Invariant manifolds near hyperbolic fixed points

Ale Jan Homburg

Abstract

In these notes we discuss obstructions to the existence of local invariant manifolds in some smoothness class, near hyperbolic fixed points of diffeomorphisms. We present an elementary construction for continuously differentiable invariant manifolds, that are not necessarily normally hyperbolic, near attracting fixed points. The analogous theory for invariant manifolds near hyperbolic equilbria of differential equations is included. For differential equations we include a construction of one dimensional invariant manifolds in a higher smoothness classes.

1 Introduction

We will consider various invariant manifolds near hyperbolic fixed points of dynamical systems obtained by iterating diffeomorphisms. As this is a local study, we may restrict to diffeomorphims on $\mathbb{R}^n$ with a fixed point at the origin 0. That is, we assume the following context: f is a smooth diffeomorphism on $\mathbb{R}^n$ with a hyperbolic fixed point at the origin 0. Recall that hyperbolic means that the eigenvalues of $Df(0)$ are away from the unit circle in the complex plane. By smooth is understood C^∞ or C^k for some (sufficiently) high value of k. Center manifolds near nonhyperbolic fixed points are not considered, we refer to [16] for this topic.

It is well known that if the origin is a hyperbolic fixed point, then near the origin there are local stable and unstable manifolds W^s, W^u characterized by

$$\begin{aligned} W^s &= \{x;\ \lim_{i\to\infty} f^i(x) = 0\}, \\ W^u &= \{x;\ \lim_{i\to-\infty} f^i(x) = 0\}. \end{aligned}$$

The tangent space T_0W^s of W^s at the origin is the sum of the generalized eigenspaces of eigenvalues with absolute value smaller than 1. For two real numbers $l < r$, let E_1 be the sum of generalized eigenspaces of all eigenvalues with absolute value in the interval (l, r). Standard invariant manifold theory gives the existence of an invariant manifold W_1 with tangent space E_1 at the origin [12]. For $(l, r) = (0, r)$ with $r < 0$ and not containing all eigenvalues with absolute value smaller than 1, W_1 is a strong stable manifold. For $(l, r) = (l, 0)$ not containing all eigenvalues inside the unit circle, W_1 is a weak stable manifold. In these cases, we refer to E_1 as a strong stable subspace or a weak stable subspace, respectively.

Similarly one can consider a union of intervals $J = (l_1, r_1) \cup \cdots \cup (l_n, r_n)$ and associated, a sum of generalized eigenspaces E_1 of all eigenvalues with absolute value in J. In general there need no longer exist a continuously differentiable manifold, invariant for X, with tangent space E_1 at the origin. We give counterexamples in

Section 3. In Section 4 we shall see that for $J \subset (0,1)$ (or $J \subset (1,\infty)$) there does exist a continuously differentiable invariant manifold W_1 with $T_0W_1 = E_1$. Note that such a manifold is a submanifold of the stable (or unstable) manifold.

The general setup for this paper is summarized in the following set of definitions and notation.

Definition 1.1. *Let Σ_1 be a subset of the spectrum Σ of $Df(0)$. Write E_1 for the sum of the generalized eigenspaces corresponding to the eigenvalues in Σ_1. Write E_2 for the sum of the generalized eigenspaces corresponding to the eigenvalues in $\Sigma\backslash\Sigma_1$. Write*

$$d^i = dim(E_i),$$

for $i = 1,2$. The eigenvalues of $Df(0)|_{E_1}$ are denoted by $\lambda_1,\ldots,\lambda_{d_1}$ and the eigenvalues of $Df(0)|_{E_2}$ by $\nu_1,\ldots,\nu_{d_2}$.

Observe that $E_1 \oplus E_2$ is a $Df(0)$-invariant splitting of $\mathbb{R}^n$.

