PROJECTS CDS

Project 1, Equidistributions: Look at the first digits of 2^n, $n \in \mathbb{N}$:

$$\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{1}6, \mathbf{3}2, \mathbf{6}4, \mathbf{1}28, \mathbf{2}56, \mathbf{5}12, \mathbf{1}024, \ldots$$

What is the distribution of these first digits? It turns out that the fraction of the first digit k in $1, \ldots, 2^N$ goes to $\log_{10}(\frac{k+1}{k})$ as $N \to \infty$. Another question: given a polynomial $p(x) = a_k x^k + \ldots + a_0$, what is the distribution of the numbers $p(n)$, $n \in \mathbb{N}$? Such questions can be answered by the study of dynamical systems, using the notion of equidistribution. Some knowledge of measure theory is required.

Literature:
M. Einsiedler, T. Ward, *Ergodic theory with a view towards number theory*, Springer Verlag, 2011. From this book: Example 1.3, Theorem 1.4, Sections 4.3, 4.4.

Assignment:
Prove the formula for the distribution of the first digits of 2^k. Consider also other powers a^k. Prove Weyl's result on the distribution of polynomial values over natural numbers.

Project 2, Julia sets: A project in complex dynamics focussing on Julia sets, the sets with nontrivial dynamics.

Literature:
R.L. Devaney, *An introduction to chaotic dynamical systems*, Westview Press, 2003. Sections 3.1-3.5.

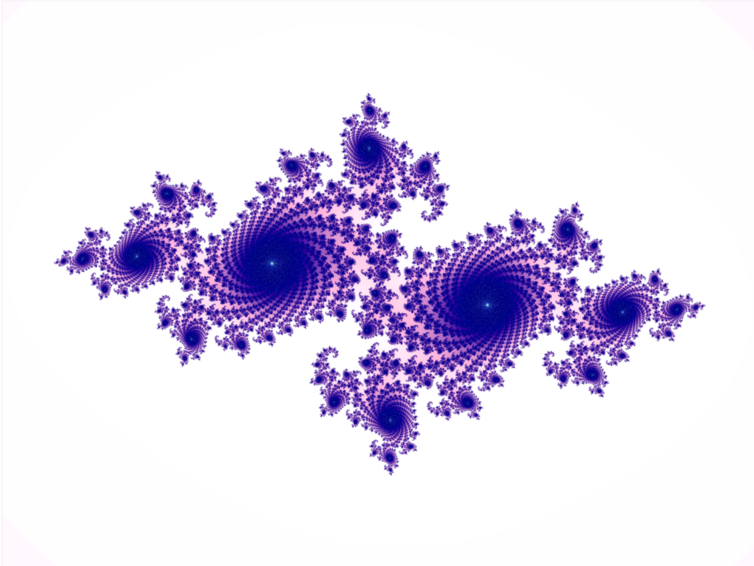
Assignment:
Prove equivalence of two definitions of Julia set in Devaney's book. What can you say about Julia sets of the logistic map $f_\mu(z) = \mu z(1-z)$?

Project 3, Fractal geometry: Fractals are sets with a fine structure, with detail on arbitrarily small scales, and an exact or approximate self-similarity. They can be assigned a fractal dimension that is often larger then its topological dimension. Cantor sets are examples of fractals.

Literature:
K. Falconer, *Fractal geometry. Mathematical foundations and applications*, Wiley, 1990. Section 13.2 on the logistic map, and earlier material for the definitions and results on dimensions.

Assignment:
Prove the dimension estimate for the invariant Cantor set of the logistic map $f_\mu(x) = \mu x(1-x)$, $\mu > 2 + \sqrt{5}$ given in Section 13.2 of Falconer's book.

Project 4, Circle dynamics: Diffeomorphism on the circle have a rotation number: the average rotation per iterate over large number of iterates. In this



Project 2: a Julia set.

project you will investigate the definition, the properties, and will discover a peculiar difference between C^1 diffeomorphisms and C^2 diffeomorphisms (or rather between C^1 diffeomorphisms whose derivatives do or do not have bounded variation).

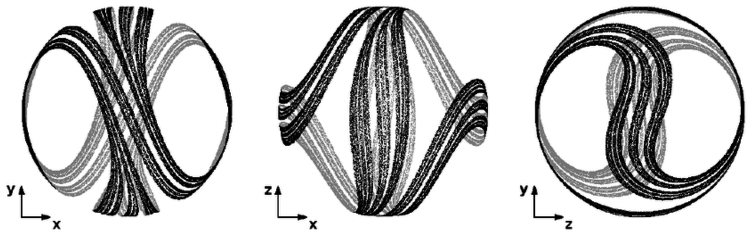
Literature:
R.L. Devaney, *An introduction to chaotic dynamical systems*, Westview Press, 2003. Section 1.14.
M. Brin, Stuck, *Introduction to dynamical Systems*, Cambridge University Press, 2002. Section 7.2

Assignment:
Prove Denjoy's theorem: A C^2 orientation preserving circle diffeomorphism with irrational rotation umber does not have wandering intervals, and there exists a C^1 orientation preserving circle diffeomorphism with irrational rotation number and with wandering intervals. In particular exercise 5 in section 1.14 of Devaney.

Project 5, Strange attractors: The notion of strange attractor was introduced by Floris Takens and David Ruelle in a paper "On the nature of turbulence". It turned out that attractors don't have to be fixed points, or periodic orbits, or manifolds, but can have a fractal appearance. This project studies a construction of strange attractors for diffeomorphsims on the torus (Derived from Anosov systems) and the sphere (Plykin attractor) and their relation.

