Chapter 1

The Hartman-Grobman theorem and Anosov diffeomorphisms

Let A be a nonsingular $n \times n$ matrix. Suppose that $\mathbb{R}^n = E^s \oplus E^u$ is an invariant splitting for A. For any $x \in \mathbb{R}^n$, we let $x = x_s + x_u$ with $x_s \in E^s$ and $x_u \in E^u$. On $\mathbb{R}^n$ we can take a product norm $|x| = \max\{|x_s|, |x_u|\}$, for given norms on E^s, E^u. We suppose that the eigenvalues of $A_s = A|_{E^s}$ have moduli less than one and that the eigenvalues of $A_u = A|_{E^u}$ have moduli greater than one. By a linear coordinate change we may assume that

$$|A_s|, |A_u^{-1}| < 1.$$

Let $\mathfrak{C}^j$ be the space of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose derivatives up to order j are bounded and uniformly continuous. The norm $|\cdot|_j$ on $\mathfrak{C}^j$ is the sup norm of all derivatives up through order j;

$$|f|_j = \sup\{|f(x)|, |Df(x)|, \dots, |D^j f(x)|;\ ; x \in \mathbb{R}^n\}.$$

Theorem 0.1. *There is $\mu_0 > 0$ such that, for any $f \in \mathfrak{C}^1_{\mu_0} = \{f \in \mathfrak{C}^1 \ ; \ |f|_1 < \mu_0\}$, there is a unique homeomorphism h with $f \mapsto h - id \in \mathfrak{C}^0$ depending continuously on f, $h(0) = id$, and $h \circ (A + f) = A \circ h$.*

Proof. Let $a = \max(|A_s|, |A_u^{-1}|) < 1$ and choose $\mu_0 > 0$ so that $a - \mu_0 > 0$, and for any $f \in \mathfrak{C}^1_{\mu_0}$, $(A + f)^{-1}$ exists and belongs to $\mathfrak{C}^1$. If $f \in \mathfrak{C}^1_{\mu_0}$, the equation $h \circ (A + f) = A \circ h$ is equivalent to either of the equations

$$h = A \circ h \circ (A + f)^{-1},$$
$$h = A^{-1} \circ h \circ (A + f).$$

We will use the first equation to define h_s and the second to define h_u (where $h = h_s + h_u$). For any continuous h, $f \in \mathfrak{C}^1_{\mu_0}$, define $T(h, f) = T(h, f)_s + T(h, f)_u$ by the relations

$$T(h, f)_s = h_s - A_s \circ h_s \circ (A + f)^{-1},$$
$$T(h, f)_u = h_u - A_u^{-1} \circ h_u \circ (A + f).$$

Denote $\mathfrak{D}^0 = \{h : \mathbb{R}^n \to \mathbb{R}^n\ ;\ h - id \in \mathfrak{C}^0\}$. It is easy to verify that $T : \mathfrak{D}^0 \times \mathfrak{C}^1_{\mu_0} \to \mathfrak{C}^0$ is continuous in h, f, and $T(id, 0) = 0$. Furthermore, $D_hT(h, f)$ exists and is continuous in h, f with

$$(D_hT(h,0)g)_s = g_s - A_s \circ g_s \circ A^{-1},$$
$$(D_hT(h,0)g)_u = g_u - A_u^{-1} \circ g_u \circ A.$$

For any $w \in \mathfrak{C}^0$, the equation $D_hT(h,0)g = w$ has a unique solution bounded above by $(1-a)^{-1}|w|$ (note that $D_hT(h,0)$ is of the form $I + L$ with $|L| < a$, so that $D_hT(h,0) = (I-L)^{-1}$ exists and equals $I + L + L^2 + L^3 + \ldots$). Thus, $D_hT(h,0)$ is an isomorphism. The implicit function theorem implies there is a unique function $h = h(f)$, continuous in f in $\mathfrak{C}^1_{\mu_0}$ (we may have to take μ_0 smaller to get this), $h(0) = id$, and $T(h(f), f) = 0$.

