The Cusp Horseshoe and its Bifurcations in the Unfolding of an Inclination-flip Homoclinic Orbit

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The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic orbit

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1 Introduction

Let X be a vector field with a homoclinic orbit to a saddle equilibrium point. Shil'nikov [Shil68] showed in a very general context that an open set of small perturbations of X has a periodic orbit with a very high period. As the size of the perturbation shrinks to 0 the Hausdorff distance between the periodic orbit and the homoclinic loop approaches 0 and the period of the periodic orbit approaches infinity. This phenomenon is often referred to as an *infinite period bifurcation* or a *homoclinic bifurcation*. Recently there has been a considerable interest in understanding the dynamics near degenerate homoclinic orbits, typically occuring in two parameter families. Suppose X is a vector field in $\mathbb{R}^3$ having a saddle point at O and a homoclinic orbit Γ asymptotic to O. Assume that the linearization of X at O has three real eigenvalues $\lambda^s, \lambda^u, \lambda^{uu}$ satisfying $\lambda^s < 0 < \lambda^u < \lambda^{uu}$. A degeneracy of Γ, known as *inclination-flip* or *critical twist*, can be characterized as follows. Generically $T|_\Gamma \mathbb{R}^3$ has a continuous subbundle with one dimensional fibers which is invariant under the linearization of the flow of X along Γ and whose fiber at O is tangent the eigendirection of λ^{uu}. This bundle, which we refer to as the *strong unstable bundle*, can be orientable or nonorientable. The corresponding homoclinic orbits are called *nontwisted* and *twisted* respectively. A point of transition between the two cases is called an *inclination-flip point* or a *critical twist point*. The analysis of the dynamics in the unfoldings of an inclination-flip point is the subject of this article.

Inclination-flip bifurcation, together with two other codimension two problems was studied by Yanagida [Yan87]. The two other problems Yanagida considered were the *resonant bifurcation*, occuring when the magnitudes of the principal eigenvalues are equal ($-\lambda^s = \lambda^u$), and the *orbit-flip bifurcation*, taking place when the homoclinic orbit Γ is tangent at O to the strong unstable direction. The results of Yanagida asserted that each of the three bifurcations led to the occurrence of double homoclinic orbits, that is homoclinic orbits consisting of two loops near Γ. The article of Yanagida was followed by a number of publications on this subject. In particular the work of Chow, Deng and Fiedler [CDF90] and Kisaka, Kokubu and Oka [KKO93a] led to a complete understanding of the resonant bifurcation. Further work has also been done on the inclination-flip bifurcation. Dumortier, Kokubu and Oka [DKO92] studied the persistence condition for inclination-flip homoclinic orbits in terms of Melnikov-like integrals. Kisaka, Kokubu and Oka [KKO93b] carried out a rigorous analysis of the homoclinic doubling for an inclination-flip homoclinic orbit in the case when $\lambda^u < -\lambda^s < 2\lambda^u$. Deng [Deng91] presented a scenario suggesting that a perturbation of an inclination-flip point would lead to the occurrence of Smale horseshoes. The work of Deng is one of the main motivations of our research and will be discussed in more detail in the sequel. Using Lin's method Sandstede [San93] has recently shown the existence of shift dynamics and n-homoclinic orbits for arbitrary n in the unfolding of an inclination-flip point in the case when $2\lambda^u > \lambda^{uu}$ and $-\lambda^s > 2\lambda^u$. Sandstede has also studied the orbit-flip bifurcation finding similar phenomena.

Inclination-flip bifurcations have also been studied in the context of $\mathbb{Z}_2$ symmetric vector fields. Rychlik [Rych90] considers the inclination-flip bifurcation for a pair of symmetry related homoclinic orbits. He assumes that the linearization of X at O has the eigenvalue configuration $\lambda^{ss} < \lambda^s < 0 < \lambda^u$ and the $\mathbb{Z}_2$ action flips the principal unstable direction and fixes the principal stable direction. He shows that arbitrarily small perturbations of this configuration have a geometric Lorenz attractor. Aronson, Golubitsky and Krupa [AGK91] and later Aronson, van Gils and Krupa [AvGK92] studied inclination-flip bifurcations of homoclinic orbits invariant under the $\mathbb{Z}_2$ (reflection) symmetry. They showed that in the unfoldings of such bifurcations there exist symmetry related pairs of homoclinic orbits tangent to the $\mathbb{Z}_2$ symmetry plane. Homburg [Hom93] proved that unfoldings of this type of homoclinic orbits lead to occurrence of $\mathbb{Z}_2$ symmetric horseshoes.

The work of Deng [Deng91] and Homburg [Hom93] provided the main motivation for this research. Deng conjectured that a suitable perturbation of a flow at an inclination-flip point would have a Smale horseshoe. He suggested that by following a circular path around the inclination-flip point one would observe the disappearence of all the periodic orbits of the horseshoe in an infinite period bifurcation. As a result of this bifurcation sequence only one periodic orbit would remain. Deng studied a one parameter family of planar maps modelling the return maps of a transverse section near the homoclinic orbit Γ and analyzed the bifurcation sequence occuring for these maps. He showed the occurrence of a number of bifurcations other than the ones associated with the disappearence of a periodic orbit through an infinite period bifurcation. Homburg [Hom93] found horseshoes and similar bifurcation sequences in the unfoldings of a codimension 1 homoclinic orbit under the assumption of a global property for the vector field X. He assumed that, simultaneously with Γ, there exists a *generalized homoclinic orbit*, that is an orbit in the unstable manifold of the saddle O which is forward asymptotic to $\Gamma \cup \{O\}$. It turns out that generalized homoclinic orbits naturally occur near inclination-flip points. In his analysis Homburg took advantage of the existence of a strong invariant foliation to reduce the dimension of the dynamics and consequently obtained a more complete description of the bifurcation sequence than Deng [Deng91]. He was, in particular, able to specify the order of the infinite period bifurcations of periodic orbits and prove that the bifurcation set had Lebesgue measure 0.

This article achieves the following two objectives. We prove that, provided that $\lambda^{uu} > 2\lambda^u$ and $-\lambda^s > 2\lambda^u$ and a number of nondegeneracy assumptions hold, then in every parameter neighborhood of an inclination-flip point there are parameter values for which the return map defined on a cross section of the flow near the homoclinic orbit Γ has a horseshoe. The second objective we achieve is proving that the bifurcation sequence conjectured by Deng occurs in an arbitratrily small neighborhood of the inclination-flip point and can be analyzed using the methods of Homburg. Sandstede [San93] has obtained similar results for the orbit-flip case using the method of Lin.

Since our analysis is only valid in a thin wedge of the parameter space many questions about the inclination-flip bifurcation remain unanswered. It is remarkable that the bifurcation sequence we analyze does not involve the occurrence of homoclinic tangencies and the related chaotic dynamics. It is, however, quite clear from the form of the return map around the homoclinic orbit that other bifurcation scenarios, leading to annihilation of a horseshoe, possibly involving chaotic dynamics, must occur. Another way which could lead to interesting bifurcation sequences would be to violate the eigenvalue conditions we impose. These issues will be discussed in more detail in Section 6.

The article is organized as follows. In Section 2 we give a rigorous definition of an inclination-flip point and discuss the form of generic unfoldings. At the end of the section we state the main theorems of the article. In Section 3 we show the existence of topological horseshoes near an inclination-flip point. Section 4 is devoted to the proof of the existence of an unstable invariant foliation on a conveniently chosen subset of a cross section transverse to the flow. In Section 5 we use the results of Section 4 to obtain a reduction of the dynamics of the return map defined on a two dimensional transverse section of the flow to the dynamics of a multivalued map of an interval. We analyze the dynamics and bifurcations of the relevant multivalued maps. The analysis in Section 5 provides the proof of the main theorems stated in Section 2. We conclude the article in Section 6 where we present some conjectures on the type of phenomena that could occur outside the region of validity of our analysis.

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2 Inclination-flip homoclinic orbit

Let X_0 be a smooth vector field on $\mathbb{R}^3$ with a hyperbolic equilibrium point O. Assume the linearization $DX_0(O)$ has real eigenvalues $\lambda^s, \lambda^u, \lambda^{uu}$ satisfying $\lambda^s < 0 < \lambda^u < \lambda^{uu}$, and hence the vector field has a one-dimensional stable manifold $W^s(O)$ and a two-dimensional unstable manifold $W^u(O)$ at O. Furthermore there exists a two-dimensional invariant manifold whose tangent space at O is spanned by the eigenvectors associated with the eigenvalues λ^s and λ^u. See [HPS77] for the existence of such an invariant manifold. Clearly it contains the stable manifold by definition. Here we call it the *extended stable manifold* and denote it by $W^{es}(O)$. Note that such an invariant manifold is not unique but has the unique tangent space at any point on the stable manifold.

We moreover assume that the vector field X_0 has a homoclinic orbit Γ based at O, namely, $\Gamma \subset W^s(O) \cap W^u(O)$. Clearly Γ is contained in the intersection of $W^{es}(O)$ and $W^u(O)$. Let $h(t)$ be a homoclinic solution of Γ, namely, $\lim_{t\to\pm\infty} h(t) = O$ and $\Gamma = \overline{\{h(t) | t \in \mathbb{R}\}}$.

Definition 1 The homoclinic orbit Γ is called an *inclination-flip homoclinic orbit* if it satisfies the following three conditions:

(CT) $W^u(O)$ and $W^{es}(O)$ are tangent along Γ;

(NR) $\lambda^u \neq |\lambda^s|$;

(PR) $\lim_{t\to-\infty} |h(t)e^{-\lambda^u t}| < +\infty, \quad \lim_{t\to+\infty} |h(t)e^{-\lambda^s t}| < +\infty.$

Remark 1 The condition (CT) makes sense since the extended stable manifold $W^{es}(O)$ has the well-defined tangent space along the homoclinic orbit Γ. This definition of the inclination-flip homoclinic orbit is equivalent to the one explained in the previous section.

In this article we assume the following two eigenvalue conditions.

(EV1) $\lambda^{uu} > 2\lambda^u$.

(EV2) $-\lambda^s > 2\lambda^u$.

Under the condition (EV1) the extended stable manifold $W^{es}(O)$ is of at least C^2-class (see [HPS77]), and hence the second order derivative of the $W^{es}(O)$ is well defined. Therefore the inclination-flip homoclinic orbit generically satisfies that

(QT) the unstable manifold $W^u(O)$ and the extended stable manifold $W^{es}(O)$ have the quadratic tangency along the homoclinic orbit Γ.

Throughout this paper an inclination-flip homoclinic orbit is always assumed to satisfy the genericity condition (QT).

Consider a smooth family of vector fields X_μ on $\mathbb{R}^3$ unfolding the vector field X_0 possessing the inclination-flip homoclinic orbit Γ. Our goal is to study the dynamics in the family near the homoclinic orbit. For this purpose, we first describe the return map along the homoclinic orbit and its perturbation. The return map is constructed by the composition of two successive mappings between cross sections as follows: Take the local coordinates (x, y, z) near the origin O in which the vector field X_μ is uniformly C^3-linearized as

$$X_\mu = \lambda^s_\mu x \frac{\partial}{\partial x} + \lambda^{uu}_\mu y \frac{\partial}{\partial y} + \lambda^u_\mu z \frac{\partial}{\partial z}.$$

This uniform smooth linearization assumption, which is guaranteed under a generic assumption for the family (see [Rych90]), is not necessary but it simplifies the arguments in the sequel. We consider the planes

$$\begin{aligned} \Sigma_1 &= \{x = 1, |y| + |z| < 1\}, \\ \Sigma_0 &= \{|x| + |y| < 1, z = 1\}. \end{aligned}$$

Rescaling the variables we may assume that these planes are contained in the neighborhood where the linearization of the vector field is valid and are transverse to the homoclinic orbit Γ.

