

MARKOV PARTITIONS

In Chapter 4 we mentioned the existence of Markov partitions for dynamically defined Cantor sets and their relation with Markov partitions for hyperbolic diffeomorphisms in two dimensions. Here, we shall discuss these points further: we construct partitions for basic sets of surface diffeomorphisms and apply them to get the corresponding structure for Cantor sets. We observe that Markov partitions also exist in higher dimensions. However, as we explain below, in two dimensions their construction is simpler (and they have nicer properties).

Let $\varphi: M \rightarrow M$ be a diffeomorphism on a compact 2-manifold and let Λ be a basic set for φ . That is, Λ is a compact, invariant, hyperbolic set which is transitive and contains a dense subset of periodic orbits; moreover, Λ has local product structure or, equivalently, Λ is the maximal invariant set in some neighbourhood of it (see Chapter 0). We suppose that Λ is not just a periodic orbit, in which case we say that Λ is *trivial*. There are three possibilities for Λ : either Λ is all of M and then M is a torus and φ is an Anosov diffeomorphism, or Λ is an attractor (repeller) or Λ is of saddle-type. The second possibility, Λ a hyperbolic attractor, which is locally the product of an interval and a Cantor set, includes Plykin's attractors on S^2 (see [GH, 1983]) and the 1-dimensional attractors in the DA (derived from Anosov) maps defined by Smale (see [S, 1967], [W, 1970]). Finally, the third possibility includes the horseshoes constructed in Chapter 2; in our context of homoclinic bifurcations, this is the most relevant case. Also, in what follows, we are going to assume that Λ is *topologically mixing*; this means that for any two open sets U, V , in Λ , we have $\varphi^n(U) \cap V \neq \emptyset$ for all sufficiently big n ; this implies that all nonzero powers of φ are topologically transitive. Our assumption is justified by the fact that we can always decompose Λ into a finite union of topologically mixing components for some power φ^m of φ , as can easily be shown.

For $x \in \Lambda$ we define local stable and unstable manifolds as in Appendix

1:

$$W_\varepsilon^s(x) = \{y \in M \mid \lim_{n \rightarrow -\infty} \rho(\varphi^n(x), \varphi^n(y)) = 0 \text{ and}$$

$$\text{for all } n \leq 0, \rho(\varphi^n(x), \varphi^n(y)) \leq \varepsilon\},$$

$$W_\varepsilon^u(x) = \{y \in M \mid \lim_{n \rightarrow +\infty} \rho(\varphi^n(x), \varphi^n(y)) = 0 \text{ and}$$

$$\text{for all } n > 0, \rho(\varphi^n(x), \varphi^n(y)) \leq \varepsilon\},$$

where ρ denotes the distance with respect to some fixed Riemannian metric. From the local product structure, we know that for $x, x' \in \Lambda$ sufficiently close, $W_\varepsilon^u(x)$ and $W_\varepsilon^s(x')$ have a unique point of intersection and that this point also belongs to Λ .

We say that x is a *boundary point* of Λ in the *unstable direction* if x is a boundary point of $W_\varepsilon^u(x) \cap \Lambda$, i.e. if x is an accumulation point only from one side by points in $W_\varepsilon^u(x) \cap \Lambda$. If x is a boundary point of Λ in the unstable direction, then, due to the local product structure, the same holds for all points in $W_\varepsilon^s(x) \cap \Lambda$. So the boundary points in the *unstable direction* are locally intersections of local *stable* manifolds with Λ . For this reason we denote the set of boundary points in the unstable direction by $\partial_s \Lambda$. The boundary points in the stable direction are defined similarly; the set of these boundary points is denoted by $\partial_u \Lambda$. Notice that if $\Lambda = M^2$ then $\partial_s \Lambda = \partial_u \Lambda = \phi$ and if Λ is a one-dimensional attractor then $\partial_s \Lambda = \phi \neq \partial_u \Lambda$.

