Bifurcations for random differential equations with bounded noise on surfaces

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Abstract

In random differential equations with bounded noise minimal forward invariant (MFI) sets play a central role since they support stationary measures. We study the stability and possible bifurcations of MFI sets. In dimensions 1 and 2 we classify all minimal forward invariant sets and their codimension one bifurcations in bounded noise random differential equations.

1 Introduction

We will consider bifurcations in a class of random differential equations (RDEs)

$$\dot{x} = f_\lambda(x, \xi_t), \tag{1}$$

as the parameter $\lambda \in \mathbb{R}$ is varied. Here x will belong to a smooth, compact surface X and ξ_t will be a realization of some noise process. We treat such random differential equations with *bounded noise* where ξ_t takes values in a closed disk $\Delta \subset \mathbb{R}^n$. We will assume some regularity conditions on the way the noise enters the equations, in particular, we assume that the range of vectors $f_\lambda(x, \Delta)$ is a convex set for each $x \in X$.

Assuming certain conditions, the RDE admits a finite number of stationary measures. In the case of bounded noise, there may exist more than one stationary measure. The supports of these measures can be identified as closures of *minimal forward invariant* (MFI) sets. See the next section for definitions and properties. MFI sets can be defined and studied by varying ξ_t over the measurable functions $\mathbb{R} \to \Delta$, without assumptions about the noise realizations. MFI sets can in particular be studied for differential inclusions [4], or in a context of control theory where the closures of the MFI sets coincide with *invariant control sets* [8].

A first result in Section 3 in this paper is a classification of the shape of MFI sets for typical RDEs on surfaces. This is worked out in Appendix B in the special setting of normal form linear vector fields in the plane.

A class of examples fitting into our context is constituted by certain degenerate Markov diffusion systems [3, 13] of the form

$$\begin{aligned} dx &= X_0(x)dt + \sum_{i=1}^{m} f_i(\eta)X_i(x)dt, \\ d\eta &= Y_0(\eta)dt + \sum_{j=1}^{l} Y_j(\eta) \circ dW_j, \end{aligned}$$

given by differential equations for the state space variable x from some manifold, driven by a stochastic process η (defined by a Stratonovich stochastic differential equation on a manifold, see e.g. [14]). Under additional conditions, such as a Lie algebra rank condition on the Y_i's and regularity conditions on the f_i's, a set-up as in this paper is achieved. See the above references and [5] which studies near invariance for degenerate Markov diffusion systems.

It was observed that under parameter variation, stationary measures of RDEs can experience dramatic changes, such as a change in the number of stationary measures or a discontinuous change in one of the supports of densities. We refer to such changes a *hard bifurcations* [12]. Given the one to one correspondence between stationary measures and the MFI sets on which they are supported, in order to study hard bifurcations, it is sufficient to study the bifurcations of MFI sets themselves. In the context of random diffeomorphisms a theory of hard bifurcations was developed in [16], see also [17] for extensions to the context of skew product systems.

A further result in this paper is the description of hard bifurcations in typical one parameter families of RDEs on surfaces. Appendix A contains an analysis of bifurcations in the easier setting of one dimension.

Hard bifurcations are related to the phenomenon of near invariance in random dynamical systems, and the resulting effect of metastability. In a control theory framework, near invariance was studied in [5, 10] for RDEs and [6] for random diffeomorphisms. The approach to explore near invariance through bifurcation theory is taken in [6, 16]. See also [5, 10] for some numerical experiments, on control systems under comparable but slightly different conditions than adopted here. For this it becomes important to describe mechanisms that result in hard bifurcations, a purpose of this paper for a class of RDEs on surfaces.

The next section introduces the set-up of this paper; providing background and fixing the class of RDEs on surfaces that we consider. Sections 3 and 4 develop the shape of MFI sets supporting the stationary measures and bifurcations thereof. In two appendices we present results for one dimensional RDEs and a class of randomly perturbed linear planar RDEs.

2 Preliminaries

As explained in the introduction, we consider a class of random differential equations (1). Here we provide the conditions, formulated as Hypotheses below that we assume. For the general framework of random differential equations we refer the reader to L. Arnold's book [1] (see also [8] for background information on control theory).

Let $\Delta \subset \mathbb{R}^2$ be the closed unit disk. We will assume that $f_\lambda(x, v)$ is a smooth vector field depending smoothly on parameters $\lambda \in \mathbb{R}$ and $v \in \Delta$, i.e. $(x, v, \lambda) \mapsto f_\lambda(x, v) \in TX$ is a C^∞ smooth function.

H1. For each x the map $\Delta \to T_x X$ given by $v \mapsto f(x, v)$ is a diffeomorphism with a strictly convex image $D(x) = f(x, \Delta)$.

The time-dependent function ξ_t represents noise, we will consider functions ξ_t as being chosen randomly from the space $\mathcal{U} = \{\xi : \mathbb{R} \to \Delta,\ \xi \text{ measurable}\}$. The flow defined by the shift:

$$\theta^t : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \qquad \theta^t(\xi_s) = \xi_{s+t},$$

is then a continuous dynamical system [8, Lemma 4.2.4] with the weak topology on $\mathcal{U}$.

Since $\xi \in \mathcal{U}$ is measurable, and f is smooth and bounded, the differential equation (1) has unique, global solutions $\Phi_\lambda^t(x, \xi)$ (in the sense of Caratheodory), i.e.:

$$\Phi_\lambda^t(x, \xi) = x + \int_0^t f(\Phi_\lambda^t(x, \xi), \xi_s)\, ds,$$

for any $\xi \in \mathcal{U}$ and all initial conditions x in X, and the solutions are absolutely continuous in t. Furthermore, solutions depend continuously on ξ in the space $\mathcal{U}$. By the assumptions, $\Phi_\lambda^t(\cdot, \xi) : X \to X$ is a diffeomorphism for any λ, ξ, and $t \geq 0$. Further, if ξ_t is continuous then Φ_λ^t is a classical solution.

Denote hereout

$$\Phi_\lambda^t(x, U) = \{y \in X \mid y = \Phi_\lambda^t(x, \xi) \text{ for some } \xi \in U\}$$

and

$$\Phi_\lambda^{-t}(x, U) = \{y \in X \mid x = \Phi_\lambda^t(y, \theta^{-t}\xi) \text{ for some } \xi \in U\}$$

for any set of functions $U \subset \mathcal{U}$.

H2. There exist $r_0 > 0$ and $t_1 > 0$ such that for each $x \in X$,

$$\Phi_\lambda^t(x, \mathcal{U}) \supset B(\Phi_\lambda^t(x, 0), r_0) \qquad \forall t > t_1.$$

We call a set $F \subset X$ *forward invariant* if

$$\Phi_\lambda^t(F, \mathcal{U}) \subset F \tag{2}$$

for all $t > 0$. Denote by $\mathcal{F}$ the collection of forward invariant sets. There is a partial ordering on $\mathcal{F}$ by inclusion, i.e. $E \prec F$ if $E \subset F$. We call $E \subset \mathcal{F}$ a *minimal forward invariant* (MFI) set if it is minimal with respect to the partial ordering $\prec$. The following elementary facts about MFI sets were shown in [12].

