A GEOMETRIC CRITERION FOR POSITIVE TOPOLOGICAL ENTROPY II: HOMOCLINIC TANGENCIES

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Abstract. In a series of important papers [GS1, GS2] Gavrilov and Shilnikov established a topological conjugacy between a surface diffeomorphism having a dissipative hyperbolic periodic point with certain types of *quadratic* homoclinic tangencies and the full shift on two symbols, thus exhibiting horseshoes near a tangential homoclinic point. In this note, which should be viewed of as an addendum to [BW], we extend this result by showing that such a diffeomorphism with a homoclinic tangency having *any* order contact, possible with *infinite* order contact, possesses a horseshoe near the homoclinic point.

I: Introduction

Homoclinic tangencies and their bifurcations play a fundamental role in Dynamical Systems [PT]. For instance, Palis has conjectured [P] that the set of (non-uniformly) hyperbolic surface diffeomorphisms together with the diffeomorphisms exhibiting homoclinic tangencies are dense in the space of all surface diffeomorphisms. Systems exhibiting homoclinic tangencies can exhibit more complicated and more subtle quasi-local behavior than systems possessing transverse homoclinic points or homoclinic points with topological crossings. For instance, Gavrilov and Shilnikov [GS1, GS2] showed that horseshoes (locally maximal hyperbolic sets) may exist near the homoclinic tangency. Newhouse [N1, N2] (see also [GST]) showed that these homoclinic tangencies typically generate secondary tangencies which persist under small perturbations, and that the Gavrilov and Shilnikov horseshoes may co-exist with infinitely many sinks in a neighborhood of the homoclinic orbit.

In a series of important papers [GS1, GS2] Gavrilov and Shilnikov established a topological conjugacy (on a closed invariant set) between a surface diffeomorphism having a dissipative hyperbolic periodic point with certain types of *quadratic* homoclinic tangencies (see Figures 2c and 2d) and the full shift on two symbols, thus exhibiting horseshoes near a tangential homoclinic point. In this note, which should be viewed of as an addendum to [BW], we extend this important result by showing that such a diffeomorphism with a homoclinic tangency having *any* order contact (possible with *infinite* order contact), possesses a horseshoe near

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the homoclinic point. Such a map has positive topological entropy and possesses infinitely many hyperbolic periodic points near the homoclinic tangency.

In [BW] the authors consider a surface diffeomorphism with a hyperbolic periodic point such that components of the stable and unstable manifolds have a *topological* crossing, possible with infinite order contact. They prove that some power of the diffeomorphism has the full shift on two symbols as a topological factor. It follows from a remarkable theorem of Katok [K1, K2] that the map possesses a horseshoe. This result extends the well known theorem of Smale [S], where one assumes that the intersection is transversal and one obtains a topological conjugacy on a closed invariant set between some power of the map and the full shift on two symbols. The conjugacy immediately implies that the map possesses a horseshoe. The idea of the construction in [BW] is to find a *horseshoe-like picture* in the dynamics and to code the dynamics as if coding on a horseshoe. One then proves that this coding map is a factor map and thus the diffeomorphism has the full shift on two symbols as a topological factor. Then Katok's theorem implies the existence of horseshoes arbitrarily close to a homoclinic tangency. Using this technique, we avoid having to verify difficult uniform contracting/expanding cone estimates to directly prove the existence of a hyperbolic set.

II: Homoclinic Tangencies

Let M denote a smooth (C^2) surface and $f\colon M \to M$ a smooth (C^2) surface diffeomorphism. Let p be a hyperbolic periodic point (which by considering some iterate of the map we will assume is a fixed point) and assume that $|\lambda\,\mu| < 1$, where $|\lambda| > 1$ and $|\mu| < 1$ are the two eigenvalues of the differential Df_p. We will call such a periodic point **dissipative.**[1] Also suppose that the map f possesses a tangential homoclinic point q (with order of tangency $2 \le 2l \le \infty$). Let $\mathcal{U}$ be a small neighborhood of the orbit of q consisting of finitely many balls (including one ball containing p). In this context *small* means that the sum of the diameters of the balls is sufficiently small. An important problem is to describe the set of points $\mathcal{U}_f$ whose orbits are entirely contained in $\mathcal{U}$.

