Resonance bifurcation from homoclinic cycles

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Differential equations that are equivariant under the action of a finite group can possess robust homoclinic cycles that can moreover be asymptotically stable. For differential equations in $\mathbb{R}^4$ there exists a classification of different robust homoclinic cycles for which moreover eigenvalue conditions for asymptotic stability are known. We study resonance bifurcations that destroy the asymptotic stability of robust 'simple homoclinic cycles' in four-dimensional differential equations. We establish that typically a periodic trajectory near the cycle is created, asymptotically stable in the supercritical case.

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1. Introduction

Consider differential equations that possess a symmetry, or more precisely that are equivariant under the action of a finite group. It is well known that such equivariant differential equations may admit homoclinic networks that are possibly asymptotically stable. We study a class of bifurcations by which asymptotically stable homoclinic networks loose their stability, through bifurcations that involve the spectra about the equilibria in the homoclinic network. We treat homoclinic networks in four-dimensional differential equations, for which a classification of the different homoclinic networks exists. Periodic attractors close to the homoclinic network appear in an unfolding (in a supercritical bifurcation scenario, in a subcritical bifurcation scenario periodic repellers are obtained). For each of the possible homoclinic networks, we describe number and type of periodic trajectories that bifurcate.

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Let G be a finite group with a linear action (representation) $x \mapsto gx$ on $\mathbb{R}^4$. By choosing an appropriate inner product on $\mathbb{R}^4$, the group G may be assumed to be a subgroup of $O(4)$. A differential equation

$$\dot{x} = f(x) \tag{1}$$

is G-equivariant when $x(t)$ is a solution to (1) if and only if $gx(t)$ is a solution to (1), for all $g \in G$. Equivalently, if

$$gf(x) = f(gx), \quad \forall g \in G.$$

For a subgroup $H \subset G$, we write $\mathrm{Fix}(H) = \{x \in \mathbb{R}^n \mid gx = x, \; g \in H\}$ for the fixed point space of H. Further, we write $G_x = \{g \in G \mid gx = x\}$ for the isotropy group of $x \in \mathbb{R}^n$.

Recall that a heteroclinic trajectory γ for (1) connecting equilibria p to q is a trajectory $\gamma(t)$, $t \in \mathbb{R}$, with $\lim_{t \to -\infty} \gamma(t) = p$ and $\lim_{t \to \infty} \gamma(t) = q$. A heteroclinic cycle Γ consists of a collection of disjoint equilibria $p_1, \ldots, p_m$ and heteroclinic trajectories h_i connecting p_i to p_{i+1}, indices taken modulo m. A connected invariant set that is a finite union of heteroclinic cycles is called a polycycle.

Definition 1. A homoclinic cycle Γ is a polycycle that is equal to a group orbit $\langle h \rangle \overline{\gamma}$, for a heteroclinic trajectory γ connecting equilibria p to hp for some $h \in G$. The element $h \in G$ is called the twist for the homoclinic cycle. A homoclinic network is a connected component of the group orbit $G\Gamma$ of a homoclinic cycle Γ.

In this context we note that the group orbit $G\Gamma$ of a homoclinic cycle Γ is connected if and only if

$$G = \langle h, G_p \rangle,$$

see Homburg et al. [9]. Throughout the paper we will tacitly assume that G satisfies this identity and thus that $G\Gamma$ is connected.

Note that the twist is well defined modulo the isotropy group G_p of p. Given a homoclinic cycle Γ with twist h and a group element $g \in G$, $g\Gamma$ defines a homoclinic cycle with conjugate twist ghg^{-1}.

Definition 2. A simple homoclinic cycle is a homoclinic cycle $\langle h \rangle \overline{\gamma}$ with γ connecting p to hp, for some $h \in G$, so that

1. the unstable manifold $W^u(p)$ is one-dimensional,
2. there is a two-dimensional fixed point subspace P of an isotropy subgroup of G, such that $\gamma \subset P$ and hp is a sink in P.

A simple homoclinic network is the group orbit of a simple homoclinic cycle.

As a consequence of the continuous dependence of stable and unstable manifolds $W^s(p)$, $W^u(p)$ on the vector field f and invariance of P, a simple homoclinic cycle persists under small G-equivariant perturbations of f. For more on robust homoclinic networks we refer to Field [7,8], Krupa [11], Chossat and Lauterbach [3]. Following Chossat et al. [2], simple homoclinic cycles (and networks) in $\mathbb{R}^4$ are distinguished in three types:

1. Type A: $h^{-1}P + P$ is not a fixed point space.
2. Type B: $h^{-1}P + P$ is a (three-dimensional) fixed point space containing the homoclinic cycle.
3. Type C: $h^{-1}P + P$ is a (three-dimensional) fixed point space not containing the homoclinic cycle.

One can also consider homoclinic networks with heteroclinic trajectories lying in three-dimensional fixed point spaces (and not in two-dimensional fixed point spaces). Although we will concentrate on simple homoclinic networks, such homoclinic networks will be included in this paper as a variant of type B homoclinic networks, see Remark 1 below.

Sottocornola classified type A simple homoclinic cycles in $\mathbb{R}^4$ up to isomorphisms of groups, see Sottocornola [18,19]. A description of possible simple homoclinic cycles of type B and C in $\mathbb{R}^4$ is contained in Krupa and Melbourne [13]. The following theorem lists the different simple homoclinic networks $G\Gamma$. It gives generators for the action of the group G (a minimal admissible group, up to isomorphisms of groups, for which the homoclinic network occurs, see Sottocornola [18,19]) and the number of connecting trajectories in the homoclinic cycles contained in $G\Gamma$. The generators for the action of G will be linear maps with matrices from the following list. Define, for $\alpha = \pm 1$,

$$A_{t,s}^{\alpha} = \begin{pmatrix} 0 & 0 & \cos(s) & -\sin(s) \\ \alpha\sin(t) & \cos(t) & 0 & 0 \\ -\alpha\cos(t) & \sin(t) & 0 & 0 \\ 0 & 0 & \sin(s) & \cos(s) \end{pmatrix}$$

(note that $\det A_{t,s}^{\alpha} = \alpha$) and the reflections

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Theorem 1. *(See Krupa and Melbourne [13], Sottocornola [17–19].) The classification of simple homoclinic cycles in $\mathbb{R}^4$ is in Table 1, listing the homoclinic network (with indices indicating its type—added for type A is a sign indicating whether or not $G \subset SO(4)$—and the number of equilibria it contains), generators of G, and for possible twists h the length (i.e. number of heteroclinic trajectories) of the corresponding homoclinic cycle $\Gamma = \langle h \rangle \overline{\gamma}$. Here γ is a heteroclinic trajectory in $P = \text{Fix}(S)$, the coordinate plane of the first two coordinates in $\mathbb{R}^4$, connecting equilibria in $h^{-1}P \cap P$ to $P \cap hP$.*

For Γ a homoclinic cycle as in the above theorem with twist h, write $\rho(\Gamma)$ for the homoclinic cycle with twist Sh.