As mentioned above, we construct a continuously differentiable invariant manifold W_1 for X with tangent space E_1 at the origin, in case the eigenvalues of $Df(0)|_{E_1}$ are smaller than (or larger than) 1 in absolute value and have different absolute values than the eigenvalues of $Df(0)|_{E_2}$. Although this follows from linearization results by Belitskii [1], there is a straightforward construction which we present in Section 4. This construction involves weak stable manifolds, strong stable manifolds and strong stable foliations, whose constructions are standard. It is not true that such a manifold W_1 always exist of higher smoothness class. Resonance conditions among the eigenvalues of $Df(0)$ obstruct the existence of smooth manifolds W_1. This will be explained in Section 2. The notion of resonance among a collection of numbers is defined as follows. A collection of numbers $\{\alpha_i\}$ is said to possess a *resonance of order* r, $r \in \mathbb{N}$, if

$$\alpha_j = \prod_i \alpha_i^{m_i},$$

for some j and for $\sum_i m_i = r$, where m_i are nonnegative integers.

Following the material on invariant manifolds near fixed points, we consider the largely analogous theory for differential equations in Section 5. We do not construct invariant manifolds of higher smoothness in these notes, with the exception of invariant curves for differential equations as this is less involved. Section 6 constructs C^m invariant curves near hyperbolic equilibria of differential equations under the absence of resonance conditions among real parts of the eigenvalues.

The existence of smooth linearizing coordinates is similarly obstructed by resonance conditions among eigenvalues. Several papers have been written on the existence of C^m linearizing coordinates, see e.g. [14, 9, 2, 11]. The existence of such coordinates of course implies the existence of C^m invariant manifolds.

2 Obstructions to C^m invariant manifolds

The following result gives necessary conditions for the existence of an invariant manifold for f with tangent space E_1 at the origin and in a prescribed smoothness class, for generic diffeomorphisms f. The proof is an arithmetic argument similar to the ones in [13, 8].

Theorem 2.1. *Suppose that for some natural numbers j, h_i with $1 \leq j \leq d_2$, $h_i \geq 0$ and $\sum_{i=1}^{d_1} h_i = m$,*

$$\nu_j = \prod_{i=1}^{d_1} \lambda_i^{h_i}. \tag{1}$$

Let $\mathcal{X}$ be the set of diffeomorphisms that have a hyperbolic attracting fixed point at the origin as above and satisfy (1). Endow $\mathcal{X}$ with the C^m topology. There is an open dense subset $\mathcal{U} \subset \mathcal{X}$ so that any $f \in \mathcal{U}$ does not possess a C^m invariant manifold W_1 with $T_0 W_1 = E_1$.

Proof. In coordinates $x = (x_1, x_2)$ on $\mathbb{R}^n$ with $x_1 \in E_1$, $x_2 \in E_2$, we can write

$$f(x_1, x_2) = (F(x_1, x_2), G(x_1, x_2)).$$

Consider a smooth coordinate transformation $h(x_1, x_2) = (x_1, x_2 + H(x_1))$. If a C^m invariant manifold W_1 with $T_0 W_1 = E_1$ exists, the coordinate change h with $\mathrm{graph}(-H) = W_1$, will be such that

$$h \circ f = (F(x_1, x_2), S(x_1, x_2) x_2) \circ h, \tag{2}$$

for some map $S(x_1, x_2) : \mathbb{R}^n \to \mathcal{L}(E_2, E_2)$ with $S(0,0) = D_2 G(0)$ (where $DG(0) = \begin{pmatrix} D_1 G(0) & D_2 G(0) \end{pmatrix}$). We will show that such a coordinate change may not exist if (1) holds. Because $h \circ f = (F, G + H \circ F)$, (2) is equivalent to

