Jankov's Theorems for Intermediate Logics in the Setting of Universal Models

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Abstract. In this article we prove two well-known theorems of Jankov in a uniform frame-theoretic manner. In frame-theoretic terms, the first one states that for each finite rooted intuitionistic frame there is a formula ψ with the property that this frame can be found in any counter-model for ψ in the sense that each descriptive frame that falsifies ψ will have this frame as the p-morphic image of a generated subframe ([12]). The second one states that **KC**, the logic of weak excluded middle, is the strongest logic extending intuitionistic logic **IPC** that proves no negation-free formulas beyond **IPC** ([13]). The proofs use a simple frame-theoretic exposition of the fact discussed and proved in [4] that the upper part of the n-Henkin model $\mathcal{H}(n)$ is isomorphic to the n-universal model $\mathcal{U}(n)$ of **IPC**. Our methods allow us to extend the second theorem to many logics L for which L and $L + \mathbf{KC}$ prove the same negation-free formulas. All these results except the last one earlier occurred in a somewhat different form in [16].

1 Introduction

In this article we give a unified purely frame-theoretic treatment of two well-known theorems of Jankov concerning intuitionistic propositional logic **IPC** and its extensions. The first one states in (intuitionistic) frame-theoretic terms the following.

Theorem 1. ([12]) *For each finite rooted frame $\mathfrak{F}$ there exists a formula ψ such that, if ψ is falsified on any descriptive frame $\mathfrak{F}'$, then $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{F}'$.*

The second one states

Theorem 2. ([13]) *If L is an intermediate logic such that $L \nsubseteq \mathbf{KC}$, then $L \vdash \theta$ and $\mathbf{IPC} \nvdash \theta$ for some negation-free formula θ.*

This implies that **KC** is the strongest intermediate logic extending **IPC** that proves the same negation-free formulas as **IPC**. We will prove the following extension of this theorem.

Theorem 3. *If an intermediate logic L is complete with respect to a class of finite rooted frames which is closed under the operation of adding a top node, then L + **KC** proves the same neagtion-free formulas as L, and if L' is an intermediate logic such that $L' \nsubseteq L + \mathbf{KC}$, then $L' \vdash \theta$ and $L \nvdash \theta$ for some negation-free formula θ.*

This means that for such a logic L, L + **KC** is the strongest intermediate logic extending L that proves the same negation-free formulas as L.

We will show that the formula ψ figuring in the first theorem can be found as a so-called de Jongh-formula ψ_w for a node w in the n-universal model for **IPC** (see Section 2) for some n. The basic line of proof of the first theorem is the following.

In the n-universal model ψ_w is falsified exactly in the downward closed set generated by w, its dual φ_w is verified exactly in the upward closed set generated by w. The formulas ψ- and φ-formulas are defined by a simultaneous induction on the points of the universal model. The important relationship here is that $\psi_w = \varphi_w \to \varphi_{w_1} \vee \cdots \vee \varphi_{w_m}$, where $w_1, \ldots, w_m$ are the immediate successors of w. Now, if ψ_w is falsified anywhere, there have to be nodes upwards where φ_w is true and $\varphi_{w_1}, \ldots, \varphi_{w_m}$ false. The generated submodel that figures in the theorem is the submodel on the points where φ_w is true.

One now calls in the fact that the n-universal model is the upper part of the n-Henkin model, and in particular that, as we will show here, w is represented in the n-Henkin model by the set of consequences of φ_w. It is then not difficult to see that the points in any model where φ_w is true and $\varphi_{w_1}, \ldots, \varphi_{w_m}$ false all correspond to one point in the n-Henkin model, and therefore to the point w in the n-universal model, and that this holds in a similar way for all the v accessible from w. The desired p-morphism now simply comes out as the function that, for each v, sends all the points with φ_v true and $\varphi_{v_1}, \ldots, \varphi_{v_k}$ false to v.

The first theorem is accompanied by an easy corollary (Theorem 27) that states that any intermediate logic stronger than **IPC** will have to prove at least one of the formulas ψ_w.

It is this accompanying theorem that in a transformed form will deliver the second Jankov theorem. Any logic stronger than **KC** will prove a negation-free formula not provable in **IPC**. This negation-free formula will be a formula ψ'_w. This formula comes from a simultaneous definition of φ' and ψ'-formulas very much parallel to the φ,ψ-definition. These formulas are negation-free variations on the φ,ψ-formulas but only for those nodes of the n-universal model that have a unique top node in which all atoms are true. The formulas use an additional propositional variable, not represented in the n-universal model, that partway simulates $\bot$. The top node is there because **KC**-frames require it, and it can be because for negation-free formulas the addition of a top node with all atoms true makes no difference. The difficulty in the proof is that the relationships between the various φ'_w, ψ'_w-formulas which were clear from the correspondence to the n-Henkin model all have to be proved directly for the φ', ψ'-formulas because that correspondence is no longer there. However, the required submodel can be generated by the φ'_w and the p-morphism can be defined as the function that sends all the points where φ'_v is true and $\varphi'_{v_1}, \ldots, \varphi'_{v_k}$ false to v. All this is expressed in Lemma 35 that thus is the transformed form of the first Jankov theorem. The transformed form of the corollary remains easy to prove and that becomes the second Jankov theorem.

The first Jankov theorem will be proved in Section 4, the second one and our extension of it in Section 5. Section 2 introduces Kripke models for **IPC** in general and n-universal models and n-Henkin models in partcular. In Section 3 the relationship between n-universal models and n-Henkin models is developed sufficiently for the proofs in Sections 4 and 5. In Section 6 we conclude our very straightforward proof that the upper part of the n-Henkin model $\mathcal{H}(n)$ is isomorphic to the n-universal model $\mathcal{U}(n)$ of **IPC**. This theorem was discussed extensively and proved in [4] in a more algebraic manner.

The article finds its place in the study of fragments of **IPC** containing only certain connectives like, for example, implication and conjunction: $[\rightarrow, \wedge]$, or implication, conjunction and negation: $[\rightarrow, \wedge, \neg]$. Fragments without disjunction are locally finite (i.e. have only finitely equivalence classes of formulas in n variables). For an overview of results, see [14], and also [9], [11]. The different finite universal models (called exact models) of $[\rightarrow, \wedge]$ and $[\rightarrow, \wedge, \neg]$ inspired the present approach to situate the locally infinite fragment $[\rightarrow, \wedge, \vee]$ in the fragment $[\rightarrow, \wedge, \vee, \neg]$ (containing all connectives).

2 Preliminaries

In this section we introduce the intuitionistic Kripke frames and models we will use, and more specifically the canonical or Henkin models arising from the standard Henkin type completeness proofs. Also the universal models will be defined in this section.

Definition 4. *A* Kripke frame *is a pair $\mathfrak{F} = \langle W, R\rangle$ consisting of a nonempty set W and a partial order R on W.*

Definition 5. *A* Kripke model *is a triple $\mathfrak{M} = \langle W, R, V\rangle$ such that $\langle W, R\rangle$ is a Kripke frame, and V is an intuitionistic valuation, which is a partial map $V : \textsc{Prop} \rightarrow \wp(W)$ satisfying the* persistence *condition: if $w \in V(p)$ and wRv, then $v \in V(p)$.*

We extend the notation $V(p)$ to formulas: $V(\varphi) = \{w \in W \mid w \models \varphi\}$. Our models will usually be n-models, i.e. models with the valuation V restricted to the atoms $p_1, \ldots, p_n$ and thereby to n-formulas, formulas formed from $p_1, \ldots, p_n$. If X is a set of elements in the frame $\mathfrak{F}$ we will write $\mathfrak{F}_X$ for the subframe of $\mathfrak{F}$ generated by X, shortening this to $\mathfrak{F}_w$ if X is a single element w; similarly for models. We call the upward closed subsets of W (with respect to the relation R) *upsets*. The set of all upsets of W is denoted by $Up(W)$.

Definition 6. *A* general frame *is a triple $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, where $\langle W, R\rangle$ is a Kripke frame and $\mathcal{P}$ is a family of upsets containing $\emptyset$ and closed under $\cap$, $\cup$*

and the following operation $\supset$*: for every* $X, Y \subseteq W$*,*

$$X \supset Y = \{x \in W : \forall y \in W(xRy \wedge y \in X \rightarrow y \in Y)\}$$

Elements of the set $\mathcal{P}$ *are called* admissible sets.

Definition 7. *A general frame* $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ *is called* refined *if for any* $x, y \in W$*,*

$$\forall X \in \mathcal{P}(x \in X \rightarrow y \in X) \Rightarrow xRy.$$

$\mathfrak{F}$ *is called* compact*, if for any families* $\mathcal{X} \subseteq \mathcal{P}$ *and* $\mathcal{Y} \subseteq \{W \setminus X : X \in \mathcal{P}\}$ *for which* $\mathcal{X} \cup \mathcal{Y}$ *has the finite intersection property,* $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$*.*

Definition 8. *A general frame* $\mathfrak{F}$ *is called a* descriptive frame *iff it is refined and compact.*

The next lemma states a basic fact of descriptive frames, for a proof, see e.g. Lemma 2.6.13 in [16].