Figure 1 illustrates four different types of homoclinic tangencies. It is not difficult to show that for cases (i) and (ii), the set $\mathcal{U}_f$ contains only the orbit of q and the fixed point p [AH, GS1, GS2]. However, for cases (iii) and (iv), the dynamics in $\mathcal{U}_f$ is much more complicated and we will show that in these cases, $\mathcal{U}_f$ contains horseshoes. We note that a precise description of $\mathcal{U}_f$ is quite difficult to provide since $\mathcal{U}_f$ may also contain non-hyperbolic orbits and infinitely many sinks.

We quickly recall the main technical result in [BW, Theorem 2.4] and the related definitions. Let $N \subseteq M$ be homeomorphic to $[-1,1]\times[-1,1]$. In the following we shall identify N with $[-1,1]\times[-1,1]$ and suppress the homeomorphism. Let $R = [-1,1]\times[-\rho,\rho]$, where $\rho \in (0,1)$.

A set $S \subset R$ will be called a **horizontal strip** if

(1) S is closed and path connected,
(2) S contains a curve joining the left edge $\{-1\}\times[-\rho,\rho]$ and the right edge $\{1\}\times[-\rho,\rho]$ of R,
(3) ∂S is a Jordan curve which is the union of a finite number of arcs all of whose endpoints lie on the left edge or the right edge of R.

[1] The case where $|\lambda\,\mu| > 1$ can be reduced to the dissipative case by considering the inverse f^{-1}.