Remark 1. One can distinguish two different homoclinic networks with heteroclinic trajectories lying in three-dimensional, but not in two-dimensional, fixed point spaces. These are direct analogues of type B homoclinic networks, with groups G generated by $A_{\pi,0}^1, SR$ or by $A_{\pi/2,0}^1, SR$.

The geometry given by the subspaces P in the definition of simple homoclinic cycle allows us to divide the spectrum of $Df(p)$ into four classes:

1. Radial eigenvalue λ_r: the negative eigenvalue restricted to $V(r) = h^{-1}P \cap P$.
2. Contracting eigenvalue λ_c: the negative eigenvalue restricted to $V(c) = h^{-1}P \ominus V(r)$.
3. Expanding eigenvalue λ_e: the positive eigenvalue restricted to $V(e) = P \ominus V(r)$.
4. Transverse eigenvalue λ_t: the negative eigenvalue restricted to $V(t) = (h^{-1}P + P)^{\perp}$.

Spectral conditions ensuring asymptotic stability of simple homoclinic networks are given in the following result by Krupa and Melbourne.

Theorem 2. *(See Krupa and Melbourne [12,13].) Let $\dot{x} = f(x)$ be a G-equivariant ODE on $\mathbb{R}^n$, for a finite group G acting linearly. Suppose it admits a simple homoclinic network $G\Gamma$. Assume there are C^1 linearizing coordinates near the equilibria in $G\Gamma$.*

Table 1
Classification of simple homoclinic cycles in $\mathbb{R}^4$.

type	homoclinic network	generators	twist	length cycle
A, $G \subset SO(4)$	$\mathcal{H}_2^{A,+}$	$A_{\pi,0}^1, S$	$A_{\pi,0}^1$	2
			$SA_{\pi,0}^1$	4
	$\mathcal{H}_6^{A,+}$	$A_{\frac{\pi}{2},0}^1, S$	$A_{\frac{\pi}{2},0}^1$	3
			$SA_{\frac{\pi}{2},0}^1$	6
	$\mathcal{H}_8^{A,+}$	$A_{\frac{\pi}{2},\frac{\pi}{2}}^1, S$	$A_{\frac{\pi}{2},\frac{\pi}{2}}^1, SA_{\frac{\pi}{2},\frac{\pi}{2}}^1$	8
	$\mathcal{H}_{48}^{A,+}$	$A_{\frac{\pi}{4},\frac{\pi}{4}}^1, S$	$A_{\frac{\pi}{4},\frac{\pi}{4}}^1$	12
			$SA_{\frac{\pi}{4},\frac{\pi}{4}}^1$	24
A, $G \not\subset SO(4)$	$\mathcal{H}_6^{A,-}$	$A_{\frac{\pi}{2},0}^{-1}, S$	$A_{\frac{\pi}{2},0}^{-1}$	6
			$SA_{\frac{\pi}{2},0}^{-1}$	3
	$\mathcal{H}_{2k^2}^{A,-}, k \geqslant 1$	$A_{\frac{\pi}{k},\frac{\pi}{k}}^{-1}, S$	$A_{\frac{\pi}{k},\frac{\pi}{k}}^{-1}$	4
			$SA_{\frac{\pi}{k},\frac{\pi}{k}}^{-1}$	2k
B	$\mathcal{H}_2^B$	$A_{\pi,0}^1, S, R$	$A_{\pi,0}^1, RA_{\pi,0}^1$	2
			$SA_{\pi,0}^1, SRA_{\pi,0}^1$	4
	$\mathcal{H}_6^B$	$A_{\frac{\pi}{2},0}^1, S, R$	$A_{\frac{\pi}{2},0}^1, RA_{\frac{\pi}{2},0}^1$	3
			$SA_{\frac{\pi}{2},0}^1, SRA_{\frac{\pi}{2},0}^1$	6
C	$\mathcal{H}_8^C$	$A_{\frac{\pi}{2},\frac{\pi}{2}}^1, S, R$	$A_{\frac{\pi}{2},\frac{\pi}{2}}^1, SA_{\frac{\pi}{2},\frac{\pi}{2}}^1$	8
			$RA_{\frac{\pi}{2},\frac{\pi}{2}}^1, SRA_{\frac{\pi}{2},\frac{\pi}{2}}^1$	4

Then $G\Gamma$ is asymptotically stable precisely if $-\lambda_c > \lambda_e$ in case $G\Gamma$ is of type A or B and $-\lambda_c - \lambda_t > 1$ in case $G\Gamma$ is of type C.

For asymptotically stable simple homoclinic networks in $\mathbb{R}^4$ two types of bifurcations through which the homoclinic network can loose its asymptotic stability, are distinguished. The transverse bifurcation occurs when the transverse eigenvalue moves through zero, see Chossat et al. [2], Driesse and Homburg [6]. In this article we consider the resonance bifurcation, which occurs when the eigenvalue condition for asymptotic stability becomes violated, see Postlethwaite and Dawes [15], Chossat and Lauterbach [3], Scheel and Chossat [16]. The symmetry, giving the existence of invariant fixed point spaces, can force heteroclinic trajectories to approach an equilibrium from directions that are not the leading directions (which would typically hold true for generic, nonsymmetric, differential equations). Compared to corresponding global bifurcation theory in generic, nonsymmetric, differential equations, this makes the bifurcation analysis more delicate. Earlier work treating the resonance bifurcation of homoclinic cycles, Chossat and Lauterbach [3], Scheel and Chossat [16], restricts to special cases where such degeneracies do not occur and the bifurcation is in its analysis similar to the resonant homoclinic bifurcation, see Chow et al. [4]. Using recent classification results for homoclinic cycles and an adapted normal form theory, we derive reduced bifurcation equations and study those in order to describe the bifurcations.

We now formulate the bifurcation theorem. A coefficient Φ appearing in the formulation will be made precise in later sections. Let

$$\dot{x} = f(x, \mu) \tag{2}$$

be a one parameter family of smooth differential equations in $\mathbb{R}^4$, each vector field being G-equivariant, and possessing a simple homoclinic cycle Γ for $\mu = 0$. Define

Table 2
Periodic trajectories branching from type A homoclinic cycles.

$G\Gamma$	$|\Gamma|$	♯ periodic trajectories	$|\rho(\Gamma)|$	♯ periodic trajectories
$\mathcal{H}_2^{A,\pm}$	2	4	4,	2
$\mathcal{H}_6^{A,\pm}$	3	8	6,	4
$\mathcal{H}_8^{A,+}$	8	4	8,	4
$\mathcal{H}_{48}^{A,+}$	12	16	24,	8
$\mathcal{H}_{2k^2}^{A,-}$	4	$2k^2$	$2k$,	$4k$

Table 3
Periodic trajectories branching from type B, C homoclinic cycles.