It remains to show that h is a homeomorphism. Consider the equation $(A+f) \circ g = g \circ A$ for $g \in \mathfrak{C}^0$, $f \in \mathfrak{C}^1$. We can repeat the same type of argument as above to obtain a unique function $g = g(f) \in \mathfrak{C}^0$, $f \in \mathfrak{C}^1_{\mu_0}$, $g(f) - id \in \mathfrak{C}^0$ continuous in f, $g(0) = id$ and $(A+f) \circ g = g \circ A$. The same type of argument also gives unique solutions to equations of the form $(A+f_1) \circ g = g \circ (A+f_2)$ for differentiable and small f_1, f_2. The combination of solutions g and h of $(A+f) \circ g = g \circ A$ and $A \circ h = h \circ (A+f)$ provides a solution $h \circ g$ of $h \circ g \circ A = Ah \circ g$ and a solution $g \circ h$ of $g \circ h \circ (A+f) = (A+f) \circ g \circ h$. These solutions are unique. Since the identity map also solves these equations, we have $h \circ g = id$ and $g \circ h = id$. Hence h is a homeomorphism. □

Consider a matrix $A \in GL(2,\mathbb{Z})$, i.e. a 2×2 matrix with integer coefficients, whose inverse also has integer coefficients. Note that $\det A = \pm 1$. A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$ induces an automorphism on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The induced map on $\mathbb{T}^2$, also denoted by A, is given by

$$A(x,y) = (ax + by, cx + dy) \mod 1.$$

Suppose that $\mathbb{R}^2 = E^s \oplus E^u$ is an invariant splitting for A. For any $x \in \mathbb{R}^2$, we let $x = x_s + x_u$ with $x_s \in E^s$ and $x_u \in E^u$. We suppose that the eigenvalue of $A_s = A|_{E^s}$ has modulus less than one and that the eigenvalue of $A_u = A|_{E^u}$ has modulus greater than one. Hence

$$|A_s|, |A_u^{-1}| < 1.$$

The induced map on $\mathbb{T}^2$ is called hyperbolic. Periodic points of A lie dense in $\mathbb{T}^2$, in fact, every point in $\mathbb{T}^2$ with rational coordinates is periodic. An example is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

with eigenvalues $\frac{1}{2}(3+\sqrt{5})$ and its reciprocal.

Let $\mathfrak{C}^j(\mathbb{T}^2)$ be the space of maps from $\mathbb{T}^2$ to $\mathbb{T}^2$ whose derivatives up to order j are bounded and uniformly continuous. The norm $|\cdot|_j$ on $\mathfrak{C}^j(\mathbb{T}^2)$ is the sup norm of all derivatives up through order j;

$$|f|_j = \sup\{|f(x)|, |Df(x)|, \ldots, |D^jf(x)|;\ ; x \in \mathbb{R}^n\}.$$

Theorem 0.2. *There is $\mu_0 > 0$ such that, for any $f \in \mathfrak{C}^1_{\mu_0}(\mathbb{T}^2) = \{f \in \mathfrak{C}^1(\mathbb{T}^2)\ ;\ |f|_1 < \mu_0\}$, there is a unique homeomorphism h in $\mathfrak{C}^0$, depending continuously on f, $h(0) = id$, such that $h \circ (A + f) = A \circ h$.*

Proof. The proof follows the argument in Section 1 to prove the Hartman-Grobman theorem. We lift functions on $\mathbb{T}^2$ to functions on its universal cover $\mathbb{R}^2$.

Following the proof of Theorem 0.1, define $T : \mathfrak{C}^0 \times \mathfrak{C}^1_{\mu_0}$ as $T(h, f) = T(h, f)_s + T(h, f)_u$ by the relations

$$T(h, f)_s = h_s - A_s \circ h_s \circ (A + f)^{-1},$$
$$T(h, f)_u = h_u - A_u^{-1} \circ h_u \circ (A + f).$$

Write

$$D^j = \{f \in \mathfrak{C}^j\ ;\ f(x, y) + (k, l) = f(x, y) \text{ for all } (k, l) \in \mathbb{Z}^2\},$$
$$E^j = \{f \in \mathfrak{C}^j\ ;\ f - id \in D^j\}.$$