Proposition 2.1. *Under assumption* **H2** *an MFI set for (1) is open and connected. The closures of distinct MFI sets are disjoint. If x is any point in an MFI set E, then E is equal to the forward orbit of x, i.e.*

$$E = O^+(x) \equiv \bigcup_{t>0} \Phi_\lambda^t(F, \mathcal{U}).$$

Suppose that we are also given a θ^t-invariant, ergodic probability measure $\mathbb{P}$ on $\mathcal{U}$, i.e. ξ_t is a sample path of the continuous-time measurable dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, \theta^t)$. The topological support of $\mathbb{P}$ may be some proper subset of $\mathcal{U}$. The flow generates a stochastic process with transition probabilities given by

$$\begin{aligned} P_\lambda^t(x, B) &:= \mathbb{P}(\{\xi \in \mathcal{U} : \Phi_\lambda^t(x, \xi) \in B\}) \\ &= \int_{\{\xi : \Phi_\lambda^t(x,\xi) \in B\}} d\mathbb{P}(\xi), \end{aligned} \tag{3}$$

for any Borel set B. Generally, the process cannot satisfy the Markov condition. However, under mild conditions the process defined by the pair (P_λ^t, ξ_t) is Markov [2].

For the current discussion only, we assume the following condition on the noise realizations.

H3. There exists $t_2 > 0$ so that the push forward of $\mathbb{P}$ via $\Phi_\lambda^t(x, \cdot)$ is equivalent to a Riemannian measure m on $\Phi_\lambda^t(x, \mathcal{U})$ for all $t > t_2$ and all $x \in X$.

We will call a probability μ on X *stationary* if it is a stationary measure for the process given by P_λ^t, i.e.

$$\iint \phi(y)\, d_y P_\lambda^t(x, y) d\mu(x) = \int \phi\, d\mu,$$

for all $\phi \in C(X, \mathbb{R})$. A stationary measure μ is called *ergodic* if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(\Phi_\lambda^t(x, \xi)) dt = \int \phi\, d\mu \tag{4}$$

for $\mu \times \mathbb{P}$-a.e. $(x, \xi) \in X \times \mathcal{U}$ and all $\phi \in C(X, \mathbb{R})$. We say that a point $x \in X$ is μ-generic if (4) holds for every $\phi \in C(X, \mathbb{R})$ and for $\mathbb{P}$-a.e. $\xi \in \mathcal{U}$. The set of generic points of a stationary ergodic measure μ is called the *ergodic basin* of μ. An ergodic stationary probability measure whose basin has positive volume will be called a *physical measure*.

Choose $x \in X$ and consider the push forward of $\mathbb{P}$ via $\Phi_\lambda^t(x, \cdot)$. The Krylov-Bogolyubov procedure [1, Theorem 1.5.8] gives that the limit points of the forward time averages are precisely the stationary measures for the process. In the current context, the authors, following [9], showed that the support of each one of these measures is exactly a MFI set:

Theorem 2.2 ([9],[12])**.** *Let (1) be a random differential equation with bounded noise on a smooth, compact manifold X whose flow satisfies* **H2** *and* **H3***. Then there are a finite number of ergodic, physical, invariant probability measures $\mu_1, \ldots, \mu_k$ on X. Each μ_i is supported on the closure of an MFI set E_i. Further, given any $x \in X$ and almost any $\xi \in \mathcal{U}$, there exists $t^* = t^*(x, \xi)$, such that $\Phi_\lambda^t(x, \xi) \in E_i$ for some i and all $t > t^*$.*

Given this theorem, to study changes in the supports of stationary measures, it is sufficient to study bifurcations of MFI sets. Notice that this requires only assumptions on f and Φ_λ^t, not on P^t. Although random differential equations form a natural set-up, MFI sets can be studied without any assumptions about (or even existence of) the probability $\mathbb{P}$ on the noise space.

We now define bifurcation of MFI sets.

Definition 2.3. *A* bifurcation *of MFI sets is said to occur in a parameterized family of random differential equations if either:*

B1 *The number of MFI sets changes.*

B2 *An MFI set changes discontinuously with respect to the Hausdorff metric.*

Definition 2.4. *We will denote by R^∞ the space of bounded noise vector fields f satisfying* **H1** *and* **H2***. We will take as a topology on R^∞ the C^∞ topology on the vector fields $f : X \times \Delta \to TX$.*

Note that we do not include **H3** in this definition. While some assumption such as **H3** is needed to ensure the existence of stationary measures, MFI sets can be studied without this assumption.

Definition 2.5. *We will say that an MFI set E for f is* stable *if there is a neighborhood $U \supset E$ such that if $\tilde{f}$ is sufficiently close to f in R^∞ then $\tilde{f}$ has exactly one MFI set $\tilde{E} \subset U$ and $\tilde{E}$ is close to E in the Hausdorff metric. We will say that $f \in R^\infty$ is* stable *if all of its MFI sets $\{E_i\}$ are stable.*

Definition 2.6. *We say that an MFI set E for* (1) *is* isolated *or* attracting *if for any proper neighborhood U ($\overline{E} \subset U$) there is an open forward invariant set $F \subset U$ such that $\overline{E} \subset F$, F*

contains no other MFI set and $\overline{\Phi^t_\lambda(F,\mathcal{U})} \subset F$ *for all* $t > 0$*. Such an* F *is called an* isolating set *for* E*.*

Proposition 2.7. *Isolated MFI sets are stable.*

Proof. Let E be an isolated MFI set for (1) at $\lambda = \lambda_0$. Suppose E is not stable. The definition of bifurcation allows only a finite number of possible scenarios. These are: (1) E disappears, (2) E merges with one or more other MFI sets, (3) E changes discontinuously in the Hausdorff topology.

We begin with (1). By disappear, we mean that there is a neighborhood $U \supset E$ such that E is the only MFI set in U for (1) at $\lambda = \lambda_0$ and there are values of λ arbitrarily close to λ_0 for which there is no MFI set of (1) that intersects U. For such λ, every point $x \in E$ must have some forward orbit that leaves U, otherwise $O^+(x)$ would be a forward invariant set in U and thus it must contain an MFI set. This contradicts E being isolated since by continuity of solutions, an isolating set F for E will be forward invariant for all λ sufficiently near λ_0.

Next consider case (2) where E merges with one or more other MFI sets. In this case points in E must once again be able to escape from a neighborhood of E after the bifurcation. By the same reasoning as in the previous case this can occur only if E is an MFI set that is not isolated. Note that by Proposition 2.1 the sets which merge must be bounded away from each other before the bifurcation. Thus the bifurcation involves not only a change in number of MFI sets E_i, but also a discontinuous change in $\bigcup_i E_i$ in the Hausdorff topology.

