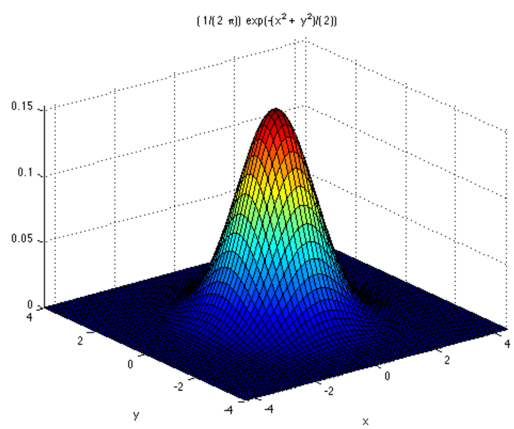
Werkcollege 5

- up to now, we were handling function $f : \mathbb{R} \to \mathbb{R}$ which maps a real number to another real number
- in this second part of the course, we consider functions with *multiple variables*, that is functions of the form $f : \mathbb{R}^n \to \mathbb{R}$

  such a function maps a tuple $(x_1, \ldots, x_n)$ to a real number
- it is more difficult to represent the graphs of such functions, but it is still possible when $n = 2$

  Example:



  Here, $f(x, y) = \frac{1}{2\pi} e^{\frac{-x^2 - y^2}{2}}$

  So each pair (x, y) of the 2D plane is mapped to $f(x, y)$, which is the “height” of the “hill” at the point of coordinate (x, y)
- *partial derivative* of a function $f(x, y)$ with respect to x: we “understand” y as a constant and we derive “normally” with respect to x
- example: partial derivative of $\cos(xy)$:

$$-y \sin(xy)$$

  Notation: $\frac{\partial \cos(xy)}{\partial x} = -y \sin(xy)$

  We use ∂ instead of a d to differientiate with the case of a function with one variable
- Similarly, we can define $\frac{\partial f}{\partial y}$
- we also use the notation $f_x(x, y)$ for $\frac{\partial f}{\partial x}$ and $f_y(x, y)$ for $\frac{\partial f}{\partial y}$
- the calculation rules are the same as in the case of derivation of a function with one variable (rule for sum, chain rule, etc)

- definition and notation for derivation of higher order

$$\frac{\partial}{\partial x}\left(\frac{f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} == f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} == f_{yy}(x,y)$$

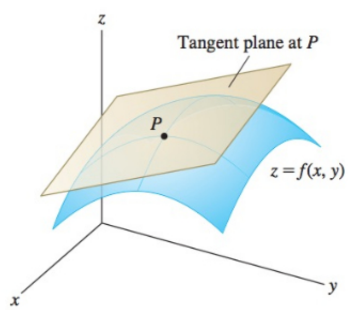
$$\frac{\partial}{\partial x}\left(\frac{f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}(x,y) = f_{yx}(x,y)$$

  We can see that the order in which we derive does not make a difference, more precisely $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

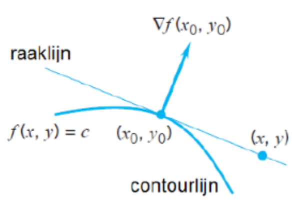
- the *gradient* $\nabla f(x,y)$ of a function with 2 variables is defined as the vector

$$\begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$$

  Interpretation: the vector $[f_x(a,b), f_y(a,b), -1]^T$ is the normal vector of the tangeant plane at the point $(a,b,f(a,b))$



  The gradient $\nabla f(a,b)$ is also the normal vector to the contour line[1] going through the point $(a,b,f(a,b))$, at the point $(a,b,f(a,b))$



- equation of the tangeant plane at the point $(a,b,f(a,b))$:

$$z = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$

  Note: since we know that $[f_x(a,b), f_y(a,b), 1]^T$ is a normal vector for that tangeant plane, this equation should be known from the Linear Algebra course

  Parametric equation of the tangeant plane at the point $(a,b,f(a,b))$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ f(a,b) \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ f_x(a,b) \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ f_y(a,b) \end{bmatrix}$$

[1] This is the line that contains all the points of the form $(x,y,f(x,y))$ with $f(x,y) = f(a,b)$. So if we look at the 3D plot of the function, we select all of the points of that surface that have "height" $f(a,b)$.

- Taylor serie of first order for a function $f(x, y)$ (with 2 variables) at the point (a, b):

$$f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$$

- Taylor series of second order for a function $f(x, y)$ (with 2 variables) at the point (a, b):

$$\begin{aligned} f(a, b) + \quad & (x - a)f_x(a, b) + (y - b)f_y(a, b) + \\ & \frac{1}{2!}((x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)) \end{aligned}$$