It is easy to obtain the following forms of successive flow-defined mappings:

$$F: \ \Sigma_1 \to \Sigma_0; \quad (1, y, z) \mapsto (z^{-\frac{\lambda^s}{\lambda^u}}, y z^{-\frac{\lambda^{uu}}{\lambda^u}}, 1), \tag{1}$$

$$G_\mu: \ \Sigma_0 \to \Sigma_1; \quad (X, Y, 1) \mapsto (1, G(X, Y; \mu)), \tag{2}$$

where $G(X, Y; \mu) = (g^1(X, Y; \mu), g^2(X, Y; \mu))$ is a diffeomorphism satisfying

$$G(0, 0; 0) = 0, \quad \frac{\partial}{\partial Y} g^1(0, 0; 0) = 0, \quad \frac{\partial^2}{\partial^2 Y} g^1(0, 0; 0) \neq 0.$$

Here the first equality corresponds to the existence of the homoclinic orbit Γ at $\mu = 0$, the second equality to the inclination-flip condition and the last inequality expresses the quadratic tangency condition (QT). The resulting return map is thus obtained by the composition of these two mappings as follows (see Figure 1): $f_\mu : \Sigma_1 \to \Sigma_1$ is simply given by

$$f_\mu(y, z) = (G_\mu \circ F)(y, z).$$

In what follows we sometimes suppress the parameter dependence in the no-

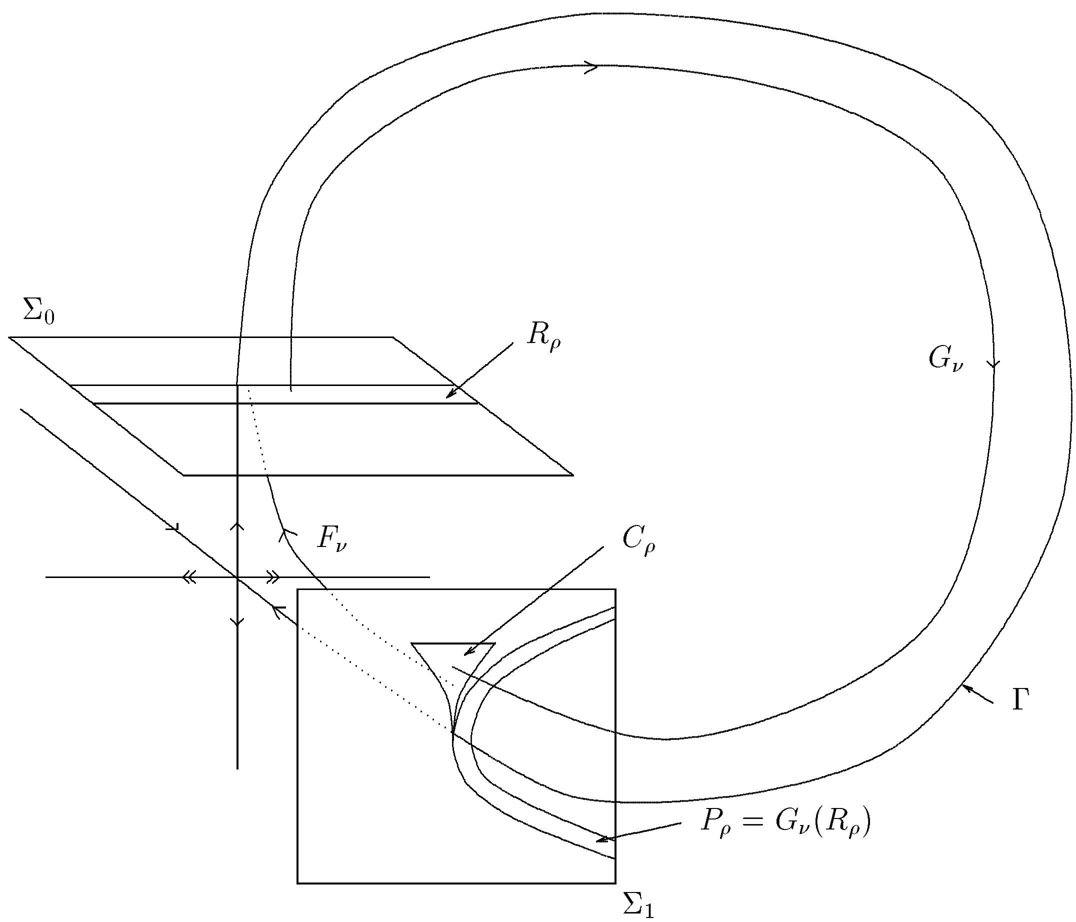


Figure 1: The return map to a section Σ_1 at an inclination-flip point.

tation if there is no confusion. We also use the following abbreviations. The derivatives of the functions $g^i(X,Y;\mu)$ $(i=1,2)$ with respect to (X,Y,μ_1,μ_2) are denoted by using the corresponding suffices. For instance, $g^2_\mu(X,Y;\mu)$ stands for $\frac{\partial}{\partial\mu}g^2(X,Y;\mu)$. Moreover, $g^{10}_{Y\mu}$ stands for $\frac{\partial^2}{\partial Y\partial\mu}g^1(0,0;0)$, the value of the corresponding derivative at $(X,Y;\mu)=(0,0;0)$.

After the perturbation by μ, the persistence condition for the inclination-flip homoclinic orbits is given by the equations

$$G(0,Y;\mu)=0 \quad \text{and} \quad g^1_Y(0,Y;\mu)=0.$$

The condition $g^{10}_Y=0$ and the fact that G is a diffeomorphism imply that $g^{20}_Y\neq 0$. By the implicit function theorem there exists $Y=Y_*(\mu)$ satisfying $g^2(0,Y_*(\mu);\mu)\equiv 0$. We express the persistence condition of the inclination-flip homoclinic orbit by the equations:

$$g^1(0,Y_*(\mu);\mu)=0 \quad \text{and} \quad g^1_Y(0,Y_*(\mu);\mu)=0.$$

If the family X_μ generically unfolds the inclination-flip homoclinic orbit Γ near $\mu=0$, then the above equalities, treated as a function of μ, have to attain a regular value at $\mu=0$, namely,

$$rank\left[\left.\frac{d}{d\mu}\right|_{\mu=0}\begin{pmatrix} g^1(0,Y_*(\mu);\mu) \\ g^1_Y(0,Y_*(\mu);\mu)\end{pmatrix}\right]=2.$$

The converse is also true, and hence we have shown the following Proposition.

Proposition 1 *The family of vector fields X_μ unfolds the inclination-flip homoclinic orbit Γ generically, if and only if $rank M=2$, where*

$$M=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}=-\begin{pmatrix} 0 \\ g^{10}_{YY}\end{pmatrix}\cdot\frac{g^{20}_\mu}{g^{20}_Y}+\begin{pmatrix} g^{10}_\mu \\ g^{10}_{Y\mu}\end{pmatrix}.$$

We assume X_μ is a generic two-parameter unfolding of X_0 and we make the change of parameters:

$$\nu=(\nu_1,\nu_2)=\left(g^1(0,Y_*(\mu);\mu),\frac{g^1_Y(0,Y_*(\mu);\mu)}{g^2_Y(0,Y_*(\mu);\mu)}\right). \tag{3}$$

In order to study the dynamics of the return map f_μ, we look at the rectangle $R_\rho=[0,\rho]\times[-1,1]$ contained in the cross section Σ_0. We will later choose ρ in such a way that the orbits which stay in a neighborhood of the homoclinic orbit Γ have to pass through the rectangle, otherwise they eventually go far from Γ. We now consider the preimage $C_\rho = F^{-1}(R_\rho)$ of R_ρ under F and the image $P_\rho = G_\mu(R_\rho)$ of R_ρ under G_μ. Clearly the way these sets intersect determines the recurrent dynamics of f_μ. A straightforward computation shows that $C_\rho = F^{-1}(R_\rho) \subset \Sigma_1$ is a cusp-shaped region whose boundary consists of two side curves

$$b_\pm = \{(\pm z^{\frac{\lambda^{uu}}{\lambda^u}}, z) | 0 \leq z \leq t_z = \rho^{-\frac{\lambda^u}{\lambda^s}}\}$$

and the top segment

$$t = \{(\rho^{-\frac{\lambda^{uu}}{\lambda^s}} y, t_z) | -1 \leq y \leq 1\}.$$

See Figure 2.

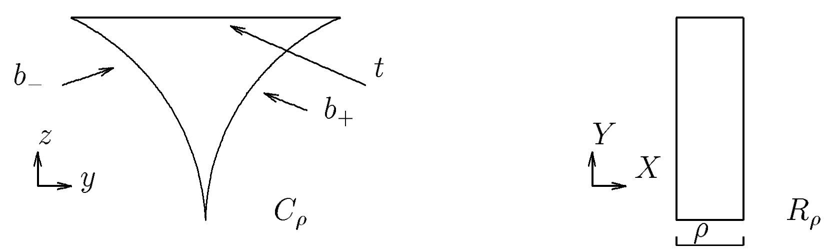


Figure 2: The cusp-shaped region C_ρ and the rectangle R_ρ.

Next we study the region $P_\rho = G_\mu(R_\rho)$. A segment given by $\{\xi\} \times [-1, 1]$ in R_ρ is mapped by G_μ to a curve $p_\xi = \{G_\mu(\xi, Y, 1) | -1 \leq Y \leq 1\}$. Letting

$$\begin{aligned} y &= g^1(\xi, Y; \mu) \\ z &= g^2(\xi, Y; \mu) \end{aligned}$$

and eliminating Y from these expressions, we obtain

$$y = \varphi(z; \xi, \mu).$$

More precisely, we consider the equation

$$\zeta = g^2(\xi, Y; \mu)$$

and solve it for Y using the implicit function theorem (recall that $g_Y^{20} \neq 0$). We obtain the function

$$Y = Y(\zeta; \xi, \mu). \tag{4}$$

satisfying

$$g^2(\xi, Y(\zeta; \xi, \mu); \mu) = \zeta \quad \text{and} \quad Y(0; 0, 0) = 0.$$

It follows that

$$\frac{\partial}{\partial \zeta} Y(\zeta; \xi, \mu) = \left[g_Y^2(\xi, Y(\zeta; \xi, \mu); \mu) \right]^{-1}$$

from which we obtain

$$\frac{\partial}{\partial \zeta} Y(0; 0, 0) = \frac{1}{g_Y^{20}},$$

and

$$\frac{\partial^2}{\partial \zeta^2} Y(0; 0, 0) = -\frac{g_{YY}^{20}}{(g_Y^{20})^3}.$$

Moreover, from the definition of $Y_*(\mu)$, we also have

$$Y_*(\mu) = Y(0; 0, \mu).$$

The function $\varphi(z; \xi, \mu)$ is now defined by

$$\varphi(z; \xi, \mu) = g^1(\xi, Y(z; \xi, \mu); \mu), \tag{5}$$

and therefore it satisfies

$$\varphi(0; 0, \mu) = g^1(0, Y(0; 0, \mu); \mu) = g^1(0, Y_*(\mu); \mu) = \nu_1. \tag{6}$$

Similarly we obtain

$$\frac{\partial}{\partial z} \varphi(0; 0, \mu) = \frac{g_Y^1(0, Y_*(\mu); \mu)}{g_Y^2(0, Y_*(\mu); \mu)} = \nu_2$$

and

$$\frac{\partial^2}{\partial z^2} \varphi(0; 0, 0) = \frac{g_{YY}^{10}}{(g_Y^{20})^2}.$$

In particular, the first order derivative $\frac{\partial}{\partial z} \varphi(0; 0, 0)$ at the bifurcation point vanishes whereas the second derivative $\frac{\partial^2}{\partial z^2} \varphi(0; 0, 0)$ does not, hence defining a parabola denoted by p_ξ. Note that the unstable manifold $W^u(O)$ intersects the cross section Σ_1 along the parabola p_0 given by $y = \varphi(z; 0, \mu)$. The set of the parabolas p_ξ forms a parabola-like region $P_\rho = G_\mu(R_\rho) \subset \Sigma_1$.