Case (3). Here $E = E_{\lambda_0}$ continues as a parameterized MFI set E_λ for λ near λ_0. We claim that for any $\varepsilon > 0$, E_{λ_0} is contained in a ε-neighborhood of E_λ for λ close enough to λ_0 (this is a lower semicontinuity property of $\lambda \mapsto E_\lambda$). Otherwise there is some point $x \in E_{\lambda_0}$ such that $x \notin E_\lambda$ for some λ near λ_0. This cannot happen due to Proposition 2.1 and continuous dependence of solutions, proving the claim. Now that the claim is established, suppose that E_λ for $\lambda > \lambda_0$ differs from E_{λ_0} in the Hausdorff topology. There must be a point $x \in E_{\lambda_0}$ which escapes a neighborhood of E_{λ_0} for $\lambda > \lambda_0$, similar to the previous cases. This can occur only if E_{λ_0} is not isolated. □

3 Random differential equations in two dimensions

This section examines the shape of MFI sets of planar RDEs (1) satisfying properties **H1** and **H2** introduced above. From **H1**, the vectors $f_\lambda(x,\xi)$ range over a strictly convex set $D^\lambda(x) \equiv f_\lambda(x,\Delta)$ that is diffeomorphic to a closed disk and has a smooth boundary, varying smoothly with x and λ. Define $K^\lambda(x)$ as the cone of positive multiples of vectors in $D^\lambda(x)$. As we are concerned with single RDEs in this section, we suppress dependence on the parameter λ.

Definition 3.1. *A point* $x \in X$ *will be called* stationary *if* $0 \in D(x)$*, i.e. there is a possible vector* *field for which x is fixed. A stationary point x is an* interior stationary point *if* $0 \in intD(x)$.

Note by **H1** and continuity of the vector fields, that an interior point must be in the interior of a set of stationary points. It is clear that an interior stationary point x cannot be on the boundary of a forward invariant set F, since there will exist $\xi \in \Delta$ for which $f(x, \xi)$ points outside of F and solutions from x that leave F. This implies the following:

Proposition 3.2. *Suppose S is a connected set of interior stationary points and F is forward invariant, then either $F \cap S = \emptyset$ or $S \subset F$. In particular, S cannot contain an MFI set as a proper subset.*

If $0 \in \mathrm{int}D(x)$, then $K(x) = \mathbb{R}^2$. Outside the closed set $P = \{x \in X \mid 0 \in D(x)\}$, the cones $K(x)$ depend smoothly on x. By **H1** if $0 \in \partial D(x)$, then $K(x)$ is an open half-plane. Consider the direction fields E_i, $i = 1, 2$, defined by the extremal half lines in the cones $\overline{K(x)}$ over the open set $R = X \setminus P$. By standard results we can integrate these two direction fields, obtaining two sets of smooth solution curves γ_i, $i = 1, 2$ in R. Denote by $\gamma_i^{\pm}$ the forward and backward portions of these curves. Note that these two sets of curves each make a smooth foliation of R. We remark that the direction fields E_i are defined on the closure of R, but may give rise to nonunique solution curves at points in the boundary of R. Further, by the assumptions, the angle between the direction fields at any point is bounded below. However, at stationary points on the boundary, the angle will be π, in which case the solution curves are tangent or coincide (but flow in opposite directions).

We will build up a description of the possible boundary components of an MFI set. To begin, for a point on the boundary either (1) $K(x)$ is less than a half plane, or, (2) $K(x)$ is an open half plane, in which case x must a stationary point, i.e. $f(x, \xi) = 0$ for some $\xi \in \partial\Delta$. We begin by classifying points of type (1).

Lemma 3.3. *If $x \in \partial E$ for an MFI set E and $K(x)$ is less than a half plane, then either:*

- *One of the local solution curves $\gamma_i(x)$ coincides locally with ∂E, or,*
- *Both backward solution curves $\gamma_i^-(x)$ belong to the boundary ∂E.*

Proof: First we claim that if $x \in \partial E$ for an MFI set E and $K(x)$ is less than a half plane, then at most one of γ_i^+ can be contained in ∂E. Suppose not. Then there is a point $x \in \partial E$ that contradicts the conditions. Let y be a point in $\gamma_1^+(x)$ near x and let z be a point in $\gamma_2^+(x)$ near x. Now consider $\gamma_2^+(y)$ and $\gamma_1^+(z)$. For all points y and z near x, $K(y)$ and $K(z)$ are close to $K(x)$. Further, $K(x)$ is not trivial, so it follows that $\gamma_2^+(y)$ and $\gamma_1^+(z)$ must intersect. The segments $\gamma_1^+(x)$, $\gamma_2^+(x)$, $\gamma_2^+(y)$ and $\gamma_1^+(z)$ form a closed curve bounding a region D. See Figure 1(a). Removing D from E results in a new forward invariant region, contradicting the assumption that E is minimal.

Next we claim that if $x \in \partial E$ for an MFI set E and $K(x)$ is less than a half plane, then at least one of γ_i^- must be contained in ∂E. Suppose not and let x be a point such that neither of γ_i^- are contained in ∂E. By the previous claim, renumbering if necessary, let $\gamma_1^+(x)$ be a forward segment that is not contained in ∂E. It follows that $\gamma_1^+(x) \subset E$. Now consider $\gamma_1^-(x)$. By assumption this arc is not contained in $\overline{E}$. Let y be a point near but not in $\gamma_2^-(x)$, also not in $\overline{E}$ and not in any backward solutions from x. By continuity and uniqueness of solutions $\gamma_2^+(y)$ will intersect ∂E at a point y'. Consider $\gamma_2^+(y')$. This segment must be in E or ∂E. If it is in ∂E then it will hit x and this would violate uniqueness of solutions for the vector field that induces γ_2. Next consider the region N bounded by ∂E, $\gamma_1^+(x)$ and $\gamma_2^+(y')$. See Figure 1(b). This region can be removed from E creating a new forward invariant set in E. This contradicts the assumption that E is minimal.

