Intuitionistic Logic
Spring 2012

Homework 5
(due Monday, 2nd April)

1. (a) Show that the following is valid: If $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to (\chi \vee \theta)$, then $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \chi$ or $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \theta$ or $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \varphi$. [3 pts]

   (b)* Give an example such that the first two alternatives of (a) do not apply, but the last one does. [2 pts]

   (c) Show, using the above, that, if $\vdash (\neg p \to q \vee r) \to \varphi \vee \psi$, then $\vdash (\neg p \to q \vee r) \to \varphi$ or $\vdash (\neg p \to q \vee r) \to \psi$. [2 pts]

2. (a) Let φ be a propositional formula containing only $\wedge, \to$ and $\bot$.
   Let $\mathfrak{M}$ be a model and w a node in $\mathfrak{M}$ such that w has proper successors (i.e., there is at least one v in $\mathfrak{M}$ with wRv and $w \neq v$). Suppose

   - φ holds in all proper successors of w (i.e., for all v with $wRv, w \neq v$, we have $\mathfrak{M}, v \models \varphi$), and
   - for all propositional variables p, we have that p is true in w iff p is true in all proper successors of w (i.e., $w \in V(p)$ iff $\forall v\ (wRv, w \neq v \Rightarrow v \in V(p))$.

     (In other words, the valuation in w is maximal for propositional variables considering persistency).

   Show that φ is true in w. [4 pts]

   (b) Show on the basis of the above that, if φ is a propositional formula not containing $\vee$ and $\vdash_{\mathbf{IPC}} \varphi \to \psi \vee \chi$, then $\vdash_{\mathbf{IPC}} \varphi \to \psi$ or $\vdash_{\mathbf{IPC}} \varphi \to \chi$. [2 pts]

3. (a) Show that, if $\Gamma \,|\, \varphi$, then $\Gamma \vdash \varphi$ [1]. [1 pt]

   (b) Show that, if $\Gamma \vdash \delta$ for each $\delta \in \Delta$, and $\Delta \vdash \gamma$ for each $\gamma \in \Gamma$ (Γ and Δ are equivalent), then $\Gamma \,|\, \varphi$ iff $\Delta \,|\, \varphi$. [1 pt]

   (c) Show that, if $\vdash \chi \to \varphi \vee \psi$ implies $\vdash \chi \to \varphi$ or $\vdash \chi \to \psi$, for all subformulas $\varphi \vee \psi$ of χ, then $\chi \,|\, \chi$. (Hint: Show first that, under this condition, $\chi \,|\, \theta$ iff $\chi \vdash \theta$, for all subformulas θ of χ). [3 pts]

[1] erratum: Add to the definition 21 of $|$ (p. 25) the clause, $\Gamma \,|\, \bot$ iff $\Gamma \vdash \bot$.