$G\Gamma$	$|\Gamma|$	♯ periodic trajectories
$\mathcal{H}_2^B$	2	4
$\mathcal{H}_6^B$	3	8
$\mathcal{H}_8^C$	4	16

$$\eta(\mu) = \begin{cases} -\lambda_c/\lambda_e, & \text{in case } \Gamma \text{ is of type A or B,} \\ (-\lambda_c - \lambda_t)/\lambda_e, & \text{in case } \Gamma \text{ is of type C.} \end{cases} \tag{3}$$

With this definition, the condition for asymptotic stability from Theorem 2 reads $\eta < 1$. We consider generically unfolding resonance bifurcations where $\eta(0) = 1$ and $\eta'(0) \neq 0$. By reparameterizing if necessary we may assume that

$$\eta(0) = 1, \qquad \eta'(0) < 0. \tag{4}$$

For a homoclinic cycle or network Γ, write $|\Gamma|$ for the number of connecting trajectories in Γ. Note that $|G\Gamma|$ is two times the number of equilibria in $G\Gamma$, as each equilibrium counts two outgoing connecting trajectories (from the two unstable separatrices).

Theorem 3. *Let $\dot{x} = f(x, \mu)$ be a one parameter family of G-equivariant smooth differential equations in $\mathbb{R}^4$, possessing for $\mu = 0$ a simple homoclinic cycle Γ. Suppose that $G\Gamma$ is asymptotically stable for $\mu < 0$ and undergoes a generically unfolding resonance bifurcation for $\mu = 0$; (4) holds.*

There is a number $\Phi \neq 0$ depending only on the differential equation for $\mu = 0$, so that the following holds.

1. *If $G\Gamma$ is of type A and $|\Phi| \neq 1$, then either a periodic trajectory close to Γ bifurcates (if $\Phi > 0$) or a periodic trajectory close to $\rho(\Gamma)$ bifurcates (if $\Phi < 0$). Table 2 lists the number of heteroclinic trajectories in the homoclinic cycles Γ and $\rho(\Gamma)$ (assuming without loss of generality $|\Gamma| \leqslant |\rho(\Gamma)|$), and the number of branching periodic trajectories near $G\Gamma$.*
2. *If $G\Gamma$ is of type B or C, Γ is such that $|\Gamma|$ is minimal (as in Table 3), and $|\Phi| \neq 1$, then a periodic trajectory close to Γ bifurcates.*

For all cases, the following holds on stability of the bifurcating periodic trajectory: If $|\Phi| < 1$, the bifurcation is supercritical and the periodic trajectory exists for $\mu > 0$ and is asymptotically stable. If $|\Phi| > 1$, the bifurcation is subcritical and the periodic trajectory exists for $\mu < 0$ and is unstable.

Remark 2. The theorem treats bifurcating periodic trajectories that make one round near a homoclinic cycle, and does not provide statements on possible other recurrent dynamics. The total number of these periodic trajectories created near a homoclinic network follows from the following observations:
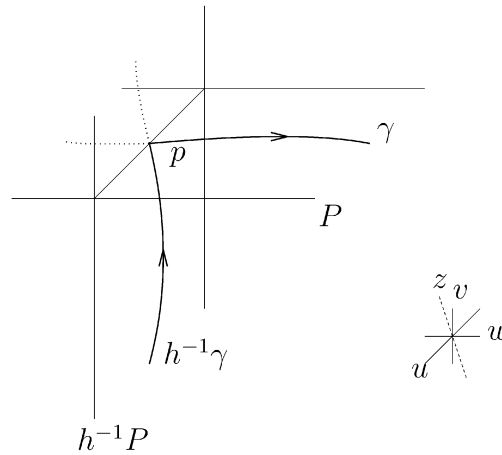
Fig. 1. The homoclinic cycle near P.

1. If $G\Gamma$ is of type A or B: for each heteroclinic trajectory γ that generates Γ, there are 2 periodic trajectories, $\zeta, S\zeta$ that pass close to it.
2. If $G\Gamma$ is of type C: for each heteroclinic trajectory γ that generates Γ, there are 4 periodic trajectories, $\zeta, S\zeta, R\zeta, SR\zeta$, that pass close to it.
3. The total number of heteroclinic trajectories in $G\Gamma$ is twice the number of equilibria in $G\Gamma$.

Remark 3. The results for homoclinic networks of type B also apply to homoclinic networks with three-dimensional instead of two-dimensional fixed point spaces (compare Remark 1).

For homoclinic networks of type B or C, three-dimensional fixed point spaces divide state space in invariant domains and obstructs trajectories to leave these domains. This explains the difference between homoclinic networks of type A and of type B or C.

In the following sections we will treat homoclinic cycles of type A, B and C separately. The organization of the proof is the same in all three cases: normal forms, asymptotic expansions, bifurcation analysis. In short, one takes a cross section Σ^{in} transverse to a heteroclinic trajectory γ in a homoclinic cycle $\Gamma = \langle h \rangle \overline{\gamma}$. By identifying Σ^{in} with $h\Sigma^{\text{in}}$, the flow from Σ^{in} to $h\Sigma^{\text{in}}$ defines a return map

$$\Pi : \Sigma^{\text{in}} \to \Sigma^{\text{in}}.$$

To derive asymptotic expressions for Π, Π is as usual computed as a composition of a local transition map through a neighborhood of an equilibrium and a global transition map. The global transition map is a diffeomorphism by the flow box theorem, the symmetry induces various terms in an expansion to vanish. For the local transition map, workable asymptotic expansions can be obtained when the vector field is written in a local normal form. We work in local coordinates

$$x = (u, v, w, z) \tag{5}$$

around p so that the u, v, w and z axes are the eigenspaces of respectively the radial, contracting, expanding and transverse eigenvalue, see Fig. 1. The fixed point space P is spanned by the u and w coordinates. The action of the twist defines local coordinates near the other equilibria in a homoclinic cycle.

Periodic trajectories are found by solving bifurcation equations for fixed points of Π, their stability is analyzed from computing the linearization of Π about a fixed point. Bifurcation equations for periodic trajectories are substantially easier to derive if one assumes the vector field near the equilibria to be smoothly equivalent to linear vector fields. By the resonance conditions among the eigenvalues,

this will typically not be possible. See also Chow et al. [4]. Much effort therefore goes into the derivation of local normal forms that are as close as possible to a linear vector field, and the computation of transition maps that have similar asymptotic expansions as one would obtain from local linear vector fields.

The simplest case, homoclinic cycles of type B, are considered first in Section 2. Then in Section 3 we consider homoclinic cycles of type C and in Section 4 the more involved case of homoclinic cycles of type A.