Now using the two claims, without loss of generality we may assume that $\gamma_1^-(x) \in \partial E$. It then follows that $\gamma_2^+ \in E$ since the two extremal vector fields at x form a non-zero angle. Then we are left with two possibilities (1) $\gamma_1^+(x) \in \partial E$ and (2) $\gamma_1^+ \notin \partial E$. In case (2) we are finished; $\gamma_1(x) \in \partial E$. In case (2), since $\gamma_1^+(x)$ cannot be in the complement of E it must be contained in E. By the smoothness of the extremal solutions, if we choose a sufficiently small disk D_ϵ at x, then $\gamma_i(x)$, partitions D_ϵ into four sectors. It follows that the sector bounded by $\gamma_2^-(x)$ and $\gamma_1^+(x)$ must contain points $y \in \partial E$ arbitrarily close to x. For such points it following from smooth dependence of solutions that $\gamma_1^-(y)$ must be contained in ∂E. Since such curves must be arbitrarily close to $\gamma_2^-(x)$ in D_ϵ, it follows that $\gamma_2^-(x) \in \partial E$. □

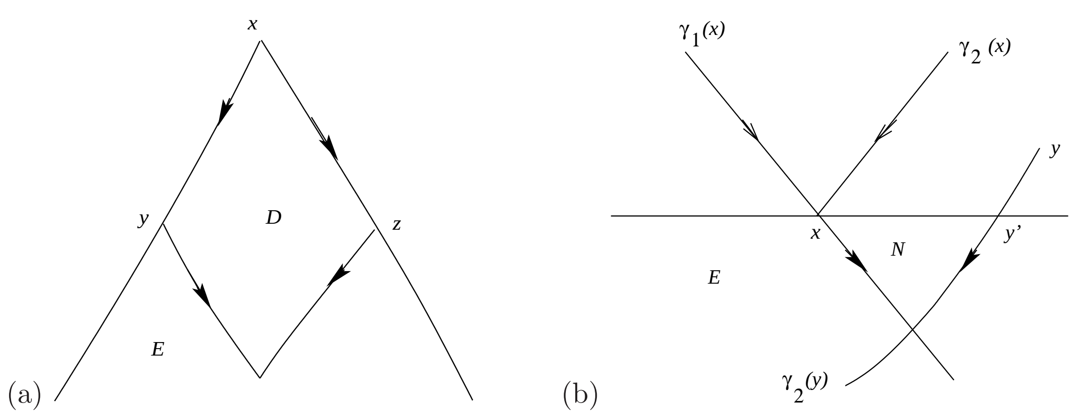

Figure 1: (a) If $D(x)$ is less than a half plane and E is an MFI set, at most one of γ_i^+ can lie on ∂E. (b) If $D(x)$ is less than a half plane and E is an MFI set, at least one of γ_i^- must lie on ∂E.

Definition 3.4. *We call a boundary point, x, of an MFI set, E,* regular *if $x \in R$ and one of $\gamma_i(x)$ coincides locally with ∂E. We call a segment of the boundary of E a* solution arc *if it consists of regular points. If both γ_i^- belong to ∂E locally, then we call x a* wedge *point.*

Lemma 3.5. *Any MFI set, E, has a differentiable boundary, except possibly at a wedge point. Further, this boundary coincides locally with solution curves of the extremal fields.*

Proof: If x is a boundary point of E and x is not a stationary point, then the conclusions hold locally at x by Lemma 3.3.

Suppose that x is a stationary point and ∂E is not differentiable at x. Then there are two sequences $\{y_i\}$ and $\{z_i\}$ in ∂E, each approaching x, but from different limiting directions. It follows that the tangent to the extremal curves at x must not coincide with one of these directions, say corresponding to $\{z_i\}$. For all $\epsilon > 0$ there is a neighborhood U of x so that, working in a local chart, E intersected with U contains a subcone $K_\epsilon(x)$ of $K(x)$ with angle $\pi - 2\epsilon$. Consider the case where $\{z_i\}$ lies in $K(x)$ for i large enough. Take ϵ small enough so that $\{z_i\}$ is contained in $K_\epsilon(x)$ for i large enough. But z_i cannot be both in the interior of E and be a boundary point of E. The case where z_i is outside of $K(x)$ is treated by a similar argument. □

In summary, the boundary of an MFI set E consists of points of the following type:

- Regular solution arc: $K(x)$ is a strictly less than a half plane at x and $\gamma_i \subset \partial E$ for either $i = 1$ or 2.
- Wedge: $K(x)$ is a strictly less than a half plane at x and $\gamma_i^- \subset \partial E$ for both $i = 1, 2$.
- Stationary point: $K(x)$ is a half plane at x.

Proposition 3.6. *If γ is a component of the boundary of an MFI set E and $K(x)$ is strictly less than a half-plane for each $x \in \gamma$, then there is a periodic noise function $\xi^* \in \mathcal{U}$ such that γ is the orbit of a periodic solution of $\dot{x} = f(x, \xi^*(t))$. Further, the Floquet multiplier of this orbit must be less than or equal to 1.*

Proof. If x is such a boundary point, then by the previous proposition, one of the backward arcs is contained in ∂E. We can cover γ by taking the union of all such arcs. By continuity of $D(x)$ with respect to x and the assumption that it contains an open set for each x, all of these arcs belong to a single extremal solution curve. This extremal solution curve is γ and is an actual solution, because the extremal vector field is achieved by the RDE.

Notice that the Floquet multiplier along this periodic orbit is defined, and since the periodic orbit must bound a region, is non-negative. For the arc to be on the boundary of an MFI set the Floquet multiplier of the orbit must be not greater than 1, otherwise interior orbits arbitrarily near the boundary would also have Floquet multipliers greater uniformly than 1 and this would push orbits out of E. □

In light of this proposition we call a component γ of ∂E without stationary points a periodic cycle. Now we turn our attention to stationary points.

Lemma 3.7. *A hyperbolic stationary point x on the boundary of a connected set of stationary points S can be in the boundary of an MFI set E only if x is not a focus, and, one of the eigendirections of $Df(x)$ is tangential to ∂E at x.*

Proof: If x is a focus it is clear then it cannot be on the boundary of an MFI set since all orbits in a neighborhood of x circulate around x.

Next suppose x is a hyperbolic saddle and neither of the eigendirections are tangential to ∂E at x. It is clear that the flow defined by the vector field that makes x stationary would carry points $y \in \partial E$ on one side of x out of E transversely and with nonzero speed.

Suppose x is a nondegenerate node and neither of the eigendirections are tangential to ∂E at x. Then the same holds for all points in S in a neighborhood of x. Consider the vector field corresponding to one of those nearby points and consider the flow generated by this field. It follows that points not on the strong stable manifold of this flow must approach the stationary point tangentially to the weak stable direction. This implies that the flow through x of any of these vector fields on one side of x will flow out of $\overline{E}$.

Suppose x is a degenerate node. Then either there is a single eigendirection, or all directions are eigendirections. If there is a single eigendirection we may argue as in the previous two cases. If all directions are eigendirections, then one of them is tangential to ∂E and we do not exclude x from being on ∂E. $\square$

Lemma 3.8. *There is an open dense set of random differential equations in R^∞ such that for any f in this set all sets of stationary points are diffeomorphic to a disk.*

Proof: The boundary

$$N = \{(x, y) \in TX \mid y \in f(x, \partial\Delta) \subset T_x X\} \tag{5}$$

of the union of sections of $f(\cdot, \xi)$ in TX over $\xi \in \Delta$, is a smooth three dimensional manifold. For random differential equations where N intersects the zero section in TX transversely, this intersection is a finite union of closed simple curves. This property is open and dense by the transversality theorem [11]. $\square$

Theorem 3.9. *There is an open and dense set $V \subset R^\infty$ such that any RDE in V is stable. Further, for any random differential equation in V, an MFI set E has piecewise smooth boundary consisting of regular curves, a finite number of wedge points, and a finite number of stationary points that belong to disks of stationary points inside E. If a component γ is a periodic cycle, it has Floquet multiplier less than one.*

Proof. By Lemma 3.8 there is an open and dense set of RDEs for which the stationary points form domains bounded by smooth closed curves. Consider one of these sets S with boundary ∂S.