The mutual position of the cusp C_ρ and the region P_ρ at $\mu = 0$ depends on the Jacobian matrix

$$DG(0,0;0) = \begin{pmatrix} g_X^{10} & g_Y^{10} \\ g_X^{20} & g_Y^{20} \end{pmatrix}$$

and the second derivative g_{YY}^{10} as in Figure 3. In fact, since the flow-induced map from Σ_0 to Σ_1 is orientation-preserving and since $g_Y^{10} = 0$ from inclination-flip condition, we always have $\det DG(0,0;0) = g_X^{10} g_Y^{20} < 0$. If $g_{YY}^{10} g_X^{10} > 0$, then the unstable manifold is on the outside boundary of the parabola region P_ρ, while it is on the inside boundary if $g_{YY}^{10} g_X^{10} < 0$. According to [Deng 1991], the former case is called the *inward twist* case and the latter is called the *outward twist* case (see Figure 3).

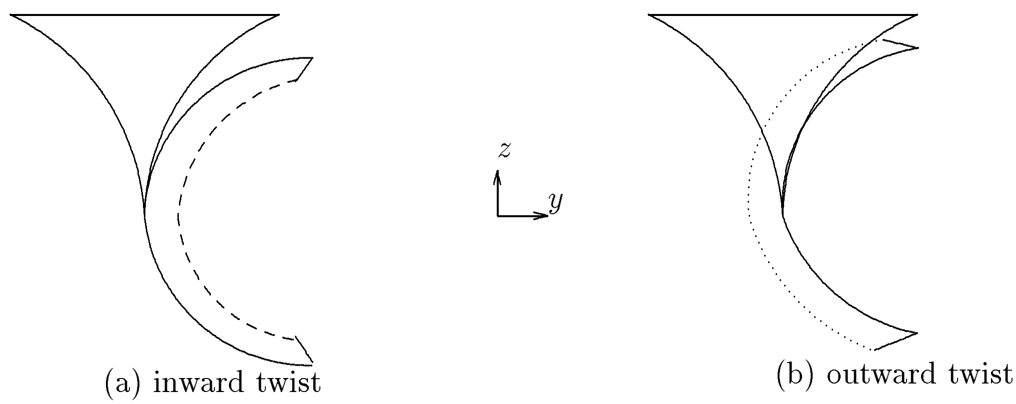


Figure 3: The regions C_ρ and P_ρ for the inward and outward twist.

In what follows we assume the parameter transformation (3) has been carried out, that is the considered objects depend on the parameters (ν_1, ν_2). We also assume that $g_{YY}^{10} > 0$. The other case is similarly treated.

We now state the two main theorems of the article.

Theorem 1 *Consider a two parameter family of vector fields X_ν having an inclination-flip bifurcation point at $\nu = 0$ corresponding to the inward twist* *case. Then there exist functions* $\nu_1^-(\nu_2) < 0 < \nu_1^+(\nu_2)$, *a neighborhood* U *of* $\overline{\Gamma}$ *and* $\varepsilon > 0$ *such that for each* $-\varepsilon < \nu_2 < 0$ *the following statements hold.*

(i) *When* $0 < \nu_1 < \nu_1^+(\nu_2)$ *the nonwandering set in* U *is the union of the singularity at* 0 *and a suspended horseshoe, namely the Poincaré map along* Γ *possesses a horseshoe.*

(ii) *As* ν_1 *decreases from* 0 *to* $\nu_1^-(\nu_2)$, *all the orbits of the suspended horseshoe disappear in a bifurcation connecting to the origin* O. *The bifurcation set is the closure of the parameter set for which there exists a homoclinic orbit in* U. *The Lebesgue measure of the bifurcation set is* 0.

(iii) *There exists an ordering of symbolic representations of periodic orbits. The order of disappearence of periodic orbits in an infinite period bifurcation agrees with the symbolic ordering.*

(iv) *Twisted homoclinic orbits correspond to isolated bifurcation values. Parameter values where nontwisted homoclinic orbits occur are isolated on the right and are accumulation points of other bifurcation points on the left.*

Theorem 2 *Consider a two parameter family of vector fields* X_ν *having an inclination-flip bifurcation point at* $\nu = 0$ *corresponding to the outward twist case. Then there exist two curves* $0 < \nu_1^-(\nu_2) < \nu_1^+(\nu_2)$, *a neighborhood* U *of* $\overline{\Gamma}$ *and* $\varepsilon > 0$ *such that for every* $-\varepsilon < \nu_2 < 0$ *the following statements hold.*

(i) *When* $0 < \nu_1^-(\nu_2) < \nu_1 < \nu_1^+(\nu_2)$ *the nonwandering set in* U *is the union of the singularity at* 0 *and a suspended horseshoe.*

(ii) *As* ν_1 *decreases from* $\nu_1^-(\nu_2)$ *to* 0 *all the orbits of the suspended horseshoe disappear in a bifurcation connecting to* O. *The bifurcation set is the closure of the parameter set for which there exists a homoclinic orbit in* U. *The Lebesgue measure of the bifurcation set is* 0.

(iii) *There exists an ordering of symbolic representations of periodic orbits. The order of disappearence of periodic orbits in an infinite period bifurcation agrees with the symbolic ordering.*

(iv) *Twisted homoclinic orbits correspond to isolated bifurcation values. Parameter values where nontwisted homoclinic orbits occur are isolated on the right and are accumulation points of other bifurcation points on the left.*

3 The existence of horseshoes

In this section we prove the existence of a horseshoe in the return map $f_\nu : \Sigma_1 \to \Sigma_1$ given in the previous section. Namely we show the following theorem:

Theorem 3 *Assume*

(EV1) $\lambda^{uu} > 2\lambda^u$,

(EV2) $-\lambda^s > 2\lambda^u$.

Then for small enough ν_1, ν_2 with

$$\nu_2 < 0, 0 < \nu_1 < k{\nu_2}^2, \tag{7}$$

for some $k > 0$ in the case of inward twist, and with

$$\nu_2 < 0, 0 < k_1{\nu_2}^R < \nu_1 < k_2{\nu_2}^2, \tag{8}$$

for some $k_1, k_2 > 0$, $R > 2$ in the case of outward twist, the nonwandering set of X_ν in a small neighborhood of $\bar{\Gamma}$ consists of the singularity O and a suspended topological horseshoe.

(*Proof.*) We only prove the inward twist case, the other case is proved in the same way. Recall $g_Y^{20} > 0$. Consider the strip $R_\rho = \{0 \le X \le \rho, |Y| \le 1\}$ in Σ_0 and its inverse image under F_ν,

$$C_\rho = \left\{ |y| \le z^{\frac{\lambda^{uu}}{\lambda^u}}, 0 < z < t_z = \rho^{-\frac{\lambda^u}{\lambda^s}} \right\}$$

in Σ_1. We shall show that, if the two eigenvalue conditions (EV1) and (EV2) hold, we can determine ρ as function of ν_2 (for small ν_2), such that, for ν_1 satisfying (7), the image $f_\nu(C_\rho)$ has a horseshoe shape intersecting C_ρ in two

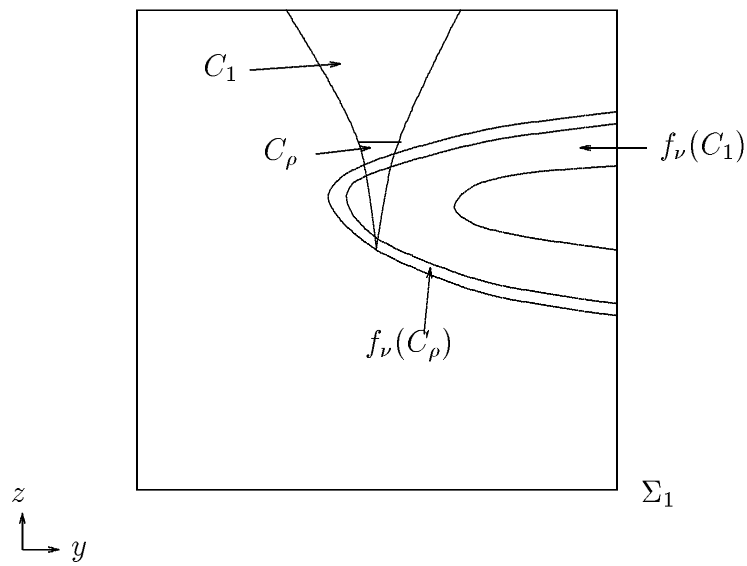


Figure 4: The position of C_ρ and $f_\nu(C_\rho)$ in Σ_1, for $\nu_1 = 0, \nu_2 < 0$.

strips, see Figure 4. Because $f_\nu(C_1 \setminus C_\rho)$ does not intersect $C_1 \setminus C_\rho$, this suffices to prove the theorem.

Recall that the image $G_\nu(\{X = \xi\})$ is of the form

$$\{(y,z) \in \Sigma_1 | y = \varphi(z;\xi,\nu) = \nu_1 + \nu_2 z + a z^2 + O(|\xi| + |z|^3 + |\nu|^2)\} \tag{9}$$

Here a is a function of the parameters, close to $\frac{g^{10}_{YY}}{(g^{20}_{Y})^2}$ for small ν_1, ν_2, which is hence positive from the assumption $g^{10}_{YY} > 0$. It follows from (9) that in a sufficiently small neighborhood of $\bar{\Gamma}$ and for ν_1, ν_2 small enough, there are positive constants α, β such that

$$G_\nu(R_\rho) \subset \{\nu_1 + \nu_2 z + \alpha z^2 \leq y \leq \nu_1 + \nu_2 z + \alpha z^2 + \beta\rho\}. \tag{10}$$

Define the map q_ξ by $q_\xi(z) = \nu_1 + \nu_2 z + \alpha z^2 + \beta\xi$. Observe that the quadratic map q_ξ has its top at $z_* = -\frac{\nu_2}{2\alpha}$. To start the computations, we suppose $\nu_1 = 0$. We first perform the computations showing that for small enough positive ν_1, the return map f_ν has a horseshoe. After these computations we derive the more precise bound (7).

The return map f_ν has a horseshoe for sufficiently small positive values of ν_1 if the following two conditions are satisfied:

$$q_\rho(z_*) \quad < \quad -z_*^{\frac{\lambda^{uu}}{\lambda^u}}, \tag{11}$$

$$q_0(t_z) \quad > \quad b_+(t_z) = (t_z)^{\frac{\lambda^{uu}}{\lambda^u}}. \tag{12}$$

This condition is sufficient for the existence of a topological horseshoe, if we observe the following two points. First, the side boundary of R_ρ given by $Y = \pm 1$ are mapped by G_ν to somewhere far to the right of the cusp-shaped region C_ρ. This is a consequence of the choice of local coordinates in the upper cross section Σ_0 in such a way that a sufficiently small neighborhood of $\Gamma \cap \Sigma_0$ contains the lines $Y = \pm 1$. This together with the continuous dependence of the image P_ρ on the parameter ν implies the claim for sufficiently small ν. Second, the image P_ρ indeed cuts the cusp C_ρ through horizontally. This is verified by checking that the preimage of the line $\{z = z_*\}$ by G_ν, namely a solution $Y = Y(z_*; \xi, \nu)$ of the equation $g^2(\xi, Y; \nu) = z_*$, sits in R_ρ disjoint from the side boundary. Since $\frac{\partial Y}{\partial \zeta}(0; 0, 0) = \frac{1}{g_Y^{20}} \neq 0$, the last assertion follows if ρ and ν are chosen sufficiently small. Thus the second condition is also verified.