Our construction of Markov partitions for Λ is based on the following simple theorem from [NP, 1973]:

THEOREM 1. *For a basic set Λ as above there is a finite number of (periodic) saddle points p_1^s, \dots, p_n^s such that*

$$\Lambda \cap \left(\bigcup_i W^s(p_i^s) \right) = \partial_s \Lambda.$$

Similarly, there is a finite number of (periodic) saddle points p_1^u, \dots, p_n^u such that

$$\Lambda \cap \left(\bigcup_i W^u(p_i^u) \right) = \partial_u \Lambda.$$

PROOF: Let $x \in \partial_s \Lambda$. We first claim that $W^s(x)$ is periodic, i.e. $x \in W^s(p)$ for some periodic point $p \in \Lambda$. If we suppose that this is not the case we reach a contradiction. In fact, let $\varphi^{n_i}(x)$ be a converging sequence of iterates of x , $n_i \rightarrow \infty$. Since $W^s(x)$ is not periodic, $\varphi^{n_i}(x)$ and $\varphi^{n_j}(x)$ are in different stable manifolds if $n_i \neq n_j$. But, by local product structure (see Chapter 0), there is $N > 0$ such that for $n_i, n_j > N$ and $n_i \neq n_j$, $W_\varepsilon^u(\varphi^{n_i}(x)) \cap W_\varepsilon^s(\varphi^{n_j}(x))$ consists of exactly one point and this point belongs to Λ . It then follows that there are arbitrarily large $n_i, n_j, n_\ell > N$ such that the points $W_\varepsilon^u(\varphi^{n_i}(x)) \cap W_\varepsilon^s(\varphi^{n_j}(x))$ and $W_\varepsilon^u(\varphi^{n_i}(x)) \cap W_\varepsilon^s(\varphi^{n_\ell}(x))$ are on different sides of $\varphi^{n_i}(x)$ in $W_\varepsilon^u(\varphi^{n_i}(x))$. Since φ^{-n} decreases distances along unstable manifolds, we see that x is accumulated from both sides in $W^u(x)$ by points of $W^u(x) \cap \Lambda$, which is a contradiction. (Observe that the same reasoning holds in higher dimensions if the unstable bundle of Λ has codimension 1). The argument for points in $\partial_u \Lambda$ is similar. We now

claim that if p is periodic and $W^s(p) \cap \partial_s \Lambda \neq \phi$, then $W^s(p) \cap \Lambda \subset \partial_s \Lambda$. Indeed, since all sets involved in the claim are invariant under φ , we can localize the question near p . But we already observed that if $x \in \partial_s \Lambda$ then the same is true for all points in its local stable manifold. This shows that $\partial_s \Lambda = \Lambda \cap \left(\bigcup_i W^s(p_i^s) \right)$, where each p_i is a periodic point. The proof that there are only finitely many such periodic points p_i^s is now clear from the arguments above. In fact, if this were not the case, one would just take an accumulation point say y of these periodic points. Now, by local product structure, the local unstable manifold of y intersects many $W^s(p_i^s)$ in points of Λ and this shows that not all $W^s(p_i^s)$ bound Λ . Thus, we reach a contradiction and the proof of the theorem is complete. \square

We note that this theorem, which is typically two-dimensional, makes the construction of Markov partitions in two dimensions simpler than in higher dimensions. Another reason why the two-dimensional case is easier is that in this dimension one can make boxes (see below), whose boundaries consist of pieces of stable and unstable separatrices. For the general construction see Bowen [B, 1975a].