Lemma 9. *A subframe of a descriptive frame* $\mathfrak{F}$ *generated by an admissible subset of* $\mathfrak{F}$ *is a descriptive frame.*

Definition 10.
1. *Let* $\mathfrak{F} = \langle W, R \rangle$ *and* $\mathfrak{G} = \langle V, S \rangle$ *be two Kripke frames. A map* f *from* W *to* V *is called a (*Kripke frame*)* p-morphism *of* $\mathfrak{F}$ *to* $\mathfrak{G}$ *if it satisfies the following conditions:*
   - *For any* $w, u \in W$*,* wRu *implies* $f(w)Sf(u)$*;*
   - $f(w)Sv'$ *implies* $\exists v \in W(wRv \wedge f(v) = v')$*.*
2. *Let* $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ *and* $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ *be two descriptive frames. We call a Kripke frame p-morphism* f *of* $\langle W, R \rangle$ *to* $\langle V, S \rangle$ *a (descriptive frame) p-morphism of* $\mathfrak{F}$ *onto* $\mathfrak{G}$*, if it also satisfies the following condition:*
   - $\forall X \in \mathcal{Q}$*,* $f^{-1}(X) \in \mathcal{P}$*.*
3. *A Kripke frame p-morphism* f *of* $\mathfrak{F}$ *to* $\mathfrak{G}$ *is called a* p-morphism *of a model* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ *to a model* $\mathfrak{N} = \langle \mathfrak{G}, V' \rangle$ *if*
   - $w \in V(p) \Longleftrightarrow f(w) \in V'(p)$ *for every* $p \in \textsc{Prop}$*.*

Definition 11.

1. *An* n-theory *is a set of* n*-formulas closed under deduction in* **IPC***.*
2. *A set of formulas* Γ *has the* disjunction property *if, for all* n*-formulas* φ, ψ*,* $\varphi \vee \psi \in \Gamma$ *implies* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$*.*
3. *The* n-canonical model *or* n-Henkin model $\mathfrak{M}_n$ *is the Kripke model with*
   - *The set* W_n *of all consistent* n*-theories with the disjunction property as its set of worlds.*
   - *The relation* $\subseteq$ *as its accessibility relation.*
   - $V(p) = \{\Gamma \in W_n \mid p \in \Gamma\}$*.*

The point of the completeness proof lies in showing that, for each n-formula φ and each $\Gamma \in W_n$, $\Gamma \vDash \varphi$ iff $\varphi \in \Gamma$, or in a different notation, $\Gamma \vDash \varphi$ iff $\Gamma \in V(\varphi)$.

Next, we recall the definition of an n-universal model. From now on, we will talk about the valuation of point w in a n-model $\mathfrak{M}$ by using the term *color*. In general, an n-*color* (or *color* if the n is clear in a context) is a 0-1-sequence $c_1 \cdots c_n$ of length n. The set of all n-colors is denoted by $\mathbf{C}^n$.

We define an ordering on the colors as follows:

$$c_1 \cdots c_n \leq c'_1 \cdots c'_n \text{ iff } c_i \leq c'_i \text{ for each } 1 \leq i \leq n.$$

We write $c_1 \cdots c_n < c'_1 \cdots c'_n$ if $c_1 \cdots c_n \leq c'_1 \cdots c'_n$ but $c_1 \cdots c_n \neq c'_1 \cdots c'_n$.

A *coloring* on a nonempty set W is a function $col : W \to \mathbf{C}^n$. Colorings on intuitionistic frames $\langle W, R \rangle$ will have to satisfy the persistence condition $uRv \Rightarrow col(u) \leq col(v)$. Under that condition colorings and valuations on frames are in one-one correspondence. Given a model $\mathfrak{M} = \langle W, R, V \rangle$, we can describe the valuation of a point by the coloring $col_V : W \to \mathbf{C}^n$, defined by $col_V(w) = c_1 \cdots c_n$, where for each $1 \leq i \leq n$,

$$c_i = \begin{cases} 1, & w \in V(p_i); \\ 0, & w \notin V(p_i). \end{cases}$$

We call $col_V(w)$ *the color of* w *under* V.

In any frame $\mathfrak{F} = \langle W, R \rangle$, we say that a subset $X \subseteq W$ *totally covers* a point $w \in W$, denoted by $w \prec X$, if X is the set of all immediate successors of w. We will just write $w \prec v$ in the case that $w \prec \{v\}$. A subset $X \subseteq W$ is called an *anti-chain* if $|X| > 1$ and for every $w, v \in X$, $w \neq v$ implies that $\neg wRv$ and $\neg vRw$. If uRv we also say that u is *under* v.

We can now inductively define the *n-universal model* $\mathcal{U}(n)$ by its cumulative layers $\mathcal{U}(n)^k$ for $k \in \omega$.

Definition 12.

- *The first layer* $\mathcal{U}(n)^1$ *consists of* 2^n *nodes with the* 2^n *different* n*-colors under the discrete ordering.*
- *Under each element* w *in* $\mathcal{U}(n)^k - \mathcal{U}(n)^{k-1}$*, for each color* $s < col(w)$*, we put a new node* v *in* $\mathcal{U}(n)^{k+1}$ *such that* $v \prec w$ *with* $col(v) = s$*, and we take the reflexive transitive closure of the ordering.*
- *Under any finite anti-chain* X *with at least one element in* $\mathcal{U}(n)^k - \mathcal{U}(n)^{k-1}$ *and any color* s *with* $s \leq col(w)$ *for all* $w \in X$*, we put a new element* v *in* $\mathcal{U}(n)^{k+1}$ *such that* $col(v) = s$ *and* $v \prec X$ *and we take the reflexive transitive closure of the ordering.*

The whole model $\mathcal{U}(n)$ is the union of its layers. It is easy to see from the construction that every $\mathcal{U}(n)^k$ is finite. As a consequence, the generated submodel $\mathcal{U}(n)_w$ is finite for any node w in $\mathcal{U}(n)$.

The 1-universal model is also called *Rieger-Nishimura ladder*, which is depicted in Figure 1.

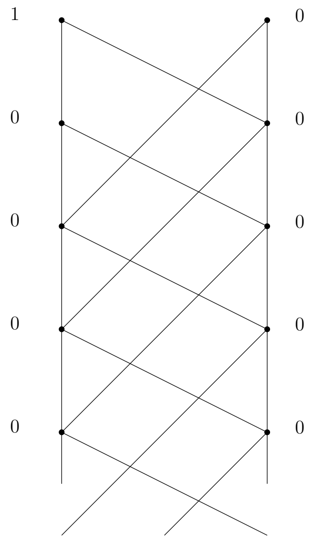

Fig. 1. Rieger-Nishimura ladder

3 *n*-universal models and *n*-Henkin models of IPC

Let $Upper(\mathfrak{M})$ denote the submodel $\mathfrak{M}_{\{w \in W | d(w) < \omega\}}$ generated by all the points with finite depth, where depth is defined as usual. It is well-known by now that the n-universal model is isomorphic to this upper part of the n-Henkin model $Upper(\mathcal{H}(n))$. N. Bezhanishvili gave in [4] an algebraic proof of this fact. In the final section, we prove it directly on the basis of two important lemmas that we already need in the next section on the first Jankov theorem. These two lemmas respectively state that every finite model can be mapped p-morphically onto a generated submodel of $\mathcal{U}(n)$ (Lemma 13), and that $\mathcal{U}(n)_w$ is isomorphic to the submodel of $\mathcal{H}(n)$ generated by the theory axiomatized by the de Jongh formula of w (Lemma 20, see Definition 15). We restrict the information on n-universal models to what we need. For more information, see [4].

Lemma 13. *For any finite rooted Kripke n-model* $\mathfrak{M}$*, there exists a unique* $w \in U(n)$ *and a p-morphism of* $\mathfrak{M}$ *onto* $\mathcal{U}(n)_w$.

For a proof of Lemma 13, see e.g. [16]. It follows as stated in the following theorem that $\mathcal{U}(n)$ is a counter-model to every n-formula that is a non-theorem of **IPC**. This shows that $\mathcal{U}(n)$ deserves being called a "universal model".

Theorem 14. *1. For any n-formula* φ, $\mathcal{U}(n) \models \varphi$ *iff* $\vdash_{\mathbf{IPC}} \varphi$.
2. For any n-formulas φ, ψ*, for all* $w \in \mathcal{U}(n)(w \models \varphi \Rightarrow w \models \psi)$ *iff* $\varphi \vdash_{\mathbf{IPC}} \psi$.

Proof. (1) $\Leftarrow$: trivial. $\Rightarrow$: Suppose $\nvdash_{\mathbf{IPC}} \varphi$. Then there exists a finite n-model $\mathfrak{M}$ and a point $w \in \mathfrak{M}$ such that $\mathfrak{M}, w \not\models \varphi$. By Lemma 13, there exists a p-morphism f of $\mathfrak{M}$ to $\mathcal{U}(n)$. Hence, $\mathcal{U}(n), f(w) \not\models \varphi$.

(2) Follows easily from (1).