The existence conditions (11) and (12) for the topological horseshoe are computed as follows:

$$q_\rho\left(\frac{-\nu_2}{2\alpha}\right) \quad < \quad -\left(\frac{-\nu_2}{2\alpha}\right)^{\frac{\lambda^{uu}}{\lambda^u}},$$

$$q_0\left(\rho^{-\frac{\lambda^u}{\lambda^s}}\right) \quad > \quad \left(\rho^{-\frac{\lambda^u}{\lambda^s}}\right)^{\frac{\lambda^{uu}}{\lambda^u}} = \rho^{-\frac{\lambda^{uu}}{\lambda^s}}.$$

Thus

$$-\frac{{\nu_2}^2}{4\alpha} + \beta\rho \quad < \quad -\left(\frac{-\nu_2}{2\alpha}\right)^{\frac{\lambda^{uu}}{\lambda^u}}, \tag{13}$$

$$\nu_2\rho^{-\frac{\lambda^u}{\lambda^s}} + \alpha\rho^{-\frac{2\lambda^u}{\lambda^s}} \quad > \quad \rho^{-\frac{\lambda^{uu}}{\lambda^s}}. \tag{14}$$

For ν_2, ρ small, by (14), in order to find the horseshoe we must have

$$-\frac{2\lambda^u}{\lambda^s} < -\frac{\lambda^{uu}}{\lambda^s}, \quad \text{i.e.} \quad \lambda^{uu} > 2\lambda^u.$$

Write $\rho = {\nu_2}^R$. By (13), using $\lambda^{uu} > 2\lambda^u$, we must require $R > 2$. Further, (14) gives

$$\alpha(-\nu_2)^{-\frac{2R\lambda^u}{\lambda^s}} > (-\nu_2)^{\frac{R\lambda^{uu}}{\lambda^s}} + (-\nu_2)^{1-\frac{R\lambda^u}{\lambda^s}}.$$

So we need

$$-\frac{2R\lambda^u}{\lambda^s} < 1 - \frac{R\lambda^u}{\lambda^s}, \quad \text{i.e.} \quad R < -\frac{\lambda^s}{\lambda^u}.$$

Since also $R > 2$, we need $-\lambda^s > 2\lambda^u$ to find the horseshoe.

It remains to find the estimate (7) on how far the existence of the horseshoe extends into the ν_1 direction. We find this bound by computing where the graph of q_ρ is tangent to the boundary $\{y = -z^{\frac{\lambda^{uu}}{\lambda^u}}\}$ of C_ρ. Because $\frac{\lambda^{uu}}{\lambda^u} > 2$, we may approximate this point by solving $q_\rho(z) = -z^{\lambda^{uu}/\lambda^u} \sim 0$ at $z = z_*$. This gives $\nu_1 - \frac{{\nu_2}^2}{4\alpha} \sim 0$, from which (7) follows. The upper bound in (8) is found in a similar way. To establish the lower bound we note that the topological horseshoe exists as soon as the parabola $(\nu_1 + \nu_2 z + \alpha z^2 + \beta\rho, z)$ cuts the y-axis at a negative value of y. Moreover $\beta < 0$ in the outward twist case. The estimate (8) follows. $\square$

Let $\kappa, \varepsilon > 0$ be small constants. In the forthcoming analysis we restrict our attention to the parameter region

$$\{-\varepsilon < \nu_2 < 0, \quad |\nu_1| < |\nu_2|^{2+\kappa}\}, \tag{15}$$

where the constants κ and ε are determined by the following proposition.

Proposition 2 *Fix $\kappa > 0$. There exists $\varepsilon > 0$ such that if ν is in the parameter region defined by (15), then $C_\rho \cap P_\rho$ is the union of two disjoint regions H_1 and H_2 (H_1 may be empty). Moreover there exists a constant $C > 0$, independent of ν and κ such that*

$$|g^1_Y(\xi, Y; \nu)| \geq C|\nu_2|, \quad (\xi, Y) \in G_\nu^{-1}(C_\rho).$$

(*Proof.*) The equation $y = \varphi(z; \xi, \nu)$ can be written in the form

$$az^2 + \nu_2 z + o(\nu_2^2 + z^2) = 0.$$

It follows that there are two solutions for z; $z = o(\nu_2)$ and $z = -\frac{1}{a}\nu_2 + o(\nu_2)$, which proves the first two assertions. To prove the estimate on $|g^1_Y(\xi, Y, ; \nu)|$

we note that from the definition of φ (see the equation (5)) it follows that

$$\frac{\partial}{\partial \zeta} Y(\zeta; \xi, \nu) \cdot g_Y^1(\xi, Y; \nu) = \frac{\partial \varphi}{\partial z}(\xi, Y; \nu).$$

The assertion follows since $\left|\frac{\partial \varphi}{\partial z}\right| = |\nu_2| + o(\nu_2)$. □

Let R be the constant introduced in Theorem 3. Let $\mathcal{P}$ denote the parameter region defined by (15) with (κ, ε) chosen so that $0 < \kappa < R - 2$ and so that Proposition 2 holds.

Remark 2 Note that for $\nu \in \mathcal{P}$, $P_1 \cap C_1 \subset C_\rho$. It follows that all the local recurrent dynamics of f_ν takes place in C_ρ. In other words, to understand the recurrent structure of the dynamics of $f_\nu|_{C_1}$ it suffices to consider $f_\nu|_{C_\rho}$.

Note that the topological horseshoe exists for some values of $\nu \in \mathcal{P}$ but its region of existence is not confined to $\mathcal{P}$. In the sequel we analyze the structure of the dynamics of $f_\nu|C_\rho$ for $\nu \in \mathcal{P}$, concentrating on the bifurcation sequence leading to the annihilation of the horseshoe. Our methods apply to a larger parameter region, but we are at this point not able to give a precise estimate of its size and can only speculate what happens at its boundary.

Showing that the topological horseshoe we found in Theorem 3 is a true horseshoe, namely, proving the hyperbolicity of the invariant sets of $f_\nu|_{C_\rho}$ is more difficult, because the angle between the expanding and contracting directions at the points of the invariant sets approaches 0 as $\nu_2 \to 0$. To remedy this fact we introduce a new cross section

$$\Sigma_\delta = \{x = \delta, |y| + |z| < 1\},$$

and decompose the local map F into two parts:

$$\begin{aligned} S_\delta : \Sigma_1 \to \Sigma_\delta; &\quad (1, y, z) \mapsto (\delta, \delta^{\frac{\lambda^{uu}}{\lambda^s}} y, \delta^{\frac{\lambda^u}{\lambda^s}} z), \\ F_\delta : \Sigma_\delta \to \Sigma_0; &\quad (\delta, y, z) \mapsto (\delta z^{-\frac{\lambda^s}{\lambda^u}}, y z^{-\frac{\lambda^{uu}}{\lambda^u}}, 1). \end{aligned}$$

Instead of $f_\nu : C_\rho \to C_\rho$ we consider the return map $f_\nu : C_{\rho,\delta} \to C_{\rho,\delta}$ where $C_{\rho,\delta} = F_\delta^{-1}(R_\rho)$. The expanding and contracting directions of the invariant set of $f_\nu|C_{\rho,\delta}$ are almost orthogonal. In the next section we prove that $f_\nu|C_{\rho,\delta}$ has a strong unstable foliation and that the distance between the leaves of the foliation is strongly contracted when f_ν is applied. These two facts imply, in particular, the hyperbolicity of the horseshoe found in Theorem 3.

4 Existence of an invariant foliation

In this section we prove the existence of an invariant foliation on the cusp $C_{\rho,\delta}$. Using this foliation we can reduce the study of the return map f to the analysis of a one-dimensional multivalued map.

Theorem 4 *Let R be the constant introduced in the proof of Theorem 3. Let D be such that R and D satisfy*

$$2 < R < -\frac{\lambda^u}{\lambda^s}, \quad \max\left\{\frac{\lambda^s}{\lambda^u - \lambda^{uu}}, \frac{-2\lambda^s}{\lambda^{uu}}\right\} < D < R.$$

Let $\alpha = \frac{\lambda^u - \lambda^{uu}}{\lambda^s}$. Then, for $\rho = |\nu_2|^R$, $\delta = |\nu_2|^D$ and $\nu \in \mathcal{P}$ there exists a foliation $\mathcal{F}^u$ on the cusp $C_{\rho,\delta}$ satisfying the following properties.

(A) *$\mathcal{F}^u$ is invariant for f in the following sense. If l is a leaf of $\mathcal{F}^u$ then the connected components of $f(l) \cap C_{\rho,\delta}$ are leaves of $\mathcal{F}^u$.*

(B) *$\mathcal{F}^u$ depends $C^{1+\alpha}$-smoothly on the base points and on the parameter ν_1. The dependence on ν_2 is continuous. The leaves of $\mathcal{F}^u$ are at least C^2 smooth.*

(C) *f contracts distances between the leaves of $\mathcal{F}^u$ and expands distances along the leaves of $\mathcal{F}^u$. More precisely there exists $\eta > 1$ such that*

1. *if $f(l_1)$ and $f(l_2)$ are in the same connected component of $f(C_{\rho,\delta}) \cap C_{\rho,\delta}$ then $\mathrm{dist}(f(l_1), f(l_2)) < \eta^{-1} \cdot \mathrm{dist}(l_1, l_2)$.*
2. *if x, $y \in l$ and $f(x)$, $f(y) \in C_{\rho,\delta}$, then $\mathrm{dist}(f(x), f(y)) > \eta \cdot \mathrm{dist}(x, y)$.*

The remainder of this section is devoted to proving Theorem 4. The proof is given using the graph transformation technique. We begin by defining the suitable graph transformation. Let $\mathcal{C}_0$ be the Banach space of continuous vector fields on $C_{\rho,\delta}$ of the form $(1, v(x))$ equipped with the supremum norm $\|v\|$. We identify this space with the space of continuous functions on $C_{\rho,\delta}$. We write $Df(p)$ as

$$Df(p) = \begin{pmatrix} A(p) & B(p) \\ C(p) & D(p) \end{pmatrix}.$$

Let v_0 be a fixed function such that $\|v_0\| \leq 1$ and the following conditions hold.

(i) $v_0(p) = \delta^{-\alpha}\frac{g^1_Y((S_\delta\circ G)^{-1}(p))}{g^2_Y((S_\delta\circ G)^{-1}(p))}$ for $p \in (S_\delta \circ G)(W^u_{loc}(O)\cap\Sigma_0)$;

(ii) $v_0(f(p)) = \frac{C(p)+D(p)v_0(p)}{A(p)+B(p)v_0(p)}$ for $p \in t$.