Now we come to the *definition* of a *Markov partition* for a basic set Λ as introduced above. Such a Markov partition consists of a finite set of boxes, i.e. diffeomorphic images of the square $Q = [-1, +1]^2$, say $B_1 = \Psi_1(Q), \dots, B_\ell = \Psi_\ell(Q)$ such that

(i) $\Lambda \subset \bigcup_i B_i$,

(ii) $B_i \cap B_j = \phi, i \neq j$, where $\overset{\circ}{B}$ denotes the interior of B ,

(iii) $\varphi(\partial_s B_i) \subset \bigcup_j \partial_s B_j$ and

$$\varphi^{-1}(\partial_u B_i) \subset \bigcup_j \partial_u B_j, \text{ where}$$

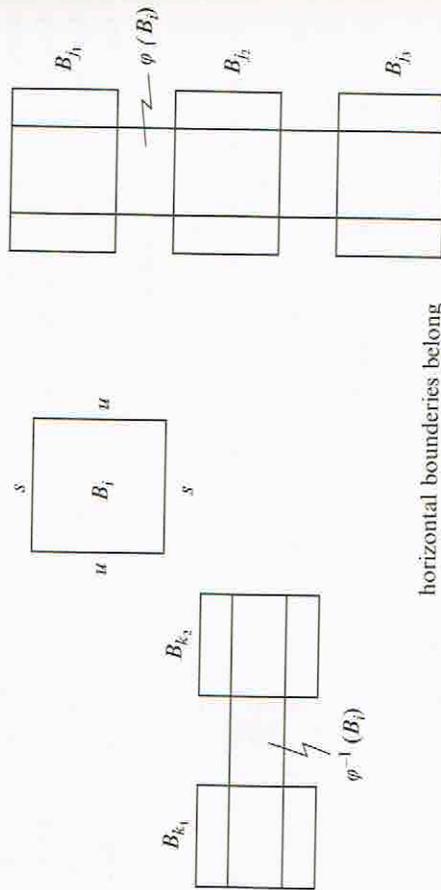
$$\partial_s B_i = \Psi_i(\{(x, y) \mid -1 \leq x \leq 1, |y| = 1\}) \text{ and}$$

$$\partial_u B_i = \Psi_i(\{(x, y) \mid |x| = 1, -1 \leq y \leq 1\}),$$

(iv) there is a positive integer n such that $\varphi^n(B_i) \cap B_j \neq \phi$ for all $1 \leq i, j \leq \ell$ (this means that the Markov partition is mixing).

The geometric consequences of the third condition are indicated in Figure A2.1.

Usually one also requires that $\varphi(B_i) \cap B_j$ is either empty or connected. For our present considerations this is not important, but one can always satisfy this last condition by taking the boxes of the Markov partition sufficiently small.



horizontal boundaries belong to ∂_s , vertice ones to ∂_u

Figure A2.1

THEOREM 2. There is a Markov partition for Λ with arbitrarily small diameter.

PROOF: Let us begin the construction of a Markov partition in case $\partial_s \Lambda \neq \emptyset \neq \partial_u \Lambda$. For this we take closed arcs

$$I_1^s, \dots, I_{n_s}^s \text{ in } W^s(p_1^s), \dots, W^s(p_{n_s}^s) \text{ and } I_1^u, \dots, I_{n_u}^u \text{ in } W^u(p_1^u), \dots, W^u(p_{n_u}^u),$$

where $p_1^s, \dots, p_{n_s}^s$ and $p_1^u, \dots, p_{n_u}^u$ are (periodic) saddle points as in the above theorem. These intervals are chosen so that for each i , $\partial I_i^s \subset \bigcup_j I_j^u$ and

$\partial I_i^u \subset \bigcup_j I_j^s$. Moreover, if $p_i^s \notin \partial_u \Lambda$ then p_i^s is contained in the interior

of I_i^s ; if $p_i^s = p_j^u \in \partial_s \Lambda \cap \partial_u \Lambda$ then this is a boundary point of both I_i^s and I_j^u . We assume, except for this last case, that the intervals I_i^s and I_j^u have disjoint boundaries. It is possible to satisfy these conditions since both $(\bigcup_j W^u(p_j^u)) \cap \Lambda$ and $(\bigcup_j W^s(p_j^s)) \cap \Lambda$ are dense in Λ . (For each periodic point $p \in \Lambda$, the components of $W^s(p) - \{p\}$ and $W^u(p) - \{p\}$ are dense in Λ , or disjoint from Λ , since we assume φ to be mixing on Λ .)

