Werkcollege 6

1. Find the stationary points of $f(x, y) = 2 - x^2 - xy - y^2$ and for each of those stationary points, specify whether it is a local minimum, local maximum or a saddle point.

$$f(x, y) = 2 - x^2 - xy - y^2$$

For this function

$$\begin{aligned} f_x &= -2x - y \\ f_y &= -x - 2y \\ f_{xx} &= -2 \\ f_{yy} &= -2 \\ f_{xy} &= -1 \end{aligned}$$

For stationary points, $-2x - y = 0$ and $-x - 2y = 0$ so again the only possibility is $(x, y) = (0, 0)$. We have

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

so that $(0, 0)$ is either a max or a min. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

2. Find the stationary points of $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$ and for each of those stationary points, specify whether it is a local minimum, local maximum or a saddle point.

Find the first partial derivatives f_x and f_y.

$f_x(x,y) = 4x + 2y - 6$

$f_y(x,y) = 2x + 4y$

The critical points satisfy the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ simultaneously. Hence.

$4x + 2y - 6 = 0$

$2x + 4y = 0$

The above system of equations has one solution at the point (2,-1).

We now need to find the second order partial derivatives $f_{xx}(x,y)$, $f_{yy}(x,y)$ and $f_{xy}(x,y)$.

$f_{xx}(x,y) = 4$

$f_{yy}(x,y) = 4$

$f_{xy}(x,y) = 2$

We now need to find D defined above.

$D = f_{xx}(2,-1)\, f_{yy}(2,-1) - f_{xy}^2(2,-1) = (\ 4\ )(\ 4\ ) - 2^2 = 12$

Since D is positive and $f_{xx}(2,-1)$ is also positive, according to the above theorem function f has a local minimum at (2,-1).

3. Find the stationary points of $f(x, y) = 4 + x^3 + y^3 - 3xy$ and for each of those stationary points, specify whether it is a local minimum, local maximum or a saddle point.

Example 1 Find and classify all the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$.

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We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let's get those.

$$= 3x^2 - 3y \qquad f_y = 3y^2 - 3x$$
$$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = -3$$

Let's first find the critical points. Critical points will be solutions to the system of equations,

$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$

This is a non-linear system of equations and these can, on occasion, be difficult to solve. However, in this case it's not too bad. We ca solve the first equation for y as follows,

$$3x^2 - 3y = 0 \quad \Rightarrow \quad y = x^2$$

Plugging this into the second equation gives,

$$3\left(x^2\right)^2 - 3x = 3x\left(x^3 - 1\right) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points.

$$x = 0: \quad y = 0^2 = 0 \quad \Rightarrow \quad (0, 0)$$
$$x = 1: \quad y = 1^2 = 1 \quad \Rightarrow \quad (1, 1)$$

So, we get two critical points. All we need to do now is classify them. To do this we will need D. Here is the general formula for D.

$$\begin{aligned} D(x, y) &= f_{xx}(x, y)\, f_{yy}(x, y) - \left[f_{xy}(x, y)\right]^2 \\ &= (6x)(6y) - (-3)^2 \\ &= 36xy - 9 \end{aligned}$$

To classify the critical points all that we need to do is plug in the critical points and use the fact above to classify them.

$(0, 0)$:

$$D = D(0, 0) = -9 < 0$$

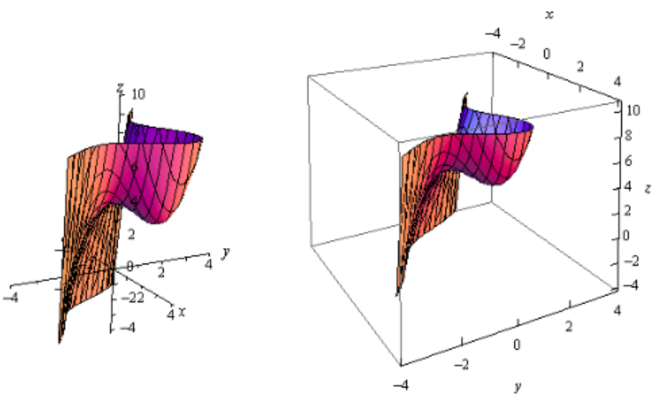
So, for $(0, 0)$ D is negative and so this must be a saddle point.

$(1, 1)$:

$$D = D(1, 1) = 36 - 9 = 27 > 0 \qquad f_{xx}(1, 1) = 6 > 0$$

For $(1, 1)$ D is positive and f_{xx} is positive and so we must have a relative minimum.

For the sake of completeness here is a graph of this function.



Notice that in order to get a better visual we used a somewhat nonstandard orientation. We can see that there is a relative minimum at $(1, 1)$ and (hopefully) it's clear that at $(0, 0)$ we do get a saddle point.

4. Find the stationary points of $f(x,y) = (x^2+y^2)e^{-y}$ and for each of those stationary points, specify whether it is a local minimum, local maximum or a saddle point.

Identify the local extrama of $f(x,y) = (x^2+y^2)e^{-y}$.

Solution

Step 1: Find the critical points.

The derivative of f is

$$Df(x,y) = \begin{bmatrix} 2xe^{-y} & (2y-x^2-y^2)e^{-y} \end{bmatrix}$$

$Df(x,y) = [0 \quad 0]$ means that $2x = 0$ and $2y - x^2 - y^2 = 0$, i.e., $x = 0$ and $y(2-y) = 0$.

The critical points are therefore $(0,0)$ and $(0,2)$.

Step 2: Classify the critical points.

The Hessian matrix is

$$Hf(x,y) = \begin{bmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & (2-4y+y^2+x^2)e^{-y} \end{bmatrix}$$

At the critical point $(0,0)$

$$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$h_{11} = 2 > 0$ and $\det(Hf) = 4 > 0$, so $(0,0)$ is a local mininum.

