17) and (C.20) gives (5.11).

The last three-factor fluctuation formula (5.12) is split up into four contributions containing the products of the kinetic and the potential parts of the energy-current density and the energy density. For the purely kinetic contribution one finds with (C.10)

$$
\frac{1}{V} \langle \mathbf{k} \cdot \mathbf{j}_\varepsilon^{\text{kin}}(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \varepsilon^{\text{kin}}(l) \rangle = -\frac{5}{2\beta^3} \left[3q_v \frac{\partial n}{\partial q_v} - \beta \frac{\partial n}{\partial \beta} - 5n \right] \mathbf{k} \qquad \text{(C.21)}
$$

in lowest order in the wave vectors.

The mixed potential–kinetic contribution becomes, upon omitting terms which vanish in the thermodynamic limit,

$$
\frac{1}{V} \langle \, [\mathbf{k} \cdot \mathbf{j}_\varepsilon^{\text{pot}}(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \varepsilon^{\text{kin}}(l) \rangle
$$

$$
= \frac{3}{2} \frac{q_v}{\beta^3} \frac{\partial n}{\partial q_v} \, [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}]
$$

$$
+ \frac{1}{\beta^2} \, \mathscr{P} \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{1}{k'^2} \, [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{k}}'\hat{\mathbf{k}}'] \sum_{\sigma_1 \sigma_2} n_{\sigma_1} n_{\sigma_2} e_{\sigma_1} e_{\sigma_2}
$$

$$
\times \left[\frac{5}{2} h_{\sigma_1 \sigma_2}^{(2)}(\mathbf{k}') + \frac{3}{2} h_{\sigma_1 \sigma_2}^{(2)}(l - \mathbf{k}') + \frac{3}{2} \sum_{\sigma_3} n_{\sigma_3} h_{\sigma_1 \sigma_2 \sigma_3}^{(3)}(-\mathbf{k}', l) \right] \qquad \text{(C.22)}
$$

For small l the integral can be expressed in thermodynamic functions when one uses (C.19) and

$$
\int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \sum_{\sigma_1 \sigma_2} n_{\sigma_1} n_{\sigma_2} e_{\sigma_1} e_{\sigma_2} h_{\sigma_1 \sigma_2}^{(2)}(\mathbf{k})
$$

$$
= 2u_v - 3\beta^{-1} n
$$

$$
= \frac{3}{4\beta} \left(6q_v \beta \frac{\partial p}{\partial q_v} - 2\beta^2 \frac{\partial p}{\partial \beta} - 3q_v \frac{\partial n}{\partial q_v} - 5n \right) \qquad \text{(C.23)}
$$

In leading order in the wave vectors one finds

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}_\varepsilon^{\text{pot}}(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \varepsilon^{\text{kin}}(l) \right\rangle$$

$$= \frac{1}{2\beta^3} \left[6 q_v \beta \frac{\partial p}{\partial q_v} - 2\beta^2 \frac{\partial p}{\partial \beta} + 9 q_v \frac{\partial n}{\partial q_v} - 8\beta \frac{\partial n}{\partial \beta} - 17 n \right] \mathbf{k}$$

$$+ \frac{3}{2} \frac{q_v}{\beta^3} \frac{\partial n}{\partial q_v} [\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}] \qquad\qquad\qquad (\text{C.24})$$

After averaging over the momenta, we can express the mixed kinetic–potential contribution to (5.12) in a well-known two-factor fluctuation formula:

$$\frac{1}{V} \left\langle [\mathbf{k} \cdot \mathbf{j}_\varepsilon^{\text{kin}}(\mathbf{k})]^* \, \mathbf{g}(\mathbf{q}) \, \varepsilon^{\text{pot}}(l) \right\rangle$$

$$= \frac{5}{2\beta^2} \frac{1}{V} \left\langle [n(l)]^* \, \varepsilon^{\text{pot}}(l) \right\rangle \mathbf{k}$$

$$= \frac{5}{2\beta^3} \left[3 q_v \frac{\partial n}{\partial q_v} - 2\beta \frac{\partial n}{\partial \beta} - 3n \right] \mathbf{k} \qquad\qquad (\text{C.25})$$

Finally, we consider the purely potential contribution to (5.12). As in the case of the corresponding three-factor fluctuation formula for the one-component plasma,[10] this contribution can be expressed in three integrals over combinations of two-, three-, and four-particle Ursell functions in such a way that the sum of two of these integrals is proportional to the two-factor fluctuation formula

$$\frac{1}{V} \left\langle [\varepsilon^{\text{pot}}(\mathbf{k})]^* \, \varepsilon^{\text{pot}}(\mathbf{k}) \right\rangle = -\frac{3}{2\beta^2} \left[2\beta^2 \frac{\partial p}{\partial \beta} + 3 q_v \frac{\partial n}{\partial q_v} - 4\beta \frac{\partial n}{\partial \beta} - n \right] \quad (\text{C.26})$$

The remaining integral is proportional to the fluctuation formula

$$\frac{1}{V} \left\langle [q_v(\mathbf{k})]^* \, \varepsilon^{\text{pot}}(\mathbf{k}) \right\rangle = \frac{3}{\beta^2} k^2 \left[\beta \frac{\partial p}{\partial q_v} - \frac{\partial n}{\partial q_v} \right] \qquad (\text{C.27})$$

One finds, in leading order in the wave vectors,

$$\frac{1}{V} \left\langle \left[\mathbf{k} \cdot \mathbf{j}_\varepsilon^{\text{pot}}(\mathbf{k}) \right]^* \mathbf{g}(\mathbf{q}) \, \varepsilon^{\text{pot}}(l) \right\rangle$$

$$= -\frac{2}{\beta^3} \left[2\beta^2 \frac{\partial p}{\partial \beta} + 3q_v \frac{\partial n}{\partial q_v} - 4\beta \frac{\partial n}{\partial \beta} - n \right] \mathbf{k}$$

$$+ 3 \frac{q_v}{\beta^3} \left[\beta \frac{\partial p}{\partial q_v} - \frac{\partial n}{\partial q_v} \right] \left[\mathbf{k} - \mathbf{k} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}} \right] \qquad\qquad \text{(C.28)}$$

By adding (C.21), (C.24), (C.25), and (C.28), we arrive at (5.12).

ACKNOWLEDGMENTS

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