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Tentamen *Chaotic Dynamical Systems*

1.[20%]

Let T_2 be the tent map on $[0,1]$,

$$T_2(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2-2x, & 1/2 \leq x \leq 1. \end{cases}$$

(a) Give the definition, from Devaney, of the statement "T_2 is chaotic on $[0,1]$";

T_2 is chaotic on $[0,1]$ if the following properties hold

i. periodic points are dense in $[0,1]$;

ii. T_2 shows sensitive dependence on initial conditions (there exists $d>0$ so that for $x \in [0,1]$ and U an open neighborhood of x, there exists $y \in U$, $n>0$ with $|T_2^n(x) - T_2^n(y)| > d$);

iii. T_2 is topologically transitive (for every relatively open U,V there is $n>0$ with $T_2^n(U) \cap V \neq \emptyset$);

(b) Prove that T_2 is chaotic on $[0,1]$.

Note that $[0,1]$ is partitioned into 2^n intervals P_i^n of length $1/2^n$ on which T_2 is monotone and maps onto $[0,1]$. On each interval P_i^n there is a point $T_2^n(y) = y$ (where the graph of T_2^n intersects the diagonal): it follows that periodic points are dense. Write $P_i^n(x)$ for the interval that contains x. Since $T_2^n(P_i^n(x)) = [0,1]$, in each $P_i^n(x)$ there exists y with $|T_2^n(y) - T_2^n(x)| > 1/2$. Sensitive dependence on initial conditions follows with $d = 1/2$. Finally for topological transitivity: each open U contains some P_i^n for n high enough, so that $T_2^n(U) = [0,1]$ intersects each open set V.

2.[30%]

Consider the family of maps

$$A_\lambda(x) = \lambda \arctan(x)$$

on the real line.

(a) Identify the bifurcation that occurs at $x=0$, $\lambda = -1$;

Period doubling bifurcation.

(b) Examine existence and stability of fixed and period points near $x=0$, for λ near -1;

Consider the maps for x near 0 and λ near -1. The origin is a fixed point for all λ and has derivative -1 for $\lambda=-1$. Note that $A_\lambda'(0) = \lambda$ so that the origin is a stable fixed point for $\lambda > -1$ and unstable for $\lambda < -1$. A calculation shows $\frac{d^3}{dx^3}(A_{-1} \circ A_{-1})(0) = -4$. Consequently, by the period doubling bifurcation theorem, there is an attracting period two orbit if the fixed point at the origin is unstable. (This is clear from a picture of $A_{-1} \circ A_{-1}(x) = x - \frac{4}{6}x^3 + O(x^4)$, just consider what $A_\lambda \circ A_\lambda$ then looks like for λ near -1.)

(c) Sketch the bifurcation diagram.

3.[20%]

Let $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ be the set of two-sided sequences of 0's and 1's.

(a) Denote $\mathbf{s} = (s_i)_{i\in\mathbb{Z}}$, $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$. Prove that

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}$$

is a metric on Σ_2;

(b) Let Σ_2 be endowed with this metric. Prove that periodic points are dense for the shift operator σ on Σ_2.

Let $\omega \in \Sigma_2$ and take a small ball B of radius ε around it. For N large enough, every sequence η with $\eta_n = \omega_n$ if $-N \le n \le N$, is contained in B. So is the period sequence η with $\eta_i = \omega_j$ with $j = i + k(2N + 1)$ such that $-N \le j \le N$. Hence periodic points lie dense.

4.[30%]

Let L_A be a hyperbolic torus automorphism on the two-torus $\mathbb{T}^2$, corresponding to the matrix $A \in GL(2, \mathbb{Z})$.

(a) Prove that periodic points are dense in $\mathbb{T}^2$;

Note that a point $(p/q, r/q)$, p, q, r integers, is mapped, by L_A, to another point $(P/q, R/q)$ with P, R integers. Since L_a is invertible, all these points with rational coordinates and denominator q are permuted. Each such point is therefore mapped back to itself by some iterate of L_A. As this holds for all q, periodic points are dense.

(b) Let $(x, y) \in \mathbb{T}^2$. Prove that both the stable and unstable manifold of (x, y) is dense in $\mathbb{T}^2$;

The unstable manifold of (x, y) is the projection (to the torus) of the line through (x, y) that is parallel to the unstable eigenvector of A (this line is not horizontal or vertical). Let y_n denote the y-coordinates of the intersections of this line with the verticals $x = n$. Then $y_{n+1} - y_n$ is irrational. Indeed, if $y_{n+1} - y_n = p/q$, then $y_{n+q} - y_n = p$ and $(n+q, y_{n+q})$ would correspond to the same point on the torus as (n, y_n). This is not possible as the unstable manifold would then be closed and could not get longer under iteration by L_A. We know that for irrational circle rotations orbits are dense (there is no need to also prove this) and thus the points $y_n \mod 1$ lie dense in $[0, 1]$. The unstable manifold of (x, y) lies therefore dense in the torus. The stable manifold is dense by a similar argument, or by noting that it is the unstable manifold of $L_{A^{-1}}$.

(c) Prove that L_A is topologically transitive.

Let U, V be open sets. Take a periodic point $(x, y) \in U$ of period k. Iterates of U under L_A^k are open sets that contain longer and longer pieces of the unstable manifold of (x, y) (which is fixed under L_A^k). Since the unstable manifold is dense, some iterate will intersect V.