Solutions to selected exercises

Ale Jan Homburg

January 6, 2016

Exercise 5.3.2. Let Λ be a hyperbolic set of $f : U \to M$. Prove that the restriction of $f|_\Lambda$ is expansive.

Recall that $f|_\Lambda$ is expansive if there is $\delta > 0$ so that for any distinct $x, y \in \Lambda$, $d(f^n(x), f^n(y)) > \delta$ for some $n \in \mathbb{Z}$. By the stable manifold theorem, Theorem 5.6.4, there is $\varepsilon > 0$ so that for all $x \in \Lambda$, both $W^s_\varepsilon(x) = \{y\ ;\ d(f^n(x), f^n(y)) < \varepsilon$ for all $n \geq 0\}$ and $W^u_\varepsilon(x) = \{y\ ;\ d(f^n(x), f^n(y)) < \varepsilon$ for all $n \leq 0\}$ are C^1 embedded discs tangent to $E^s(x)$ and $E^u(x)$ respectively. Moreover, for ε small enough, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$ equals the point x. That is,

$$\{y\ ;\ d(f^n(x), f^n(y)) < \varepsilon \text{ for all } n \in \mathbb{Z}\} = \{x\}.$$

A second possible solution uses the shadowing theorem. As we need a stronger statement than Corollary 5.3.2, we first formulate the shadowing theorem as deducted from Theorem 5.3.1. There is an open set O of Λ and positive δ_0 with the following property: for every $\varepsilon > 0$ there is $\delta > 0$ so that for any δ-orbit $\{x_n\}$ in O there exists a continuous map $\psi : \{x_n\} \to O$ with $\psi(x_{n+1}) = f(\psi(x_n))$ (i.e. $\{\psi(x_n)\}$ is an orbit) and $d(\psi(x_n), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$. Moreover, ψ is unique in the sense that if $\psi' : \{x_n\} \to O$ satisfies $\psi'(x_{n+1}) = f(\psi'(x_n))$ and $d(\psi'(x_n), x_n) < \delta_0$ for all $n \in \mathbb{Z}$, then $\psi' = \psi$.

To apply this theorem, assume that $f|_\Lambda$ is not expanding. Then for any $\sigma > 0$ there are distinct $x, y \in \Lambda$ with $d(f^n(x), f^n(y)) \leq \sigma$. Take $\sigma \leq \delta_0$ and let $x_n = f^n(x)$. Consider maps $\psi, \psi' : \{f^n(x)\} \to \Lambda$ given by $\psi(f^n(x)) = f^n(x)$ and $\psi'(f^n(x)) = f^n(y)$. Then the above theorem assures that $\psi = \psi'$, i.e. $x = y$.

Exercise 5.3.5. Show that every minimal hyperbolic set consists of exactly one periodic orbit.

Let x be a point in the minimal hyperbolic set Λ. Then x is recurrent and so, given $\delta > 0$, there is an iterate $f^n(x)$ within distance δ of x. Let $x_i = f^i(x)$ for $0 \leq i < n$ and extend periodically: $x_{kn+i} = x_i$ for all $k \in \mathbb{Z}$. Since $d(x_n, f^n(x)) < \delta$ we get that $\{x_n\}$ is a δ-orbit. By the shadowing theorem, Corollary 5.3.2, for $\varepsilon > 0$ there is $\delta > 0$ so that a δ-orbit that is contained in a δ-neighborhood of Λ, is ε-shadowed by a real orbit. Moreover, by the uniqueness expressed in Theorem 5.3.1 (see also the solution to Exercise 5.3.2 above), a periodic δ-orbit is shadowed by a periodic orbit. We conclude that in any neighborhood of the minimal hyperbolic set Λ, there exists a periodic orbit. If Λ is locally maximal, sufficiently nearby periodic orbits are part of Λ, so that Λ equals a periodic orbit. In general the statement of the exercise is wrong.

