ON THE UNIQUENESS OF THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

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The energy-momentum tensor in a polarized medium. as derived from microscopic theory. can be looked upon as the sum of a material part. defined at zero field. and a field part. It can then be shown that both Minkowski's and Abraham's tensors are unjustifiable.

The macroscopic relativistic energy-momentum conservation laws in dipole substances have been derived [1] from the corresponding microscopic laws. In this way a statistical expression for the (conserved and symmetrical) energy-momentum tensor $T^{\alpha\beta}$ in terms of atomic parameters has been found. In this expression a part appears, which reads in the rest frame of the medium ($i,j = 1,2,3$; $g^{ii} = 1$; $g^{ij} = 0$, $i \neq j$):

$$T^{\alpha\beta}_{(f)} = \begin{pmatrix} \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 & (\boldsymbol{E}\times\boldsymbol{H})^i \\ (\boldsymbol{E}\times\boldsymbol{H})^i & -E^iD^j - H^iB^j + (\frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 - \boldsymbol{B}\cdot\boldsymbol{M})g^{ij} \end{pmatrix}; \tag{1}$$

it depends on the Maxwell fields only. ($T^{00}_{(f)}$ and $c\,T^{0i}_{(f)}$ are the energy density and flow; $c^{-1}T^{i0}_{(f)}$ and $T^{ij}_{(f)}$ are the momentum density and flow.) The remaining part $T^{\alpha\beta}_{(m)} = T^{\alpha\beta} - T^{\alpha\beta}_{(f)}$, which contains bulk material, velocity fluctuation and interatomic electromagnetic interaction correlation terms, is directly related to the statistical thermodynamical expressions. For the electric dipole case the second line of (1) was proposed already by Lorentz [2] and Einstein-Laub [3].

If instead of $T^{\alpha\beta}_{(m)}$ one defines a similar tensor $T^{\alpha\beta}_{(m')}$ for the system with same temperature T and density ρ, but at zero fields, a corresponding field tensor $T^{\alpha\beta}_{(f')} = T^{\alpha\beta} - T^{\alpha\beta}_{(m')}$ is obtained. Under the conditions 1°: local equilibrium in the rest frame and non-relativistic velocity fluctuations; 2°: linear relations $\boldsymbol{P} = \kappa\boldsymbol{E}$ and $\boldsymbol{M} = \chi\boldsymbol{B}$ it reads in the rest frame

$$T^{00}_{(f')} = \tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H} + \tfrac{1}{2}\boldsymbol{E}^2 T\,\partial\kappa/\partial T + \tfrac{1}{2}\boldsymbol{B}^2\,T\partial\chi/\partial T, \tag{2}$$

$$T^{0i}_{(f')} = T^{i0}_{(f')} = (\boldsymbol{E}\times\boldsymbol{H})^i, \tag{3}$$

$$T^{ij}_{(f')} = -E^iD^j - H^iB^j + (\tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H} - \tfrac{1}{2}\boldsymbol{E}^2\rho\,\partial\kappa/\partial\rho - \tfrac{1}{2}\boldsymbol{B}^2\rho\,\partial\chi/\partial\rho)g^{ij} + \\ - \tfrac{1}{5}(P^iP^j + M^iM^j) + \tfrac{1}{15}(\boldsymbol{P}^2 + \boldsymbol{M}^2)g^{ij}. \tag{4}$$

For induced dipole media the last two terms of eq. (2) are negligible; for permanent dipole media, which obey the Langevin-Debye laws (with $\kappa(\kappa+3)^{-1}$ and $\chi(3-2\chi)^{-1}$ inversely proportional to T), the field energy density (2) becomes

$$T^{00}_{(f')} = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{H}^2) - \tfrac{1}{6}(\boldsymbol{P}^2 + \boldsymbol{M}^2). \tag{5}$$

Using the Clausius-Mosotti formula for $\kappa(\rho)$ and $\chi(\rho)$ one gets for the momentum flow

$$T^{ij}_{(f')} = -E^i D^j - H^i B^j + (\tfrac{1}{2}E^2 + \tfrac{1}{2}H^2 - \tfrac{1}{6}P^2 - \tfrac{1}{6}M^2)g^{ij} - \tfrac{1}{5}(P^i P^j + M^i M^j) + \tfrac{1}{15}(P^2 + M^2)g^{ij}. \qquad (6)$$

Consequently for induced dipole media with $\kappa \ll 1$ and $\chi \ll 1$ the tensor $T^{\alpha\beta}_{(f')}$ reduces to

$$T^{\alpha\beta}_{(f')} = \begin{pmatrix} \tfrac{1}{2}E\cdot D + \tfrac{1}{2}B\cdot H & (E\times H)^i \\ (E\times H)^i & -E^i D^j - H^i B^j + (\tfrac{1}{2}E^2 + \tfrac{1}{2}H^2)g^{ij} \end{pmatrix}, \qquad (7)$$

where the first line occurs also in the tensors proposed by Minkowski [4] and Abraham [5]; the momentum density $c^{-1}D\times B$ of Minkowski is not found, it remains $c^{-1}E\times H$, as in Abraham's tensor; the momentum flow (field pressure) is different from both Minkowski's expression $-E^i D^j - H^i B^j + (\tfrac{1}{2}E\cdot D + \tfrac{1}{2}B\cdot H)g^{ij}$ and Abraham's expression which is the symmetrical part thereof.

In the past the discussion focused mainly on the relative merits of Minkowski's and Abraham's tensors, while Lorentz's and Einstein-Laub's expressions were hardly taken into account, even in Pauli's review [6]. The widely adopted argument of von Laue [7] in favour of Minkowski was invalidated by Tang and Meixner [8].

Thus it has been shown here that starting from the field tensor (1), based on microscopic theory, one can define - under two limiting conditions - a field tensor (2)-(4) which corresponds to a material tensor at zero fields. In a more special case it takes the form (7), which shows some similarity with the tensors of Abraham and Minkowski. However even in this case differences between the results obtained from microscopic theory and those of Minkowski and Abraham remain, such that the latter cannot be justified.

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