We define the graph transformation as follows (compare [Rob89]).

$$\Gamma(v)(p) = \begin{cases} v_0(p) & p \in C_{\rho,\delta}\setminus f(C_{\rho,\delta}) \\ \frac{C(q)+D(q)v(q)}{A(q)+B(q)v(q)} & q = f^{-1}(p) \text{ with } p \in f(C_{\rho,\delta}). \end{cases}$$

Note that for a general v, Γv is not necessarily continuous as a function of p. Therefore we restrict ourselves to the subspace $\mathcal{C}_0'$ of $\mathcal{C}_0$ consisting of the functions v with $\|v\| \le 1$ and satisfying the condition:

$$v(f(p)) = \frac{C(p)+D(p)v_0(p)}{A(p)+B(p)v_0(p)} \quad \text{for} \quad p \in t.$$

We will show that $\Gamma(\mathcal{C}_0') \subset \mathcal{C}_0'$. To prove this property we observe that Γ can be written as a composition of two transformations, one induced by F_δ and the other one by $H_\delta = S_\delta \circ G$. We now define the transformation $\Phi = \Phi_\nu$ carrying vector fields of the form $(1, v)$ on $C_{\rho,\delta}$ to vector fields of the form $(w, 1)$ on R_ρ.

$$(\Phi v)(p) = \begin{cases} 0 & \text{for } p = (0, Y), \\ \frac{F^2_z(q)+F^2_y(q)v(q)}{F^1_z(q)+F^1_y(q)v(q)} & q = F_\delta^{-1}(p) \quad \text{for } p \ne (0, Y). \end{cases}$$

We have the following lemma.

Lemma 1 *If $v \in \mathcal{C}_0$, then $(\Phi v)(p)$ is continuous. If moreover $v \in C^{1+\alpha}$, then $(\Phi v)(p) \in C^{1+\alpha}$ and $\frac{\partial(\Phi v)}{\partial X}(0, Y) = 0$.*

(*Proof.*) A straightforward computation shows that

$$\Phi v(X, Y) = -\frac{\lambda^s}{\lambda^u}X^{1+\alpha}\frac{v(F_\delta^{-1}(X, Y))}{\delta^\alpha - \frac{\lambda^{uu}}{\lambda^u}YX^\alpha v(F_\delta^{-1}(X, Y))}.$$

Recall that $X \le \rho$, $\rho = |\nu_2|^R$ and $\delta = |\nu_2|^D$. Hence, by the assumption $R < D$, $X \ll \delta$. $\square$

We now prove a technical lemma which will be used in verifying a number of inequalities required for the proof of the Theorem 4. We write $Df^{-1}(p)$ in the following form:

$$Df^{-1}(p) = \begin{pmatrix} A'(p) & B'(p) \\ C'(p) & D'(p) \end{pmatrix}.$$

Lemma 2 *Let $p \in C_{\rho,\delta} \cap f(C_{\rho,\delta})$. Then the expressions:*

$$|D'(p)^{-1}|, \qquad \left|\frac{\det Df^{-1}(p)}{D'(p)}\right|,$$

$$\left|\frac{A'(p)}{D'(p)}\right|, \qquad \left|\frac{B'(p)}{D'(p)}\right|, \qquad \left|\frac{C'(p)}{D'(p)}\right|$$

converge to 0 *as* $\nu_2 \to 0$.

(*Proof.*) The proof is a straightforward calculation based on the estimate in Proposition 2 and is left to the reader. □

The following proposition implies the existence of a continuous direction field on $C_{\rho,\delta}$ which is invariant under f.

Proposition 3 *Suppose that $(\nu_1, \nu_2) \in \mathcal{P}$. Then $\Gamma(\mathcal{C}'_0) \subset \mathcal{C}'_0$ and Γ is a contraction on $\mathcal{C}'_0$.*

(*Proof.*) The graph transformation Γ is a composition of Φ and a transformation induced by the smooth map $H_\delta = S_\delta \circ G$. This and the statement of Lemma 1 imply that $\Gamma(\mathcal{C}'_0) \subset \mathcal{C}'_0$. We now show that Γ is a contraction on $\mathcal{C}'_0$ (compare ([Moser73]). Let v, $\hat{v} \in \mathcal{C}'_0$. A straightforward computation shows that

$$\|\Gamma v - \Gamma \hat{v}\| \leq \sup_{C_{\rho,\delta} \cap f^{-1}(C_{\rho,\delta})} \left\| \frac{\det Df(q)}{(A(q) + B(q)v(q))(A(q) + B(q)\hat{v}(q))} \right\| \cdot \|v - \hat{v}\|.$$

The equality $[Df(p)]^{-1} = Df^{-1}(f(p))$ implies that

$$\left|\frac{\det Df}{(A + Bv)(A + B\hat{v})}\right| = \left|\frac{\det Df^{-1}}{D'}\right| \cdot \frac{1}{|1 - \frac{B'}{D'}\hat{v}| \cdot |D'| \cdot |1 - \frac{B'}{D'}v|}. \tag{16}$$

It follows from Lemma 2 that the right hand side of (16) converges to 0 as $\nu_2 \to 0$ independently of v and $\hat{v}$. The proposition follows. $\square$

We now state a lemma which will be useful in the proof of the existence of a smooth invariant foliation. For proof see [Hen81] or [Hom93].

Lemma 3 *Let $C^0(E,F)$ be the space of bounded continuous maps $E \to F$, between two complete metric spaces E and F, equipped with the supremum norm.*

Let $C_N^{1+\alpha}(E,F) \subset C^0(E,F)$ be the C^1 maps $f : E \to F$, such that Df is α-Hölder with Hölder constant N: $\|Df(x) - Df(y)\| \leq N\|x-y\|^\alpha$.

Let $Lip_1(E,F) \subset C^0(E,F)$ be the Lipschitz continuous maps $f : E \to F$, with Lipschitz constant 1: $\|f(x) - f(y)\| \leq \|x - y\|$.

Then the set $C_N^{1+\alpha}(E,F) \cap Lip_1(E,F)$ is closed in $C^0(E,F)$.

Let N be a constant and α as in the statement of Theorem 4. We define the space $S_{\alpha,N}$ as the space of elements $v \in \mathcal{C}_0$ which are $C^{1+\alpha}$, and Dv is Hölder continuous with constant N. Lemma 3 implies that $S_{\alpha,N}$ is closed in $\mathcal{C}_0$. Assume that $v_0 \in S_{\alpha,N}$ (v_0 is the function used in the definition of the graph transform) and let $S'_{\alpha,N}$ be the subspace of $S_{\alpha,N}$ consisting of the elements $v \in \mathcal{C}'_0$ whose derivatives satisfy the conditions:

(iii)

$$\frac{\partial v}{\partial(y,z)}(p) = \frac{\partial(\Gamma v_0)}{\partial(y,z)}(p) \quad \text{for} \quad p \in H_\delta(\rho, Y)$$

(iv)

$$\frac{\partial v}{\partial(y,z)}(p) = DH_\delta(0,Y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for} \quad p = H_\delta(0,Y).$$

The proof of part (A) of Theorem 4 is based on the following proposition.

Proposition 4 *There exist $N > 0$ and $\nu_2^* > 0$ such that for $(\nu_1,\nu_2) \in \mathcal{P}$ the graph transformation Γ maps $S'_{\alpha,N}$ into $S'_{\alpha,N}$.*

(*Proof*) Let $\Psi : \mathbb{R}^3 \to \mathbb{R}$ and $\Theta : \mathbb{R}^3 \to \mathbb{R}$ be defined as follows.

$$\Psi(X,Y,\omega) = \left(-\frac{\lambda^s}{\lambda^u}\right) \frac{X^{1+\alpha}\omega}{\delta^\alpha - (\frac{\lambda^{uu}}{\lambda^u})YX^\alpha\omega}$$

and

$$\Theta(X,Y,\omega) = \delta^\alpha \frac{g_Y^2(X,Y) - g_X^2(X,Y)\cdot\Psi(X,Y,\omega)}{g_Y^1(X,Y) - g_X^1(X,Y)\cdot\Psi(X,Y,\omega)}.$$

A straightforward computation shows that

$$\Gamma v(y,z) = \Theta(H_\delta^{-1}(y,z), v(f^{-1}(y,z))). \tag{17}$$

Let $(\Gamma v)'$ denote $\frac{\partial(\Gamma v)}{\partial(y,z)}$. Using the above formula we prove that

$$(\Gamma v)' = \left(\frac{\partial\Theta}{\partial(X,Y)} + \frac{\partial\Theta}{\partial\omega}\cdot\frac{\partial v}{\partial(y,z)}\cdot DF_\delta^{-1}\right)\circ H_\delta^{-1}\cdot DH_\delta^{-1}. \tag{18}$$

We first show that $\|(\Gamma v)'\|$ converges to 0 uniformly in v as $\nu_2 \to 0$. We begin by obtaining an estimate for $\frac{\partial\Theta}{\partial(X,Y)}$.

$$\frac{\partial\Theta}{\partial(X,Y)} = \delta^\alpha\left\{\frac{\partial}{\partial(X,Y)}\left[\frac{g_Y^2}{g_Y^1 - g_X^1\Psi}\right] - \frac{\partial}{\partial(X,Y)}\left[\frac{g_X^2}{g_Y^1 - g_X^1\Psi}\right]\Psi\right\}$$
$$-\delta^\alpha\left\{\frac{g_X^2}{g_Y^1 - g_X^1\Psi}\cdot\frac{\partial\Psi}{\partial(X,Y)}\right\}.$$

The inequality $|X| \le \rho$ implies that

$$\Psi(X,Y,\omega) = O(\delta^{-\alpha}X^{1+\alpha}) \text{ and } \frac{\partial\Psi}{\partial(X,Y)}(X,Y,\omega) = O(\delta^{-\alpha}X^\alpha).$$

We now estimate the terms $\frac{\partial}{\partial(X,Y)}\left[\frac{g_*^2}{g_Y^1 - g_X^1\Psi}\right]$, where $* = X$ or Y.

$$\begin{aligned}\frac{\partial}{\partial(X,Y)}\left[\frac{g_*^2}{g_Y^1 - g_X^1\Psi}\right] &= \frac{\frac{\partial g_*^2}{\partial(X,Y)}}{g_Y^1 - g_X^1\Psi} - \frac{g_*^2}{(g_Y^1 - g_X^1\Psi)^2}\left(\frac{\partial g_Y^1}{\partial(X,Y)} - \frac{\partial g_X^1}{\partial(X,Y)}\Psi\right)\\ &+ \frac{g_*^2 g_X^1}{(g_Y^1 - g_X^1\Psi)^2}\cdot\frac{\partial\Psi}{\partial(X,Y)}.\end{aligned}$$

Using the inequality $|\Psi| < X < |\nu_2|$ and Proposition 2, we conclude that

$$\left\|\frac{\partial\Theta}{\partial(X,Y)}\right\| = \delta^\alpha\cdot O(\nu_2^{-2}).$$

Since $\|DH_\delta^{-1}\| = O(\delta^{-\frac{\lambda^u}{\lambda^s}})$, it follows that

$$\left\| \frac{\partial \Theta}{\partial (X,Y)} DH_\delta^{-1} \right\| = \delta^{-\frac{\lambda^{uu}}{\lambda^s}} \cdot O(\nu_2^{-2}).$$

The inequality $\frac{-2\lambda^s}{\lambda^{uu}} < D$ implies that this expression converges to 0 as $\nu_2 \to 0$. This way we have estimated the first term in $(\Gamma v)'$.