We shall prove that, if the arcs $I_1^s, \dots, I_{n_s}^s$ and $I_1^u, \dots, I_{n_u}^u$ are sufficiently long, then they "divide" Λ according to a Markov partition with an arbitrarily small diameter. To be more precise, we fix $\varepsilon > 0$ sufficiently small and say that $x \in \Lambda$ is ε -enclosed by the above arcs if

$$W_\varepsilon^s(x) \cap \left(\bigcup_j I_j^u \right)$$

and if

$$W_\varepsilon^u(x) \cap \left(\bigcup_j I_j^s \right)$$

contains x or contains points on both sides of x . We claim that if all I_j^u and I_j^s are extended at least to length ℓ from the saddle points p_j^u, p_j^s along $W^u(p_j^u), W^s(p_j^s)$ in both directions (as long as the corresponding branch of the separatrix contains points of Λ), then, for ℓ sufficiently big, they enclose all points of Λ .

We prove this claim by contradiction. Let x_i be a point of Λ which is not yet enclosed when $\ell = i$ for $i \in N$. By compactness, $\{x_i\}$ has an accumulation point, say \bar{x} . We show that for some finite ℓ_0 there is a neighbourhood of \bar{x} such that all of its points are enclosed whenever $\ell \geq \ell_0$. We have to distinguish between $\bar{x} \in \partial_u \Lambda$ and $\bar{x} \notin \partial_u \Lambda$ and also between $\bar{x} \in \partial_s \Lambda$ and $\bar{x} \notin \partial_s \Lambda$. We consider the case $\bar{x} \in \partial_u \Lambda, \bar{x} \notin \partial_s \Lambda$; the other cases can be treated similarly.

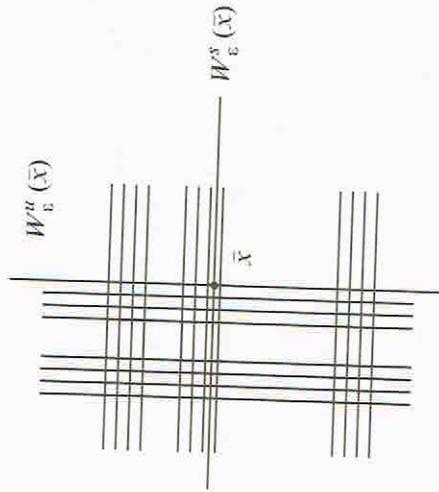


Figure A2.2

In Figure A2.2 we indicate local stable and unstable manifolds of points $x \in \Lambda$ near \bar{x} . Since $\bar{x} \in \partial_u \Lambda, \bar{x} \in \bigcup_j W^u(p_j^u)$ then, for some ℓ_1 and $\ell \geq \ell_1$,

\bar{x} belongs to $\bigcup_j I_j^u$. Since $(\bigcup_j W^u(p_j^u)) \cap \Lambda$ and $(\bigcup_j W^s(p_j^s)) \cap \Lambda$ are both

dense in Λ , for some ℓ_2 and $\ell \geq \ell_2, \bigcup_j I_j^s$ and $\bigcup_j I_j^u$ contain segments I_j^s, I_j^u as indicated in Figure A2.3.

For all $\ell \geq \ell_0 = \max(\ell_1, \ell_2)$, there is clearly a full neighbourhood of \bar{x} in which all the points are enclosed. This gives the required contradiction and completes the proof of the claim.