$$G(x_1, x_2) + H(F(x_1, x_2)) = S(x_1, x_2 + H(x_1))(x_2 + H(x_1)). \tag{3}$$

Expand

$$\begin{aligned} H(x_1) &= H_2 x_1^2 + \cdots + H_m x_1^m + o(|x_1^m|), \\ F(x_1, x_2) &= D_1 F(0) x_1 + F_2(x_2) x_1^2 + \cdots + F_m(x_2) x_1^m + o(|x_1^m|), \\ G(x_1, x_2) &= G(0, x_2) + D_1 G(0) x_1 + G_2(x_2) x_1^2 + \cdots + G_m(x_2) x_1^m + o(|x_1^m|), \end{aligned}$$

with $H_i \in \mathcal{L}^i_{\mathrm{sym}}(E_1, E_2)$, $F_i(x_2) \in \mathcal{L}^i_{\mathrm{sym}}(E_1, E_1)$, $G_i(x_2) \in \mathcal{L}^i_{\mathrm{sym}}(E_1, E_2)$. Then (3) reads

$$\begin{aligned} &G(x_1, x_2) + H_2 [F(x_1, x_2)]^2 + \cdots + H_m [F(x_1, x_2)]^m = \\ &\qquad S(x_1, x_2 + H(x_1))[x_2 + H_2 x_1^2 + \cdots + H_m x_1^m] + o(|(x_1, x_2)|^m). \end{aligned} \tag{4}$$

Suppose we can inductively determine $H_1, \ldots, H_{k-1}$ with $k < m$. Comparing the coefficients of x_1^k in (4), gives

$$G_k(x_2) x_1^k + H_k [D_1 F(0) x_1]^k = D_2 G(0) H_k x_1^k + l.o.t., \tag{5}$$

where the lower order terms contain H_i with $i < k$. It is clear that (5) can be solved for H_k, for all possible maps G_k, precisely if the linear map $L : \mathcal{L}^k_{\mathrm{sym}}(E_1, E_2) \to \mathcal{L}^k_{\mathrm{sym}}(E_1, E_2)$,

$$L(H_k) x_1 = H_k \left[D_1 F(0) x_1\right]^k - D_2 G(0) H_k x_1^k \tag{6}$$

is invertible. We claim that the eigenvalues of L equal

$$\nu_j - \prod_{i=1}^{d_1} \lambda_i^{h_i},$$

where $1 \leq j \leq d_2$, $h_i \geq 0$ and $\sum_{i=1}^{d_1} h_i = k$. The theorem follows from the claim. In establishing the claim, we may assume that $Df(0)$ is linearizable over $\mathbb{C}$, because eigenvalues depend continuously on the matrix. So, writing $x_1 = (x_{1,1}, \ldots, x_{1,d_1})$, $x_2 = (x_{2,1}, \ldots, x_{2,d_2})$, we may assume

$$\begin{aligned} D_1F(0)x_1 &= (\lambda_1 x_{1,1}, \ldots, \lambda_{d_1} x_{1,d_1}), \\ D_2G(0)x_2 &= (\nu_1 x_{2,1}, \ldots, \nu_{d_2} x_{2,d_2}). \end{aligned}$$

Writing

$$(H_k x_1^k)_i = \sum H_{i,m_1,\ldots,m_{d_1}} x_{1,1}^{m_1} \cdots x_{1,d_1}^{m_{d_1}}$$

for the i^{th} coordinate of $H_k x_1^k$, where $\sum_{i=1}^{d_1} m_i = k$, (5) yields

$$\nu_i H_{i,m_1,\ldots,m_{d_1}} = \lambda_1^{m_1} \cdots \lambda_{d_1}^{m_{d_1}} H_{i,m_1,\ldots,m_{d_1}}.$$