To estimate the second term, we observe that

$$\frac{\partial \Theta}{\partial \omega} = \delta^\alpha \cdot \frac{\det DG \cdot \frac{\partial \Psi}{\partial \omega}}{(g_Y^1 - g_X^1 \Psi)^2}.$$

and

$$\frac{\partial \Psi}{\partial \omega} = \left(-\frac{\lambda^u}{\lambda^s} \right) \frac{\delta^\alpha X^{1+\alpha}}{(\delta^\alpha - \frac{\lambda^{uu}}{\lambda^u} X^\alpha Y \omega)^2}.$$

It follows that $\frac{\partial \Theta}{\partial \omega} = O(X^{1+\alpha} \cdot \nu_2^{-2})$. Moreover

$$DF_\delta^{-1} = \delta^{\frac{\lambda^u}{\lambda^s}} X^{-\frac{\lambda^u}{\lambda^s} - 1} \cdot \begin{pmatrix} c_1 \cdot \delta^{-\alpha} X^\alpha Y & c_2 \cdot \delta^{-\alpha} X^{\alpha+1} \\ c_3 & 0 \end{pmatrix},$$

where c_1, c_2 and c_3 are almost constant terms. We now have

$$\begin{aligned} \left\| \frac{\partial \Theta}{\partial \omega} \cdot \frac{\partial v}{\partial (X,Y)} \cdot Df^{-1} \right\| &\leq (\text{const.}) \left\| \frac{\partial \Theta}{\partial \omega} \right\| \cdot \left\| DF_\delta^{-1} \right\| \cdot \left\| DH_\delta^{-1} \right\| \\ &= O(X^{\alpha - \frac{\lambda^u}{\lambda^s}} \nu_2^{-2}) \leq O(\rho^{\frac{-\lambda^{uu}}{\lambda^s}} \nu_2^{-2}). \end{aligned}$$

The conditions $R > D$ and $D(\frac{-\lambda^{uu}}{\lambda^s}) > 2$, together with the preceding computation, imply that $\|(\Gamma v)'\|$ converges to 0 uniformly in v as $\nu_2 \to 0$. We observe that $(\Gamma v)'(y,z)$ consists of terms of the form

$$h(y,z,v(y,z)) \cdot l(y,z,v(y,z)),$$

where h and l are α-Hölder as functions of (y,z,ω), or

$$k(y,z,v(y,z)) \cdot \frac{\partial v}{\partial (y,z)},$$

where k is α-Hölder as a function of (y,z,ω) and $\|k\| \ll 1$. Therefore, for sufficiently large N, $(\Gamma v)'(y,z)$ is α-Hölder as a function of (y,z). Hence

$\Gamma(S'_{\alpha,N}) \subset S'_{\alpha,N}$. $\square$

(*Proof of Theorem 4* .) (A) Proposition 3 and the Contraction Mapping Theorem imply that Γ has a unique fixed point $u \in \mathcal{C}'_0$. The fixed point u is a direction field invariant under Γ. By integrating u we obtain the required invariant foliation $\mathcal{F}^u$.

(B) Choose ν_2^* and N, so that the assertion of Proposition 4 holds. Let $v \in S'_{\alpha,N}$. Clearly $u = \lim_{n\to\infty} \Gamma^n(v)$. Proposition 4 and Lemma 3 imply that $u \in S'_{\alpha,N}$. Hence u is $C^{1+\alpha}$. It follows that $\mathcal{F}^u$ is $C^{1+\alpha}$. Differentiation along the leaves produces u, hence the leaves are at least $C^{2+\alpha}$. To prove the continuous dependence of u on the parameters (ν_1, ν_2) we modify the definition of $\mathcal{C}'_0$ to functions depending on (ν_1, ν_2) and observe that the estimates similar to the ones obtained in the proof of Proposition 3 hold. To prove the $C^{1+\alpha}$ dependence on ν_1 we redo the proof of Proposition 4 for functions depending on (y, z, ν_1). The proof is analogous. Note that this method of proof may fail for the parameter ν_2.

(C) follows from the fact that the expressions $|C'u + D'|$ and $|A + Bu|$ uniformly diverge as $\nu_2 \to 0$. This property is the consequence of Lemma 2 via the following equalities.

$$|C'u + D'| = |D'| \cdot \left| 1 + \frac{C'}{D'} u \right|$$

$$|A + Bu| = |A| \cdot \left| 1 + \frac{B}{A} u \right| = \left| \frac{D'}{\det Df^{-1}} \right| \cdot \left| 1 - \frac{B'}{D'} u \right| .$$

$\square$

5 Multivalued maps

Consider the foliation $\mathcal{F}^u$ whose existence was proved in Section 4. Let I be the intersection of the line $\{y = 0\}$ in Σ_δ with the cusp shaped region $C_{\rho,\delta}$. It is clear that I intersects the leaves of $\mathcal{F}^u$ transversally. Let τ denote the projection of the leaves of $\mathcal{F}^u$ onto I and let $\pi_\nu = \tau \circ f_\nu \circ \tau^{-1}$. ¿From the considerations in Section 4 it follows that π_ν is in general not a well-defined map. Indeed, there exist leaves l of $\mathcal{F}^u$ such that $f_\nu(l) \cap C_{\rho,\delta}$ consists of two leaves. Clearly $\pi_\nu(\tau(l))$ has two values. In other words π_ν is a union of two maps: $z \to \xi_\nu(z)$, which assigns to z the larger value in $\pi_\nu(z)$ and $z \to \eta_\nu(z)$ which assigns to z the smaller value in $\pi_\nu(z)$. The domain of η_ν is, in general, only a subinterval of I. We denote this interval by K_ν (see Figures 5, 6). Note that π_ν is only defined on $(0, t_z]$. We extend this definition by requiring that $\pi_\nu(0) = \lim\limits_{z\to 0}\pi_\nu(z)$.

The goal of this section is to characterize the recurrent dynamics of the vector field X_ν near the homoclinic orbit Γ and to analyze its bifurcations in the parameter region mentioned in Theorems 1 and 2. It follows from Remark 2 that this is equivalent to analyzing the recurrent dynamics of $f_\nu|C_{\rho,\delta}$. Note that each trajectory of π_ν remaining in I corresponds to a unique trajectory of $f_\nu|C_{\rho,\delta}$ and each trajectory of f_ν which remains for all positive time in $C_{\rho,\delta}$ corresponds to a trajectory of π_ν. Consequently Theorems 1 and 2 follow from the results for π_ν which we describe below. Theorem 5 and Proposition 6 have been first formulated and proved by Homburg in [Hom93]. We include these results for completeness, but only sketch the proofs, referring the reader to the source for details. We begin with the following proposition.

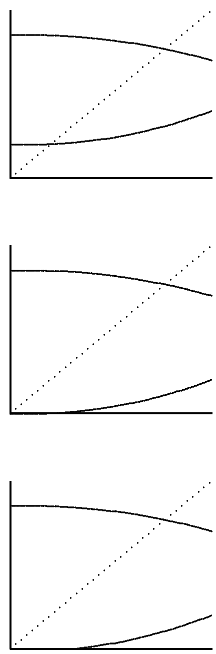
Proposition 5 *The maps ξ_ν, η_ν are $C^{1+\alpha}$ with $|\eta'_\nu(z)|, |\xi'_\nu(z)| = o(1)$ with respect to ν_2. In the case of inward twist $\xi'_\nu(z) < 0$ and $\eta'_\nu(z) > 0$. In the case of outward twist $\xi'_\nu(z) > 0$ and $\eta'_\nu(z) < 0$.*

(*Proof.*) The statements about the signs of the derivatives follow from the definition of inward and outward twist and from Proposition 1. The statement about the size of the derivatives follows from Theorem 4. $\square$

Note that $I = [0, t_\nu]$, where $t_\nu = |\nu_2|^{(R-D)\frac{-\lambda^s}{\lambda^u}}$. It follows from Proposition 5 that in the case of inward twist $K_\nu = [a_\nu, t_\nu]$, where $a_\nu = \eta^{-1}(0)$. In the case of outward twist $K_\nu = [0, b_\nu]$ $(b_\nu = \eta^{-1}(0))$. See Figures 5, 6.

We concentrate on the case of inward twist. At the end of the section we prove a result extending the analysis to the case of outward twist.

Fix ν_2 and let $n \in \mathbb{N}$. Consider the set $\pi^n_\nu(0)$. When $\nu_1 > 0$ this set consists of 2^n points. As ν_1 is varied a point $p \in \pi^n_\nu(0)$ varies defining a curve $p(\nu_1)$. More precisely, let $\sigma : \{1, \cdots, n\} \to \{\xi_\nu, \eta_\nu\}$ and define $p(\nu_1) = \sigma(n) \circ \sigma(n-1) \circ \cdots \circ \sigma(1)(0)$. Suppose at least one element of the sequence $\{\sigma(1), \cdots, \sigma(n)\}$ equals η_ν. Then for $\nu_1 < 0$ and $|\nu_1|$ large enough $p(\nu_1)$ no longer exists. This is a consequence of the fact that as ν_1 decreases the interval K_ν of definition of η_ν shrinks and eventually becomes empty. For

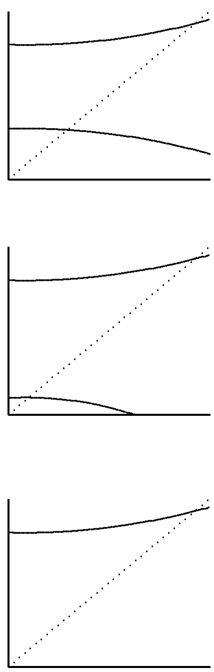


(a) $\nu_1^+ > \nu_1 > 0$

(b) $\nu_1 = 0$

(c) $\nu_1 < 0$

Figure 5: The map π_ν in the inward twist case.



(a) $\nu_1^+ > \nu_1 > \nu_1^-$

(b) $\nu_1^- > \nu_1 > 0$

(c) $\nu_1 = 0$

Figure 6: The map π_ν in the outward twist case.

example $\eta_\nu(0)$ does not exist for $\nu_1 < 0$. Assume $p(\nu_1)$ is a curve corresponding to an element of $\pi_\nu^n(0)$. We now estimate the rate of change of $\eta_\nu(p(\nu_1))$ as ν_1 is being varied.

Lemma 4 *There exists a positive constant C independent of n such that if $p(\nu_1)$ is a curve corresponding to an element of $\pi_\nu^n(0)$ then*

$$\frac{d}{d\nu_1}\eta_\nu(p(\nu_1)) > C.$$

(*Proof.*) Fix $z_0 \in I$ and let $l_{z_0} = \{(y, h(y))\}$ be a leaf of $\mathcal{F}^u$ with $h(0) = z_0$. The curve transforms under F_δ to the curve

$$\tilde{l}_{X_0} = \{(\tilde{h}(Y), Y)\}, \quad X_0 = F_\delta(z_0, 0).$$

It follows from the definition of F_δ that $\tilde{h}(Y)$ is implicitly defined by the equation

$$X = \delta \cdot [h(\delta^{\frac{\lambda^{uu}}{\lambda^s}} X^{-\frac{\lambda^{uu}}{\lambda^s}} Y)]^{-\frac{\lambda^u}{\lambda^s}}.$$

Implicit differentiation and the boundedness of $\frac{\partial h}{\partial \nu_1}$ and $\frac{\partial h}{\partial y}$ for the function $h(y; z_0, \nu)$ imply that $\left|\frac{\partial \tilde{h}}{\partial \nu_1}\right|$ becomes arbitrarily small as $\nu_2 \to 0$. The curve $S_\delta \circ G_\nu(\tilde{l}_{X_0})$ intersects the line $y = 0$ at the points $\eta_\nu(z_0) < \xi_\nu(z_0)$. The quantity $\delta^{-\frac{\lambda^u}{\lambda^s}} \cdot \eta_\nu(z_0)$ is obtained by solving the following two equations for z.

$$g^1(\tilde{h}(Y), Y, \nu) = 0$$

$$g^2(\tilde{h}(Y), Y, \nu) = z.$$

We solve the second equation for Y as a function of z. The remaining equation is

$$g^1(\tilde{h}(Y(z)), Y(z), \nu) = 0. \tag{19}$$

The fact that $\left|\frac{\partial \tilde{h}}{\partial \nu_1}\right| \to 0$ as $\nu_2 \to 0$ implies that $g^1(\tilde{h}(Y(z)), Y(z), \nu)$ can be written in the following form (see also the equations (6) and (9)).

$$g^1(\tilde{h}(Y(z)), Y(z), \nu) = \nu_1 + \nu_2 z + az^2 + o(|\nu_1| + |z|^2 + |\nu|^2).$$

It follows that (19) has two solutions z_+ and z_-, with $z_- = o(\nu_1)$. Implicit differentiation implies

$$\frac{dz_-}{d\nu_1} = -\frac{1}{\nu_2}(1 + o(\nu_1)).$$

Hence there exist constants $0 < C_- < C_+$ and $\beta > 0$, independent of z_0, such that

$$C_-|\nu_2|^{-\beta} < \frac{\partial \eta_\nu}{\partial \nu_1}(z_0) < C_+|\nu_2|^{-\beta}. \tag{20}$$

We also have

$$\frac{dz_+}{d\nu_1} = \frac{1}{\nu_2}(1 + o(\nu_1)). \tag{21}$$

We choose C_+ so that $\frac{\partial \xi_\nu}{\partial \nu_1}(z_0) < C_+|\nu_2|^{-\beta}$. Note that we can choose the constants C_- and C_+ so that the estimates (20) and (21) hold independent of z_0.