2. Simple homoclinic cycles of type B

We assume in this section the conditions of Theorem 3 for a simple homoclinic network of type B. That is, given a G-equivariant differential equation $\dot{x} = f(x, \mu)$, depending on a parameter μ, with for $\mu = 0$ a simple homoclinic cycle $\Gamma = \langle h \rangle \overline{\gamma}$ of type B. There is in particular an invariant three-dimensional subspace Q, that contains the homoclinic network $G\Gamma$ and is attracting near $G\Gamma$. Consequently, a bifurcating periodic trajectory lies inside Q and we may restrict the differential equations to Q.

As outlined at the end of the previous section, the main bifurcation result, Theorem 3, for simple homoclinic cycles of type B is proved in two steps: first we discuss asymptotic expansions for a first return map Π and then we use these to solve for the existence of a periodic trajectory. The asymptotic expansions apply in a suitable local normal form near the equilibria. We will start deriving this normal form. The analysis is fairly straightforward for simple homoclinic cycles in case B, and serves as an introduction to the more involved reasoning needed for type A and C simple homoclinic cycles.

We mention that the analysis in this section generalizes to homoclinic networks with three instead of two-dimensional fixed point spaces (compare Remarks 1 and 3). For this, in the coordinates (u, v, w) below one interprets u as a two-dimensional coordinate. Details are left to the reader.

2.1. Normal forms

Recall from the introduction that we express the differential equations $\dot{x} = f(x, \mu)$ restricted to Q, locally around p, as

$$\dot{u} = \lambda_r u + F^u(u, v, w),$$
$$\dot{v} = \lambda_c v + F^v(u, v, w),$$
$$\dot{w} = \lambda_e w + F^w(u, v, w), \tag{6}$$

for functions F^u, F^v, F^w that vanish together with derivatives at the origin. The coordinates u, v, w give respectively the radial, contracting and expanding directions. We do not incorporate the dependence on the parameter μ in the notation. The local differential equations will be transformed into normal form by a smooth (C^∞) coordinate transformation and a smooth time reparametrization (i.e. multiplication by a smooth positive function). Recall that such transformations define a smooth equivalence.

Proposition 4. *The differential equations* (6) *are locally smoothly equivalent to differential equations of the form*

$$\dot{u} = \lambda_r u + P(u, v, w),$$
$$\dot{v} = \lambda_c v + v w Q(u, v, w),$$
$$\dot{w} = \lambda_e w, \tag{7}$$

where $P(u, v, w) = \mathcal{O}(|(u, v)||(u, v, w)|)$, $Q(u, v, w) = \mathcal{O}(|(u, v)|)$. *These equations are G-equivariant and smoothly dependent on the parameter μ.*

Proof. We first remark that G-equivariance will follow from the constructions, as the coordinate changes can be made to be G-equivariant. Invariance of $h^{-1}P = \{w = 0\}$ and $P = \{v = 0\}$ implies that $F^w(u, v, 0) = 0$ and $F^v(u, 0, w) = 0$. We write $F^u(u, v, w) = \tilde{P}(u, v, w)$, $F^v(u, v, w) = v\tilde{Q}(u, v, w)$ and $F^w(u, v, w) = w\tilde{R}(u, v, w)$. It follows that the differential equations can be written as

$$\dot{u} = \lambda_r u + \tilde{P}(u, v, w),$$

$$\dot{v} = \lambda_c v + v\tilde{Q}(u, v, w),$$

$$\dot{w} = \lambda_e w + w\tilde{R}(u, v, w), \tag{8}$$

where $\tilde{P}(u, v, w) = \mathcal{O}(|(u, v, w)|^2)$, $\tilde{Q}(u, v, w) = \mathcal{O}(|(u, v, w)|)$ and $\tilde{R}(u, v, w) = \mathcal{O}(|(u, v, w)|)$.

Note that G_p is generated by a reflection. The unstable manifold $W^u(p)$ is G_p-symmetric. It is easily seen that it can be straightened by a smooth G_p-equivariant coordinate change. In the new coordinates, $\tilde{P}(0, 0, w) = 0$, implying $\tilde{P}(u, v, w) = \mathcal{O}(|(u, v)||(u, v, w)|)$. Multiplying the vector field with the G-symmetric function $(1 + \tilde{R}/\lambda_e)^{-1}$, we obtain $\dot{w} = \lambda_e w$. To get the desired form we remove the terms $v\tilde{Q}(0, 0, w)$, $v\tilde{Q}(u, v, 0)$ from the equation for $\dot{v}$.

Consider first the term $v\tilde{Q}(0, 0, w)$. By symmetry, $\tilde{Q}(0, 0, w) = \mathcal{O}(w^2)$ (in equations without symmetry, a term vw from the equation for $\dot{v}$ can be removed by a polynomial coordinate change $\bar{v} = v + avw$ for suitable a). To remove the term $v\tilde{Q}(0, 0, w)$ from the equation for $\dot{v}$, we apply an argument in Ovsyannikov and Shil'nikov [14]. Consider a change of coordinates

$$\bar{v} = v + r(w)v.$$

Compute the differential equation for $\dot{\bar{v}}$,

$$\dot{\bar{v}} = \dot{v} + \dot{r}(w)v + r(w)\dot{v}$$

$$= \lambda_c v + v\tilde{Q}(u, v, w) + r(w)\lambda_c v + r(w)v\tilde{Q}(u, v, w) + \dot{r}(w)v$$

$$= \lambda_c \bar{v} + \bar{v}\big[\tilde{Q}\big(u, \bar{v}/(1 + r(w)), w\big) + \dot{r}(w)/(1 + r(w))\big],$$

and solve $\tilde{Q}(0, 0, w) + \dot{r}(w)/(1 + r(w)) = 0$ along the unstable manifold $\{u = v = 0\}$. Considering r as a variable, the unstable manifold of the system

$$\dot{r} = \mathcal{O}(w^2),$$

$$\dot{w} = \lambda_e w \tag{9}$$

solves r as a C^∞-function of w.

Similarly, to remove the term $v\tilde{Q}(u, v, 0)$, first remove the terms vu from the equation for $\dot{v}$ by a coordinate change of the form $\bar{v} = v + avu$. As there are no terms v^2 in the equation for $\dot{v}$ (by symmetry), this makes $Q(u, v, 0)$ of quadratic order. Another application of the argument of Ovsyannikov and Shil'nikov removes the term $v\tilde{Q}(u, v, 0)$ from the differential equation for $\dot{v}$. Apply for this a coordinate change

$$\bar{v} = v + vr(u, v) = v\big(1 + r(u, v)\big).$$

Compute the differential equation for $\dot{\bar{v}}$,

$$\dot{\bar{v}} = \lambda_c \bar{v} + \bar{v}\big[\tilde{Q}\big(u, \bar{v}/(1 + r(u, v)), 0\big) + \dot{r}(u, v)/(1 + r(u, v))\big].$$