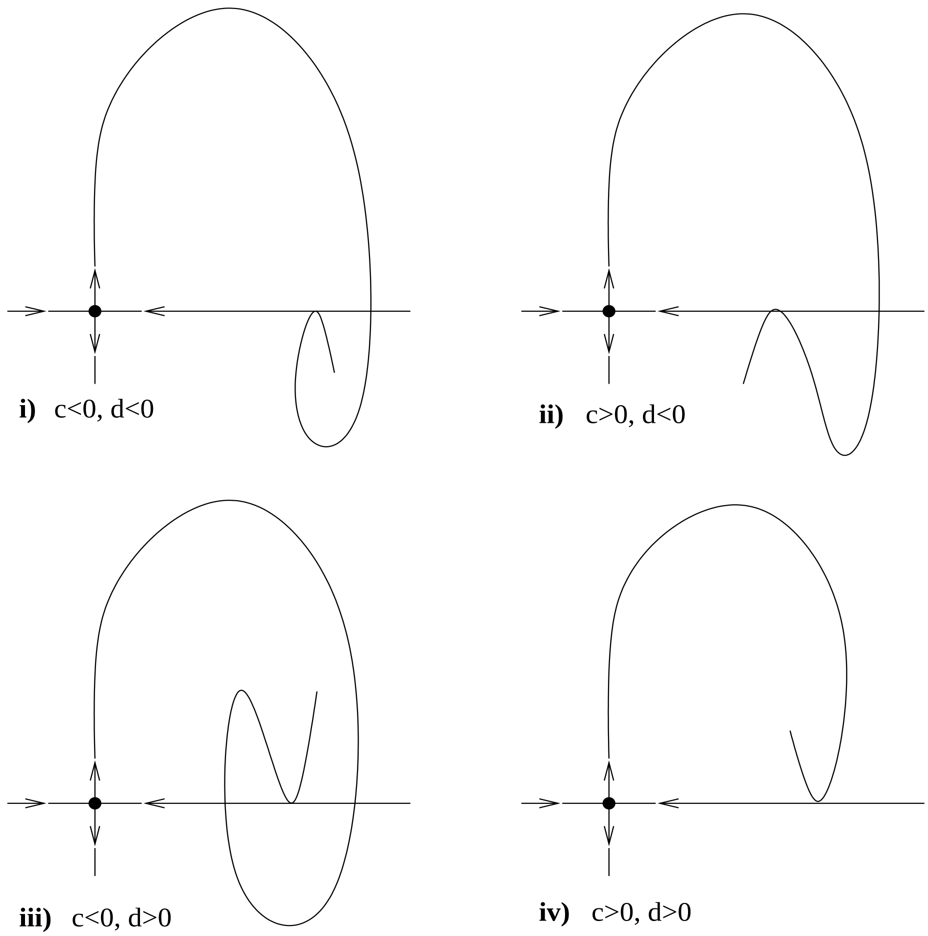

Figure 1: Four types of homoclinic tangencies

It is easily seen that ∂S contains exactly two curves joining the left edge $\{-1\} \times [-\rho, \rho]$ and the right edge $\{1\} \times [-\rho, \rho]$ of R, and S lies in the region of R bounded by these curves (see pictures in [BW]). We shall call the curve on which the second coordinate is larger c_{upper} and the other curve c_{lower}.

Definition. Let n be a positive integer and S a horizontal strip. We shall say that f^n **stretches** S **across** R if $f^n S \subset \operatorname{Int} N$, $f^n(\partial S \cap \operatorname{Int} R) \subset N \setminus R$, and f^n maps c_{upper} and c_{lower} into opposite components of $N \setminus R$.

Theorem BW 2.4. *Suppose N contains two disjoint closed horizontal strips S_0 and S_1 that are stretched across R by f^{n_0} for some $n_0 \geq 1$. Then f^{n_0} has the full two shift as a topological factor.*

It is convenient to work in $C^{1+\alpha}$ *linearizing* coordinates. Let $f\colon M \to M$ be a surface diffeomorphism with a hyperbolic fixed point p and homoclinic tangency q having $2l$-order contact. This means that components of the stable and unstable manifolds of p, $W^s(p)$ and $W^u(p)$, are tangent at q and the tangency has order $2l$. By choosing a suitable basis for the tangent space T_pM at p, we may think of $df(p)$ as a linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$, contracts the first $\mathbb{R}$ by a factor of μ and expands the second $\mathbb{R}$ by a factor of λ. By the Hartman-Grobman theorem, there is a neighborhood N of p and a homeomorphism h of N into $\mathbb{R}^2$ with $h(p) = (0,0)$ such that if $x \in N$ and $f(x) \in N$, then $h(f(x)) = L(h(x))$. One can choose h arbitrarily close to the identity by chosing the neighborhood N sufficiently small. Thus the *type* of homoclinic tangency, i.e., case (i), (ii), (iii) or (iv), is unchanged under these orientation preserving homeomorphims.

It also follows from a theorem of Belitski [B] that the homomorphism h may be chosen to be a $C^{1+\alpha}$ diffeomorphism for some $0 < \alpha < 1$, and we will use this $C^{1+\alpha}$ linearization in our proof. It was previously shown by Hartman [H] that the homomorphism h may be chosen to be a C^1 diffeomorphism.

We may assume that N and h have been chosen so that h is $C^{1+\alpha}$ and $h(N) = D(1)\times D(1)$, where $D(r)$ is the closed disc of radius r about the origin in $\mathbb{R}$. We may also assume that $D(1) \times \{0\}$ and $\{0\} \times D(1)$ lie in $W^s(p) \cap N$ and $W^u(p) \cap N$ respectively. We also assume that the point of homoclinic tangency q lies in N. In order to simplify our notation, we shall henceforth identify N with $D(1) \times D(1)$ and suppress the homeomorphism h. Distances in N will be measured with respect to the product of the Euclidean metric on $D(1)$.

We may assume that a point of homoclinic tangency has coordinates $(q, 0)$ and lies on $W^s(p) \cap N$, and that some preimage has coordinates $(0, r)$, where $r = f^{-n_0}(q, 0)$ lies on $W^u(p) \cap N$. We choose small neighborhoods $V \subset N$ of $(r, 0)$ and $W \subset N$ of $(q, 0)$ and we wish to study iterates f^{n+n_0} of the map f restricted to W by decomposing the map $f^{n+n_0}\colon W \to N$ into the linear action of $f^n \asymp L^n\colon W \to V$ defined by $L^n(x, y) = (\mu^n x, \lambda^n y)$ and a *global* mapping $G\colon V \to W$ defined by