Lemma 3.5 gives differentiability of the boundary of E away from wedge points. By applying perturbations to the family $f(x, v)$, $v \in \Delta$, that do not move the stationary points, the RDE in a neighborhood of a boundary point of S can be altered. Doing this, standard theory of differential equations gives an open and dense set U of RDEs so that the following hold on the boundary of S:

1. nonhyperbolic stationary points are isolated, and have one dimensional center directions that are tangential to ∂S,

2. hyperbolic nodes with equal eigenvalues are isolated and are degenerate nodes (have a single eigendirection), and the eigendirection is not tangent to ∂S,

3. nondegenerate hyperbolic nodes (with different eigenvalues) with an eigendirection tangent to ∂S are isolated.

To clarify the statement on nonhyperbolic stationary points on the boundary of S, note that such a stationary point x_0 for (1) at $\xi_t \equiv v_0$ can not be a center, as orbits in a neighborhood of x_0 would then circulate around x_0, and the kernel of $Df(x_0, v_0)$ equals the one dimensional center directions. The assumed transversality of N (see (5)) with the zero section in TX then implies that the center directions are tangential to ∂S.

Consider a RDE from U. A stationary point in the boundary of E cannot be a focus, and at a hyperbolic stationary point x in the boundary of E, $T_x\partial E$ equals an eigendirection at the stationary point x (Lemma 3.7). The above properties imply that stationary points are isolated points in the boundary of E.

It remains to identify an open and dense set V of stable RDEs. We construct V as the subset of RDEs from U, for which in addition

4. there is no equilibrium on the boundary of E that has an unstable direction transverse to ∂E,

5. any periodic cycle on the boundary has Floquet multiplier less than one.

We need to show stability of RDEs from V; openness is a consequence.

Clearly, an MFI set E from a RDE in V with a periodic cycle as boundary component is isolated near this boundary.

Let γ be a boundary component of an MFI set E containing a stationary point. For a stationary point on ∂E, the normal direction has to be stable for it to occur on the boundary of an MFI set. Since the RDE is in U, the stationary points and wedge points on γ are isolated and connected by solution arcs of the extremal flow. Consider one such arc, connecting a stationary point p to either another stationary point or a wedge point (which we call q in both cases).

Let x be a point close to p outside E. The cone $K(x)$ is acute. Draw the solution curves of the extremal flows starting at x. These give two curves going in different directions, following ∂E closely. Each of these curves will either enter E near a wedge point, or end at a stationary point. These curves define a neighborhood of E, except near stationary points that are endpoints of arcs of the extremal flow on ∂E, see Figure 2. Near a stationary point q that occurs as an endpoint of an arc of the extremal flow on ∂E, similarly take a point y and draw the solution curves of the extremal flow starting at y. Because q is normally attracting, these solution curves enter E near q.

The union of the constructed curves define a neighborhood of E that is forward invariant. Stability of the RDEs in V follows from Proposition 2.7. As a consequence, V is open.

Finally we show denseness of V. Consider any RDE. A small perturbation puts the RDE into U. If the RDE is not in V, one of the items defining V does not hold. If item 5. does not hold, i.e. if a periodic cycle with Floquet multiplier one occurs on the boundary of an MFI set, then a small perturbation of the RDE, obtained by a small perturbation of the extremal vector field yielding the periodic cycle, makes the Floquet multiplier smaller than one. If item 4. is violated, a small perturbation near a regular point of the boundary curve of the extremal flow that flows to the equilibrium, see also Figure 3, ensures that the equilibrium no longer lies on the boundary of the MFI set. □

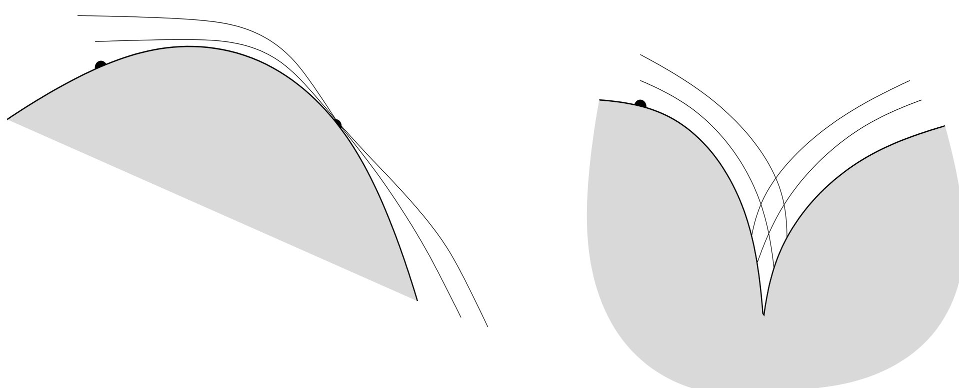

Figure 2: Extremal flow lines near a wedge or stationary point, used to construct a forward invariant neighborhood of E.

4 Bifurcations in two dimensions

In this section we will study bifurcations of MFI sets on surfaces. Appendix A contains an analysis of bifurcations in the easier setting of RDEs on circles.

Definition 4.1. *A* one-parameter family *of RDEs in R^∞ is a mapping from an interval* $(0, 1)$ *given by $\lambda \mapsto f_\lambda$ that is smooth in λ in R^∞.*

Theorem 4.2. *There exists an open dense set $\mathcal{O}$ of one-parameter families of RDEs in R^∞ such that the only bifurcations that occur are one of the following:*

1. *Two sets of stationary points collide at a stationary point on the boundary ∂E which undergoes a saddle-node bifurcation.*
2. *An MFI set E collides with a set of stationary points outside E at a saddle-point p.*
3. *The Floquet multiplier of a non-isolated periodic cycle becomes one and then the cycle disappears.*

Proof. Consider the family of RDEs (1), depending on a real parameter λ. From Theorem 3.9, a family can undergo a loss of stability only at parameter values for which the random differential equation is outside an open, dense $V \subset R^\infty$.

