1.[25%]

A map $f : J \to J$ is expansive if there exists $\nu > 0$ such that, for any $x, y \in J$, $x \neq y$, there exists $n > 0$ so that $|f^n(x) - f^n(y)| > \nu$.

(a) Show that the circle doubling map $f(x) = 2x \mod 1$ on $\mathbb{R}/\mathbb{Z}$ is expansive.

(b) Is the logistic map $f(x) = 4x(1 - x)$ expansive?

2.[25%]

Let σ be the shift map on the space $\Sigma_N = \{1, \ldots, N\}^{\mathbb{N}}$, endowed with the product topology.

(a) How many periodic points of some period k does σ have in Σ_N?

(b) Show that there exists a dense orbit for σ.

3.[25%]

Let T_2 be the tent map on $[0, 1]$,

$$T_2(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2 - 2x, & 1/2 \leq x \leq 1 \end{cases}.$$

Prove that T_2 is chaotic on $[0, 1]$.

4.[25%]

Discuss the phase portrait and the bifurcation diagram for $S_a(x) = (1 + a^2)x - x^3$, a near 0.