From this the claim follows. ∎

Note that Theorem 2.1 involves resonance conditions among the eigenvalues, and not their absolute values. It follows from the above computations that the following system $x(k+1) = f(x(k))$ with $Df(0)$ possessing real eigenvalues does not allow a C^m invariant manifold tangent to $\{(x_{2,1}, \ldots, x_{2,d_2}) = 0\}$ if $\nu_j = \lambda_1^{m_1} \cdots \lambda_{d_1}^{m_{d_1}}$ and $c \neq 0$:

$$\left\{ \begin{array}{rcll} x_{1,i}(k+1) &=& \lambda_i x_{1,i}(k), & 1 \leq i \leq d_1, \\ x_{2,h}(k+1) &=& \nu_j x_{2,h}(k), & 1 \leq h \leq d_2, h \neq j, \\ x_{2,j}(k+1) &=& \nu_j x_{2,j}(k) + c\,(x_{1,1}(k))^{m_1} \cdots (x_{1,d_1}(k))^{m_{d_1}}, & \end{array} \right.$$

In particular,

$$\left\{ \begin{array}{rcl} x_1(k+1) &=& \lambda_1 x_1(k), \\ x_2(k+1) &=& \nu_1 x_2(k) + c\,(x_1(k))^2, \end{array} \right.$$

with $\nu_1 < \lambda_1 < 1$ and $\nu_1 = \lambda_1^2$ does not possess a C^2 weak stable manifold tangent to the x_1-axis at the origin.

3 Obstructions to C^1 invariant manifolds

In this section we give a specific example of a diffeomorphism without a continuously differentiable invariant manifold near an invariant subspace for the linearization. Consider the system of difference equations

$$\left\{ \begin{array}{rcl} x_{1,1}(k+1) &=& \lambda_1 x_{1,1}(k), \\ x_{1,2}(k+1) &=& \lambda_2 x_{1,2}(k), \\ x_2(k+1) &=& \nu_1 x_2 + c x_{1,1}(k) x_{1,2}(k), \end{array} \right. \tag{7}$$

with $0 < \lambda_1 < \nu_1 < 1 < \lambda_2$ and $\nu_1 = \lambda_1 \lambda_2$. This system possesses a two dimensional stable manifold, containing a one dimensional strong stable manifold, as well as a one dimensional unstable manifold. The results in Section 2 show that, as a consequence of the resonance of second order, there is no C^2 invariant manifold with E_1 as tangent space at the origin. Note that such a manifold would contain the unstable manifold and the strong stable manifold. The following result shows that there is even no C^1 invariant manifold with E_1 as tangent space at the origin.

Theorem 3.1. *The system of difference equations (7) has no continuously differentiable invariant manifold W_1 with $T_0 W_1 = E_1$.*

Proof. The proof is inspired by [6, Exercise 5.50], treating C^1 linearizability of diffeomorphisms. As $\nu_1 = \lambda_1\lambda_2$, (7) is solved by

$$\begin{aligned} x_{1,1}(k) &= \lambda_1^k x_{1,1}(0), \\ x_{1,2}(k) &= \lambda_2^k x_{1,2}(0), \\ x_2(k) &= \nu_1^k x_2(0) + ck\nu_1^{k-1} x_{1,1}(0) x_{1,2}(0). \end{aligned}$$

We will show that there is no Lipschitz continuous map $G : E_1 \to E_2$ so that graph(G) is an invariant manifold. Suppose that graph(G) is a Lipschitz continuous invariant manifold. The unstable manifold, i.e. the $x_{1,2}$ axis, lies on graph(G). This means that $G(0, x_{1,2}) = 0$.

Consider an orbit piece $x(k)$ with $k \in [0, \tau]$ and $x_{1,2}(\tau) = 1$, $x_{1,1}(0) = \varepsilon$. Note that $x_{1,2}(0) = \lambda_2^{-\tau}$ and $x_{1,1}(\tau) = \lambda_1^\tau \varepsilon$. Take the orbit piece on graph(G), so that $x_2(k)$ is determined given $x_k(t)$. For τ large and ε small, $x(0)$ lies close to the origin. By Lipschitz continuity of G and $G(0, 1) = 0$, there is K (depending on ε, τ but with uniformly bounded absolute value) such that $x_2(\tau) = K x_{1,1}(\tau)$. A direct computation shows that $x_2(0) = \varepsilon \left(\lambda_2^{-\tau} K - c\tau\lambda_2^{-\tau}/\nu_1\right)$. Writing $\delta = \lambda_2^{-\tau}$, this gives

$$G(\varepsilon, \delta) = x_2(0) = \varepsilon \left(\delta K + c\delta \ln(\delta)/(\nu_1\lambda_2)\right).$$