$$G(x,y) = (q + ax + b(y-r) + O(x^{1+\alpha}) + O(|y-r|^{1+\alpha}), cx + g(y-r) + O(x^{1+\alpha})),$$

where $a, b, c \in \mathbb{R}$. If the homoclinic tangency has order of contact $2 \leq 2l < \infty$ the function $g(y-r) = d(y-r)^{2l} + O(x^{2l(1+\alpha)}) + O(|y-r|^{2l(1+\alpha)})$, $d > 0$, and if the tangency has infinite order contact the function $g(y-r)$ is infinitely flat at $y = r$ (derivatives of all orders vanish) and has constant sign on $V \cap W^u(p)$ except at $y = r$. Clearly $G(0, r) = (q, 0)$. We note that case (iii) corresponds to $c < 0$ and case (iv) corresponds to $c > 0$.

Consider the family of small rectangles R_n near $(q, 0)$ with vertices $(q \pm \epsilon, r/\lambda^n \pm \omega^n/2)$, where $\epsilon > 0$ is sufficiently small, $\mu < \omega < 1/\lambda$ (to be chosen later), and n sufficiently large to insure that $L(R_n) \subset V$. Let us study the image $f^{n_0+n}(R_n) = (G \circ L^n)(R_n)$ (see Figure 2).

We make the following three observations which will imply that for cases (iii) and (iv), for ω sufficiently close to $1/\lambda$ and $n = n(\omega)$ sufficiently large, the image $f^{n_0+n}(R_n)$ intersects R_n in a horseshoe-like picture (see Figure 2). Clearly the two shaded regions in Figure 2 are blowups of the actual (much small) regions around q and r.

(1) The abscissa of points in $f^{n_0+n}(R_n)$ is contained in $q \pm a\mu^n(q \pm \epsilon) + b(\pm\lambda^n\omega^n/2) + O(|\mu^n, \lambda^n\omega^n|^{1+\alpha})$. Since $|\lambda\omega| < 1$, it follows that for n sufficiently large the abscissa of points in $f^{n_0+n}(R_n)$ is contained in $q \pm \epsilon$.

(2) The ordinate of points in $f^{n_0+n}(R_n \cap \{y = r/\lambda^n\})$ is contained in $c(q\pm\epsilon)\mu^n + O(\mu^{(1+\alpha)n})$. Our assumptions imply that $c(q\pm\epsilon)\mu^n + O(\mu^{(1+\alpha)n}) << r/\lambda^n - \omega^n/2$ for n sufficiently large. Using (1) we see that the points in $f^{n_0+n}(R_n \cap \{y = r/\lambda^n\})$ lie *far* below R_n.

(3) The ordinate of points in $f^{n_0+n}(R_n \cap \{y = r/\lambda^n \pm \omega^n/2\})$ is at least $c(q \pm \epsilon)\mu^n + g(\pm\lambda^n\omega^n/2) + O(\mu^{(1+\alpha)n})$. We can choose ω sufficiently close to $1/\lambda$ and n sufficiently large to insure that

$$q/\lambda^n + \omega^n/2 < c(q \pm \epsilon)\mu^n + g(\pm\lambda^n\omega^n/2) + O(\mu^{(1+\alpha)n}).$$