Before treating bifurcations of MFI sets, we consider sets of stationary points of (1). Write $N^0_\lambda \subset X$ for the boundary of the set of stationary points of the RDE (1); N^0_λ is the intersection of the manifold

$$N_\lambda = \{(x, y) \in TX \mid y \in f_\lambda(x, \partial\Delta) \subset T_xX\}$$

with the zero section in TX. Recall (see Lemma 3.8) that for an open and dense set of random differential equations, the sets of stationary points form disks with smooth boundaries. Likewise, invoking the jet transversality theorem [11], the following statements can be seen to hold for one parameter families. For an open and dense set of families, the set

$$M_\lambda = \{(x, y, \lambda) \in TX \times I \mid y \in f_\lambda(x, \partial\Delta) \subset T_xX\} \tag{6}$$

intersects the product of the zero section in TX and I transversely. Note that at nontransverse intersections of M_λ with the zero section in TX, N_{λ_0} has a two dimensional tangent space to the zero section. Here the quadratic terms are nondegenerate for an open and dense set of families. Thus, for an open and dense set $\mathcal{P}$ of one parameter families, N^0_λ consists of finitely many smooth closed curves except at isolated parameter values, where it either has a component with the shape of a smooth figure eight or a component that is an isolated point. This results in one of two following qualitative changes in the set of stationary points:

1. two sets of stationary points merge, at the bifurcation the sets touch at a single point.

2. a set of stationary points is created, at the bifurcation forming a single point.

Now, as formulated in the proof of Theorem 3.9, there are five conditions that define V. Conditions 1.-3., defining an open set $U \subset R^{\infty}$ of which V is a subset, deal with sets of stationary points. A violation of one or more of the conditions 1.-3. may result in a bifurcation only if the stationary points involved lie on the boundary of an MFI set. First, via an arbitrarily small local perturbation of the family (1) one assures that violation of conditions 1.-3. occurs only at isolated parameter values, and only one of these conditions will be violated at such a parameter value.

Violation of condition 2. or 3. may lead to a qualitative change of the boundary flow (on the boundary of an MFI set), but will not lead to a bifurcation. The argument to prove stability of a random differential equation in the proof of Theorem 3.9 applies at and near such a parameter value.

Violation of conditions 1., 4. or 5. leads to bifurcations in generic one parameter families, as discussed below.

1. Collision of two sets of stationary points. Consider the situation of an isolated nonhyperbolic stationary point on the boundary of an MFI set, but where condition 1. is violated. The manifold N_λ has two dimensional tangent directions to the zero section in TX. The scenario where two sets of stationary points, one of them contained in an MFI set, collide and merge results in an explosion of an MFI set by Proposition 3.2.

Next we consider the scenario in which a set of stationary points is created. Suppose x_0 is an isolated stationary point in (1) with $\xi_t \equiv v_0$ at $\lambda = \lambda_0$. Then x_0 will, for an open and dense set of families, be a saddle-node point with one dimensional center directions: in the (λ, v)-parameter space (with v from an open neighborhood of Δ), $\{\lambda_0\} \times \partial\Delta$ will be a closed curve that is tangent to a surface of saddle-node bifurcations for the vector fields $f_\lambda(\cdot, v)$. The tangency will be from the side of the surface for which no stationary points exist near x_0. (For the merging of two sets of stationary points likewise one typically finds a tangency from the side of the surface with two stationary points near x_0.) Hence there are one dimensional center directions for the vector field $f_{\lambda_0}(\cdot, v_0)$ at x_0. Since x_0 is on the boundary of an MFI, $f_{\lambda_0}(\cdot, v_0)$ further possesses one dimensional normally hyperbolic stable directions at x_0. The MFI set E will, for an open and dense set of families (1) be tangent to the center directions at x_0 (note that the extremal curves defining the boundary of E may be perturbed in a deleted neighborhood of x_0 to achieve this). Arguments as in the proof of Theorem 3.9 establish stability of the family (1).

2. Collision of stationary points with regular flow. Consider the situation where condition 5. is not satisfied. The stationary point must be a saddle, connected with an external disk of saddles. γ collides with this external disk of saddles. This results in a bifurcation of E. This is a saddle colliding with a solution arc. To show that a saddle colliding with a solution arc of the boundary of E results in a bifurcation consider the following. At the moment of collision, the incoming portion of the arc must approach the saddle-point. This can only occur along one of the branches of the stable manifold of the point. For parameter values nearby, the arc will be first on one side of the stable manifold and then on the other side. Once it is on the side away from E, points on the arc can be carried away from E by the vector field giving rise to the saddle point. A bifurcation occurs.

There is one remaining possibility for a qualitative change of the boundary flow of an MFI set, where a stationary point (a node) disappears and a wedge point appears.

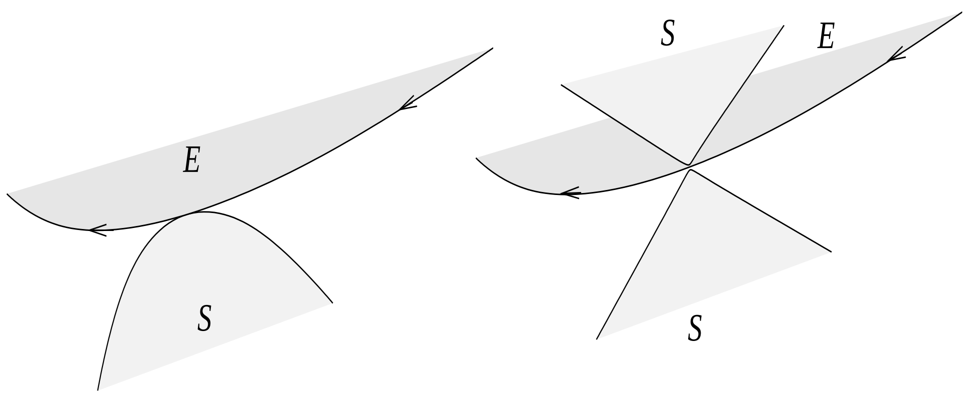

Figure 3: Pictures of bifurcations, a collision of stationary points with regular flow on the left and collision of sets of stationary points on the right.

3. Saddle-node bifurcation of boundary periodic orbit. Finally, consider the case where the component involved in the bifurcation is a periodic cycle. If the velocity of this orbit is bounded away from zero and the Floquet multiplier is less than one, then this periodic orbit will persist under small perturbations. This follows from the usual theory of periodic orbits of differential equations. The extremal vector field giving rise to the periodic orbits varies smoothly and so it will have stable periodic orbits near γ which will bound approximately the same area as γ and so a bifurcation cannot take place.

On the other hand, if the Floquet multiplier becomes greater than one then a bifurcation takes place because the period curve γ disappears. Again from the usual theory of periodic solutions of differential equations, for generic families, if the Floquet multiplier passes through 1 the periodic orbit of the extremal vector field disappears and it will have orbits escaping from a neighborhood of the original periodic orbit, cf. for instance [15]. $\square$

A Appendix: random differential equations in one dimension

If X is one dimensional, i.e. a circle, then by Proposition 2.1 an MFI set is either the entire circle or an open interval. In this context we may assume that the set of possible noise values is $\Delta = [-\epsilon, \epsilon]$. Also note that under our assumptions for each x, $f(x, \Delta)$ is a closed interval with endpoints $f(x, -\epsilon)$ and $f(x, \epsilon)$. Thus there is an envelope of all possible vector fields which are bounded below and above by $f(\cdot, -\epsilon)$ and $f(\cdot, \epsilon)$. Denote by $f_-(\cdot)$ and $f_+(\cdot)$ the upper and lower vector fields.