Now, considering r as a variable, r is obtained from the local stable manifold of the system

$$\dot{r} = -\tilde{Q}\left(u, \bar{v}/(1+r), w\right)(1+r) = \text{h.o.t.},$$

$$\dot{u} = \lambda_r u + \text{h.o.t.},$$

$$\dot{v} = \lambda_c v + \text{h.o.t.} \qquad \square$$

2.2. Asymptotic expansions

In this section we obtain asymptotic expressions for a first return map on a cross section. We refer to e.g. Homburg et al. [10] for similar computations. Define cross-sections in a neighborhood of p by

$$\Sigma^{\text{in}} = \left\{\left|(u,v)\right| = \delta, |w| < \delta\right\}, \qquad \Sigma^{\text{out}} = \left\{|w| = \delta, \left|(u,v)\right| < \delta\right\},$$

for $\delta > 0$ small. We scale u, v and w so that

$$\Sigma^{\text{in}} = \left\{\left|(u,v)\right| = 1, |w| < 1\right\}, \qquad \Sigma^{\text{out}} = \left\{|w| = 1, \left|(u,v)\right| < 1\right\}.$$

Identifying Σ^{in} with $h\Sigma^{\text{in}}$ through the twist map h, the transition map from Σ^{in} to $h\Sigma^{\text{in}}$, yields a "first return map" $\Pi : \Sigma^{\text{in}} \mapsto \Sigma^{\text{in}}$.

Proposition 5. *Assume normal form coordinates near p given by Proposition 4. Then $\Pi : \Sigma^{\text{in}} \mapsto \Sigma^{\text{in}}$ has asymptotic expansions*

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \xrightarrow{\Pi} \begin{pmatrix} u^* + \mathcal{O}(w^{\Theta}) \\ v^* + \mathcal{O}(w^{\Theta}) \\ \gamma v w^C + \mathcal{O}(w^{C+\Theta}) \end{pmatrix},$$

for $C = -\lambda_c/\lambda_e$ and some $\Theta > 0$.

The expansions apply to derivatives as well: taking derivatives of the higher order terms on the right-hand side with respect to u, v and μ does not alter the order, while derivatives with respect to w can be taken in the arguments of the order symbol.

Remark 4. The group G is either $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$ or $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$. With twist $h = A_{\pi,0}^1$ (with length 2 for the homoclinic cycle) for $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$, and twist $h = A_{\pi/2,0}^1$ (with length 3 for the homoclinic cycle) for $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$, the coefficient γ in Proposition 5 is necessarily positive. This is due to the existence of fixed point spaces of codimension one, so that along trajectories the perpendicular one-dimensional coordinate cannot change sign.

Define $x^* = (u^*, v^*, 0) = \Gamma \cap \Sigma^{\text{in}}$. For initial points $x_0 = (u_0, v_0, w_0)$ in Σ^{in} close to x^* the trajectory will intersect Σ^{out} after time $\tau = -\log w_0/\lambda_e$. Using the variation of constants formula, we compute exponential expansions for the coordinates u and v in trajectories. The solutions for u and v depend on τ, u_0, v_0 and so we write $u = u(t, \tau, u_0, v_0)$ and $v = v(t, \tau, u_0, v_0)$. Before computing exponential expansions, we derive initial bounds for u and v.

Lemma 6. *For δ small enough and $k \geq 0$ there are constants $C_k > 0$ such that, for $0 \leq t \leq \tau$,*

$$\left|\frac{d^k}{d(t, u_0, v_0)^k} u(t, \tau, u_0, v_0)\right| \leq C_k e^{-\omega t},$$

$$\left|\frac{d^k}{d(t, u_0, v_0)^k} v(t, \tau, u_0, v_0)\right| \leq C_k e^{\lambda_c t}, \tag{10}$$

where $(u_0, v_0, w_0) \in \Sigma^{\text{in}}$, $\omega = \min\{-\tilde{\lambda}_c, -\lambda_r\}$ and $t \in [0, \tau]$.

Proof. To simplify notation we omit writing dependency of u and v on the variables (τ, u_0, v_0). The rescaling of (u, v, w) by a factor $1/\delta$ implies that

$$\left| P(u, v, w) \right| \leqslant C\delta \big(|u| + |v| \big) \big(|u| + |v| + |w| \big),$$

$$\left| vwQ(u, v, w) \right| \leqslant C\delta |v||w| \big(|u| + |v| \big), \tag{11}$$

for some $C > 0$. Expressions for $(u(t), v(t))$, $0 \leqslant t \leqslant \tau$, are obtained from the variation of constants formula:

$$u(t) = u_0 e^{\lambda_r t} + \int_0^t e^{\lambda_r (t-s)} P\big(u(s), v(s), w(s) \big) \, ds,$$

$$v(t) = v_0 e^{\lambda_c t} + \int_0^t e^{\lambda_c (t-s)} v(s) w(s) Q\big(u(s), v(s), w(s) \big) \, ds. \tag{12}$$

Here, $w(s) = e^{\lambda_e (s-\tau)}$. The right-hand side of (12) defines a map $\mathscr{S} = (\mathscr{S}_1, \mathscr{S}_2)$ on a space of continuous functions (u, v) defined for $0 \leqslant t \leqslant \tau$. Let

$$B_C^{\omega, \lambda_c} = \big\{ (u, v) : [0, \tau] \to \mathbb{R}^2 \,\big|\, \left| u(t) \right| < Ce^{-\omega t}, \; \left| v(t) \right| < Ce^{\lambda_c t} \big\},$$

endowed with the supnorm

$$\big\| (u, v) \big\| = \|u\| + \|v\| = \sup_{t \in [0, \tau]} \left| u(t) \right| + \sup_{t \in [0, \tau]} \left| v(t) \right|.$$

We claim that there exists $C > 0$ such that $\mathscr{S}$ maps B_C^{ω, λ_c} into itself and $\mathscr{S}$ is a contraction on B_C^{ω, λ_c}. Using the estimates on P and vwQ we get

$$\mathscr{S}_1(u, v)(t) \leqslant (u_0 + C\tilde{C}\delta) e^{-\omega t} < Ce^{-\omega t},$$

$$\mathscr{S}_2(u, v)(t) \leqslant (v_0 + C\tilde{C}\delta) e^{\lambda_c t} < Ce^{-\lambda_c t},$$

for C large enough and δ small enough. This proves the first part of the claim. To prove that $\mathscr{S}$ is a contraction we show that

$$\big\| \mathscr{S}(u_1, v_1) - \mathscr{S}(u_2, v_2) \big\| < \big\| (u_1, v_1) - (u_2, v_2) \big\|.$$

We first estimate

$$\big\| \mathscr{S}(u_1, v_1) - \mathscr{S}(u_2, v_2) \big\| < \big\| \mathscr{S}(u_1, v_1) - \mathscr{S}(u_2, v_1) \big\| + \big\| \mathscr{S}(u_2, v_1) - \mathscr{S}(u_2, v_2) \big\|.$$