□

For any node w in an n-model $\mathfrak{M}$, if $w \prec \{w_1, ..., w_m\}$, then we let
$prop(w) := \{p_i \,|\, w \models p_i, 1 \leq i \leq n\}$,
$notprop(w) := \{q_i, \,|\, w \not\models q_i, 1 \leq i \leq n\}$,
$newprop(w) := \{r_j, \,|\, w \not\models r_j$ and $w_i \models r_j$ for each $1 \leq i \leq m$, for $1 \leq j \leq n\}$[3].

Here $newprop(w)$ denotes the set of atoms which are about to be true in w, i.e. they are true in all of w's proper successors. Next, we define the formulas φ_w and ψ_w, which were first introduced in [7], and which were extensively discussed and named de Jongh formulas in [4].

Definition 15. *Let w be a point in $\mathcal{U}(n)$. We inductively define its* de Jongh formulas *φ_w and ψ_w.*

If $d(w) = 1$, then let

$$\varphi_w := \bigwedge prop(w) \wedge \bigwedge\{\neg p_k \,|\, p_k \in notprop(w),\ 1 \leq k \leq n\},$$
$$\psi_w := \neg\varphi_w.$$

If $d(w) > 1$, and $\{w_1, ..., w_m\}$ is the set of all immediate successors of w, then define

$$\varphi_w := \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \varphi_{w_i}),$$
$$\psi_w := \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}.$$

The most important properties of the de Jongh formulas are revealed in the next theorem. It was first proved in [7].

Theorem 16. *For every $w \in \mathcal{U}(n) = \langle U(n), R, V\rangle$, we have that*

- $V(\varphi_w) = R(w)$, *where* $R(w) = \{u \in \mathcal{U}(n) \,|\, wRu\}$;
- $V(\psi_w) = U(n) \setminus R^{-1}(w)$, *where* $R^{-1}(w) = \{u \in \mathcal{U}(n) \,|\, uRw\}$.

An easy lemma that is needed in the proof of Jankov's theorem in the next section is the following.

Lemma 17. *If $u, v \in \mathcal{U}(n)$ and vRu, then $\vdash_{\mathbf{IPC}} \varphi_u \to \varphi_v$ and $\not\vdash_{\mathbf{IPC}} \varphi_v \to \varphi_u$.*

Proof. Immediate from Theorem 16 and Theorem 14. □

Definition 18.

[3] Note that if w is an endpoint, $newprop(w) = notprop(w)$.

– *We write* $Cn_n(\varphi) = \{n\text{-formula } \psi \mid \vdash_{\mathbf{IPC}} \varphi \to \psi\}$*, but we may leave the* n *out if it is clear from the context.*
– *We write* $Th_n(\mathfrak{M}, w) = \{n\text{-formula } \varphi \mid w \models \varphi\}$*, but we may leave out the* $\mathfrak{M}$ *and* n *if they are clear from the context.*

Corollary 19. *For any point* w *in* $\mathcal{U}(n)$, $Th_n(w) = Cn_n(\varphi_w)$.

Proof. By Theorem 16, $Th_n(w) \supseteq Cn_n(\varphi_w)$. For the other direction, let ψ be an n-formula such that $\mathcal{U}(n), w \models \psi$. By Theorem 16 again, we have that $\mathcal{U}(n) \models \varphi_w \to \psi$, thus by Theorem 14, $\vdash_{\mathbf{IPC}} \varphi_w \to \psi$, i.e. $\psi \in Cn_n(\varphi_w)$. □

The next lemma expresses the essence of the fact that the upper part of the n-Henkin model is isomorphic to the n-universal model. We will pursue this in the last section. For the time being the lemma will come in very useful in the proof of the first Jankov Theorem, the main theorem of the next section.

Lemma 20. *For any* $w \in \mathcal{U}(n)$*, let* φ_w *be a de Jongh formula. Then we have that* $\mathcal{H}(n)_{Cn(\varphi_w)} \cong \mathcal{U}(n)_w$.

Proof. Let $\mathcal{U}(n) = \langle U(n), R, V\rangle$ and $\mathcal{H}(n) = \langle H(n), R', V'\rangle$. Define a map $f : U(n)_w \to \mathcal{H}(n)_{Cn(\varphi_w)}$ by taking

$$f(v) = Cn(\varphi_v).$$

We show that f is an isomorphism.

First for any $v \in U(n)$, by Corollary 19, we have that $v \in V(p)$ iff $Cn(\varphi_v) \in V'(p)$ and that

$$\begin{aligned} uRv &\iff \mathcal{U}(n), v \models \varphi_u \text{ (by Theorem 16)} \\ &\iff \varphi_u \in Cn(\varphi_v) \text{ (by Corollary 19)} \\ &\iff Cn(\varphi_u) \subseteq Cn(\varphi_v) \\ &\iff f(u)R'f(v). \end{aligned}$$

This makes f into a homomorphism.

Now, suppose $u \neq v$; w.l.o.g. we may assume that $\neg uRv$, which by Theorem 16 means that $\mathcal{U}(n), u \not\models \varphi_v$. Thus, $\varphi_v \notin Cn(\varphi_u)$ by Corollary 19, and so $f(u) = Cn(\varphi_u) \neq Cn(\varphi_v) = f(v)$. Hence, f is injective.

It remains to show that f is surjective. That is, to show that for any $\Gamma \in \mathcal{H}(n)_{Cn(\varphi_u)}$ (i.e. any n-theory $\Gamma \supseteq Cn_n(\varphi_u)$ with the disjunction property) there exists v with uRv such that $\Gamma = Cn(\varphi_v)$. We prove this by induction on the depth of u.

$d(u) = 1$. It suffices to show that if $Cn(\varphi_u) \subseteq \Gamma$, then $\Gamma = Cn(\varphi_u)$. This is clear from the fact that $\theta \in Cn(\varphi_u)$ iff $\vdash_{\mathbf{IPC}} \varphi_u \to \theta$ iff (by Corollary 19), because this shows that $Cn(\varphi_u)$ is maximal consistent.

$d(u) = k + 1$. Let $\{u_1, ..., u_m\}$ be the set of all immediate successors of u. Suppose $Cn(\varphi_u) \subseteq \Gamma$. If $Cn(\varphi_{u_i}) \subseteq \Gamma$ for some $1 \leq i \leq m$, then by induction hypothesis, $\Gamma = Cn(\varphi_v)$ for some $v \in R(u_i)$, i.e. $v \in R(u)$. So, we can assume

$Cn(\varphi_{u_i}) \nsubseteq \Gamma$ for all $1 \leq i \leq m$. Thus $\Gamma \nvdash \varphi_{u_i}$ for each $1 \leq i \leq m$. Take any $\theta \in \Gamma$. We then have also $\theta \wedge \varphi_u \in \Gamma$. So,

$$\theta \wedge \varphi_u \nvdash \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}.$$

Since $\mathcal{U}(n)$ is universal, there exists a $u' \in U(n)$ such that

$$\mathcal{U}(n), u' \models \theta \wedge \varphi_u \text{ and } \mathcal{U}(n), u' \not\models \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}.$$

By Theorem 16, $u' = u$, which implies that $\mathcal{U}(n), u \models \theta$. By Corollary 19, $\theta \in Cn(\varphi_u)$. Therefore $\Gamma = Cn(\varphi_u)$. □

We end this section by a corollary which follows from the correspondence between $\mathcal{H}(n)$ and $\mathcal{U}(n)$, and which plays a crucial role in our proof of Jankov's theorem.

Corollary 21. *Let $\mathfrak{M}$ be any model and w be a point in $\mathcal{U}(n) = \langle W, R, V \rangle$. For any point x in $\mathfrak{M}$, if $\mathfrak{M}, x \models \varphi_w$, then there exists a unique point v satisfying*

$$\mathfrak{M}, x \models \varphi_v,\ \mathfrak{M}, x \not\models \varphi_{v_1}, \cdots, \mathfrak{M}, x \not\models \varphi_{v_m},$$

where $v \prec \{v_1, \cdots, v_m\}$, and wRv.

Proof. Note that $Th_n(\mathfrak{M}, x)$ is a node in $\mathcal{H}(n) = \langle W', R', V' \rangle$. $\mathfrak{M}, x \models \varphi_w$ implies that $Th_n(\mathfrak{M}, x) \vdash_{\mathbf{IPC}} \varphi_w$ and $Cn_n(\varphi_w) R' Th_n(\mathfrak{M}, x)$. Thus, by Lemma 20, $Th_n(\mathfrak{M}, x) = Cn_n(\varphi_v)$ for a unique point $v \in W$. Moreover we have wRv. So $\mathfrak{M}, x \models \varphi_v$.

By Lemma 16, we have that $\nvdash_{\mathbf{IPC}} \varphi_v \to \varphi_{v_i}$ for each $1 \leq i \leq m$, so $\varphi_{v_i} \notin Cn_n(\varphi_v) = Th_n(\mathfrak{M}, x)$, i.e. $\mathfrak{M}, x \not\models \varphi_{v_i}$. □

4 Jankov's Theorem for extensions of IPC

In [12], Jankov proved

Theorem 22. *For each finite subdirectly irreducible Heyting-algebra $\mathfrak{H}$ there exists a formula ψ such that, if ψ is falsified on any Heyting-algebra $\mathfrak{H}'$, then $\mathfrak{H}$ is a subalgebra of a homomorphic image of $\mathfrak{H}'$.*

The formula ψ used in the proof of the theorem contained a direct description of $\mathfrak{H}$. De Jongh proved in [7] the same theorem with regard to the de Jongh formulas defined above. Here we transform the latter proof, which made an algebraic detour, into a purely frame-theoretic one in the following form.