Exercise 5.7.3. Let p be a hyperbolic fixed point of f. Suppose $W^s(p)$ and $W^u(p)$ intersect transversally at q. Prove that the union of p with the orbit of q is a hyperbolic set Λ.

Write $\mathcal{O}(q)$ for the orbit of q. Note that its closure $\overline{\mathcal{O}(q)}$ equals $\{p\} \cup \mathcal{O}(q)$. We claim that the hyperbolic splitting $E^s(x) \oplus E^u(x)$, $x \in \overline{\mathcal{O}(q)}$, is given by $E^s(x) = T_x W^s(p)$ and $E^u(x) = T_x W^u(p)$. To see this, note first that the splitting is invariant under Df.

Without loss of generality, there is $\lambda < 1$ such that

$$\|Df(p)|_{E^s(p)}\| < \lambda \text{ and } \|Df^{-1}(p)|_{E^u(p)}\| < \lambda.$$

Now consider $r \in \mathcal{O}(q)$ and let $v \in E^u(r)$. Let V denote a small neighborhood of p. Given V there is a finite number N of points in $\mathcal{O}(q)$ that are outside V. Assume that $f^i(r) \notin V$ for $i \in \mathbb{Z} \cap [R, R+N-1]$. We have

$$\lim_{i\to\infty} E^u(f^i(r)) = E^u(p)$$

by Theorem 5.7.2. Obviously also $\lim_{i\to-\infty} E^u(f^i(r)) = E^u(p)$. Consequently, for V small enough and $f^i(r) \in V$, $\|Df(f^i(r))v\| \geq (1/\lambda)\|v\|$ if $v \in E^u(f^i(r))$. As $\|Df^N(f^R(r))v\| \geq C\|v\|$ for some $C > 0$ if $v \in E^u(f^R(r))$, we get that $\|Df^n(r)v\| \geq C(1/\lambda)^{n-R}\|v\|$ for $v \in E^u(r)$, $n \geq R$. Hence there exists $c > 0$ so that

$$\|Df^n(r)v\| \geq c(1/\lambda)^n\|v\|$$

for $v \in E^u(r)$, $n \geq 0$. The action of $Df^n(r)$ on $E^s(r)$ is treated similarly by noting that for the inverse diffeomorphism f^{-1} stable and unstable directions are swapped.

Exercise 5.11.3. Prove that $\leq$ is a transitive relation.

Write f for the diffeomorphism on a n-dimensional manifold that generates the dynamical system. Suppose $x \leq y$ and $y \leq z$ for hyperbolic periodic points x, y, z of f. For simplicity we assume that x, y, z are fixed points. So $W^s(x) \pitchfork_q W^u(y)$ and $W^s(y) \pitchfork_p W^u(z)$ at some points q, p. Note that e.g. $\dim W^s(x) + \dim W^u(y) \geq n$, so that $\dim W^s(y) + \dim W^u(y) = n$ gives

$$\dim W^s(x) \geq \dim W^s(y).$$

Take a local submanifold V of $W^s(x)$ containing q with the same dimension as $W^s(y)$, so that V is transverse to $W^u(y)$ at q. By the lambda lemma, Theorem 5.7.2, backward iterates of V accumulate on $W^s(y)$. In more detail this means the following. Let $W^s_\delta(y)$ be a local stable manifold of y. Then $W^s(y) = \cup_{n\leq 0} f^n(W^s_\delta(y))$. For N large enough, the point p will lie in $f^{-N}(W^s_\delta(y))$. Now given $\varepsilon > 0$, a sufficiently high iterate of V contains a submanifold that lies within distance ε of the disc $f^{-N}(W^s_\delta(y))$ measured in the C^1 distance. Since $W^s(y)$ intersects $W^u(z)$ transversally in a point p, it follows that a high iterate of V intersects $W^u(z)$ transversally in a point r close to p. This implies that $W^s(x)$ intersects $W^u(z)$ transversally in r: we find that $x \leq z$.