We now estimate $\frac{d}{d\nu_1}\eta_\nu(p(\nu_1))$. Let $\sigma : \{1, \cdots, n\} \to \{\xi_\nu, \eta_\nu\}$ be the sequence defining $p(\nu_1)$. Let $p_j = \sigma(j) \circ \sigma(j-1) \circ \cdots \circ \sigma(1)$. In particular $p_n = p$. Differentiating η_ν with respect to ν_1 gives

$$\frac{d}{d\nu_1}\eta_\nu(p(\nu_1)) = \frac{\partial \eta_\nu}{\partial \nu_1}(p(\nu_1)) + \frac{\partial \eta_\nu}{\partial z}(p(\nu_1)) \cdot \frac{d}{d\nu_1}p(\nu_1).$$

$\frac{d}{d\nu_1}p(\nu_1)$ can be expressed as follows.

$$\frac{d}{d\nu_1}p(\nu_1) = \sum_{j=1}^{n-1} \frac{\partial \sigma(n-j)}{\partial \nu_1}(p_{n-j-1}(\nu_1)) \cdot \left(\prod_{k=0}^{j-1} \frac{\partial \sigma(n-k)}{\partial z}(p_{n-k-1}(\nu_1)) \right).$$

Let

$$r = \sup_{z \in I} \left| \frac{\partial \pi_\nu}{\partial z}(z) \right|.$$

Theorem 4 implies that r is arbitrarily small provided that ν_2 is small enough. Hence

$$\frac{d}{d\nu_1}p(\nu_1) \le C_+|\nu_2|^{-\beta}(r + r^2 + \cdots + r^n) \le C_+|\nu_2|^{-\beta} \cdot \frac{r}{1-r}.$$

It follows that for small enough ν_2 there exists $C > 0$, independent of n, such that

$$\frac{\partial}{\partial \nu_1} \eta_\nu(p(\nu_1)) > C.$$

$\square$

Similar arguments as used in the proof of Theorem 3 imply that for every small enough ν_2 there exists $\nu_1^-(\nu_2) < 0$ such that for every $\nu_1 < \nu_1^-(\nu_2)$ $K_\nu = \emptyset$. Moreover $|\nu_1^-(\nu_2)| < O(|\nu_2|^R)$, which implies that $(\nu_1^-(\nu_2), \nu_2) \in \mathcal{P}$. In the subsequent analysis we fix ν_2 and let ν_1 decrease from 0 to ν_1^-.

We analyze the non-wandering set and the bifurcation set of π_ν. Observe that π_ν^{-1} is a well-defined map reminiscent of the quadratic map of the interval. We can define symbolic dynamics of π_ν^{-1} in the following way. Given $x \in I$ let $S(x)$ be the infinte sequence of the letters U and D such that

$$S_j(x) = \begin{cases} \text{U} & \text{if} \quad \pi_\nu^{-j}(x) \in \xi_\nu(I) \\ \text{D} & \text{if} \quad \pi_\nu^{-j}(x) \in \eta_\nu(I). \end{cases}$$

We consider the well known ordering on the set of sequences. Let σ, τ be two sequences. Let j be the first integer such that $\sigma_j \neq \tau_j$. Then

$$\sigma \prec \tau \quad \text{if} \quad \begin{cases} \sigma_0, ..., \sigma_{j-1} \text{ contain an even number of U's, } \sigma_j = \text{D and } \tau_j = \text{U}, \\ \sigma_0, ..., \sigma_{j-1} \text{ contain an odd number of U's, } \sigma_j = \text{U and } \tau_j = \text{D}. \end{cases}$$

Note that every periodic point of π_ν has a well defined symbolic sequence given by the corresponding sequence of π_ν^{-1}. A periodic orbit of period n has n different symbolic sequences. We will refer to the minimal of these sequences as the sequence of the periodic orbit. Fix a small ν_2. Consider a periodic orbit of π_ν with a given symbolic sequence. As ν_1 decreases from 0 the leftmost point on the orbit approaches 0. The value of ν_1 for which a_ν is an element of the periodic orbit marks the parameter point for which the periodic orbit disappears; when ν_1 further decreases a periodic orbit with this itinerary no longer exists. We refer to periodic orbits of π_ν which contain 0 as *homoclinic orbits*. Clearly such periodic orbits correspond to homoclinic orbits of the vector field and the disappearance of these periodic orbits correspond to the bifurcations of homoclinic orbits. We hence refer to them as homoclinic bifurcations. We have the following lemma.

Lemma 5 *Fix $\nu_2 < 0$. Let γ_1, γ_2 be periodic orbits of π_ν and let σ_1, σ_2 be minimal symbolic sequences corresponding to γ_1 and γ_2. Then, as ν_1 decreases, γ_1 disappears first in a homoclinic bifurcation if $\sigma_1 \prec \sigma_2$. Moreover, the homoclinic bifurcations unfold generically, that is a homoclinic orbit with a given itinerary exists for a unique value of ν_1.*

(*Proof.*) $\sigma_1 \prec \sigma_2$ if and only if the leftmost point of the orbit γ_1 is left of the leftmost point of γ_2. The second statement of the lemma follows from Lemma 4. $\square$

Remark 3 Homoclinic orbits of π_ν whose symbolic sequences contain an odd number of U's correspond to twisted homoclinic orbits of the vector field. Homoclinic orbits of π_ν whose symbolic sequences contain an even number of U's correspond to nontwisted homoclinic orbits of the vector field.

Let $J = (\eta_\nu(\xi_\nu(0)), \xi_\nu^2(0))$ ($J = (\eta_\nu(0), \xi_\nu(0))$ in the case of outward twist). Note that $\pi_\nu^m(J) \cap J = \emptyset$ for all positive integers m. It follows that $\pi_\nu^m(J) \cap \pi_\nu^k J = \emptyset$ for all choices of positive integers (m, k). In other words J is a wandering interval. Let ω_ν denote the nonwandering set of π_ν. It is clear that $\omega_\nu \subset I \setminus \bigcup_{j\in\mathbb{N}}(\pi_\nu^j(J))$. Moreover Proposition 5 implies that $\bigcup_{j\in\mathbb{N}}(\pi_\nu^j(J))$ is dense in I. Hence $\omega_\nu \subset \mathrm{cl}(\bigcup_{j\in\mathbb{N}}(\pi_\nu^j(\partial J)))$. Fix ν_2 and let ν_1 decrese from 0 to $\nu_1^-(\nu_2)$. It is not hard to see that for the values of ν such that $0 \in \bigcup_{j\in\mathbb{N}}(\pi_\nu^j(J))$ f_ν is structurally stable. Moreover, when 0 passes through a point in $\mathrm{cl}(\bigcup_{j\in\mathbb{N}}(\pi_\nu^j(\partial J)))$, a trajectory of π_ν^{-1} with a certain symbolic sequence ceases to exist. Let

$$\mathcal{B}_{\nu_2} = \{\nu_1 \mid 0 \in \mathrm{cl}(\bigcup_{j\in\mathbb{N}} \pi_\nu^j(\partial J))\}$$

It is clear from the above discussion that $\mathcal{B}_{\nu_2}$ is the set for values of ν_1 for which $f_\nu|C_{\rho,\delta}$ undergoes a bifurcation. We have the following theorem.

Theorem 5 *Fix ν_2. Then*

(i) *$\mathcal{B}_{\nu_2}$ equals the closure of the set of homoclinic bifurcations.*

(ii) *The homoclinic bifurcations corresponding to twisted homoclinic orbits are isolated in $\mathcal{B}_{\nu_2}$. The homoclinic bifurcations corresponding to nontwisted homoclinic orbits are isolated on the left side and are accumulation points of elements of $\mathcal{B}_{\nu_2}$ from the right side.*

(iii) *Let $\hat{\nu}_1$ be a bifurcation point corresponding to a nontwisted homoclinic orbit. There exists a converging sequence of isolated bifurcation points $\cdots > \hat{\nu}_1^k > \hat{\nu}_1^{k-1} > \cdots > \hat{\nu}_1$ corresponding to twisted homoclinic orbits.*

(iv) *$\mathcal{B}_{\nu_2}$ is the union of a Cantor set and the set of isolated bifurcation values. The Lebesgue measure of $\mathcal{B}_{\nu_2}$ is 0.*

(*Proof.*) (i) This follows from the definition of $\mathcal{B}_{\nu_2}$.
(iii) Consider a bifurcation point $\hat{\nu}_1$ corresponding to a bifurcation from a nontwisted homoclinic orbit. For this value 0 is a periodic point of π_ν and the symbolic sequence of the corresponding periodic orbit contains an even number of U's. It follows that 0 is the right boundary of an interval I^n in $\pi_\nu^{n-2}(J)$, where n is the period of the periodic orbit. Hence there exists $\hat{\nu}_1^1 > \hat{\nu}_1$ where a homoclinic bifurcation takes place where 0 is the left boundary of I^n. Clearly this bifurcation corresponds to a twisted homoclinic orbit. Moreover 0 is the right boundary of an interval $I^{2n} \subset \pi_\nu^{2n-2}(J)$. Repeating the above procedure we find $\hat{\nu}_1^2 > \hat{\nu}_1^1$ at which 0 is the left boundary of I^{2n}, thus obtaining another bifurcation point corresponding to a twisted homoclinic orbit. Repeating this procedure we obtain the desired sequence of bifurcations. Note that in this way we account for all possible twisted homoclinic orbits.
(ii) It follows from the proof of (iii) that the bifurcation points corresponding to twisted homoclinic bifurcations are isolated in $\mathcal{B}_{\nu_2}$ and the bifurcation points corresponding to nontwisted homoclinic orbits are isolated on the left. Suppose $\hat{s}$ is a point of nontwisted homoclinic bifurcation and let σ be the symbolic sequence of the corresponding periodic orbit. We write $\sigma = A^\infty$, where A is a finite sequence. Consider the sequences $A^m U^\infty$ for π_ν^{-1}. Since $\sigma \prec A^m U^\infty$ it follows that there exist trajectories of π_ν^{-1} having $A^m U^\infty$ as their itineraries. When m converges to infinity these trajectories come arbitrarily close to 0. Moreover it is easy to see that every point in I is a starting point of a trajectory converging to p_ν. It follows that there exist trajectories of π_ν which are backward and forward asymptotic to p_ν and pass arbitrarily close to 0. Clearly these trajectories disappear in a sequence of bifurcation points converging to $\hat{\nu}_1$.
(iv) It follows from the proofs of (ii) and (iii) that $\mathcal{B}_{\nu_2}$ is the union of a Cantor set given by the closure of the bifurcation points corresponding to nontwisted homoclinic orbits and a set of isolated bifurcation points corresponding to

twisted homoclinic orbits. To prove the assertion that the Lebesgue measure of $\mathcal{B}_{\nu_2}$ is 0 we first observe that the Lebesgue measure of ω_ν is 0. By Lemma 4, 0 moves through ω_ν with positive speed. The assertion follows. $\square$

We now describe the structure of ω_ν at a point which is not a bifurcation value. In the statement of the following proposition we refer to a periodic sequence as *even* if it contains an even number of U's.