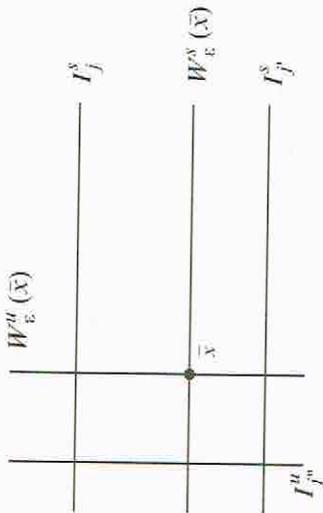


Figure A2.3

From now on we assume that the arcs I_j^s and I_j^u are so long that all points of Λ are ε -enclosed. For $\Lambda - \left(\bigcup_j (I_j^s \cup I_j^u) \right)$, we define an equivalence relation: $x \sim x'$ if we can join x by x' without crossing any of the arcs I_j^s or I_j^u . Now, it is easy to see that one can construct for each equivalence class a box, containing that equivalence class in its interior and whose boundary consists of segments of $\bigcup_j I_j^s$ and of $\bigcup_j I_j^u$ and such that the interior of the box contains no points of $\bigcup_j I_j^s$ or of $\bigcup_j I_j^u$. These boxes, one for each equivalence class, form a Markov partition. Notice that $\varphi(\partial_s B_i) \subset \bigcup_j \partial_s B_j$ because $\varphi\left(\bigcup_j I_j^s\right) \subset \bigcup_j I_j^s$, and $\varphi^{-1}(\partial_u B_i) \subset \bigcup_j \partial_u B_j$ because $\varphi^{-1}\left(\bigcup_j I_j^u\right) \subset \bigcup_j I_j^u$. Also, for φ small, the Markov partition has small diameter.

To conclude the proof of the theorem, we have now to consider the cases $\partial_u \Lambda = \partial_s \Lambda = \phi$ (Anosov case) and $\partial_u \Lambda \neq \phi = \partial_s(\Lambda)$ (one-dimensional attractors). In the first case, we take closed intervals I^s and I^u contained in $W^s(p)$ and $W^u(p)$, respectively, where p is a fixed (periodic) point. Again, if these intervals are long enough they ε -enclose all points in M and form a Markov partition with small diameter. In the remaining case we just consider the points $p_1^s, \dots, p_{n_s}^s$ and closed intervals $I_1^s, \dots, I_{n_s}^s$ together with a unique interval I^u in the unstable manifold of a periodic point and then repeat the construction above. \square

REMARK 1: In the case of a basic set of saddle type, we observe that although points of Λ may lie in the common boundary of two boxes, there is for each $x \in \Lambda$ a unique box B such that $x \in \overset{\circ}{B} \cap \Lambda$. In fact, in this case we can obtain a Markov partition with all the boxes disjoint by replacing each box B by the smallest box \tilde{B} such that

- $\tilde{B} \cap \Lambda = C\ell(\overset{\circ}{B} \cap \Lambda)$,
- $\partial \tilde{B}$ consists of parts of local stable and local unstable manifolds of the periodic points $p_1^s, \dots, p_{n_s}^s$ and $p_1^u, \dots, p_{n_u}^u$.

REMARK 2: For a diffeomorphism C^r near φ , our construction yields nearby Markov partitions for the corresponding nearby basic set. This is due to the following facts. First of all, periodic points close to $p_1^s, \dots, p_{n_s}^s$ and $p_1^u, \dots, p_{n_u}^u$ have their stable and unstable manifolds bounding the nearby basic set. Secondly, on compact parts these manifolds are C^r -close to the original ones for φ .

Now we show how to construct Markov partitions for dynamically defined Cantor sets induced by basic sets as in Chapter 4. We choose a saddle point $p \in \Lambda$ and consider $W^s(p) \cap \Lambda$, or, more precisely for an arc I^s in $W^s(p)$ we consider $I^s \cap \Lambda$. We take I^s so long that it passes through all the boxes of the Markov partition. So, for each box B , $B \cap I^s$ consists of a number (at least one) of arcs in the s -direction passing from one component of $\partial_u B$ to the other component of $\partial_u B$. (See Figure A2.4.)