Theorem 23. *For each finite rooted frame $\mathfrak{F}$ there exists a formula ψ such that, if ψ is falsified on any descriptive frame $\mathfrak{F}'$, then $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{F}'$.*

By modern insights in the duality between Heyting-algebras and descriptive frames, it is equivalent to the original theorem.

We have set the stage in the previous section in such a manner that the analogies between the proof of the Jankov theorem and the proof of our central Lemma 35 for the Jankov Theorem on **KC** (Theorem 37) in the next section will come out as clearly as possible.

One of the things we will need in the proof of Jankov's theorem is that under certain conditions a Kripke frame p-morphism from a descriptive frame to a finite frame is almost automatically also a descriptive frame p-morphism. The next lemma states the necessary conditions.

Lemma 24. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle W', R', \mathcal{P}'\rangle$ be two descriptive frames with W' finite. Let f be a (Kripke frame) p-morphism from the Kripke frame $\langle W, R\rangle$ to the Kripke frame $\langle W', R'\rangle$ such that $f^{-1}(R(w))$ is an admissible set for any $w \in W'$. Then f is also a (descriptive frame) p-morphism from the descriptive frame $\mathfrak{F}$ to the descriptive frame $\mathfrak{G}$.*

Proof. It suffices to show that for any $X \in \mathcal{P}'$, $f^{-1}(X) \in \mathcal{P}$. Observing that $X = \bigcup\limits_{w\in X} R(w)$, we obtain that

$$f^{-1}(X) = f^{-1}(\bigcup_{w\in X} R(w)) = \bigcup_{w\in X} f^{-1}(R(w)),$$

which implies $f^{-1}(X) \in \mathcal{P}$ since $f^{-1}(X)$ is a finite union of admissible sets. □

The following useful lemma was introduced (as Theorem 3.2.16) and discussed in [4]. It says that for any finite rooted frame $\mathfrak{F}$ an n can be found so that an isomorphic copy of $\mathfrak{F}$ occurs in $\mathcal{U}(n)$ as the frame of a generated submodel.

Lemma 25. *For any finite rooted frame $\mathfrak{F} = \langle W', R'\rangle$, there exists a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ on $\mathfrak{F}$ such that $\mathfrak{M}$ is isomorphic to a generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$ for some n.*

Proof. We introduce a propositional variable p_w for every point w in W, and define a valuation V by letting $V(p_w) = R(w)$. Put $n = |W|$. By Lemma 13, there exists a p-morphism f from the model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ onto a generated submodel $\mathcal{U}(n)_w$. By the construction, we know that different points of $\mathfrak{M}$ have different colors, thus f is injective, i.e. $\mathfrak{M}$ is isomorphic to $\mathcal{U}(n)_w$. □

Note that the underlying Kripke frame of $\mathcal{U}(n)_w = \langle W, R, V\rangle$ described in the previous lemma can be viewed as the general frame $\mathfrak{U} = \langle W, R, Up(W)\rangle$, which is a descriptive frame since W is finite.

Theorem 26 (Jankov). *For every finite rooted frame $\mathfrak{F}$, let ψ_w be the de Jongh formula of w in the model $\mathcal{U}(n)_w$ described in Lemma 25. Then for every descriptive frame $\mathfrak{G}$,*

$$\mathfrak{G} \not\models \psi_w \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Proof. The direction from right to left is obvious, since $\mathfrak{F} \not\models \psi_w$ follows immediately from Theorem 16.

For the other direction, suppose $\mathfrak{G} \not\models \psi_w$. Then there exists a model $\mathfrak{N}$ on $\mathfrak{G}$ such that

$$\mathfrak{N} \not\models \varphi_w \to \varphi_{w_1} \vee \cdots \vee \varphi_{w_m}, \tag{1}$$

where $w \prec \{w_1, \cdots, w_m\}$. Consider the generated submodel $\mathfrak{N}' = \mathfrak{N}_{V'(\varphi_w)} = \langle W', R', \mathcal{P}', V' \rangle$ of $\mathfrak{N}$. Note that since $V'(\varphi_w)$ is admissible, by Lemma 9, $\langle W', R', \mathcal{P}' \rangle$ is a descriptive frame. Define a map $f : W' \to W$ by taking $f(x) = v$ iff

$$\mathfrak{N}', x \models \varphi_v,\ \mathfrak{N}', x \not\models \varphi_{v_1}, \cdots, \mathfrak{N}', x \not\models \varphi_{v_k}, \tag{2}$$

where $v \prec \{v_1, \cdots, v_k\}$.

Note that for every $x \in W'$, $\mathfrak{N}', x \models \varphi_w$, thus by Corollary 21, there exists a unique $v \in R(w)$ satisfying (2). So f is well-defined.

We show that f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Suppose $x, y \in W'$ with $xR'y$, $f(x) = v$ and $f(y) = u$. Since $\mathfrak{N}', x \models \varphi_v$, we have that $\mathfrak{N}', y \models \varphi_v$. By Corollary 21, there exists a unique point $u' \in W$ such that u' and y satisfy (2), moreover, vRu'. So, since u and y also satisfy (2), by the uniqueness, $u' = u$ and vRu.

Next, suppose $x \in W'$ and $v, u \in W$ such that $f(x) = v$ and vRu. We now show that there exists $y \in W'$ with $xR'y$ such that

$$\mathfrak{N}', y \models \varphi_u,\ \mathfrak{N}', y \not\models \varphi_{u_1}, \cdots, \mathfrak{N}', y \not\models \varphi_{u_l} \tag{3}$$

where $u \prec \{u_1, \cdots, u_l\}$. This will give us the required $f(y) = u$. We will prove this directly if u is an immediate successor of v, i.e. one of the v_i. For u with vRu in general there is a chain $v = u_0 R u_1 \ldots R u_k = u$ with u_{i+1} each time an immediate successor of u_i, so that the result for u follows by induction along this chain.

Since x and v satisfy (2), and φ_v implies by its definition that

$$\bigvee_{i=1}^{l} \psi_{v_i} \to \bigvee_{i=1}^{l} \varphi_{v_i}, \tag{4}$$

we must have that

$$\mathfrak{N}', x \not\models \psi_u, \tag{5}$$

because u is one of the v_i. From (5) the existence of y with $xR'y$ satisfying (3) immediately follows. Hence, we have shown that f is a (Kripke frame) p-morphism.

To show that f is surjective it is sufficient to note that, by (1), there exists $x \in W'$ such that (2) holds for x and w, i.e. $f(x) = w$. Then, for every node $v \in W$, we have that wRv. Since f is a (Kripke frame) p-morphism, there exists $y \in R'(x) \subseteq W'$ such that $f(y) = v$.

It remains to show that f is a (descriptive frame) p-morphism between the two descriptive frames. In view of Lemma 24, it is sufficient to show that for any $v \in X$, $f^{-1}(R(v)) = V'(\varphi_v)$ which is an admissible set.

Indeed, for every $x \in f^{-1}(R(v))$, there exists $u \in R(v)$ such that $f(x) = u$ and so $\mathfrak{N}', x \models \varphi_u$. Applying Lemma 17 gives $\mathfrak{N}', x \models \varphi_v$, and so $x \in V'(\varphi_v)$. On the other hand, for every $x \in V'(\varphi_v)$, by Corollary 21, there exists a unique $u \in R(v)$ such that $f(x) = u$, thus $x \in f^{-1}(R(v))$.

Hence f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Then since $\mathfrak{F} \cong \langle W, R, \mathcal{P} \rangle$, $\mathfrak{F}$ is a p-morphic image of $\langle W', R', \mathcal{P}' \rangle$, which is a generated subframe of $\mathfrak{G}$. □

As announced in the introduction we conclude this section with the corollary of ([7], [8]) that says that any intermediate logic that really extends **IPC** will prove at least one of the ψ_w. This is a very useful theorem for applications, e.g. to give charcaterizations of **IPC** (see e.g. [7], [8]). We will not apply it directly in this paper, but, as said before, we will use an adapted form of it in the next section.

Theorem 27. *If L is an intermediate logic strictly extending* **IPC***, i.e.* **IPC** $\subset L \subseteq$ **CPC***, then there exists $n \in \omega$ and w in $\mathcal{U}(n)$ such that $L \vdash \psi_w$.*

Proof. Suppose χ is a formula satisfying

$$L \vdash \chi \text{ and } \mathbf{IPC} \nvdash \chi.$$

Then there exists a finite rooted frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \chi$. By Lemma 25, there exists a model $\langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$ for some generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$. Consider the de Jongh formula ψ_w. Suppose $L \nvdash \psi_w$. Then there exists a descriptive frame $\mathfrak{G}$ of L such that $\mathfrak{G} \not\models \psi_w$. By Theorem 26, $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Thus, $\mathfrak{F}$ is an L frame. Since $L \vdash \chi$, we have that $\mathfrak{F} \models \chi$, which gives us a contradiction. □

5 Jankov's Theorem for KC

The logic **KC**, called the logic of weak excluded middle, and also Jankov's Logic, is the intermediate logic axiomatized by $\neg\varphi \vee \neg\neg\varphi$. **KC** is complete with respect to finite rooted frames with unique top points. From that fact it is not difficult to show that **KC** proves exactly the same negation-free formulas as **IPC**.

