Filtration

Dick de Jongh

March 14, 2012

Let $\mathfrak{M}$ be a model and Σ a (usually finite) set of formulas.

Definition 0.1.

1. $w \Longleftrightarrow_\Sigma v$ iff, for all $\varphi \in \Sigma\,(w \models \varphi \Leftrightarrow v \models \varphi)$
2. $|w| = \{v \in W \mid v \Longleftrightarrow_\Sigma w\}$
3. $W^f = W_\Sigma = \{|w| \mid w \in W\}$
4. $|w| \in V^f(p)$ iff $w \in V(p)$
5. $\mathfrak{M}^\mathfrak{f} = < W^f, R^f, V^f >$ is a *filtration* (of $\mathfrak{M}$ w.r.t. Σ) if
   (a) If wRv, then $|w|R^f|v|$,
   (b) If $|w|R^f|v|$ and $w \models \varphi$ (for $\varphi \in \Sigma$), then $v \models \varphi$.

Proposition 0.2. *If $\mathfrak{M}^\mathfrak{f}$ is a filtration of $\mathfrak{M}^\mathfrak{f}$ w.r.t. Σ, then, for each $\varphi \in \Sigma$, $\mathfrak{M}^\mathfrak{f}, |w| \models \varphi$ iff $\mathfrak{M}, w \models \varphi$.*

Proof. Only the induction step for $\to$:

$w \models \varphi \to \psi \iff \forall v(wRv \Rightarrow (v \models \varphi \Rightarrow v \models \psi))$

$w \models \varphi \to \psi \iff \forall v(wRv \Rightarrow (|v| \models \varphi \Rightarrow |v| \models \psi))$ (by IH)

This is implied by $\forall v(|w|R^f|v| \Rightarrow (|v| \models \varphi \Rightarrow |v| \models \psi))$ (i.e. $|w| \models \varphi \to \psi$) according to (a). On the other hand, assume, $\forall v(wRv \Rightarrow (|v| \models \varphi \Rightarrow |v| \models \psi))$, and $|w|R^f|v|$ and $|v| \models \varphi$. then, by IH, $v \models \varphi$. Since we have $w \models \varphi \to \psi$, we have by (b), $v \models \varphi \to \psi$ and hence $v \models \psi$ and, by IH, $|v| \models \psi$. So, $|w| \models \varphi \to \psi$. $\dashv$

If $\mathfrak{M}$ is a canonical model, then we can replace $|w|$ (which is $\{\Gamma \mid \Gamma \cap \Sigma = w \cap \Sigma\}$) by $w \cap \Sigma$ (we can do that because they uniquely determine each other), then W_Σ becomes $\{w \cap \Sigma \mid w \in W\}$, the set of DP-theories in Σ. That is a good move, because it makes not only $R = \subseteq$ but $R^f = \subseteq$ as well (easy, by (a) and (b) and using $w \models \varphi \Leftrightarrow \varphi \in w$).