Compute

$$\big\| \mathscr{S}_1(u_1, v_1) - \mathscr{S}_1(u_2, v_1) \big\| = \sup_{t \in [0, \tau]} \int_0^t e^{-\lambda_r (s-t)} \big| P\big(u_1(s), v_1(s), w(s) \big) - P\big(u_2(s), v_1(s), w(s) \big) \big| \, ds$$

$$= \sup_{t \in [0, \tau]} \int_0^t e^{-\lambda_r (s-t)} \left| \frac{d}{du} P\big(p(s), v_1(s), w(s) \big) \right| \big| u_1(s) - u_2(s) \big| \, ds$$

$$\leqslant \sup_{t\in[0,\tau]} \int_0^\tau e^{-\lambda_r(s-t)} \left| \frac{d}{du} P\big(p(s), v_1(s), w(s)\big) \right| \big|u_1(s) - u_2(s)\big|\, ds$$

$$\leqslant \|u_1 - u_2\| \left\| \frac{d}{du} P(p, v_1, w) \right\| \int_0^\tau e^{-\lambda_r(s-\tau)}\, ds,$$

where $p(s)$ is a bounded function obtained by applying the mean value theorem. By (11), $\|\frac{d}{du} P(p, v_1, w)\| < C\delta$ for some $C > 0$. We get

$$\big\| \mathscr{S}_1(u_1, v_1) - \mathscr{S}_1(u_2, v_1) \big\| < 1/8 \|u_1 - u_2\|,$$

for δ small enough. In the same way we can estimate

$$\big\| \mathscr{S}_1(u_2, v_1) - \mathscr{S}_1(u_2, v_2) \big\| < 1/8 \|v_1 - v_2\|,$$

$$\big\| \mathscr{S}_2(u_1, v_1) - \mathscr{S}_2(u_2, v_1) \big\| < 1/8 \|u_1 - u_2\|,$$

$$\big\| \mathscr{S}_2(u_2, v_1) - \mathscr{S}_2(u_2, v_2) \big\| < 1/8 \|v_1 - v_2\|,$$

for δ small enough. Combining these estimates establishes

$$\big\| \mathscr{S}(u_1, v_1) - \mathscr{S}(u_2, v_2) \big\| < 1/2 \big\| (u_1, v_1) - (u_2, v_2) \big\|$$

and so $\mathscr{S}$ is a contraction on B_C^{ω, λ_c}. This now implies that there is a fixed point of $\mathscr{S}$ in the space of continuous functions (u, v) defined on $[0, \tau]$ satisfying the estimates. Estimates on the derivatives of $|u(t, \tau, u_0, v_0)|$, $|v(t, \tau, u_0, v_0)|$ with respect to u_0, v_0, t are obtained by similar reasoning as above, by differentiating the integral formulas accordingly. □

The estimates on u and v yield the following asymptotic expression for $v(\tau)$.

Lemma 7. *For* $\theta = \min\{-\lambda_r, -\lambda_c, \lambda_e\}$ *the asymptotic expression*

$$v(\tau) = v_0 e^{\lambda_c \tau} + \mathcal{O}\big(e^{(\lambda_c - \theta)\tau}\big) \tag{13}$$

holds.

Proof. Using the estimates of Proposition 6 and also $Q(u(s), v(s), w(s)) = \mathcal{O}(|(u, v)|)$, we compute

$$\left| \int_0^\tau e^{-\lambda_c s} v(s) w(s) Q\big(u(s), v(s), w(s)\big)\, ds \right| \leqslant C \int_0^\tau e^{\lambda_e(s-\tau) - \omega s}\, ds$$

$$= \mathcal{O}\big(e^{\lambda_e \tau}\big) + \mathcal{O}\big(e^{-\omega \tau}\big)$$

$$= \mathcal{O}\big(e^{-\theta \tau}\big).$$

By the variation of constants formula for $v(t)$ given in (12) we then have

$$v(\tau) = v_0 e^{\lambda_c \tau} + \mathcal{O}\big(e^{(\lambda_c - \theta)\tau}\big). □$$

Lemmas 6 and 7 give that the transition map $\Pi^{\text{loc}} : \Sigma^{\text{in}} \mapsto \Sigma^{\text{out}}$ has an expansion

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{O}(w^{\Omega}) \\ v w^C + \mathcal{O}(w^{(C+\Theta)}) \\ 1 \end{pmatrix}, \tag{14}$$

where $C = -\lambda_c/\lambda_e$, $\Omega = -\omega/\lambda_e$ and $\Theta = -\theta/\lambda_e$.

Write $\Pi^{\text{far}} : \Sigma^{\text{out}} \to \Sigma^{\text{in}}$ for the transition map from Σ^{out} to $h\Sigma^{\text{in}}$ with an identification of Σ^{in} and $h\Sigma^{\text{in}}$ through the twist h. By invariance of P and $W^u(p) \cap h\Sigma^{\text{in}} = (u^*, v^*, 0)$, Π^{far} at lowest order equals

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} u^* + \tilde{\alpha} u + \alpha v \\ v^* + \tilde{\beta} u + \beta v \\ \gamma v \end{pmatrix}. \tag{15}$$

We obtain $\Pi = \Pi^{\text{far}} \circ \Pi^{\text{loc}}$, proving Proposition 5.

2.3. Bifurcation analysis

The differential equations depend smoothly on a parameter μ so that the homoclinic network $G\Gamma$ (recall that we suppress dependence on μ from the notation) is asymptotically stable if $\mu < 0$ and unstable if $\mu > 0$. The unfolding condition (4), $\frac{\partial}{\partial \mu} \eta(\mu)|_{\mu=0} < 0$, enables a reparametrization $\eta = 1 - \mu$ for μ close to zero. The first return map, see Proposition 5, can now be rewritten as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \overset{\Pi_\mu}{\longmapsto} \begin{pmatrix} u^*(\mu) + \mathcal{O}(w^{\sigma(\mu)}) \\ v^*(\mu) + \mathcal{O}(w^{\sigma(\mu)}) \\ \Psi(v, \mu) w^{1-\mu} + \mathcal{O}(w^{1-\mu+\sigma(\mu)}) \end{pmatrix}. \tag{16}$$

Write

$$\Pi_\mu(u, v, w) = \Pi(u, v, w, \mu) = \big(\Pi^u(u, v, w, \mu), \Pi^v(u, v, w, \mu), \Pi^w(u, v, w, \mu)\big).$$

Proposition 8. *Assume we have a family of differential equations in $\mathbb{R}^3$ depending on μ with robust homoclinic cycles Γ which are stable for $\mu < 0$ and unstable for $\mu > 0$. Define*

$$\Psi^*(\mu) = \Psi\big(v^*(\mu), \mu\big).$$

If $\Psi^*(0) < 1$ then there is an asymptotically stable periodic trajectory close to the homoclinic cycle for $\mu > 0$. If $\Psi^*(0) > 1$ then there is an unstable periodic trajectory for $\mu < 0$.