Literature:
Y. Coudène, Ergodic Theory and Dynamical Systems, Springer Verlag, 2013. Chapter 9.
J. Palis, W. de Melo, Geometric theory of dynamical systems, Springer Verlag, 1982. Chapter4, section 4, examples 5,6 (pages 165–170).
Y. Coudène, Pictures of hyperbolic dynamical systems, *Notices Amer. Math. Soc.* **53**, 2006, `http://www.ams.org/notices/200601/fea-coudene.pdf`

Assignment:
Prove the existence of strange attractors for diffeomorphisms on the torus and on the sphere. Plykin's example is on a disk with three holes. What about more holes?



Project 5: A strange attractor.

Project 6, Sard's theorem: Sard's theorem gives information on the set of critical values of a map. It is a very useful result that is used a lot in differential topology and in dynamical systems.

Literature:
J. Milnor, *Topology from the differentiable viewpoint*, Princeton University Press, 1997.
S.N. Chow, J. Hale, *Methods of bifurcation theory*, Springer Verlag, 1982.
R. Devaney, *An introduction to chaotic dynamical systems*, Westview Press, 2003.

Assignment:
Prove Sard's theorem and apply to prove the Brouwer fixpoint theorem. Give a proof of Proposition 15.7 in Devaney by applying Sard's theorem.

Project 7, Time series analysis: A project with a practical component.

Literature:
H. Kantz, T. Schreiber, *Nonlinear time series analysis*, Cambridge University Press, 2000. Chapter 2.

Assignment:
Create two artificial time series. The first, $\{\eta_n, n = 1, \ldots, 4096\}$ contains uniformly distributed random numbers (use your favorite random number generator) in the interval $[0, 1]$. The second series, $\{s_n, n = 1, \ldots, 4096\}$, is based on the deterministic evolution of x_n which follows the rules $x_0 = 0.1$ and $x_{n+1} = 1 - 2x_n^2$. The values x_n are not measured directly but through the nonlinear observation function $s_n = \arccos(-x_n)/\pi$. Compare the mean, variance and the power spectra of the two time series. Explain.

Project 8, Bifurcation theory: This project focusses on the Hopf bifurcation in families of differential equations. In this bifurcation small amplitude oscillations are born from a steady state.

Literature:
Yu. A. Kuznetsov, *Elements of applied bifurcation theory*, Springer-Verlag, 2004.

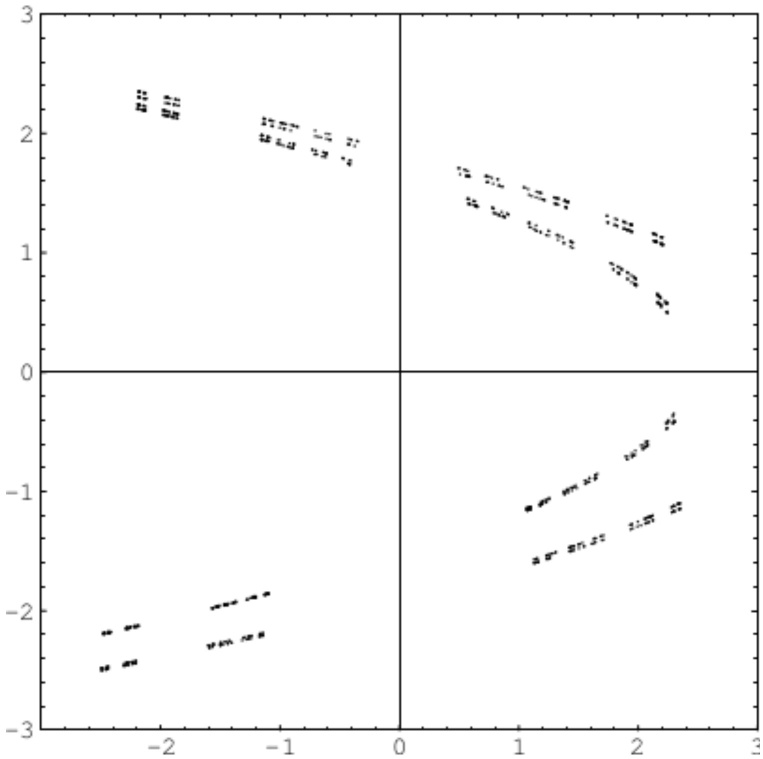
A. de Roos, *Modeling population dynamics*, reader, 2014,
`https://staff.fnwi.uva.nl/a.m.deroos/downloads/pdf_readers/syllabus.pdf`.
Section 6.1.2, 7.4.

Assignment:
Study the proof of the Hopf bifurcation theorem for two dimensional vector fields. Analyse the Hopf bifurcation in the Rosenzweig-MacArthur predator prey model (see the reader by de Roos), in particular determine super- or subcriticality.

Project 9, Homoclinic tangles: Horseshoes, i.e. dynamics on invariant sets that is topologically conjugate to the standard Smale horseshoe model, are abundant. They exists whenever a hyperbolic fixed point has transversally intersecting stable and unstable manifolds.

Literature:
M. Brin, Stuck, *Introduction to dynamical Systems*, Cambridge University Press, 2002. Section 5.8.

Assignment:
Prove the statement on horseshoes due to "transverse homoclinic orbits". Prove the existence of horseshoes in the Hénon map $H_{a,b}(x, y) = (a - by - x^2, x)$ for some suitable values of a, b.



Project 9: A horseshoe in the Hénon map.