Theorem 28. *For any negation-free formula φ,* **KC** $\vdash \varphi$ *iff* **IPC** $\vdash \varphi$.

Proof. It suffices to show the direction "⇒". Suppose **IPC** $\nvdash \varphi$ for any negation-free formula φ. Then there exists a finite rooted model $\mathfrak{M}$ such that $\mathfrak{M} \not\models \varphi$. Now construct a new model $\mathfrak{M}'$ by adding a new top node t to $\mathfrak{M}$ and making every propositional variable true at t. It is not hard to see that $\mathfrak{M}, w \models \psi$ iff $\mathfrak{M}', w \models \psi$ for all nodes w in $\mathfrak{M}$ and all negation-free formulas ψ. Therefore $\mathfrak{M}' \not\models \varphi$, which by the completeness of **KC** means that **KC** $\nvdash \varphi$. □

Jankov proved in [13] that **KC** is the strongest intermediate logic that has this property. Another proof can be obtained by using canonical formulas (see [6],

[1]). In this section, we give a frame-theoretic alternative proof of Jankov's Theorem. The basic idea of the proof comes from adapting the proof of Theorem 26 combined with Theorem 27 to the special case of **KC**-frames.

We start with defining formulas φ'_w and ψ'_w which are negation-free modifications of the de Jongh formulas. To a certain extent they will play the same role on the generated submodels of $\mathcal{U}(n)$ with a unique top node satisfying all atoms as the de Jongh formulas play on all rooted generated submodels of $\mathcal{U}(n)$. First, we introduce some terminology.

For any finite set X of formulas with $|X| > 1$, let

$$\Delta X = \bigwedge\{\varphi \leftrightarrow \psi \mid \varphi, \psi \in X\}.$$

For the case $|X| = 1$ or $|X| = 0$, we set $\Delta X = \top$.

Let $\mathcal{U}(n)_{w_0} = \langle W, R, V\rangle$ be a generated submodel with a largest element t in $\mathcal{U}(n)$ such that

- $t \models p_1 \wedge \cdots \wedge p_n$;
- $col(w) \neq col(v)$ for all $w, v \in W$ such that $w \neq v$.

To make it easy to state the next sequence of lemmas, definitions in a consistent manner we call such a model in the following a *top model*, and say that w_0 *g*enerates a top model. Let r be a new propositional variable (to be identified with p_{n+1} so that we can talk about $p_1, \ldots, p_n, r$-models as $n+1$-models).

Definition 29. *We inductively define the formulas φ'_w and ψ'_w for every $w \in W$ in a top model.*

If $d(w) = 1$,

$$\varphi'_w = p_1 \wedge \cdots \wedge p_n,$$
$$\psi'_w = \varphi'_w \to r.$$

If $d(w) = 2$, let q be an arbitrary propositional letter in $notprop(w)$. Define

$$\varphi'_w = \bigwedge prop(w) \wedge \Delta\, notprop(w) \wedge ((q \to r) \to q)^4,$$
$$\psi'_w = \varphi'_w \to q.$$

If $d(w) > 2$ and $w \prec \{w_1, \cdots, w_m\}$, then let

$$\varphi'_w := \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi'_{w_i} \to \bigvee_{i=1}^{m} \varphi'_{w_i}),$$

$$\psi'_w := \varphi'_w \to \bigvee_{i=1}^{m} \varphi'_{w_i}.$$

4 Note that in the definition, it does not matter which $q \in notprop(w)$ is chosen. Note also that $notprop(w) = newprop(w)$.

We will prove for the φ'_w and ψ'_w formulas a lemma (Lemma 35) which is analogous to Theorem 26 for the φ_w and ψ_w formulas. It is good to note already that the φ'_w and ψ'_w formulas cannot be evaluated in $\mathcal{U}(n)$, since there is one propositional variable to many in them. Nevertheless, we will be able to follow the general line of the argument in the previous section.

It is worth remarking that, for $d(w) = 2$, ψ'_w is a generalized form of Peirce's Law $(((q \to r) \to q) \to q)$.

Lemma 30. *$\mathbf{IPC} \vdash \varphi'_w[r/\bot]$[5] $\leftrightarrow \varphi_w$ and $\mathbf{IPC} \vdash \psi'_w[r/\bot] \leftrightarrow \psi_w$.*

Proof. We prove this by induction on $d(w)$.

$d(w) = 1$. Trivial.

$d(w) = 2$. $\varphi'_w[r/\bot] = \bigwedge prop(w) \wedge \Delta\, notprop(w) \wedge ((q \to \bot) \to q)$.

First note that $(q \to \bot) \to q)$ is equivalent to $\neg\neg q$. On the other hand,

$$\vdash \varphi_w \leftrightarrow \bigwedge prop(w) \wedge (\bigvee notprop(w) \vee \neg(p_1 \wedge \cdots \wedge p_n) \to p_1 \wedge \cdots \wedge p_n)$$

$$\vdash \varphi_w \leftrightarrow \bigwedge prop(w) \wedge (\bigvee notprop(w) \to p_1 \wedge \cdots \wedge p_n) \wedge (\neg(p_1 \wedge \cdots \wedge p_n) \to p_1 \wedge \cdots \wedge p_n)$$

Under the assumption $\bigwedge prop(w)$, $\bigvee notprop(w) \to p_1 \wedge \cdots \wedge p_n$ is equivalent to $\Delta\, notprop(w)$. Furthermore, $\neg(p_1 \wedge \cdots \wedge p_n) \to p_1 \wedge \cdots \wedge p_n$ is equivalent to $\neg\neg(p_1 \wedge \cdots \wedge p_n)$ and hence to $\neg\neg p_1 \wedge \cdots \wedge \neg\neg p_n$. This, in its turn is under the assumptions $\bigwedge prop(w)$ and $\Delta\, notprop(w)$ equivalent to $\neg\neg q$. So, indeed, $\vdash \varphi_w \leftrightarrow \varphi'_w[r/\bot]$ and

$$\begin{aligned}
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi'_w[r/\bot] \to q)\\
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi'_w[r/\bot] \to p_1 \wedge \cdots \wedge p_n)\\
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi_w \to p_1 \wedge \cdots \wedge p_n)\\
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi_w \to \varphi_t)\\
&\vdash \psi'_w[r/\bot] \leftrightarrow \psi_w.
\end{aligned}$$

$d(w) > 2$. This is proved easily by applying the induction hypothesis to the successors w_i $(1 \leq i \leq m)$ of w. □

Obviously, we could have defined φ'_w and ψ'_w slightly differently but equivalently in such a manner that this lemma would have been a complete triviality, but that would have meant a much less intuitive and pleasing definition of the φ'_w and ψ'_w for w of depth 2. One corollary we will use later in the proof of Theorem 37 is the following.

Corollary 31. *Let w_0 generate a top model in $\mathcal{U}(n)$. Then, for any point w in $\mathcal{U}(n)_{w_0}$, $\nvdash_{\mathbf{IPC}} \psi'_w$.*

Proof. By Theorem 16, $\mathcal{U}(n)_{w_0} \not\models \psi_w$, thus, by the Lemma 30, the underlying frame of $\mathcal{U}(n)_{w_0}$ falsifies ψ'_w. Hence $\nvdash_{\mathbf{IPC}} \psi'_w$. □

[5] We write $\varphi[p/\psi]$ for the formula obtained by replacing all occurrences of p in φ by ψ.

The next lemma is an analogue of Lemma 17 that was crucial in our proof of Jankov's Theorem. The property that was proved for the φ_w formulas in that lemma was an easy consequence of Theorem 14. We do not have such a theorem for the φ'_w formulas however. Here we prove the corresponding theorem directly from the construction of the φ'_w and ψ'_w formulas.

Lemma 32. *Let w_0 generate a top model in $\mathcal{U}(n)$ and let w, v be two nodes in W with wRv. Then we have that $\vdash_{\mathbf{IPC}} \varphi'_v \to \varphi'_w$.*

Proof. We prove the lemma by induction on $d(v)$.

If $d(v) = 1$, then $\varphi'_v = p_1 \wedge \cdots \wedge p_n$. Since wRv, we have that $prop(w) \subseteq \{p_1, \cdots, p_n\}$ and

$$\vdash \varphi'_v \to \bigwedge prop(w). \tag{6}$$

We show that $\vdash \varphi'_v \to \varphi'_w$ by induction on $d(w)$.