Proposition A.1. *If (a, b) is an MFI set then for any $x \in (a, b)$,*

$$0 \in int(f(x, \Delta)).$$

Proof. If not, then there is $x \in (a, b)$ such that either $f(x, \Delta) \leq 0$ or $f(x, \Delta) \geq 0$. In the first case the forward invariance of (a, b) implies that (a, x) is forward invariant. In the second case we obtain that (x, b) is forward invariant. Either case contradicts the minimality of (a, b). □

Proposition A.2. *If (a, b) is an MFI set then*

$$f(a, \xi) \geq 0 \quad \text{and} \quad f(b, \xi) \leq 0 \tag{7}$$

for all $\xi \in \Delta = [-\epsilon, \epsilon]$ and that $f_-(a) = 0$ and $f_+(b) = 0$. Further, $f'_-(a) \leq 0$ and $f'_+(b) \leq 0$.

Proof. The inequalities (7) are necessary for a and b to be boundary points of an MFI set. The claim that $f_-(a) = f_+(b) = 0$ follows from **H1**. The final claim $f'_-(a) \leq 0$ and $f'_+(b) \leq 0$ then follows from the assumption that f is C^1. □

We can distinguish the following types for endpoints a and b based on the properties of f'. We say that a is *hyperbolic* if $f'_-(x) \neq 0$ and similarly for b. Otherwise, a or b is said to be non-hyperbolic.

Proposition A.3. *Given any f satisfying* **H1**, **H2** *suppose that (a, b) is an MFI set with both a and b hyperbolic. Then (a, b) is isolated with some isolating neighborhood W. Further, if $\hat{f}$ is sufficiently close to f in the C^1 topology, then $\hat{f}$ has a unique MFI set $(\hat{a}, \hat{b})$ inside W. Further, $\hat{a}$ and $\hat{b}$ are close to a and b respectively and are each hyperbolic.*

Proof. If a is hyperbolic it follows that $f(x, \xi) > 0$ for all x in some neighborhood (c, a) and all $\xi \in \Delta$. Similarly, there is a neighborhood (b, d) on which $f(x, \xi) < 0$. It follows that $W = (c, d)$ is an isolating neighborhood for (a, b).

Now let $\delta > 0$ be sufficiently small so that $f'_-(x) > f'_-(a)/2$ for all $x \in [a - \delta, a + \delta]$. If $\hat{f}$ is within $f'_-(a)/2$ of f in the C^1 topology then the conclusion holds. □

Definition A.4. *We say that a one-parameter family of vector fields* $g_\lambda(x)$ generically unfolds a quadratic saddle-node *point at* $x^* = a$ *or* b *of an MFI set* (a, b)*, if* $g(x^*) = 0$*,* $g'(x^*) = 0$*,* $g''(x^*) \neq 0$ *and* $\partial g_\lambda(x^*)/\partial\lambda \neq 0$*. A RDE (1) on the circle* generically unfolds a quadratic saddle-node *at* x^* *if one of the extremal vector fields* $f_\lambda(\cdot, \pm\epsilon)$ *generically unfolds a quadratic saddle-node at* x^**.*

(a) (b)

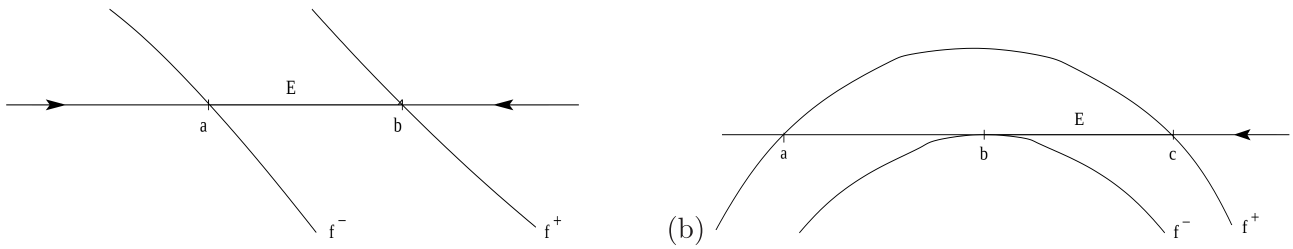

Figure 4: (a) A stable one dimensional MFI set. Both endpoints of $E = (a, b)$ are hyperbolic. (b) A random saddle-node in one dimension. $E = (b, c)$ is minimal forward invariant.

Theorem A.5. *In a generic one-parameter family of one-dimensional bounded noise random differential equations (1) the only codimension one bifurcation of an MFI set is the generic unfolding of a quadratic saddle-node.*

Proof. By Proposition A.3 an MFI set (a, b) is stable if a and b are both hyperbolic. Thus a bifurcation can occur only if hyperbolicity is violated at one of the endpoints. For codimension one hyperbolicity cannot be violated at both the endpoints simultaneously.

If the stationary point is odd, a standard argument shows that the bifurcation is not codimension 1. If the stationary point is of even order ≥ 4, then standard arguments show that the family is not generic. □

B Appendix: MFI sets for perturbations of linear normal forms

In order to demonstrate that MFI sets with non-trivial boundaries can occur, we will consider the MFI sets associated with perturbations of linear normal form vector fields in the plane.

Consider a linear vector field where the origin is asymptotically stable. The equations can then be reduced to a normal form $\dot{x} = -\Lambda x$ where the matrix Λ has one of the following forms:

$$\Lambda_1 = \begin{pmatrix} 1 & -b \\ b & 1 \end{pmatrix}. \quad \Lambda_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where $0 < a < 1$. Then consider the perturbed system $\dot{x} = -\Lambda x + \xi_t$ where $\xi_t \in \Delta = B_\epsilon(0)$.

Example 1. Perturbed linear stable focus

Consider a linear vector field $\dot{x} = -\Lambda_1 x$ where Λ_1 is as above. If we perturb this normal form by additive noise uniformly distributed in a ball of radius ϵ, then for each $\xi \in \Delta$ there is a stationary point x^* given by $x^* = \Lambda_1^{-1}\xi$. One can calculate directly that these points form a ball of radius $\epsilon/\sqrt{1+b^2}$ in the phase plane. However, by plugging into the equation one finds that this set is not forward invariant since at the boundary vectors are possible which have a positive projections onto the outward normal. By considering

$$(\Lambda_1 x + \xi) \cdot x \leq 0$$

one finds that the ball of radius ϵ is the minimal forward invariant set. At each point on the boundary of this disk, there is an extremal vector tangent to the boundary in the counterclockwise direction. Thus the boundary consists of a single periodic cycle.

