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Deeltoets *Chaotic Dynamical Systems*
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1.[25%]

(a) Suppose f is a diffeomorphism on the real line. Prove that all hyperbolic periodic points are isolated.

(b) Show via an example that hyperbolic periodic points need not be isolated for smooth maps on the line.

Let p be a hyperbolic periodic point. Note that f is either orientation preserving or reversing. If f is orientation preserving, it can have only fixed points. Since f^2 is orientation preserving, the period of p is at most 2. By replacing f by f^2 if necessary we may assume p is a fixed point, and in fact any periodic point is a fixed point. We have $f'(p) \neq 1$. By the inverse function theorem applied to $g(x) = f(x) - x$, p is an isolated solution of $f(x) - x = 0$.

For $f_\mu(x) = \mu x(1-x)$ with $\mu > 2 + \sqrt{5}$ we know that $|f'| > 1$ on $[0,1] \cap f^{-1}([0,1])$. Moreover, the maximal invariant set of f_μ in $[0,1]$ is a Cantor set with dense periodic points. All of these periodic orbits are hyperbolic.

2.[25%]

A point p is recurrent for f if, for any open interval J about p, there exists $n > 0$ such that $f^n(p) \in J$.

(a) Show the existence of a non-periodic recurrent point for $f_\mu(x) = \mu x(1-x)$ when $\mu > 2 + \sqrt{5}$.

(b) Show the existence of a non-wandering point for f_μ which is not recurrent.

Recall that f_μ on the maximal invariant set in $[0,1]$ is topologically conjugate to the left shift on $\{0,1\}^{\mathbb{N}}$. Recall also that this admits points with dense orbits. These points are recurrent and not periodic.

Take $p = 1$: this point is mapped onto the fixed point at 0 and hence not recurrent. Every point in the maximal invariant set in $[0,1]$, in particular p, is a nonwandering point: any open interval U containing p is eventually mapped onto $[0,1]$: $[0,1] \subset f^n(U)$ for n large.

3.[25%]

Construct an interval map that has periodic points of period 2^j, $j < l$ but not period 2^l.

Take a map that is a contraction, and thus possesses a global attracting fixed point, and apply the doubling construction (see Figure 10.8 in the book) $l-1$ times. This yields a continuous interval map with periodic points of all periods 2^j, $j < l$.

4.[25%]

The Schwarzian derivative of a function f at x is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Prove that if f has negative Schwarzian derivative, also the n-fold composition f^n has negative Schwarzian derivative, for each $n > 0$.

A calculation, see the proof of Proposition 11.3 in the book, shows $Sf \circ g < 0$ if $Sf < 0$ and $Sg < 0$. By induction $Sf^n < 0$ if $Sf < 0$.