By letting $\tau \to \infty$ or equivalently $\delta \to 0$, it follows that G cannot be Lipschitz continuous at $(\varepsilon, 0)$. ∎

4 Existence of C^1 invariant manifolds

The following theorem on invariant submanifolds of the stable manifold also follows as a corollary of a result by Belitskii [1] on the existence of C^1 linearizing coordinates.

Theorem 4.1. *Let X be a smooth diffeomorphism on $\mathbb{R}^n$ with a hyperbolic fixed point at 0. Suppose*

$$\begin{aligned} |\lambda_i| &< 1, & (8) \\ |\nu_j| &\neq |\lambda_i|, & (9) \end{aligned}$$

for all natural numbers $1 \le j \le d_2$, $1 \le i \le d_1$. Then X has a C^1 invariant manifold W_1 with $T_0 W_1 = E_1$.

Proof. Write

$$E_1 = \bigcap_{i=1}^{r} F^i \oplus G^i,$$

where F^i is a strong stable subspace of $Df(0)$ and G^i is a weak stable subspace of $Df(0)$, so that $F^1 \supset F^2 \supset \cdots \supset F^r$ and $G^1 \subset G^2 \subset \cdots \subset G^r$. Let $\mathcal{G}^i$ be a weak stable manifold of X with $T_0\mathcal{G}^i = G^i$. Let $\mathcal{F}^i$ be a strong stable foliation of the stable manifold of X with $T_0\mathcal{F}_0^i = F^i$. See [12] and the remarks below for the construction of local weak stable manifolds and strong stable foliations. Then

$$W_1 = \bigcap_{i=1}^{r} \bigcup_{x \in \mathcal{G}^i} \mathcal{F}_x^i \tag{10}$$

is a C^1 invariant manifold of X with $T_0W_1 = E_1$, because the r manifolds which are intersected in (10) are mutually transverse. ∎

We briefly recall the construction of local weak stable manifolds and strong stable foliations, see [5], [12] for more details. By restriction of f to the local stable manifold, we may assume that the spectrum of $Df(0)$ lies inside the unit circle. Suppose E_1 is a weak stable manifold of $Df(0)$. Using a test function, we may assume that f is globally close to $Df(0)$. Then

$$\mathcal{G}^i = \lim_{i\to\infty} f^i(E_1) \tag{11}$$

defines a weak stable manifold. A weak stable manifold is not unique but depends on the choice of test function in the construction.

Now suppose E_1 is a strong stable subspace of $Df(0)$. A strong stable foliation $\mathcal{F}$ of the stable manifold of f with $T_0\mathcal{F}_0 = E_1$ is obtained as

$$\mathcal{F} = \lim_{i\to\infty} f^{-i}(\mathcal{F}^0), \tag{12}$$

where $\mathcal{F}^0$ is some smooth trial foliation of the stable manifold close to the affine foliation with leaves parallel to E_1. The limit foliation is uniquely determined. The strong stable manifold W_1 with $T_0W_1 = E_1$ equals the leaf $\mathcal{F}_0$.

5 Invariant manifolds for differential equations

For systems of differential equations one can likewise consider invariant manifolds other then normally hyperbolic ones. There are direct analogs to the above derived results for diffeomorphisms. We collect the statements in this section.