Example 2. Perturbed linear stable node with repeated eigenvalues but distinct eigenvectors

In this case it is easy to see that the set of stationary points is just the disk of radius ϵ. Using the radial symmetry and equation $(\Lambda_2 x + \xi) \cdot x \leq 0$ we find that this disk is forward invariant and thus is the MFI set. Thus the boundary consists of a single closed arc of stationary points. Note that this case is unstable in the sense that the normal form can be made into any of the other three normals forms by arbitrarily small linear perturbations.

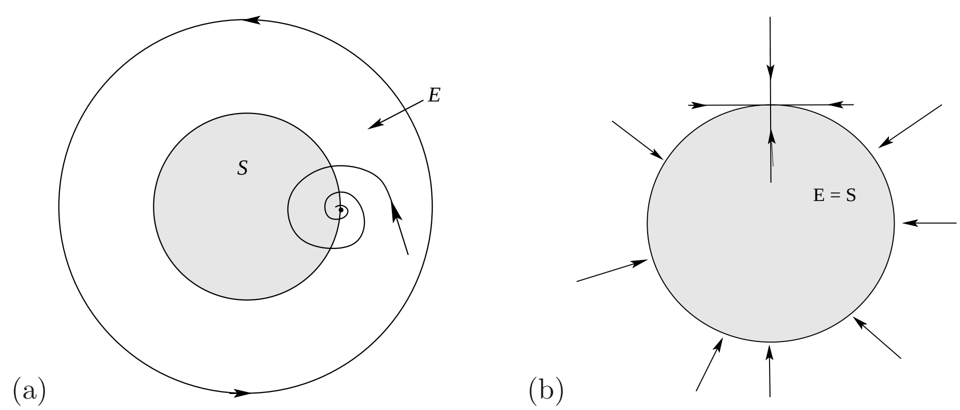

Figure 5: (a) Set of equilibria S (grey) **!** and MFI set E for a perturbed linear stable focus. (b) The set **!** of equilibria S (grey) coincides with the MFI set E for a perturbed linear radially symmetric node.

Example 3. Perturbed linear stable node with distinct eigenvalues

In this case solving the equation $x^* = \Lambda_3^{-1}\xi^*$ for each $\xi^* \in \Delta$ we obtain an elliptical disk of stationary points with minor axis ϵ in the x_1 direction and major axis ϵ/a in the x_2 direction, in other words, solutions of $x_1^{*2} + a^2 x_2^{*2} \leq \epsilon^2$. Denote this ellipse by S. In terms of ξ^* the normal vector

n to the ellipse is $n = \langle \xi_1^*, \xi_2^* \rangle$. We note that the expression for all possible vectors at x^* in terms of ξ^* is $\dot{x} = -\xi^* + \xi$. We find that $n \cdot \dot{x}$ may be rewritten as $-|v^*|^2 + v^* \cdot v$ where $v^* = \langle \xi_1^*, \sqrt{a}\xi_2^* \rangle$ and $v = \langle \xi_1, \sqrt{a}\xi_2 \rangle$ and $\xi_1^2 + \xi_2^2 = \epsilon^2$. Thus $v^* \cdot n$ is non-positive only at the apexes of the ellipse $x^* = (\pm\epsilon, 0)$ and $x^* = (0, \pm\epsilon/a)$. Thus this ellipse of stationary points is not forward invariant and by Proposition 3.2 it cannot contain an MFI set. However it is easy to check that the rectangle $R = [-\epsilon, \epsilon] \times [-\epsilon/a, \epsilon/a]$ is forward invariant. The rectangle R cannot be MFI by Lemma 3.5 since its boundary has corners. Thus there is an MFI set E that is strictly contained in R and strictly contains the ellipse S of stationary points. Now since $R \setminus S$ contains no stationary points, the boundary of E in this region must consist of solutions arcs. Since the apexes of the ellipse S are also on the boundary of R it follows that they are on the boundary of E. Since E is connected and bounded by differentiable curve there must be solution arcs connecting the apexes. The direction of these connecting orbits is from the major axes to the minor axes. This can be seen by considering that the noise ξ_t must approach the noise giving rise to the stationary points as those points are approached both in forward and backward time. Only the stationary points at the minor axes have solution arcs nearby that approach along the directions tangential to the ellipse, thus these are the limits in forward time. See figure 6(a).

(a)

(b)

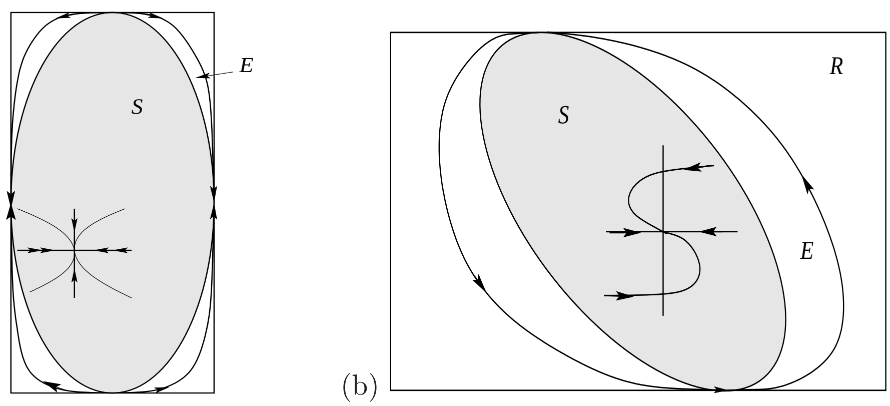

Figure 6: (a) Set of equilibria S (grey) and MFI set E for a perturbed linear generic node. (b) Set of equilibria S (grey) and MFI set E for a perturbed linear degenerate node.

Example 4. Perturbed stable nodes with a single eigenvector

Solving for the set of stationary points $x^* = \Lambda_4^{-1}\xi$ yields the ellipse $(x+y)^2 + y^2 \leq \epsilon^2$. Investigating the projection of the vector fields on the outward normal as in Case 3, we find that the projection is non-positive only at the points $(\epsilon, -\epsilon)$ and $(-\epsilon, \epsilon)$. Thus this ellipse of stationary points, S, is not forward invariant. Again by Proposition 3.2 it cannot contain an MFI set. However, the rectangle $R = [-2\epsilon, 2\epsilon] \times [-\epsilon, \epsilon]$ is easily shown to be forward invariant. Thus R contains an MFI set E which in turn contains S. Analogous in Case 3 the points $(\epsilon, -\epsilon)$ and $(-\epsilon, \epsilon)$ are on the boundary of E and connected by solution arcs in $R \setminus S$. By a similar argument as in Case 3, the direction of both of these arcs is counter-clockwise. See figure 6(b).

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