One checks that $T : E^0 \times D^1_{\mu_0} \to E^0$ is well defined. To see this, note that because $A^{-1}((x, y) + (k, l)) = A^{-1}((x, y)) + A^{-1}((k, l))$, f is small in the C^1 topology, and $(A + f)^{-1}$ is $\mathbb{Z}^2$-periodic, we have

$$(A + f)^{-1}((x, y) + (k, l)) = (A + f)^{-1}((x, y)) + A^{-1}((k, l)).$$

Likewise, for $h - id \in D^j$,

$$h_s((A + f)^{-1}((x, y)) + A^{-1}((k, l))) = h_s((A + f)^{-1}((x, y))).$$

So

$$\begin{aligned} A_s \circ h_s \circ (A + f)^{-1}((x, y) + (k, l)) &= A_s \circ h_s((A + f)^{-1}((x, y)) + A^{-1}((k, l))) \\ &= A_s \circ h_s \circ (A + f)^{-1}((x, y)) \end{aligned}$$

and, likewise,

$$A_u^{-1} \circ h_u \circ (A + f)((x, y) + \mathbb{Z}^2) = A_u^{-1} \circ h_u \circ (A + f)((x, y)).$$

The remainder of the proof follows the arguments for Theorem 0.1. $\square$

Chapter 2

From stable manifolds to structural stability

Let $f : M \to M$ be a diffeomorphism on a compact manifold M. Define the stable set $W^s(p)$ and the unstable set $W^u(p)$ by

$$W^s(p) = \{x \in M \; ; \; f^n(x) \to p \text{ as } n \to \infty\},$$
$$W^u(p) = \{x \in M \; ; \; f^n(x) \to p \text{ as } n \to -\infty\}.$$

We state the fundamental stable manifold theorem, originating with work of Jacques Hadamard and Oskar Perron.

Theorem 0.3. *Supppose p is a hyperbolic fixed point of a diffeomorphism f. Then $W^s(p)$ is a manifold, injectively immersed in M, with $T_pW^s(p) = E^s(p)$. We call $W^s(p)$ the stable manifold. Likewise, $W^u(p)$ is a manifold, injectively immersed in M, with $T_pW^u(p) = E^u(p)$. We call $W^u(p)$ the unstable manifold.*

Proof. We present a variant of Perron's proof. We will construct a local stable manifold $W^s_{loc}(p)$ near p for f. Then $W^s(p) = \cup_{n\in\mathbb{Z}} f^n(W^s_{loc})$ is the orbit of the local stable manifold.

Let us make this precise. Working in a chart near p, we may assume that p is the origin in $\mathbb{R}^n$. For given small $\delta > 0$, let

$$W^s_{loc} = \{u \in \mathbb{R}^n \; ; \; \|f^n(x)\| < \delta \text{ for all } n \in \mathbb{N}\}.$$

Take coordinates $u = (x, y) \in E^s(0) \times E^u(0)$. Write further π_s and π_u for the coordinate projections $\pi_s(x, y) = x$ and $\pi_u(x, y) = y$. Let e^μ be a bound for the eigenvalues of $Df(0)|_{E^s(0)}$; $\operatorname{spec} Df(0)|_{E^s(0)} < e^\mu < 1$, Likewise, let $1 < e^\lambda < \operatorname{spec} Df(0)_{E^u(0)}$. Replacing f by an iterate of f, if necessary, we may assume $\|Df(0)v_s\| < e^\mu\|v_s\|$ for all $v_s \in E^s(0)$ and $\|Df(0)v_u\| > e^\lambda\|v_u\|$ for all $v_u \in E^u(0)$.

Let $\mathfrak{B}(\mathbb{N}, \mathbb{R}^n)$ be the space of bounded sequences $\mathbb{N} \to \mathbb{R}^n$, endowed with the supnorm

$$\|\gamma\|_0 = \sup_{n\in\mathbb{N}} \|\gamma_n\|,$$

where $\|(x,y)\| = \max\{\|x\|, \|y\|\}$ is a box norm on $\mathbb{R}^n = E^s(0) \times E^u(0)$, for given norms on $E^s(0)$ and $E^u(0)$. Define the map $\Gamma : E^s(0) \times \mathfrak{B}(\mathbb{N}, \mathbb{R}^n) \to \mathfrak{B}(\mathbb{N}, \mathbb{R}^n)$ by

$$\Gamma(x_0, \gamma)(n) = \begin{cases} (x_0, \pi_u f^{-1}(\gamma(1))), & \text{if } n = 0, \\ (\pi_s f(\gamma(n-1)), \pi_u f^{-1}(\gamma(n+1))), & \text{if } n > 0. \end{cases}$$

Write $D^s_\delta = \{x_0 \in E^s \ ; \ \|x_0\| \le \delta\}$ and $\mathfrak{B}_\delta = \{\gamma \in \mathfrak{B}(\mathbb{N}, \mathbb{R}^n) \ ; \ \|\gamma\|_0 \le \delta\}$ We claim that for δ small enough,

(i) $\Gamma(D^s_\delta \times \mathfrak{B}_\delta) \subset D^s_\delta \times \mathfrak{B}_\delta$,

(ii) Γ is a contraction on $\{x_0 \in E^s \ ; \ \|x_0\| \le \delta\} \times \mathfrak{B}_\delta$.