Proposition 6 *Let* $(\nu_1, \nu_2) \in \mathcal{P}$, $\nu_1 \notin \mathcal{B}_{\nu_2}$. *Suppose* γ_0 *is a periodic orbit of* f_ν *whose symbolic sequence is the minimal symbolic sequence among even periodic sequences of existing periodic orbits. Then* ω_ν *is a hyperbolic basic set consisting of the closure of the intersections of* $W^s(\gamma_0)$, $W^u(\gamma_0)$ *in* $C_{\rho,\delta}$ *together with a finite set of periodic orbits.*

(*Proof.*) Observe that there are at most finitely many periodic orbits with symbolic sequences smaller than the one of γ_0. Using similar arguments as in the proof of Theorem 5 we can show that there exist connecting orbits from γ_0 to every periodic orbit with larger symbolic sequence and connecting orbits from every periodic orbit to γ_0. If the symbolic sequence of $\hat{\gamma}$ is smaller then that of γ_0 then there are no connecting orbits from $\hat{\gamma}_0$ to periodic orbits with larger symbolic sequences. The proposition follows. $\square$

We now consider the case of outward twist and show how it can be understood using the methods developed for the case of inward twist. Fix $\nu_2 < 0$. It follows from Theorem 3 and the arguments analogous to the ones used in the proof of Proposition 6 that f_ν has a hyperbolic horseshoe for $\nu_1 > \nu_1^-(\nu_2) > 0$. Let q_ν be the fixed point of ξ_ν. The number $\nu_1^-(\nu_2)$ can be characterized as the value of ν_1 when $\eta_\nu(q_\nu) = 0$. We now describe a transformation taking the family of multivalued maps π_{ν_1} to a family of multivalued maps satisfying the conditions of Proposition 5 applying to the case of inward twist and the condition of Lemma 4. Note that these are the only results required to prove Theorem 5 and Proposition 6. The transformed family describes all the bifurcations occuring in the original family except for the last one determined by $\eta_\nu(q_\nu) = 0$.

Proposition 7 *Fix* ν_2 *small and suppose* π_ν *is the family of multivalued maps arising in the case of outward twist,* $0 < \nu_1 \leq \nu_1^-(\nu_2)$. *Then there* *exists a family* $\tilde{\pi}_{\nu_1}$ *and a diffeomorphism* $h : [0, b_\nu] \to [0, 1]$ *such that* $\pi_{\nu_1} = h^{-1} \circ \tilde{\pi}_{\nu_1} \circ h$. *The family* $\tilde{\pi}_{\nu_1}$ *satisfies the conditions of Proposition 5 applying to the case of inward twist and the condition of Lemma 4. For* $\nu_1 \in (0, \nu_1^-(\nu_2))$ *the set of homoclinic orbits of* $\tilde{\pi}_{\nu_1}$ *is the image under* h *of the set of homoclinic orbits of* π_{ν_1}.

(*Proof.*) The diffeomorphism h is obtained by composing the reflection through the midpoint of $[0, b_\nu]$ with a rescaling (see Figure 7). A straightforward computation shows that $|\tilde{\eta}'_\nu(z)|, |\tilde{\xi}'_\nu(z)| = o(1)$ with respect to ν_2, where $\tilde{\xi}_\nu$ and $\tilde{\eta}_\nu$ are the two component mappings of $\tilde{\pi}_{\nu_1}$. Also $\tilde{\xi}'_\nu(z) < 0$ and $\tilde{\eta}'_\nu(z) > 0$, $z \in [0, 1]$. Using similar arguments as in the proof of Lemma 4 one can show that the condition postulated in Lemma 4 is satisfied, that is there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every curve $p(\nu_1) \in \tilde{\pi}^n(0)$ $\frac{\partial}{\partial \nu_1} \eta_\nu(p(\nu_1)) > C$.

We now establish the correspondence between the sets of homoclinic bifurcations. Note that $h(b_\nu) = 0$. Hence, if π_{ν_1} has a homoclinic orbit then, since b_ν must be its element, the image of this homoclinic orbit is a homoclinic orbit of $\tilde{\pi}_{\nu_1}$. If $\tilde{\pi}_{\nu_1}$ has a homoclinic orbit then b_ν must be an element of the corresponding periodic orbit of π_{ν_1}. If $\nu_1 < \nu_1^-(\nu_2)$ then $b_\nu < q_\nu$, which implies that this periodic orbit must be homoclinic. □

Corollary 1 *It follows from Proposition 7 that the dynamics and bifurctions of the family* π_{ν_1} *in the case of outward twist correspond to the dynamics and bifurcations of a family* $\tilde{\pi}_{\nu_1}$ *to which the results of Theorem 5 and Proposition 5 apply. Note that the order of the bifurcations of* $\tilde{\pi}_{\nu_1}$ *is given in terms of decreasing* ν_1, *just as described in Theorem 5. Here* ν_1 *must decrease from* $\nu_1^-(\nu_2)$ *to* 0. *Hence, if* ν_1 *is viewed as increasing for the original family* π_{ν_1}, *then the order of bifurcations is reversed.*

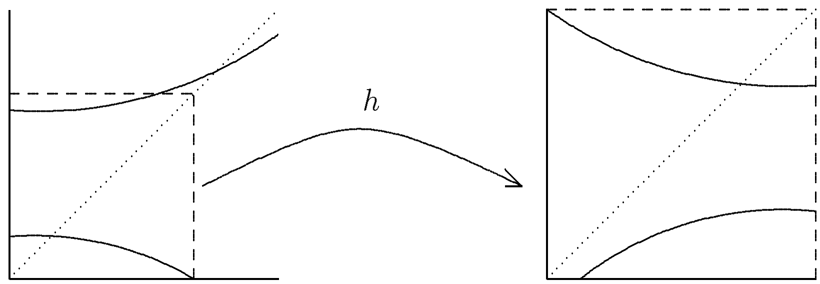
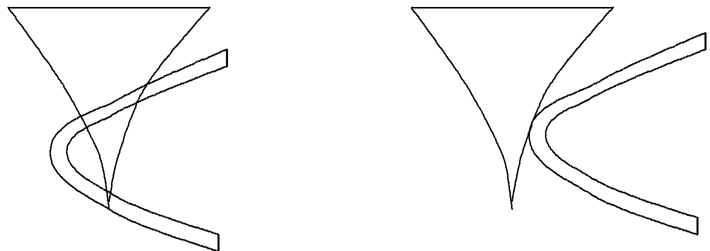


Figure 7: The conjugacy map h.

6 Remarks and conjectures on the dynamics and bifurcations for parameters outside the region of the existence of the invariant foliation.

Through most of the article we have assumed the parameters (ν_1, ν_2) are in the set $\mathcal{P}$, where we could prove the existence of the invariant foliation $\mathcal{F}^u$. Outside of the parameter region $\mathcal{P}$ the non-wandering set of the map f_ν may be non hyperbolic and we expect that complicated dynamics will occur. To see this consider the case of inward twist, fix $\nu_2 < 0$ and let ν_1 increase from 0. According to our results for small values of ν_1 relatively to ν_2 the return map f_ν has a horseshoe. When ν_1 is large enough a tangency will develop between the side of the cusp and $G(W^u_{loc}(O)) \cap \Sigma_0$, see Figure 8. It is clear that for this value of the parameters the horseshoe can no longer exist, so somewhere along the parameter path it has been annihilated. The mechanism of the disappearance of the horseshoe occurs far away from the singularity and thus is likely to involve phenomena leading to complicated dynamics, in particular tangencies between stable and unstable manifolds of periodic orbits and, related to it, occurrence of infinitely many periodic sinks and Hénon-like attractors. At present there exists no systematic study of this mechanism of horseshoe annihilation although a considerable amount of information is available, see [PT93] and the references therein. Hence there are two horseshoe annihilation mechanisms present, one which has been an-



(a) $\nu_1 = 0$

(b) $\nu_1 > 0$ and beyond the region of existence of the horseehoe

Figure 8: Tangency between C_ρ and $G(W^u_{loc}(O)) \cap \Sigma_0$.

alyzed in this paper, that is annihilation of orbits of the horseshoe through homoclinic bifurcations involving the saddle equilibrium of the vector field X and the second one arising through heteroclinic and homoclinic tangencies of invariant manifolds of the orbits in the horseshoe. Figure 9 represents the conjectured bifurcation diagram for the case of inward twist (a similar conjecture can be made for the case of outward twist). The part of the diagram occuring for negative ν_1 has been established in this paper. For ν_2 negative and ν_1 positive we conjecture the existence of two bifurcation lines. The one more to the left would correspond to the first heteroclinic tangency of the stable and unstable foliations of the horseshoe. The second one would correspond to a saddle-node bifurcation of periodic orbits. In the region between the two bifurcation lines complicated dynamics would occur involving homoclinic and heteroclinic tangencies, Hénon-like attractors, and infinitely many periodic sinks. In the region between the saddle-node bifurcation line and the line $\nu_2 = 0$ the non-wandering set of of $f_\nu | C_\rho$ would be empty.

Another relevant question is what happens when the eigenvalue condition (EV2) no longer holds. Let $\lambda = -\frac{\lambda^s}{\lambda^u}$ and suppose λ is varied as the third system parameter. Let (ν_1, ν_2) be in the region of the existence of the horseshoe for $\lambda > 2$. It follows from the methods used in the proof of Theorem 3 that for $\lambda < 2$ the horseshoe can no longer exist. [KKO93a] contains partial information on this case. Hence a bifurcation sequence leading to the destruction of the horseshoe must take place as (ν_1, ν_2) is kept fixed and λ varies from a value larger than 2 to a value less than 2.

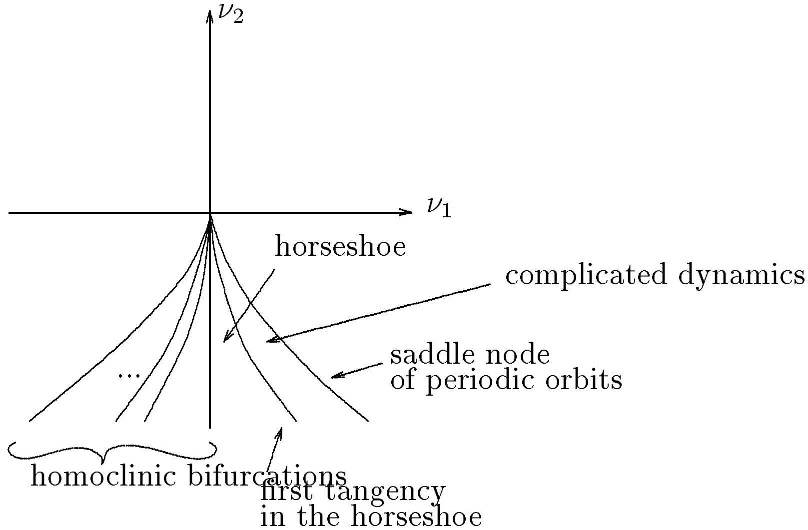


Figure 9: Conjectured bifurcation diagram.

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