Proof. We find a fixed point of (16) which corresponds to finding a solution of

$$u - \Pi^u(u, v, w, \mu) = 0, \qquad v - \Pi^v(u, v, w, \mu) = 0, \qquad w - \Pi^w(u, v, w, \mu) = 0. \tag{17}$$

Note that

$$J = \left| \begin{pmatrix} 1 - \Pi^u_u & -\Pi^u_v \\ -\Pi^v_u & 1 - \Pi^v_v \end{pmatrix} \right|_{(u^*, v^*, 0, 0)} = 1.$$

By the implicit function theorem, see e.g. Berger [1], we can find expressions for u and v in terms of w and μ that satisfy $u(w, \mu) = u^*(\mu) + \mathcal{O}(w^{\sigma}(\mu))$ and $v(w, \mu) = v^*(\mu) + \mathcal{O}(w^{\sigma}(\mu))$. Substitution in (17) yields

$$w = \Pi^w\big(u(w, \mu), v(w, \mu), w, \mu\big) = \Psi^*(\mu)w^{1-\mu} + \text{h.o.t.} \tag{18}$$

The assumption $\Psi(0) < 1$ implies that $\Psi(\mu) < 1$ for μ small and so there is a fixed point

$$\bar{w}(\mu) \approx \big(\Psi^*(\mu)\big)^{1/\mu}.$$

This proves that there is a flat branch of periodic trajectories. We claim that this branch is asymptotically stable. We will show that the magnitude of the real parts of the eigenvalues of the linearization of Π_μ about the fixed point are smaller than one. Compute

$$\left(\frac{d\Pi}{d(u, v, w)}\right)\Big|_{(\bar{u}, \bar{v}, \bar{w})} = \begin{pmatrix} \mathcal{O}(\Psi^*(\mu)^{\frac{\sigma(\mu)}{\mu}}) & \mathcal{O}(\Psi^*(\mu)^{\frac{\sigma(\mu)}{\mu}}) & \mathcal{O}(\Psi^*(\mu)^{\frac{\sigma(\mu)-1}{\mu}}) \\ \mathcal{O}(\Psi^*(\mu)^{\frac{\sigma(\mu)}{\mu}}) & \mathcal{O}(\Psi^*(\mu)^{\frac{\sigma(\mu)}{\mu}}) & \mathcal{O}(\Psi^*(\mu)^{\frac{\sigma(\mu)-1}{\mu}}) \\ \mathcal{O}(\Psi^*(\mu)^{\frac{1}{\mu}-1}) & \mathcal{O}(\Psi^*(\mu)^{\frac{1}{\mu}-1}) & 1-\mu \end{pmatrix}. \tag{19}$$

Write (D_{ij}) for the right-hand side and compute the characteristic polynomial $k(\lambda, \mu)$

$$k(\lambda, \mu) = D_{13}\left|\begin{pmatrix} D_{21} & D_{22} - \lambda \\ D_{31} & D_{32} \end{pmatrix}\right| - D_{23}\left|\begin{pmatrix} D_{11} - \lambda & D_{12} \\ D_{31} & D_{32} \end{pmatrix}\right| + D_{33}\left|\begin{pmatrix} D_{11} - \lambda & D_{12} \\ D_{21} & D_{22} - \lambda \end{pmatrix}\right|$$

$$= \mathcal{O}\big(\Psi^*(\mu)^{\frac{\sigma-\mu}{\mu}}\big) + (1 - \mu - \lambda)\big(\lambda^2 + \mathcal{O}\big(\Psi^*(\mu)^{\frac{\sigma}{\mu}}\big)\big)$$

$$= (1 - \mu - \lambda)\lambda^2 + \text{h.o.t.}$$

Note that the higher order terms are flat terms. Two eigenvalues will be close to zero for μ small by continuity of solutions. Compute

$$\frac{d}{d\lambda}k(\lambda, \mu)|_{\lambda=1, \mu=0} = -1.$$

By the implicit function theorem there is a smooth solution $\lambda = \lambda(\mu)$ with $\lambda(0) = 1$ that exists for μ small. The derivative of $k(\lambda(\mu), \mu)$ with respect to μ equals zero, i.e.

$$2\lambda\lambda'(1 - \mu - \lambda) + \lambda^2(-1 - \lambda') + \text{h.o.t.} = 0.$$

For $\mu = 0$ this reduces to $-1 - \lambda'(0) = 0$ and thus $\lambda'(0) = -1$ from which it follows that the third eigenvalue is smaller than 1 for $\mu > 0$ small enough. This proves the claim that the trajectory is asymptotically stable. In the case that $\Psi(0) > 1$ there is a fixed point for $\mu < 0$. This point is repelling, because for small enough initial values the w-coordinate of the first return map blows up. $\quad\square$

3. Simple homoclinic cycles of type C

We follow the setup of the previous section treating homoclinic cycles of type B. The vector field can now not be reduced to a three-dimensional one. Consider the homoclinic cycle Γ of type C, generated by a heteroclinic trajectory γ and a twist

$$h = A^1_{\pi/2, \pi/2}R \tag{20}$$

as in Theorem 1. Note that $h \notin SO(4)$ and $|\Gamma| = 4$.

3.1. Normal forms

Write the differential equation $\dot{x} = f(x, \mu)$ around p in local coordinates, cf. (5),

$$\dot{u} = \lambda_r u + F^u(u, v, w, z),$$

$$\dot{v} = \lambda_c v + F^v(u, v, w, z),$$

$$\dot{w} = \lambda_e w + F^w(u, v, w, z),$$

$$\dot{z} = \lambda_t z + F^z(u, v, w, z). \tag{21}$$

Proposition 9. *The differential equations* (21) *are locally smoothly equivalent to differential equations of the form*

$$\dot{u} = \lambda_r(\mu)u + P(u, v, w, z, \mu),$$

$$\dot{v} = \lambda_c(\mu)v + vwQ(u, v, w, z, \mu) + zR(u, v, w, z, \mu),$$

$$\dot{w} = \lambda_e(\mu)w,$$

$$\dot{z} = \lambda_t(\mu)z + zwS(u, v, w, z, \mu), \tag{22}$$

for smooth functions P, Q, R and S satisfying

$$P(x), Q(x), S(x) = \mathcal{O}\big(|(u, v, z)|\big).$$

In case $\lambda_c \leqslant \lambda_t < 0$, $R(x) = \mathcal{O}(|(u, v, z)|)$. *In case* $\lambda_t < \lambda_c < 0$, $R(x) = \mathcal{O}(|w|)$. *The equations are G-equivariant and smoothly dependent on the parameters.*