The following context is assumed: X is a smooth vector field on $\mathbb{R}^n$ with a hyperbolic equilibrium at the origin 0. Hyperbolicity means that the eigenvalues of $DX(0)$ are away from the imaginary axis. The time t flow of X will be denoted by φ_t. The setup and notation introduced in Definition 1.1 will be assumed.

In the current context the relevant notion of resonance conditions among eigenvalues is expressed by the following definition. A collection of numbers $\{\alpha_i\}$ is said to possess a *resonance of order* r, $r \in \mathbb{N}$, if

$$\alpha_j = \sum m_i \alpha_i,$$

for some j and for $\sum m_i = r$, where m_i are nonnegative integers.

The role of resonance conditions among eigenvalues is expressed by the following result, giving obstructions to the existence of C^m invariant manifolds. We use notation as provided before in Definition 1.1. The proof of Theorem 2.1 applied to the time 1 flow φ_1 yields the following result.

Theorem 5.1. *Suppose that for some natural numbers j, h_i with $1 \leq j \leq d_2$, $h_i \geq 0$ and $\sum_{i=1}^{d_1} h_i = m$,*

$$\nu_j = \sum_{i=1}^{d_1} h_i \lambda_i. \tag{13}$$

Let $\mathcal{X}$ be the set of vector fields that have a hyperbolic sink in 0 as above and satisfy (13). Endow $\mathcal{X}$ with the C^m topology. There is an open dense subset $\mathcal{U} \subset \mathcal{X}$ so that any $X \in \mathcal{U}$ does not possess a C^m invariant manifold W_1 with $T_0 W_1 = E_1$.

The following vector field with a linearization involving real eigenvalues does not possess a C^m invariant manifold tangent to $\{(x_{2,1}, \ldots, x_{2,d_2}) = 0\}$ if $\nu_j = m_1 \lambda_1 + \cdots + m_{d_1} \lambda_{d_1}$ and $c \neq 0$:

$$\begin{cases} \dot{x}_{1,i} = \lambda_i x_{1,i}, & 1 \leq i \leq d_1, \\ \dot{x}_{2,h} = \nu_j x_{2,h}, & 1 \leq h \leq d_2, h \neq j, \\ \dot{x}_{2,j} = \nu_j x_{2,j} + c x_{1,1}^{m_1} \cdots x_{1,d_1}^{m_{d_1}}, & \end{cases}$$

There are more necessary conditions for the existence of invariant manifolds, as expressed by the following result copying Theorem 3.1. Consider the system of differential equations

$$\begin{cases} \dot{x}_{1,1} = \lambda_1 x_{1,1}, \\ \dot{x}_{1,2} = \lambda_2 x_{1,2}, \\ \dot{x}_2 = \nu_1 x_2 + c x_{1,1} x_{1,2}, \end{cases} \tag{14}$$

with $\lambda_1 < \nu_1 < 0 < \lambda_2$ and $\nu_1 = \lambda_1 + \lambda_2$.

Theorem 5.2. *The system of differential equations given by (14) has no continuously differentiable invariant manifold W_1 with $T_0 W_1 = E_1$.*

Proof. As $\nu_1 = \lambda_1 + \lambda_2$, (7) is solved by

$$\begin{aligned} x_{1,1}(t) &= e^{\lambda_1 t} x_{1,1}(0), \\ x_{1,2}(t) &= e^{\lambda_2 t} x_{1,2}(0), \\ x_2(t) &= e^{\nu_1 t} x_2(0) + c t e^{\nu_1 t} x_{1,1}(0) x_{1,2}(0). \end{aligned}$$

We copy the reasoning of Theorem 7, including it for completeness. We will show that there is no Lipschitz continuous map $f : E_1 \to E_2$ so that graph(f) is an invariant manifold. Suppose that graph(f) is a Lipschitz continuous invariant manifold. The unstable manifold, i.e. the $x_{1,2}$ axis, lies on graph(f). This means that $f(0, x_{1,2}) = 0$.