By continuity of derivatives of Df and Df^{-1} one has the following. For $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\|\pi_s f(x,y) - \pi_s f(\bar{x}, \bar{y})\| < (e^\mu + \varepsilon)\|(x,y) - (\bar{x}, \bar{y})\|,$$
$$\|\pi_u f^{-1}(x,y) - \pi_u f^{-1}(\bar{y}, \bar{y})\| < (e^{-\lambda} + \varepsilon)\|(x,y) - (\bar{x}, \bar{y})\|$$

for all $(x,y), (\bar{x}, \bar{y})$ with $\|(x,y)\|, \|(\bar{x}, \bar{y})\| < \delta$. Pick ε so that both $e^\mu + \varepsilon < 1$ and $e^{-\lambda} + \varepsilon < 1$. To prove that Γ is a contraction in the second coordinate, compute

$$\begin{aligned} &\|\Gamma(x_0, \gamma_1) - \Gamma(x_0, \gamma_2)\|_0 \\ &= \sup_{n \in \mathbb{N}} \|\Gamma(x_0, \gamma_1)(n) - \Gamma(x_0, \gamma_2)(n)\| \\ &\le \sup_{n \in \mathbb{N}} \max\{\|\pi_s f(\gamma_1(n-1)) - \pi_s f(\gamma_2(n-1))\|, \|\pi_u f^{-1}(\gamma_1(n+1)) - \pi_u f^{-1}(\gamma_2(n+1))\|\} \\ &\le \sup_{n \in \mathbb{N}} \max\{e^\mu + \varepsilon, e^{-\lambda} + \varepsilon\} \|\gamma_1 - \gamma_2\|_0 \\ &\le \kappa \|\gamma_1 - \gamma_2\|_0 \end{aligned}$$

for $\kappa = \max\{e^\mu + \varepsilon, e^{-\lambda} + \varepsilon\} < 1$. A similar estimate establishes the first item.

By Theorem 0.7, Γ is differentiable, so that by the uniform contraction theorem, the fixed point of Γ depends differentiable on x_0. The correspondence $x_0 \mapsto \eta_{x_0}(0)$ defines a smooth manifold.

Let η_{x_0} be the fixed point of $\Gamma(x_0, \cdot)$, $\|x_0\| < \delta$. We claim that η_{x_0} is a positive orbit of f, for ε small enough (we can take ε smaller if needed). Write $\eta_{x_0}(n) = (x_n, y_n)$. Given $n \in \mathbb{N}$, we have

$$\pi_s f(x_n, y_n) = x_{n+1}, \ \pi_u f^{-1}(x_{n+1}, y_{n+1}) = y_n.$$

This defines (x_{n+1}, y_{n+1}) as function of x_n, y_n: with

$$F(x_n, y_n, x_{n+1}, y_{n+1}) = (\pi_s f(x_n, y_n) - x_{n+1}, \pi_u f^{-1}(x_{n+1}, y_{n+1}) - y_n)$$

we have

$$F(0,0,0,0) = (0,0)$$

and

$$D_{x_{n+1}, y_{n+1}} F(0,0,0,0)$$

has bounded inverse. Hence, applying the implicit function theorem, for (x_n, y_n) near $(0,0)$, one finds (x_{n+1}, y_{n+1}) as a smooth function of (x_n, y_n), equal to $(0,0)$ if $(x_n, y_n) = (0,0)$. Now let $(\bar{x}_{n+1}, \bar{y}_{n+1}) = f(x_n, y_n)$. We must show that $(x_{n+1}, y_{n+1}) = (\bar{x}_{n+1}, \bar{y}_{n+1})$. Observe that $(\bar{x}_{n+1}, \bar{y}_{n+1}) = f(x_n, y_n)$ implies that $f^{-1}(\bar{x}_{n+1}, \bar{y}_{n+1}) = (x_n, y_n)$ and thus

$$\pi_s f(x_n, y_n) = \bar{x}_{n+1}, \pi_u f^{-1}(\bar{x}_{n+1}, \bar{y}_{n+1}) = y_n.$$

