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Deeltoets *Chaotic Dynamical Systems*
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1.[25%] Let Σ_2 denote the space of sequences $(s_j)_{j\in\mathbb{N}}$ with $s_j = 0$ or 1, as usual endowed with the product topology. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the (left) shift map. Let Σ' consist of all sequences in Σ_2 satisfying: if $s_j = 0$ then $s_{j+1} = 1$.

(a) Show that σ preserves Σ'.

(b) Show that periodic points of $\sigma|_{\Sigma'}$ are dense in Σ'.

(c) Find and prove a recursive formula for the number of fixed points of σ^n in terms of the number of fixed points of σ^{n-1} and σ^{n-2}.

The forbidden code in Σ' is 00. The shift of a code without 00 obviously does not contain 00 and so lies in Σ'. To prove that periodic points lie dense, take any sequence $t \in \Sigma'$ and let U be a neighborhood of it. We must find a periodic point in U, which the following argument achieves. For n large enough, any code from Σ' that starts with $t_0 \ldots t_n$ is in U. Now take for s the periodic code that repeats the block $t_0 \ldots t_n 1$ (the symbol 1 is included to avoid the possibility that $t_n t_0 = 00$). Finally, to obtain the recursive formula, recall that the number N_n of fixed points of σ^n equals the trace of A^n with $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $A^2 = A^1 + A^0$ and by induction $A^{n+2} = A(A^{n+1}) = A(A^n + A^{n-1}) = A^{n+1} + A^n$ for $n \geq 1$. So $N_{n+2} = N_{n+1} + N_n$.

2.[25%]

(a) Prove that two different lifts of a continuous circle map must differ by an integer.

(b) Prove that if F is a lift of a continuous circle map f, then so is $F(x) + k$ where $k \in \mathbb{Z}$.

Write f for the circle map on $\mathbb{R}/\mathbb{Z}$, let π be the canonical projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$; $\pi(x) = x \mod 1$. A lift F satisfies $\pi \circ F = f \circ \pi$. If F and G are two lifts, then $\pi \circ F = \pi \circ G$, i.e. $F(x) = G(x) \mod 1$. So $F(x)$ and $G(x)$ differ an integer, and since F and G are continuous this integer is the same for all x. For the second part, if F is a lift then $F(x) + k \mod 1 = F(x) \mod 1$, so $F(x) + k$ is a lift as well.

3.[25%] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. A point p is recurrent for f if, for any open interval J about p, there exists $n > 0$ such that $f^n(p) \in J$. Consider the map $f_\mu(x) = \mu x(1-x)$ when $\mu > 2 + \sqrt{5}$ and let Λ_μ denote the maximal invariant set of f_μ in $[0, 1]$. Show the existence of a point in Λ_μ that is not recurrent.

Take $p = 1$, then $f^n(p) = 0$ for all $n \geq 1$ and p lies in Λ_μ and is not recurrent. Several variants of this observation are possible.

4.[25%] Let $f_\lambda : \mathbb{R} \to \mathbb{R}$ be a smooth one-parameter family of functions with $f_{\lambda_0}(0) = 0$ and $f'_{\lambda_0}(0) \neq 1$.

(a) Show that there is a neighborhood N of λ_0 and a curve $p : N \to \mathbb{R}$ with $p(\lambda_0) = 0$ and $f_\lambda(p(\lambda)) = p(\lambda)$ for $\lambda \in N$.

(b) Give an expression for $p'(\lambda_0)$.

Consider the equation $G(\lambda, x) = f_\lambda(x) - x$. We seek solutions to $G(\lambda, x) = 0$ near $G(\lambda_0, 0) = 0$. Since $\frac{\partial}{\partial x} G \neq 0$ in $(\lambda_0, 0)$ by the assumption, the implicit function theorem gives the function p so that $G(\lambda, p(\lambda)) = 0$. Differentiate $\frac{d}{d\lambda} G(\lambda, p(\lambda)) = \frac{\partial}{\partial \lambda} f_\lambda(p(\lambda)) + \left[f'_\lambda(p(\lambda)) - 1\right] p'(\lambda) = 0$. Hence $p'(\lambda_0) = -\frac{\partial}{\partial \lambda} f_\lambda(0) / \left[f'_\lambda(0) - 1\right]$ calculated in $\lambda = \lambda_0$.