Consider an orbit piece $x(t)$ with $t \in [0, \tau]$ and $x_{1,2}(\tau) = 1$, $x_{1,1}(0) = \varepsilon$. Note that $x_{1,2}(0) = e^{-\lambda_2 \tau}$ and $x_{1,1}(\tau) = e^{\lambda_1 \tau} \varepsilon$. Take the orbit piece on graph(f), so that $x_2(t)$ is defined by $x_1(t)$. For τ large and ε small, $x(0)$ lies close to the origin. By Lipschitz continuity of f and $f(0,1) = 0$, there is k (depending on ε, τ but with uniformly bounded absolute value) such that $x_2(\tau) = k x_{1,1}(\tau)$. A direct computation shows that $x_2(0) = \varepsilon \left(e^{-\lambda_2 \tau} k - c \tau e^{-\lambda_2 \tau} \right)$. Writing $\delta = e^{-\lambda_2 \tau}$, this gives

$$f(\varepsilon, \delta) = x_2(0) = \varepsilon \left(\delta k + c \delta \ln(\delta) / \lambda_2 \right).$$

By letting $\tau \to \infty$ or equivalently $\delta \to 0$, it follows that f can not be Lipschitz continuous at $(\varepsilon, 0)$. ∎

Finally, the construction in Section 4 to prove Theorem 4.1 also gives a C^1 invariant manifold W_1 with $T_0 W_1 = E_1$ under the eigenvalue conditions: $\mathrm{Re}(\lambda_i) < 0$ and $\mathrm{Re}(\nu_j) \neq \mathrm{Re}(\lambda_i)$.

6 C^m invariant curves for differential equations

Invariant manifolds of smoothness class C^m can be reduced to C^1 invariant manifolds for an induced flow on a space of $(m-1)$-jets. This is relatively straightforward for invariant curves of differential equations, and leads to the following existence result.

Theorem 6.1. *Let X be a smooth vector field on $\mathbb{R}^n$ with a hyperbolic sink in 0. Assume that* $\dim E_1 = 1$. *Suppose*

$$Re(\nu_j) \neq h\lambda_1, \tag{15}$$

for all natural numbers j, h with $1 \leq j \leq d_2$ and $1 \leq h < m$. Then X has a C^m invariant manifold W_1 with $T_0 W_1 = E_1$.

Proof. The manifold W_1 will be the graph of a map from E_1 to E_2. We will identify such maps with their graphs. Let $C^k(E_1, E_2)$ denote the set of C^k maps from E_1 to E_2. For $W \in C^1(E_1, E_2)$, we define the Nash blow-up $\mathcal{B}W \in C^0(E_1, \mathcal{L}(E_1, E_2))$ [3] by

$$\mathcal{B}W(x) = DW(x). \tag{16}$$

If $W_1 \in C^m(E_1, E_2)$, then $\mathcal{B}^{m-1}W_1 \in C^1(E_1, \mathcal{L}^{m-1}(E_1, E_2))$.

Starting at $E_1^{(0)} = E_1$, $E_2^{(0)} = E_2$ and $U^{(0)} = X$ define inductively vector spaces $E_1^{(i)}, E_2^{(i)}$ and a vector field $U^{(i)}$ on $E_1^{(i)} \times E_2^{(i)}$ by

$$\begin{aligned} E_1^{(i)} &= E_1^{(i-1)} \times E_2^{(i-1)}, \\ E_2^{(i)} &= \mathcal{L}(E_1, E_2^{(i-1)}), \\ U_t^{(i)}(x, \alpha) &= (U_t^{(i-1)}(x), \beta), \\ \text{graph}(\beta) &= DU_t^{(i-1)}(x)\text{graph}(\alpha). \end{aligned}$$

Let

$$\begin{aligned} \mathcal{P}^i(E_1, E_2) &= \mathcal{L}(E_1, E_2) \times \ldots \times \mathcal{L}^i(E_1, E_2), \\ \mathcal{J}^i(E_1, E_2) &= E_1 \times E_2 \times \mathcal{P}^i(E_1, E_2). \end{aligned}$$

Since there is a natural embedding $\mathcal{J}^i(E_1, E_2) \hookrightarrow E_1^{(i)} \times E_2^{(i)}$, the vector field $U^{(i)}$ induces a vector field $X^{(i)}$ on $\mathcal{J}^i(E_1, E_2)$.