So (x_{n+1}, y_{n+1}) and $(\bar{x}_{n+1}, \bar{y}_{n+1})$ satisfy the same equation. By uniqueness of solutions, it follows that $(x_{n+1}, y_{n+1}) = (\bar{x}_{n+1}, \bar{y}_{n+1})$.

We conclude that η_{x_0} is a positive orbit of f that stays in a δ-neighborhood of 0. We must show that $\eta_{x_0}(n)$ converges to 0 as $n \to \infty$. For this we note invariance of the cone $\|y\| \leq \|x\|$: for $\delta > 0$ small enough and $\|(x,y)\| \leq \delta$, $\|y\| \leq \|x\|$, we have $\|\pi_u f((x,y))\| \leq \|\pi_s f((x,y))\|$. This follows since $\|\pi_u Df(0)(x,y)\| \leq \|\pi_s Df(0)(x,y)\|$ and f is C^1. Now $x_0 \mapsto \eta_{x_0}(0)$ is a smooth manifold that is tangent to $E^s(0)$ at the origin. So η_{x_0} is contained in the cone $\|y\| \leq \|x\|$ for x_0 small. From $x_{n+1} = \pi_s f(x_n, y_n)$ and $\|y_n\| \leq \|x_n\|$ we get $\|x_{n+1}\| \leq (e^\mu + \delta)\|(x_n, y_n)\| \leq (e^\mu + \delta)\|x_n\|$ so that $\lim_{n\to\infty}(x_n, y_n) = 0$. □

The next result yields structural stability of diffeomorphisms restricted to maximal invariant hyperbolic sets, and generalizes Theorem 0.2 for hyperbolic torus automorphisms.

Consider a diffeomorphism $f : M \to M$ on a compact manifold M. Recall that a compact invariant set Λ is called (uniformly) hyperbolic if there is a splitting $T_xM = E^s(x) \oplus E^u(x)$, depending continuously on x, and constants $C \geq 1$, $0 < \lambda < 1$, $\mu > 1$, that

$$\|Df^n(x)v\| \leq C\lambda^n\|v\|, \ v \in E^s(x),$$
$$\|Df^n(x)v\| \geq \frac{1}{C}\mu^n\|v\|, \ v \in E^u(x).$$

The hyperbolic set Λ is called maximal invariant if $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$ for some open neighborhood U of Λ.

Theorem 0.4. *Let $f : M \to M$ be a diffeomorphism with a maximal invariant hyperbolic set Λ: there is an open neighborhood U of Λ so that*

$$\Lambda = \cap_{n\in\mathbb{Z}} f^n(U).$$

Then there is a neighborhood V of f in the C^1 topology, so that for any $g \in V$,

$$\Lambda(g) = \cap_{n\in\mathbb{Z}} g^n(U)$$

is a hyperbolic set of g.

Proof. The theorem follows by finding a robust way of constructing the stable an unstable subspaces $E^s(x), E^u(x)$. This is done using invariant (stable and unstable) cone fields. For simplicity we assume $C = 1$ in the definition of hyperbolicity. For $x \in \Lambda$, write $v \in T_xM$ as $v = v_s + v_u$ with $v_s \in E^s(x), v_u \in E^u(x)$. For $\alpha > 0$ and $x \in \Lambda$ define cones

$$K^s_\alpha(x) = \{v \in T_xM \ ; \ |v_u| \leq \alpha|v_s|\},$$
$$K^u_\alpha(x) = \{v \in T_xM \ ; \ |v_s| \leq \alpha|v_u|\}.$$