$d(w) = d(v) + 1 = 2$. Then for any $p, q \in notprop(w) \subseteq \{p_1, \cdots, p_n\}$ we have that

$$\vdash p_1 \wedge \cdots \wedge p_n \to (p \leftrightarrow q) \text{ and } \vdash p_1 \wedge \cdots \wedge p_n \to ((q \to r) \to q).$$

It follows that

$$\vdash \varphi'_v \to \mathit{\Delta}\, notprop(w) \text{ and } \vdash \varphi'_v \to ((q \to r) \to q).$$

Together with (6), we obtain

$$\vdash \varphi'_v \to \bigwedge prop(w) \wedge \mathit{\Delta}\, notprop(w) \wedge ((q \to r) \to q)$$

i.e. $\vdash \varphi'_v \to \varphi'_w$.

$d(w) > 2$. Let $w \prec \{w_1, \cdots, w_k\}$. Then for any immediate successor w_i of w, since $d(w_i) < d(w)$ by induction hypothesis, we have that $\vdash \varphi'_v \to \varphi'_{w_i}$. This implies that $\vdash \varphi'_v \to \bigvee_{i=1}^{k} \varphi'_{w_i}$ and that

$$\vdash \varphi'_v \to (\bigvee newprop(w) \vee \bigvee_{i=1}^{k} \psi'_{w_i} \to \bigvee_{i=1}^{k} \varphi'_{w_i}). \tag{7}$$

Together with (6), we obtain

$$\vdash \varphi'_v \to \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{k} \psi'_{w_i} \to \bigvee_{i=1}^{k} \varphi'_{w_i}) \tag{8}$$

i.e. $\vdash \varphi'_v \to \varphi'_w$.

If $d(v) = 2$, then since $prop(w) \subseteq prop(v)$, clearly (6) holds. We show $\vdash \varphi'_v \to \varphi'_w$ by induction on $d(w)$.

$d(w) = d(v) + 1$. Then $v = w_i$ and $\varphi'_v = \varphi'_{w_i}$ for some immediate successor w_i of w, hence $\vdash \varphi'_v \to \bigvee_{i=1}^{k} \varphi'_{w_i}$ and (7) follows. Together with (6), we obtain (8) i.e. $\vdash \varphi'_v \to \varphi'_w$.

$d(w) > d(v) + 1$. For any immediate successor w_i of w, by the induction hypothesis, we have that $\vdash \varphi'_v \to \bigvee_{i=1}^{k} \varphi'_{w_i}$, which implies (7). Together with (6), we obtain (8) i.e. $\vdash \varphi'_v \to \varphi'_w$.

If $d(v) > 2$, then clearly $prop(w) \subseteq prop(v)$ gives (6). By a similar argument as above, we can show that (7) holds, thus, (8) i.e. $\vdash \varphi'_v \to \varphi'_w$ holds. □

Next, we want to prove for the φ'_w formulas an analogue to Corollary 21. But we will have to do this in two steps. First, we show that nodes that make φ'_w true have the right color.

Theorem 33. *Let $\mathfrak{M} = \langle W', R', V' \rangle$ be any $n+1$-model and let w_0 generate a top model in $\mathcal{U}(n)$. Put $V_n = V'|\{p_1, \dots, p_n\}$. For any point w in $\mathcal{U}(n)_{w_0}$ and any point x in $\mathfrak{M}$, if*

$$\mathfrak{M}, x \models \varphi'_w,\ \mathfrak{M}, x \not\models \varphi'_{w_1}, \cdots, \mathfrak{M}, x \not\models \varphi'_{w_m}, \tag{9}$$

where $w \prec \{w_1, \cdots, w_m\}$, then $col_{V_n}(x) = col_V(w)$.

Proof. We prove the lemma by induction on $d(w)$. In the following discussion we restrict attention to n-formulas and n-atoms all the time.

$d(w) = 1$, i.e. $w = t$. Then (9) means that $\mathfrak{M}, x \models p_1 \wedge \cdots \wedge p_n$. Also, $\mathcal{U}(n)_{w_0}, t \models p_1 \wedge \cdots \wedge p_n$. So $col_{V_n}(x) = col_V(w)$.

$d(w) = 2$. Then (9) implies that

$$\mathfrak{M}, x \models \bigwedge prop(w). \tag{10}$$

This means that all atoms true in w are true in x. From (9) we also have that

$$\mathfrak{M}, x \models \Delta\, notprop(w). \tag{11}$$

So, either all atoms false in w are false in x, or all are true in x. But, in this case, in (9) $m = 1$ and $w_1 = t$, so

$$\mathfrak{M}, x \not\models p_1 \wedge \cdots \wedge p_n. \tag{12}$$

This implies that all atoms false in w are false in x: $col_{V_n}(x) = col_V(w)$.

$d(w) > 2$. This is the induction step. As in the previous case we have that all atoms true in w are true in x. Now (9)

$$\mathfrak{M}, x \not\models \psi'_{w_i}, \tag{13}$$

for all immediate successor w_i of w, i.e. for each immediate successor w_i of w, there exists $y_i \in R'(x)$ such that y_i and w_i satisfy (9). Since $d(w_i) < d(w)$, by the induction hypothesis, we have that $col_{V_n}(y_i) = col_V(w_i)$. So, all atoms false in at least one of the w_i are false in x. On the other hand, (9) also implies

$$\mathfrak{M}, x \not\models \bigvee newprop(w), \tag{14}$$

So, all atoms true in all w_i but not in w are also false in x. We have $col_{V_n}(x) = col_V(w)$. □

This is the point where the requirement we made at the beginning of this section that all the nodes of $\mathcal{U}(n)_{w_0}$ have distinct colors plays an essential role. Without this assumption we were not able to prove the required analogue of Corollary 21 that now follows.

Lemma 34. *Let $\mathfrak{M} = \langle W', R', V' \rangle$ be any $n+1$-model and let w_0 generate a top model in $\mathcal{U}(n)$. For any node w in $\mathcal{U}(n)_{w_0}$ and any node x in $\mathfrak{M}$, if $\mathfrak{M}, x \models \varphi'_w$, then there exists a unique point $v \in \mathcal{U}(n)_{w_0}$ satisfying*

$$\mathfrak{M}, x \models \varphi'_v,\ \mathfrak{M}, x \not\models \varphi'_{v_1}, \cdots, \mathfrak{M}, x \not\models \varphi'_{v_m}, \tag{15}$$

where $v \prec \{v_1, \cdots, v_m\}$, and wRv.

Proof. Suppose $\mathfrak{M}, x \models \varphi'_w$. We show that there exists $v \in R(w)$ satisfying (15) by induction on $d(w)$.

$d(w) = 1$. Then trivially $v = w$ satisfies (15).

$d(w) > 1$. If for all immediate successor w_i of w, $\mathfrak{M}, x \not\models \varphi'_{w_i}$, then w satisfies (15). Now suppose that for some immediate successor w_{i_0} of w, $\mathfrak{M}, x \models \varphi'_{w_{i_0}}$. Since $\mathfrak{M}, x \models \varphi'_{w_{i_0}}$ and $d(w_{i_0}) < d(w)$, by the induction hypothesis, there exists $v \in W$, such that $w_{i_0}Rv$ and v satisfies (15). And clearly, wRv.

Next, suppose $v' \in \mathcal{U}(n)_{w_0}$ also satisfies (15). By Theorem 33,

$$col_V(v') = col_{V_n}(x) = col_V(v),$$

which by the property of $\mathcal{U}(n)_{w_0}$ means that $v' = v$. □

Let $\mathfrak{F}$ be a finite rooted frame with a largest element x_0. By Lemma 25, there exists a model $\langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$ for some generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$. Obviously, $\mathcal{U}(n)_w$ has a top point t, and, by the proof of Lemma 25 we can assume that distinct points of $\mathcal{U}(n)_w$ have distinct colors, and that $t \models p_1 \wedge \cdots \wedge p_n$: $\mathcal{U}(n)_w$ is a top model. The next lemma is a modification of the Jankov-de Jongh Theorem (Theorem 26) proved in the previous section. Both the statement of the lemma and its proof are generalized from those of Theorem 26.

Lemma 35. *Let $\mathfrak{F}$ be a finite rooted frame $\mathfrak{F}$ with a largest element, and let $\mathcal{U}(n)_w$ be a top model with $\mathfrak{F}$ as its frame. Then for every descriptive frame $\mathfrak{G}$,*

$$\mathfrak{G} \not\models \psi'_w \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Proof. $\Leftarrow$: Let $\mathcal{U}(n)_w = \langle W, R, \mathcal{P}, V\rangle$. Suppose $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. By Theorem 16, $\mathcal{U}(n)_w \not\models \psi_w$, thus $\mathfrak{F} \not\models \psi_w$. By Lemma 30, we know in that case that $\mathfrak{F} \not\models \psi'_w$. Then $\mathfrak{G} \not\models \psi'_w$ follows immediately.

$\Rightarrow$: Suppose $\mathfrak{G} \not\models \psi'_w$. Then there exists a model $\mathfrak{N}$ on $\mathfrak{G}$ such that $\mathfrak{N} \not\models \psi'_w$. Consider the generated submodel $\mathfrak{N}' = \mathfrak{N}_{V'(\varphi'_w)} = \langle W', R', \mathcal{P}', V'\rangle$ of $\mathfrak{N}$. Since $V'(\varphi'_w)$ is admissible, by Lemma 9, $\mathfrak{N}'$ is descriptive. Define a map $f : W' \to W$ by taking $f(x) = v$ iff

$$\mathfrak{N}', x \models \varphi'_v,\ \mathfrak{N}', x \not\models \varphi'_{v_1}, \cdots, \mathfrak{N}', x \not\models \varphi'_{v_k}, \tag{16}$$

where $v \prec \{v_1, \cdots, v_k\}$.

