Intuitionistic Logic
Spring 2012

Homework 9
(due Thursday, 31 May)

Exercise 1.

1. Show that φ has the disjunction property (i.e., if $\vdash \varphi \to \chi \vee \theta$, then $\vdash \varphi \to \chi$ or $\vdash \varphi \to \theta$) iff, for each two models $\mathfrak{M} \models \varphi$ and $\mathfrak{M}' \models \varphi$ there exists a rooted model $\mathfrak{N} \models \varphi$ such that $\mathfrak{M}$ and $\mathfrak{M}'$ are generated submodels of $\mathfrak{N}$.

   Hint: For the difficult direction one cannot simply put a new root below $\mathfrak{M}$ and $\mathfrak{M}'$, that is to say, one can, but then one has to add a third model above that new root. For the identity of that third model you can ask for another hint. [4 pts]

2. Show that φ has the disjunction property iff, for each two finite models $\mathfrak{M} \models \varphi$ and $\mathfrak{M}' \models \varphi$ there exists a finite rooted model $\mathfrak{N} \models \varphi$ such that $\mathfrak{M}$ and $\mathfrak{M}'$ are generated submodels of $\mathfrak{N}$.

   Hint: Like before, but do exercise (a) first. [4 pts]

3. Show that n-formula φ has the disjunction property iff, for each two elements $u \models \varphi$ and $v \models \varphi$ in $\mathcal{U}(n)$ there exists $w \in \mathcal{U}(n)$ such that wRu and wRv and $w \models \varphi$. [4 pts]

Exercise 2.

Let us call w an *intersection node* in a model $\mathfrak{M}$ if w has proper successors and $w \models p$ for a propositional variable p iff $u \models p$ for all proper successors of u.

Let φ contain only $\wedge, \bot, \to$ (φ is *disjunction free*). Prove the following:

For any finite $\mathfrak{M}$, if we define $\overline{\mathfrak{M}}$ to the submodel of $\mathfrak{M}$ based on the set of its elements that are not intersection nodes, then for all u in the domain of $\overline{\mathfrak{M}}$, $\overline{\mathfrak{M}}, u \models \varphi$ iff $\mathfrak{M}, u \models \varphi$.

*N*ote: The submodel is not a generated submodel of course. [4 pts]

Exercise 3.

1. Extend Corollary 21 of deJongh-Yang by proving the following statement (using the previous results of the paper):

   Moreover, if

   $$\mathfrak{M}, x \models \varphi_w,\ \not\models \varphi_{w_1}, \cdots \not\models \varphi_{w_l},$$

   then, for each u such that wRu in $\mathcal{U}(n)$, there exists y with $xR'y$ such that

   $$\mathfrak{M}, y \models \varphi_u,\ \not\models \varphi_{u_1}, \cdots \not\models \varphi_{u_s}$$

   Here $\{w_1, \dots, w_l\}$ are the immediate successors of w and $\{u_1, \dots, u_s\}$ the immediate successors of u.

   Hint: Try to express this by one formula. [4 pts]

2. Show that, if

   $$\mathfrak{M}, x \models \varphi_w,\ \not\models \varphi_{w_1}, \cdots \not\models \varphi_{w_l},$$

   then there exists a p-morphism from $\mathfrak{M}_x$ onto $\mathcal{U}(n)_w$. [4 pts]