We may extend these cones to cone fields in $T_U M$, possible taking a smaller neighborhood U of Λ. Note that there exists $\nu > 1$ so that for α small enough, $x \in \Lambda$,

$$Df(x)K^u_\alpha(x) \subset K^u_{\alpha/\nu}(f(x)), \tag{0.1}$$

$$|Df(x)v| \geq \nu|v| \text{ for } v \in K^u_\alpha. \tag{0.2}$$

Similarly for K^s_α. It follows that $E^u(x)$ is obtained as

$$E^u(x) = \cap_{n\geq 0} Df^n(f^{-n}(x))K^u_\alpha(f^{-n}(x))$$

and $E^s(x)$ is obtained as

$$E^s(x) = \cap_{n\geq 0} Df^{-n}(f^n(x))K^u_\alpha(f^n(x)),$$

for some small $\alpha > 0$. Estimates (0.1) and (0.2) apply for any g sufficiently close to f in the C^1 topology, with $x \in \Lambda(g)$. This proves the theorem. □

Theorem 0.5. *Let $f : M \to M$ be a diffeomorphism with a maximal invariant set*

$$\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$$

which is hyperbolic. There exists $\varepsilon > 0$ so that any g which is ε close to f in the C^1 topology, possesses a maximal invariant hyperbolic set $\Lambda(g)$ in U. Moreover, $f|_\Lambda$ and $g|_{\Lambda(g)}$ are topologically conjugate: there exists a homeomorphism $h = h(g) : \Lambda \to \Lambda(g)$ with

$$g \circ h = h \circ f.$$

The dependence $g \mapsto h(g)$ is continuous in the C^0 topology and $h(f) = id$.

Proof. We follow ideas that are contained in the proof of Theorem 0.3. Along Λ, there is a continuous splitting $T_\Lambda M = E^s \oplus E^u$ in stable and unstable subspaces. For simplicity assume $C = 1$ in the definition of hyperbolicity. For $x \in M$ and $\delta > 0$ small, the ball $B_\delta(x)$ of radius δ about x is well defined. Cover Λ with finitely many balls $U_j = B_\delta(y_j)$ of small radius $\delta > 0$ with the property that for any $x \in \Lambda$, the ball $B_{\delta/2}(x)$ is contained in one of U_j's. Working in a chart, we may assume $U_j \subset \mathbb{R}^m$. Then we have coordinate projections π_s to $E^s(y_j)$ with kernel $E^u(y_j)$ and π_u to $E^u(y_j)$ with kernel $E^s(y_j)$.

Fix an orbit $x(i+1) = f(x(i))$, $i \in \mathbb{Z}$, of f in Λ. Suppose $x(i) \in U_{j(i)}$ and $B_{\delta/2}(x(i)) \subset U_{j(i)}$. Write $\mathfrak{B}(\mathbb{Z}, M)$ for the space of sequences $\mathbb{Z} \mapsto M$ endowed with the C^0 topology and let $\mathfrak{B} = \{\gamma \in \mathfrak{B}(\mathbb{Z}, M)\ ;\ \gamma(i) \in B_{\delta/2}(x(i))\}$. Define $\Gamma : \mathfrak{B} \to \mathfrak{B}(\mathbb{Z}, M)$ by

$$\Gamma(\gamma)(n) = (\pi_s f(\gamma(n-1)), \pi_u f^{-1}(\gamma(n+1))).$$

Likewise, for a diffeomorphism $g : M \to M$ close to f in the C^1 topology, define $\Gamma_g : \mathfrak{B} \to \mathfrak{B}(\mathbb{Z}, M)$ by

$$\Gamma_g(\gamma)(n) = (\pi_s g(\gamma(n-1)), \pi_u g^{-1}(\gamma(n+1))).$$

We make the following claims. For sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ so that for any diffeomorphism $g : M \to M$ that is ε close to f in the C^1 topology,

(i) $\Gamma_g(\mathfrak{B}) \subset \mathfrak{B}$,

(ii) Γ_g is a contraction on $\mathfrak{B}$,

(iii) the fixed point of Γ_g is the orbit $z(i+1) = g(z(i))$ of g with $z(i) \in B_{\delta/2}(x(i))$.