Note that for every $x \in \mathfrak{N}'$, $\mathfrak{N}', x \models \varphi'_w$, thus by Lemma 34, there exists a unique $v \in R(w)$ satisfying (16). So f is well-defined.

We show that f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}'\rangle$ onto $\langle W, R, \mathcal{P}\rangle$. Suppose $x, y \in \mathfrak{N}'$ with $xR'y$, $f(x) = v$ and $f(y) = u$. Since $\mathfrak{N}', x \models \varphi'_v$, we have that $\mathfrak{N}', y \models \varphi'_v$. By Lemma 34, there exists a unique point $u' \in W$ such that u' and y satisfy (16), moreover vRu'. So, since u and y also satisfy (16), by the uniqueness, $u' = u$ and vRu.

Next, suppose $x \in \mathfrak{N}'$ and $v, u \in W$ such that $f(x) = v$ and vRu. We show that there exists $y \in \mathfrak{N}'$ such that $f(y) = u$ and $xR'y$.

The only interesting case to consider is $d(v) = 2$ and $u \neq v$. In this case $u = t$. Since $f(x) = v$, v and x satisfy (16), so

$$\mathfrak{N}', x \models \bigwedge prop(v) \wedge \Delta\, notprop(v) \wedge ((q \to r) \to q). \tag{17}$$

Note that

$$\vdash_{\mathbf{IPC}} ((q \to r) \to q) \to \neg\neg q.$$

Thus, $\mathfrak{N}', x \models \neg\neg q$, which means there exists $y \in W'$ such that $xR'y$ and $\mathfrak{N}', y \models q$. Since

$$\mathfrak{N}', y \models \bigwedge prop(v) \wedge \Delta\, notprop(v),$$

we have that $\mathfrak{N}', y \models p_1 \wedge \cdots p_n$, i.e. $f(y) = u$.

The surjectivity of f follows in the same way as in the proof of theorem 26.

By applying Lemma 32, Lemma 34 and using the same argument as that in the proof of Theorem 26, we can show that for every $v \in X$, $f^{-1}(R(v)) = V'(\varphi'_v)$, which is an admissible set. Therefore by Lemma 24, we obtain $f^{-1}(X) \in \mathcal{P}'$.

Hence, f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}'\rangle$ onto $\langle W, R, \mathcal{P}\rangle$. Then since $\mathfrak{F} \cong \langle W, R, \mathcal{P}\rangle$, $\mathfrak{F}$ is a p-morphic image of $\langle W', R', \mathcal{P}'\rangle$, which is a generated subframe of $\mathfrak{G}$. □

One may wonder how the formulas φ'_w, ψ'_w behave in the $n+1$-Henkin model. Let us make a remark about this without proof.

Remark 36. For any w in $\mathcal{U}(n)$ that generates a top model there exists a unique w' in $\mathcal{U}(n+1)$ with φ'_w true in w', ψ'_w false in w'. $R(w')$ consists of a copy of $R(w)$ with r false throughout with its top replaced by the Rieger-Nishimura ladder for r with $p_1, \ldots, p_n$ true everywhere. The p-morphism mapping $R(w')$ onto $R(w)$ is an isomorphism on the copy of $R(w)$ and maps the ladder onto the single top.

We are now ready to prove Jankov's theorem on **KC**, which shows that **KC** is the strongest extension of **IPC** that proves the same negation-free formulas as **IPC**.

Theorem 37 (Jankov). *If L is an intermediate logic such that $L \not\subseteq \mathbf{KC}$, then $L \vdash \theta$ and $\mathbf{IPC} \nvdash \theta$ for some negation-free formula θ.*

Proof. We follow the idea of the proof of Theorem 27. Suppose χ is a formula satisfying

$$L \vdash \chi \text{ and } \mathbf{KC} \nvdash \chi.$$

Then there exists a finite rooted **KC**-frame $\mathfrak{F}$ with a largest element such that $\mathfrak{F} \not\models \chi$. Using Lemma 25 as before we can stipulate a model $\langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$ for some top model $\mathcal{U}(n)_w$ in $\mathcal{U}(n)$.

Consider the formula ψ'_w. Suppose $L \nvdash \psi'_w$. Then there exists a descriptive frame $\mathfrak{G}$ of L such that $\mathfrak{G} \not\models \psi'_w$. By Lemma 35, $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Thus, $\mathfrak{F}$ is an L-frame. Since $L \vdash \chi$, we have that $\mathfrak{F} \models \chi$, which leads to a contradiction.

Hence, $L \vdash \psi'_w$. We have that $\mathbf{IPC} \nvdash \psi'_w$ by Corollary 31, and ψ'_w is negation-free, thus $\theta = \psi'_w$ is a formula as required. □

The above proof of this theorem can straightforwardly be generalized to prove a similar theorem for many intermediate logics L for which L and $L + \mathbf{K}C$ prove the same negation-free formulas.

Theorem 38. *If an intermediate logic L is complete with respect to a class of finite rooted frames which is closed under the operation of adding a top node, then $L + \mathbf{KC}$ is the strongest logic extending L that proves the same negation-free formulas as L.*

Proof. We first show that $L + \mathbf{KC}$ proves the same negation-free formulas as L, that is, we show that for any negation-free formula φ,

$$L \vdash \varphi \iff L + \mathbf{KC} \vdash \varphi.$$

It suffices to show the direction $\Longleftarrow$. Suppose $L \nvdash \varphi$, i.e. there exists a finite rooted model $\mathfrak{M}$ on a finite L-frame $\mathfrak{F}$ such that $\mathfrak{M} \not\models \varphi$. Now, construct a new model $\mathfrak{M}'$ by adding a new top node t to $\mathfrak{M}$ and making every propositional variable true at t. By the same argument as that in the proof of Theorem 28, it can be shown that $\mathfrak{M}' \not\models \varphi$. By the assumption on L and the completeness of **KC**, the underlying frame of $\mathfrak{M}'$ is an $L + \mathbf{KC}$-frame. thus we have shown that $L + \mathbf{KC} \nvdash \varphi$.

Next, suppose L' is an intermediate logic such that $L' \not\subseteq L + \mathbf{KC}$. We will show that $L' \vdash \theta$ and $L + \mathbf{KC} \nvdash \theta$ for some negation-free formula θ. Let χ be a formula satisfying

$$L' \vdash \chi \text{ and } L + \mathbf{KC} \nvdash \chi.$$

Observe that

$$L \nvdash \bigwedge_{p \in Prop(\chi)} (\neg p \vee \neg\neg p) \to \chi,$$

where $Prop(\chi)$ is the set of all propositional variables occurring in χ. Then there exists a finite rooted n-model $\mathfrak{M}$ on an L-frame $\mathfrak{F}$ with root r such that $n = |Prop(\chi)|$,

$$\mathfrak{M}, r \models \bigwedge_{p \in Prop(\chi)} (\neg p \vee \neg\neg p) \text{ and } \mathfrak{M}, r \not\models \chi.$$

Let E be the set of all endpoints of $\mathfrak{M}$. It is not hard to see that the former of the above implies that the n-colors of points in E are all the same. Therefore, the model $\mathfrak{M}'$ obtained from $\mathfrak{M}$ by identifying all the points in E is a p-morphic image of $\mathfrak{M}$. Clearly, the underlying frame $\mathfrak{F}'$ of $\mathfrak{M}'$ is an $L + \mathbf{KC}$-frame and $\mathfrak{F}' \not\models \chi$.

Using Lemma 25 we can stipulate a model $\langle \mathfrak{F}', V \rangle$ on $\mathfrak{F}'$ such that $\langle \mathfrak{F}', V \rangle \cong \mathcal{U}(n)_w$ for some top model $\mathcal{U}(n)_w$ in $\mathcal{U}(n)$. Consider the formula ψ_w. We know that $\mathcal{U}(n)_w \not\models \psi_w$, thus the $L+\mathbf{KC}$-frame $\mathfrak{F}' \not\models \psi_w$, which means that $L+\mathbf{KC} \nvdash \psi_w$. It then follows from Lemma 30 that $L + \mathbf{KC} \nvdash \psi'_w$.

On the other hand, by an argument similar to that in the proof of Theorem 37, we can show that $L' \vdash \psi'_w$. Thus, ψ'_w is the required negation-free formula.

□

The above theorem applies to a number of well-known logics. In the first place, the logics complete w.r.t. finite frames with splittings less than $n + 1$ (introduced in [10] and called $\mathbf{T}_n$ in [6]) for $n > 1$. Further, to the logics that just restrict the width of frames (called $\mathbf{BW}_n$ in [6]). And also to the Kuznetsov-Gerciu logic **KG** and extensions of this logic with the right properties like the Rieger-Nishimura logic **RN** (see [2]). All these logics of course prove negation-free formulas that are not provable in **IPC**, in fact most of them are axiomatized by such formulas.