At the critical point $(0,2)$

$$Hf(0,2) = \begin{bmatrix} e^{-2} & 0 \\ 0 & -2e^{-2} \end{bmatrix}$$

$h_{11} = e^{-2} > 0$ and $\det(Hf) = -2e^{-4} < 0$ so $(0,2)$ is a saddle point.

There is a typo in the last Hessian matrix. The element h_{11} should be $2e^{-2}$ (instead of e^{-2}) and the determinant is then $-4e^{-4}$. This implies that $(0,2)$ is indeed a saddle point.

5. Find the stationary points of the following functions.

(a) $f(x_1,\ldots,x_n) = \sum_{i=1}^n e^{x_i^2}$

Answer. We have

$$\frac{\partial f}{\partial x_j} = 2x_j e^{x_j^2}$$

Since $e^{x_j^2} > 0$, we have

$$\frac{\partial f}{\partial x_j} = 0 \quad \text{iff} \quad x_j = 0$$

So there is only one stationary point which is $(0, \ldots, 0)$.

(b) $f(x_1, \ldots, x_n) = \ln\left(\prod_{i=1}^n x_i\right)$, where $x_1 > 0, \ldots, x_n > 0$

Answer.

We have

$$\ln\left(\prod_{i=1}^n x_i\right) = \Sigma_{i=1}^n \ln(x_i).$$

So

$$\frac{\partial f}{\partial x_j} = \frac{1}{x_j}$$

Since $x_j > 0$, then

$$\frac{1}{x_j} > 0$$

and $\frac{\partial f}{\partial x_j}$ is never equal to 0. So f does not admit any stationary point.

(c) $f(x_1, \ldots, x_n) = \sum_{i=2}^n x_1(x_i - 2)^2$

Answer. We have

$$\frac{\partial f}{\partial x_1} = \sum_{i=2}^n (x_i - 2)^2$$

Since $(x_i - 2)^2 > 0$, then $\frac{\partial f}{\partial x_1}$ is equal to 0 at a point $(p_1, \ldots, p_n)$ iff

$$p_2 - 2 = 0 \text{ and } p_3 - 2 = 0, \ldots \text{ and } p_n - 2 = 0.$$

So the first constraint $\frac{\partial f}{\partial x_1}(p_1, \ldots, p_n) = 0$ for stationary points is equivalent to

$$p_2 = 2, p_3 = 2, \ldots, p_n = 2 \tag{1}$$

Now if $j \neq 1$, we have

$$\frac{\partial f}{\partial x_j} = 2x_1(x_j - 2)$$

So if $j \neq 1$, then $\frac{\partial f}{\partial x_j}(p_1, \ldots, p_n) = 0$ iff

$$p_1(p_j - 2) = 0.$$

Since $p_j = 2$ (from (1)), then we already have that $p_1(p_j - 2) = 0$. So the stationary points are points of the form

$$(p_1, 2, \ldots, 2)$$

and the set of stationary points is given by

$$\{(p_1, 2, \ldots, 2) : p_1 \in \mathbb{R}\}$$

6. Find the Cartesian coordinates corresponding to the following polar coordinates:

- $(r, \theta) = (2, \pi)$
- $(r, \theta) = (1, \frac{\pi}{2})$

Solution.

- The cartesian coordinates are $(-2, 0)$.
- The cartesian coordinates are $(0, 1)$.

In both cases, it is easier to plot the points and derive directly the cartesian coordinates, instead of using the formula.

7. Find the polar coordinates corresponding to the following cartesian coordinates:

   - $(x, y) = (0, 2)$
   - $(x, y) = (1, 1)$

   Solution.

   - The polar coordinates are $(2, \frac{\pi}{2})$.
   - The polar coordinates are $(\sqrt{2}, \frac{\pi}{4})$.

8. Describe the following areas using polar coordinates:

   - $A = \{(x, y) \mid x^2 + y^2 \leq 16\}$
   - $A = \{(x, y) \mid x < 0, y > 0\}$
   - $A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 16, x > 0\}$

   Solution.

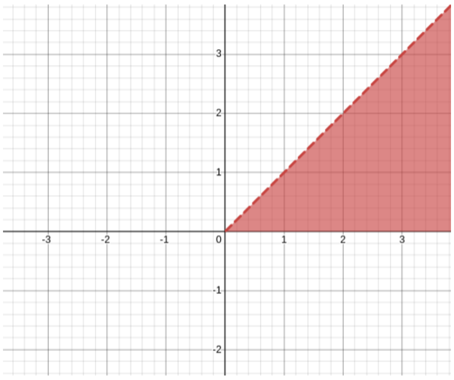
   - The area A contains all the points inside the circle of radius 4. It is described as follows in polar coordinates: $\{(r, \theta) \mid 0 \leq r \leq 4\}$.
   - The area A contains all the points in the second quadrant. It is described as follows in polar coordinates: $\{(r, \theta) \mid \frac{\pi}{2} < \theta < \pi\}$.
   - The area A contains all points in the first and second quadrants, that are between the circle of radius 4 and the circle of radius 1. It is described as follows in polar coordinates: $\{(r, \theta) \mid 1 \leq r \leq 4, 0 < \theta < \pi\}$.

9. Describe the following areas using cartesian coordinates:

   - $\{(r, \theta) \mid 0 \leq r \leq 3\}$
   - $\{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{4}\}$

   Solution.

   - The area A contains all the points inside the circle of radius 3. It is described as follows in cartesian coordinates: $\{(x, y) \mid x^2 + y^2 \leq 9\}$.
   - The area A is the following shaded area



It is described as follows in cartesian coordinates: $\{(x, y) \mid y \leq x, x \geq 0, y \geq 0\}$.