The claims rely on closeness of f to $Df(x(i))$, closeness of $E^s(x(i))$ and $E^u(x(i))$ to $E^s(y_{j(i)})$ and $E^u(y_{j(i)})$, and closeness of g to f in the C^1 topology. Consider first projections $\hat{\pi}_s : \mathbb{R}^m \to E^s(x(i))$ and $\hat{\pi}_u : \mathbb{R}^m \to E^u(x(i))$ with $\hat{\pi}_s + \hat{\pi}_u = id$. For $\varepsilon > 0$ there is $\delta > 0$ so that for $x_1, x_2 \in B_{\delta/2}(x(i-1))$,

$$
\begin{aligned}
&|\hat{\pi}_s f(x_1) - \hat{\pi}_s f(x_2)| \leq \\
&\qquad |\hat{\pi}_s(f(x_1) - x(i) - Df(x(i-1))(x(i-1) - x_1))| + |\hat{\pi}_s(f(x_2) - x(i) - Df(x(i-1))(x(i-1) - x_2))| \\
&\qquad +|\hat{\pi}_s Df(x(i-1))(x_1 - x_2)| \leq \\
&\qquad 2\varepsilon|x_1 - x_2| + \lambda|x_1 - x_2|.
\end{aligned}
$$

A similar estimate applies to $\hat{\pi}_u f^{-1}$. We conclude that for $\varepsilon > 0$ there is $\delta > 0$ so that for $x_1, x_2 \in B_{\delta/2}(x(i-1))$, $y_1, y_2 \in B_{\delta/2}(x(i+1))$,

$$
\begin{aligned}
|\hat{\pi}_s f(x_1) - \hat{\pi}_s f(x_2)| &\leq (\lambda + 2\varepsilon)|x_1 - x_2|, \\
|\hat{\pi}_u f^{-1}(y_1) - \hat{\pi}_u f^{-1}(y_2)| &\leq (1/\mu + 2\varepsilon)|y_1 - y_2|.
\end{aligned}
$$

We replace the projections $\hat{\pi}_s, \hat{\pi}_u$ by π_s, π_u. From the continuous dependence of the splitting E^s, E^u along Λ, we get the following. For $\varepsilon > 0$ there exists $\delta > 0$ so that for $x_1, x_2 \in B_{\delta/2}(x(i-1))$,

$$
\begin{aligned}
&|\pi_s f(x_1) - \pi_s f(x_2)| \leq \\
&\qquad |\pi_s f(x_1) - \hat{\pi}_s f(x_1)| + |\pi_s f(x_2) - \hat{\pi}_s f(x_2)| + |\hat{\pi}_s f(x_1) - \hat{\pi}_s f(x_2)| \leq \\
&\qquad 2\varepsilon|x_1 - x_2| + (\lambda + 2\varepsilon)|x_1 - x_2|.
\end{aligned}
$$

With a similar estimate for the unstable coordinate, we get that for $\varepsilon > 0$ there is $\delta > 0$ so that for $x_1, x_2 \in B_{\delta/2}(x(i-1))$, $y_1, y_2 \in B_{\delta/2}(x(i+1))$,

$$
\begin{aligned}
|\pi_s f(x_1) - \pi_s f(x_2)| &\leq (\lambda + 4\varepsilon)|x_1 - x_2|, \\
|\pi_u f^{-1}(y_1) - \pi_u f^{-1}(y_2)| &\leq (1/\mu + 4\varepsilon)|y_1 - y_2|.
\end{aligned}
$$

Finally and in the same manner, for g within ε of f in the C^1 topology, again with $x_1, x_2 \in B_{\delta/2}(x(i-1))$, $y_1, y_2 \in B_{\delta/2}(x(i+1))$,

$$
\begin{aligned}
|\pi_s g(x_1) - \pi_s g(x_2)| &\leq (\lambda + 6\varepsilon)|x_1 - x_2|, \\
|\pi_u g^{-1}(y_1) - \pi_s g^{-1}(y_2)| &\leq (1/\mu + 6\varepsilon)|y_1 - y_2|.
\end{aligned}
$$

For ε small, both $\lambda + 6\varepsilon < 1$ and $1/\mu + 6\varepsilon < 1$. The first two claims follow. The observation that the fixed point is an orbit, follows as in the proof of Theorem 0.3.

Define a map $h : \Lambda \to U$ that maps $x(k) \in \Lambda$ to the fixed point of $\Gamma_g(\{x(i)\})(k)$. The construction shows that $h \circ f = g \circ h$. We leave it to the reader to show that h is continuous. Vice versa, recall that the maximal invariant set of g in U is hyperbolic by Theorem 0.4. Given an orbit $\{y(i)\}$ of g in $h(\Lambda)$ for g sufficiently close to f, $\Gamma_f(\{y(i)\})$ provides a nearby orbit of f in Λ. So the map $j : g(\Lambda) \to M$ that maps $y(k) \in g(\Lambda)$ to the fixed point of $\Gamma_f(\{y(i)\})(k)$ is the inverse of h and is continuous. It follows that h is a homeomorphism. $\square$

Chapter 3

An auxiliary differentiability result

Theorem 0.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Define*

$$N : \mathfrak{C}^0([0,1], \mathbb{R}^n) \to \mathfrak{C}^0([0,1], \mathbb{R}^m)$$

by $N(\phi)(t) = f(\phi(t))$. Then N is C^1.

