All exercises count equally

1. Brin & Stuck 4.5.6

2. For a continuous map $T : X \to X$ of a compact metric space (X, d), define the invertible extension $\tilde{T} : \tilde{X} \to \tilde{X}$ as follows. Let

   - $\tilde{X} = \{x \in X^{\mathbb{Z}} \mid x(k+1) = T(x(k)) \text{ for all } k \in \mathbb{Z}\}$;
   - $\tilde{T}(x)(k) = x(k+1)$ for all $k \in \mathbb{Z}$ and $x \in \tilde{X}$;

   with metric $d(x, y) = \sum_{k\in\mathbb{Z}} 2^{-|k|} d(x(k), y(k))$. Write $\pi : \tilde{X} \to X$ for the map sending x to $x(0)$. Prove the following.

   (a) $\tilde{T}$ is a homeomorphism of a compact metric space, and $\pi : \tilde{X} \to X$ is a topological factor map.

   (b) If (Y, S) is any homeomorphism of a compact metric space with the property that there is a topological factor map $(Y, S) \to (X, T)$, then $(\tilde{X}, \tilde{T})$ is a topological factor of (Y, S).

   (c) $\pi_* \mathcal{M}_{\tilde{T}}(\tilde{X}) = \mathcal{M}_T(X)$.

   (d) $\pi_* \mathcal{M}^e_{\tilde{T}}(\tilde{X}) = \mathcal{M}^e_T(X)$.

3. Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. We say that T is totally ergodic if T^n is ergodic for all $n \geq 1$. Given $K \geq 1$ define a space $X^{(K)} = X \times \{1, \ldots, K\}$ with measure $\mu^{(K)} = \mu \times \nu$ defined on the product σ-algebra $\mathcal{B}^{(K)}$, where $\nu(A) = \frac{1}{K}|A|$ is the normalized counting measure defined on any subset $A \subset \{1, \ldots, K\}$, and a $\mu^{(K)}$-preserving transformation $T^{(K)}$ by

   $$T^{(K)}(x, i) = \begin{cases} (x, i+1) & \text{if } \ 1 \leq i < K, \\ (Tx, 1) & \text{if } \ i = K \end{cases}$$

   for all $x \in X$. Show that $T^{(K)}$ is ergodic with respect to $\mu^{(K)}$ if and only if T is ergodic with respect to μ, and that $T^{(K)}$ is not totally ergodic if $K > 1$.

4. Show that a weak*-limit of ergodic measures need not be an ergodic measure by the following steps. Start with a point x in the full 2-shift $\sigma : X \to X$ with the property that any finite block of symbols of length l appears in x with asymptotic frequency $\frac{1}{2^l}$ (such points certainly exist; indeed the ergodic theorem says that

almost every point with respect to the $(1/2, 1/2)$ Bernoulli measure will do). Write $(x_1 \dots x_n 0 \dots 0)^\infty$ for the point $y \in \{0,1\}^{\mathbb{Z}}$ determined by the two conditions

$$y|_{[0,2n-1]} = x_1 \dots x_n 0 \dots 0$$

and $\sigma^{2n}(y) = y$. Now for each n construct an ergodic σ-invariant measure μ_n supported on the orbit of y. Show that μ_n converges to some limit ν that is not ergodic.

5. Brin & Stuck 5.2.3

6. Brin & Stuck 5.4.1

7. Brin & Stuck 5.4.4