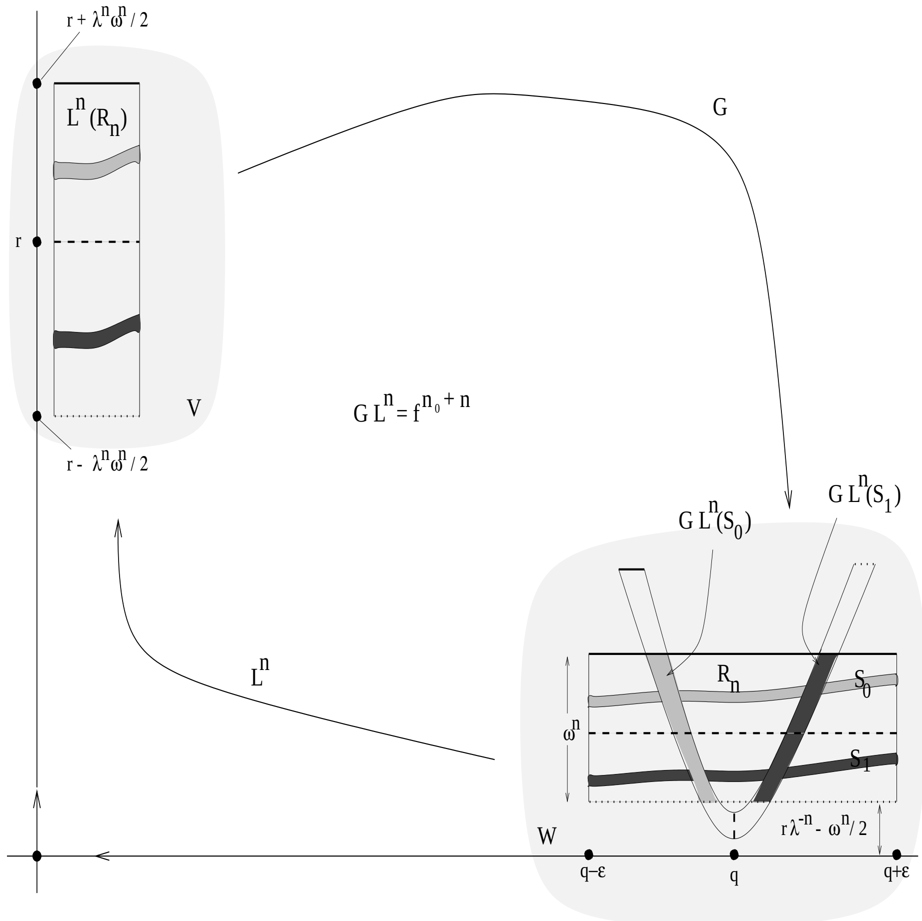

Figure 2: Creation of the horseshoe

As illustrated in Figure 2, it is easy to find two disjoint closed horizontal strips $S_0 = S_0(n)$ and $S_1 = S_1(n)$ contained in R_n with $S_0(n)$ lying in the top half of R_n and $S_1(n)$ lying in the bottom half of R_n, such that the image under f^{n_0+n} of these two strips are strethed across R_n. We have thus proven the following proposition.

Proposition 1. *Let $f : M \to M$ be a surface diffeomorphism, p a hyperbolic periodic point and q a point of homoclinic tangency of the type illustrated in Figure 1 (iii) or Figure 1 (iv). For ω sufficiently close to $1/\lambda$ there exists $n = n(\omega)$ sufficiently large such that R_n contains two disjoint horizontal strips S_0 and S_1 that are stretched across R_n by f^{n_0+n}.*

Applying Theorem BW 2.4, we obtain the following theorem.

Theorem 1. *Let $f : M \to M$ be a surface diffeomorphism, p a hyperbolic periodic point and q a point of homoclinic tangency of the type illustrated in Figure 1 (iii) or Figure 1 (iv). For ω sufficiently close to $1/\lambda$ and $n = n(\omega)$ sufficiently large, there is a closed invariant set contained in R_n on which f^{n_0+n} has the full two shift as a topological factor.*

Since the topological entropy of a topological factor of a map is no less than the topological entropy of the map, Theorem 1 immediately implies that the topological entropy of f^{n_0+n} restricted to R_n is positive (more precisely, at least $\log 2$). Applying Katok's theorem [K1, K2] on the existence of horseshoes which carry most of the entropy for a surface diffeomorphism, we obtain the existence of horseshoes near the homoclinic tangency.

Corollary 1. *Let $f : M \to M$ be a surface diffeomorphism, p a hyperbolic periodic point and q a point of homoclinic tangency of the type illustrated in Figure 1 (iii) or Figure 1 (iv). For ω sufficiently close to $1/\lambda$ and $n = n(\omega)$ sufficiently large, there are horseshoes contained in R_n. Furthermore, for any $\delta > 0$, the map f^{n_0+n} restricted to R_n possesses a horseshoe which carries topological entropy at least $\log 2 - \delta$.*

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