It is further to be noted that in the above proof it is shown that, if L is a complete logic, then $L + \mathbf{KC}$ is complete as well. The restriction to finite frames is not essential. As far as we know this result is new.

6 Some properties of $\mathcal{U}(n)$ and $\mathcal{H}(n)$

In this section we conclude in Theorem 39 the almost finished proof of section 2 that $\mathcal{U}(n)$ is isomorphic to the upper part of $\mathcal{H}(n)$. After that, we sharpen this result by giving a quick proof that these two models are even more "connected": every infinite upset of $\mathcal{H}(n)$ has an infinite intersection in $\mathcal{U}(n)$, or in other words, if an upset X generated by a point in the n-Henkin model has a finite intersection with its upper part, the n-universal model, then X lies completely in $\mathcal{U}(n)$. Both results were proved before in [4].

Theorem 39. *$Upper(\mathcal{H}(n))$ is isomorphic to $\mathcal{U}(n)$.*

Proof. Let $\mathcal{U}(n) = \langle U(n), R, V\rangle$. Define a function $f : \mathcal{U}(n) \to Upper(\mathcal{H}(n))$ by taking

$$f(w) = Cn(\varphi_w).$$

We show that f is an isomorphism. From the proof of Lemma 20 we know that

$$\mathcal{U}(n)_w \cong Upper(\mathcal{H}(n))_{f(w)}.$$

It then suffices to show that f is a bijection.

Let w, v be two distinct points of $\mathcal{U}(n)$. W.l.o.g. we may assume that $\neg wRv$, thus by Theorem 16, $\mathcal{U}(n), w \models \varphi_w$ but $\mathcal{U}(n), v \not\models \varphi_w$. We know from the proof of Lemma 20 that

$$\mathcal{U}(n)_w \cong Upper(\mathcal{H}(n))_{f(w)} \text{ and } \mathcal{U}(n)_v \cong Upper(\mathcal{H}(n))_{f(v)},$$

thus $Upper(\mathcal{H}(n))_{f(w)} \ncong Upper(\mathcal{H}(n))_{f(v)}$, so $f(w) \neq f(v)$.

For any point x in $Upper(\mathcal{H}(n))$, by Lemma 13, there exists a unique w_x such that $\mathcal{U}(n)_{w_x}$ is a p-morphic image of $Upper(\mathcal{H}(n))_x$, which by Corollary 19 implies that

$$Th(x) = Th(w_x) = Cn(\varphi_{w_x}),$$

therefore $f(w_x) = x$. □

We call $w \in X$ a *border point* of an upset X of $\mathcal{U}(n)$, if $w \notin X$ and all successors v of w with $v \neq w$ are in X. Denote the set of all border points of X by $B(X)$. An upset X is uniquely characterized by its set of border points. Note that all endpoints $\mathcal{U}(n)$ which are not in X are in $B(X)$. The concept of border point was developed and studied in [5].

Fact 40 *If X is finite, then $B(X)$ is also finite.*

Proof. Since X is finite, there exists $k \in \omega$ such that $X \subseteq U(n)^k$. Observe that $B(X) \subseteq U(n)^{k+1}$, which means that $B(X)$ is finite, since $U(n)^{k+1}$ is finite. □

The next lemma shows the syntactic side of the connection of upsets and their border points.

Lemma 41. *If $X = \{v_1, \cdots, v_k\}$ is a finite anti-chain in $\mathcal{U}(n)$ and $B(\mathcal{U}(n)_X) = \{w_1, \cdots, w_m\}$, then $\vdash_{\mathbf{IPC}} (\varphi_{v_1} \vee \cdots \vee \varphi_{v_k}) \leftrightarrow (\psi_{w_1} \wedge \cdots \wedge \psi_{w_m})$.*

Proof. In view of Theorem 14, it is sufficient to show that $\mathcal{U}(n) \models (\varphi_{v_1} \vee \cdots \vee \varphi_{v_k}) \leftrightarrow (\psi_{w_1} \wedge \cdots \wedge \psi_{w_m})$. By Theorem 16, it is then sufficient to show that

$$x \in R(v_1) \cup \cdots \cup R(v_k) \text{ iff } x \notin R^{-1}(w_1) \cup \cdots \cup R^{-1}(w_m).$$

For $\Rightarrow$: Suppose $x \in R(v_1) \cup \cdots \cup R(v_k) = U(n)_X$. If $x \in R^{-1}(w_i)$ for some $1 \leq i \leq m$, then since $U(n)_X$ is upward closed, we have that $w_i \in U(n)_X$, which contradicts the definition of $B(\mathcal{U}(n)_X)$.

For $\Leftarrow$: Suppose $x \notin R(v_1) \cup \cdots \cup R(v_k) = U(n)_X$. We show by induction on $d(x)$ that $x \in R^{-1}(w_i)$ for some $1 \leq i \leq m$.

$d(x) = 1$. Then x is an endpoint which is a border point. Thus, $x = w_i$ for some $1 \leq i \leq m$ and so $x \in R^{-1}(w_i)$.

$d(x) > 1$. The result holds trivially if x is a border point. Now suppose there exists $y \in R(x)$ such that $y \notin U(n)_X$. Since $d(y) < d(x)$, by the induction hypothesis, there exists $1 \leq i \leq m$ such that $y \in R^{-1}(w_i)$. Thus, $x \in R^{-1}(w_i)$. □

Theorem 42. *Let Γ be a point in $\mathcal{H}(n)$, i.e. Γ is an n-theory with the disjunction property. If $R(\Gamma) \cap \mathcal{U}(n)$ is finite, then $R(\Gamma) = R(\Gamma) \cap \mathcal{U}(n)$.*

Proof. Suppose $X = R(\Gamma) \cap \mathcal{U}(n)$ is finite. Then the set $B(X)$ of border points of X is finite. Let $B(X) = \{w_1, \cdots, w_m\}$. Suppose $\Gamma \nvdash \psi_{w_i}$ for some $1 \leq i \leq m$. Then there exists a descriptive frame $\mathfrak{G}$ such that $\mathfrak{G} \models \Gamma$ and $\mathfrak{G} \not\models \psi_{w_i}$. Since the underlying frame $\mathfrak{F}$ of $\mathcal{U}(n)_{w_i}$ is finite rooted, by Theorem 26, the latter implies that $\mathfrak{F}$ is a p-morphic image of a generated submodel of $\mathfrak{G}$. Thus, $\mathfrak{F} \models \Gamma$ and so $\mathcal{U}(n)_{w_i} \models \Gamma$, which is impossible since $w_i \in B(X)$ and $w_i \notin R(\Gamma) \cap \mathcal{U}(n)$.

Hence, we conclude that $\Gamma \vdash \psi_{w_i}$ for all $1 \leq i \leq m$. Let Y be the antichain consisting of all least points of X. Then by Lemma 41, $\Gamma \vdash \varphi_w$ for some $w \in Y$, which by Theorem 16 means that $\Gamma \in R(w)$, so $\Gamma \in \mathcal{U}(n)$, therefore $R(\Gamma) = R(\Gamma) \cap \mathcal{U}(n)$. □

Corollary 43. *Every infinite upset of $\mathcal{H}(n)$ has an infinite intersection with $\mathcal{U}(n)$.*

Proof. Let X be an infinite upset of $\mathcal{H}(n)$. Note that

$$X = \bigcup_{i \in I} R(\Gamma_i)$$

for some set $\{\Gamma_i\}_{i \in I}$. There are two cases.

Case 1: for all $i \in I$, $R(\Gamma_i)$ is finite, i.e. $d(\Gamma_i) < \omega$. Thus, each $R(\Gamma_i)$ lies in $Upper(\mathcal{H}(n)) = \mathcal{U}(n)$ by Theorem 39, therefore $X \cap \mathcal{U}(n) = X$ is infinite.

Case 2: there exists $i_0 \in I$ such that $R(\Gamma_{i_0})$ is infinite. Then $R(\Gamma_{i_0}) \cap \mathcal{U}(n)$ is infinite, since otherwise by Theorem 42 we would have that $R(\Gamma_{i_0}) = R(\Gamma_{i_0}) \cap \mathcal{U}(n)$, which would make $R(\Gamma_{i_0})$ finite. Hence, we have that $X \cap \mathcal{U}(n) \supset R(\Gamma_{i_0}) \cap \mathcal{U}(n)$ is infinite. □

7 Concluding remarks

Study of the n-universal model turned out to shed new light on the negationless fragment of **IPC** and enabled us to give a new proof of Jankov's theorem on the relationship of this fragment with the logic **KC** and to generalize this theorem to a large class of extensions of **IPC**. We expect further results stemming from the study of n-universal models. In the first place we intend to study such models for NNIL-formulas, formulas with no nesting of implications on the left (see [15] for details on NNIL-formulas). This study was already initiated in [16] in connection with results of [4] that show that these formulas are an alternative to $[\wedge, \rightarrow]$-formulas for axiomatizing subframe logics. The newest results on subframe logics as axiomatized by $[\wedge, \rightarrow]$-formulas can be found in [3]. Another promising area is the study of Zakharyaschev's canonical formulas in the context of n-universal models. A recent algebraic approach can be found in [1] which also stresses fragments of **IPC**.

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