By (15), applying an argument as used in the proof of theorem 2.1, we may take coordinates on $\mathbb{R}^n$ so that the following holds.

- 0 is a singularity of $X^{(m-1)}$,
- $E_1 \times E_2 \times \mathcal{L}(E_1, E_2) \times \ldots \times \mathcal{L}^{m-1}(E_1, E_2)$ is a $DX^{(m-1)}(0)$ invariant splitting,
- $DX^{(m-1)}(0)\big|_{\mathcal{L}^p(E_1,E_2)}$ has eigenvalues

  $$\nu_j - p\lambda_1,$$

  where $1 \leq j \leq d_2$.

Define

$$
\begin{aligned}
M^{(m-1)} \quad = \quad & \big\{(x_1, x_2, A_1, \dots, A_{m-1}) \in \mathcal{J}^{m-1}(E_1, E_2), \\
& (\dot{x}_1, \dot{x}_2, \dot{A}_1, \dots, \dot{A}_{m-1}) = X^{(m-1)}(x_1, x_2, A_1, \dots, A_{m-1}), \\
& (\dot{x}_2, \dot{A}_1, \dots, \dot{A}_{m-2}) = (A_1\dot{x}_1, A_2\dot{x}_1, \dots, A_{m-1}\dot{x}_1)\big\}. \qquad (17)
\end{aligned}
$$

So, if $W_1 \in C^m(E_1, E_2)$ is an invariant manifold of X, then

$$
(x_1, W_1(x_1), DW_1(x_1), \dots, D^{m-1}W_1(x_1)) \quad \in \quad M^{(m-1)}
$$

for $x_1 \in E_1$. $M^{(m-1)}$ is thus an invariant manifold of $X^{(m-1)}$. Denote by

$$
\Pi : \mathcal{J}^{m-1}(E_1, E_2) \mapsto E_1 \times \mathcal{L}^{m-1}(E_1, E_2)
$$

the coordinate projection

$$
\Pi(x_1, x_2, A_1, \dots, A_{m-1}) = (x_1, A_{m-1}).
$$

From the properties of $X^{(m-1)}$ it is clear that $\Pi : M^{(m-1)} \mapsto E_1 \times \mathcal{L}^{m-1}(E_1, E_2)$ is locally injective. Define

$$
Y^{(m-1)} \quad = \quad \Pi_* \left(X^{(m-1)}|_{M^{(m-1)}} \right). \qquad (18)
$$

Then 0 is a hyperbolic singularity of $Y^{(m-1)}$, $E_1 \times \mathcal{L}^{m-1}(E_1, E_2)$ is a $DY^{(m-1)}(0)$ invariant splitting and $DY^{(m-1)}(0)$ has eigenvalues

$$
\lambda_1, \nu_j - (m-1)\lambda_1,
$$

where $1 \leq j \leq d_2$. A C^1 invariant manifold $W_1^{(m-1)}$ of $Y^{(m-1)}$ with tangent space E_1 at 0 is obtained as in the previous section, see Theorem 4.1. By construction this yields a C^m invariant manifold W_1 for X. ■

If $\nu_j = (m-1)\lambda_1$ for some $1 \leq j \leq d_2$ is the lowest order resonance among eigenvalues, then the linearization of the induced vector field $Y^{(m-1)}$ has multiple eigenvalues, and is typically not linearizable. This provides a geometric explanation to the nonexistence of a C^m invariant manifold W_1 tangent to E_1 at the origin.

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