Proof. For $\phi \in \mathfrak{C}^0([0,1], \mathbb{R}^n)$, $\psi \in \mathfrak{C}^0([0,1], \mathbb{R}^n)$, define $B(\phi)\psi$ in $\mathfrak{C}^0([0,1], \mathbb{R}^m)$ by

$$B(\phi)\psi(t) = Df(\phi(t))\psi(t).$$

One checks that B is a bounded linear map. We claim that $DN(\phi) = B(\phi)$. Namely, compute for $t \in [0,1]$,

$$\begin{aligned}(N(\phi+\psi) - N(\phi) - B(\phi)\psi)(t) &= f(\phi(t) + \psi(t)) - f(\phi(t)) - Df(\phi(t))\psi(t) \\ &= \int_0^1 (Df(\phi(t) + s\psi(t)) - Df(\phi(t)))\psi(t)\, ds.\end{aligned}$$

Since $\phi([0,1])$ is compact, Df is uniformly continuous on $\phi[0,1]$. For $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_1 \in \phi([0,1])$ and $|x_2 - x_1| < \delta$, then $\|Df(x_2) - Df(x_1)\| < \varepsilon$. Therefore,

$$\begin{aligned}|(N(\phi+\psi) - N(\phi) - B(\phi)\psi)(t)| &= \left| \int_0^1 (Df(\phi(t) + s\psi(t)) - Df(\phi(t)))\psi(t)\, ds \right| \\ &\le \varepsilon |\psi(t)| \\ &\le \varepsilon |\psi|\end{aligned}$$

Taking the supremum over $t \in [0,1]$ we get $|N(\phi+\psi) - N(\phi) - B(\phi)\psi| \le \varepsilon|\psi|$. Since ε is arbitrary, this proves that $DN(\phi) = B(\phi)$.

Finally we establish that $B : \mathfrak{C}^0([0,1], \mathbb{R}^n) \to L(\mathfrak{C}^0([0,1], \mathbb{R}^n), \mathfrak{C}^0([0,1], \mathbb{R}^m))$ is continuous. Let $\phi_1 \in \mathfrak{C}^0([0,1], \mathbb{R}^n)$. We will show that B is continuous at ϕ_1. Note that

$$(B(\phi_2) - B(\phi_1))\psi(t) = (Df(\phi_2(t)) - Df(\phi_1(t)))\psi(t).$$

Let E be a compact neighborhood of $\phi_1([0,1])$. Let $\varepsilon > 0$. As above, there is $\delta > 0$ such that if $x_2, x_1 \in E$, $|x_2 - x_1| < \delta$, then $\|Df(x_2) - Df(x_1)\| < \varepsilon$. Then, if $|\phi_2 - \phi_1| < \delta$,

$$|(B(\phi_2) - B(\phi_1))\psi(t)| \leq \|Df(\phi_2(t)) - Df(\phi_1(t))\||\psi(t)| \leq \varepsilon|\psi|.$$

Taking the supremum over $t \in [0,1]$ we get $|(B(\phi_2) - B(\phi_1))\psi| \leq \varepsilon|\psi|$. Therefore $\|B(\phi_1) - B(\phi_2)\| \leq \varepsilon$ when $\phi_2 - \phi_1| < \delta$. Since ε is arbitrary, this shows that B is continuous at ϕ_1. □

The same argument works for the Nemytskii operator on the space $\mathfrak{B}(\mathbb{N}, \mathbb{R}^n)$ of bounded sequences in $\mathbb{R}^n$.

Theorem 0.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Define*

$$N : \mathfrak{B}(\mathbb{N}, \mathbb{R}^n) \to \mathfrak{B}(\mathbb{N}, \mathbb{R}^m)$$

by $N(\phi)(t) = f(\phi(t))$. Then N is C^1.