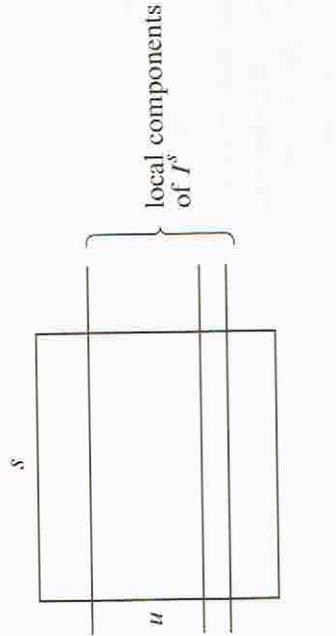


Figure A2.4

This fact, that one box B can be crossed several times by I^s , is inconvenient because it makes the definition of a projection of Λ into $\Lambda \cap I^s$ ambiguous. For this reason we shall describe how to refine the Markov partition for a fixed I^s , so as to obtain a new Markov partition in which each box is crossed exactly once by I^s .

Let $I_1, I_2 \subset I^s$ be components of $B \cap I^s$. Let $\varphi(I_1)$ and $\varphi(I_2)$ be contained in the boxes B_1 and B_2 ; the component of $I^s \cap B_k$ containing $\varphi(I_i)$ is denoted by \tilde{I}_i . If $B_1 = B_2$, we start again with $B_1 = B_2$ instead of B and I_i instead of I_i . We repeat this until we get $B_1 \neq B_2$: this must finally happen since under positive iterations of φ , I_1 and I_2 get more and more separated in the u -direction.

We have now the situation in Figure A2.5.

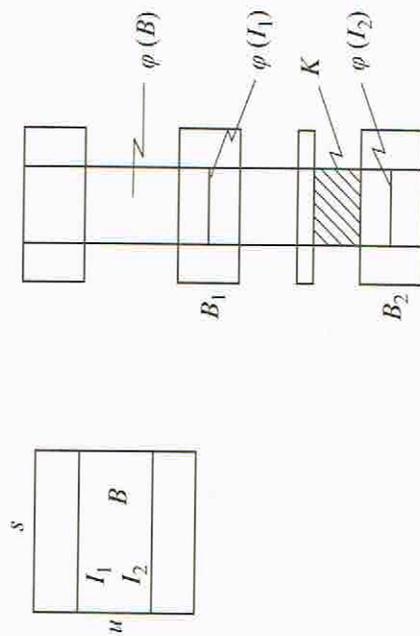


Figure A2.5

The region K of $\varphi(B)$, indicated in Figure A2.5 is just a connected component of $\varphi(B) - (\bigcup_j B_j)$, where $\bigcup_j B_j$ is the union of all the boxes of the Markov partition. Now we refine our Markov partition by removing from B the strip $\varphi^{-1}(K)$, thus splitting B in two smaller boxes. It is not hard to see that we obtain in this way a new partition in which I_1 and I_2 are not in the same box any more. So we diminished the set of pairs of components of $I^s \cap (\bigcup_j B_j)$ by one. Repeating this construction sufficiently often we obtain a Markov partition in which each box is crossed exactly once by I^s .

Next, we define a projection $\pi: \Lambda \rightarrow \Lambda \cap I^s$ by taking in each box the projection along local unstable manifolds into the intersection of I^s with that box. Then we define the expanding map $\Psi: I^s \cap \Lambda \rightarrow I^s \cap \Lambda$ as $\Psi = \pi \circ \varphi^{-N}$ (this is the same as in Chapter 4 except for the parametrization $\alpha: \mathbb{R} \rightarrow W^s(p)$). Extending the projection π from Λ to the union of the boxes of Λ , we get the map Ψ defined on a set of intervals $\{K_1, \dots, K_k\}$ in I^s ; each

interval is mapped by Ψ diffeomorphically onto one of the intersections of I^s with a box of the Markov partition of Λ . The intervals K_1, \dots, K_k form the Markov partition of $I^s \cap \Lambda$ with expanding map Ψ .

From these constructions, it is clear that indeed the ideas of the example in Chapter 4, namely the construction of Markov partitions when the basic set is a horseshoe, carry over to the general situation of a two-dimensional basic set.