Proof. As in the proof of Proposition 4, G-equivariance will follow since the coordinate changes can be made to be G-equivariant. Recall that in the case of a robust homoclinic type C cycle there are invariant subspaces $Q = h^{-1}P + P = \{z = 0\}$ and the cycle does not entirely lie inside Q. After a coordinate transformation which straightens the local stable and unstable manifolds we have the invariant spaces $\{u = v = z = 0\}$ and $\{w = 0\}$ in addition to $\{w = z = 0\}$, $\{v = z = 0\}$ and $\{z = 0\}$. By a reparametrization of time we may assume that $\dot{w} = ew$. We can write the differential equations (21) locally around p as

$$\dot{u} = \lambda_r u + \tilde{P}(u, v, w, z),$$

$$\dot{v} = \lambda_c v + v\tilde{Q}(u, v, w, z) + z\tilde{R}(u, v, w, z),$$

$$\dot{w} = \lambda_e w,$$

$$\dot{z} = \lambda_t z + z\tilde{S}(u, v, w, z), \tag{23}$$

where $\tilde{P} = \mathcal{O}(|(u, v, z)|)$, $\tilde{Q} = \mathcal{O}(|(u, v, w, z)|)$, $\tilde{R} = \mathcal{O}(|(u, w, z)|)$ and $\tilde{S} = \mathcal{O}(|(u, v, w, z)|)$. These equations are smoothly dependent on the parameters.

As in the proof of Proposition 4 we apply the argument from Ovsyannikov and Shil'nikov [14]. We start with coordinate changes that can be applied independent of the relative size of λ_t, λ_c. Consider a change of coordinates

$$\bar{v} = v + p(w)v.$$

We demonstrate that a smooth function $p(w)$ can be found that removes terms $v Q (0, 0, w, 0)$ from the equation for $\dot{v}$. The differential equation for $\dot{\bar{v}}$ takes the form

$$
\begin{aligned}
\dot{\bar{v}} &= \dot{v}(1 + p) + v\dot{p} \\
&= \big(\lambda_c v + v\tilde{Q}(u, v, w, z) + z\tilde{R}(u, v, w, z)\big)(1 + p) + v\dot{p} \\
&= \lambda_c \bar{v} + \bar{v}\big(\tilde{Q}\big(u, \bar{v}/(1 + p), w, z\big) + \dot{p}/(1 + p)\big) + z\tilde{R}\big(u, \bar{v}/(1 + p), w, z\big).
\end{aligned}
$$

Along the local stable manifold $\{u, v, z = 0\}$ we demand the identity $\dot{p} + (1 + p)\tilde{Q}(0, 0, w, 0) = 0$. With an initial (polynomial) coordinate change that removes terms vw from the equation for $\dot{v}$, $\tilde{Q}(0, 0, w, 0)$ is a quadratic function. Thus p as a function of w is found as the local unstable manifold of the system

$$
\dot{w} = \lambda_e w,
$$

$$
\dot{p} = -(1 + p)\tilde{Q}(0, 0, w, 0) = \text{h.o.t.}
$$

Similar arguments remove terms $v\tilde{Q}(u, v, 0, z)$ from the equation for $\dot{v}$, here one considers a coordinate change of the form $\bar{v} = v + p(u, v, z)v$ and obtains p from the local stable manifold of a system of differential equations for u, v, z (along $\{w = 0\}$) and p. Likewise one removes terms $z\tilde{S}(0, 0, w, 0)$ and $z\tilde{S}(u, v, 0, z)$ from the equation for z. The identity $P(0, 0, w, 0) = 0$ is a consequence of the fact that the local unstable manifold has been straightened.

We continue with an additional smooth coordinate change valid for $\lambda_c \leqslant \lambda_t < 0$, removing $z\tilde{R}(0, 0, w, 0)$ from the differential equation for $\dot{v}$. Consider for this the change of coordinates

$$
\bar{v} = v + p(w)z.
$$

Compute the differential equation for $\bar{v}$:

$$
\begin{aligned}
\dot{\bar{v}} &= \dot{v} + z\dot{p} + \dot{z}p \\
&= \lambda_c v + vwQ(u, v, w, z) + z\tilde{R}(u, v, w, z) + z\dot{p} + \lambda_t zp + zwpS(u, v, w, z) \\
&= \lambda_c \bar{v} + \bar{v}wQ(u, \bar{v} - pz, w, z) \\
&\quad + z\big((\lambda_t - \lambda_c)p + \tilde{R}(u, \bar{v} - pz, w, z) + \dot{p} + wpS(u, \bar{v} - pz, w, z) - wpQ(u, \bar{v} - pz, w, z)\big).
\end{aligned}
$$

Along the local unstable manifold $\{u, v, z = 0\}$ we require $\dot{p} + (\lambda_t - \lambda_c)p + wQ(0, 0, 0, w) + p\tilde{R}(0, 0, 0, w) - pwQ(0, 0, 0, w) = 0$.

Note that terms zw from the equation for $\dot{v}$ can be removed by a polynomial coordinate change whenever $\lambda_t + \lambda_e \neq \lambda_c$ (which is guaranteed by $\lambda_c \leqslant \lambda_t$). With an initial coordinate change that removes terms zw from the equation for $\dot{v}$, we get $\dot{p} = (\lambda_c - \lambda_t)p + \mathcal{O}(|x|^2)$. Since $\lambda_c - \lambda_t \leqslant 0$, p as a smooth function of w is found as the local unstable manifold of the system of differential equations for w and p,

$$
\dot{w} = \lambda_e w,
$$

$$
\dot{p} = (\lambda_c - \lambda_t)p + \mathcal{O}\big(|x|^2\big).
$$

The removal of the term $z\tilde{R}(u, v, 0, z)$ in case $\lambda_t < \lambda_c$ proceeds similarly. $\quad\square$

3.2. Asymptotic expansions

In this section we obtain asymptotic expressions for a first return map on a cross-section. Take a cross-section, which by using rescaled coordinates we may assume to be near p,

$$\Sigma^{\text{in}} = \big\{ |(u, v)| = 1;\ |w|, |z| \leqslant 1 \big\}.$$

Identifying Σ^{in} with $h\Sigma^{\text{in}}$ through the twist map h, a "first return map" $\Pi : \Sigma^{\text{in}} \mapsto \Sigma^{\text{in}}$ is obtained. Recall the notation $T = -\lambda_t/\lambda_e$, $C = -\lambda_c/\lambda_e$.

Proposition 10. *Assume normal form coordinates near p given by Proposition 9. Then $\Pi : \Sigma^{\text{in}} \mapsto \Sigma^{\text{in}}$ has asymptotic expansions*

$$\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} \xmapsto{P} \begin{pmatrix} u^* + \mathcal{O}(w^\Omega) \\ v^* + \mathcal{O}(w^\Omega) \\ \gamma_1 z w^{-T} + \mathcal{O}(z w^{-T+\Omega}) \\ \gamma_2 v w^C + \mathcal{O}(w^{C+\Omega}) + \mathcal{O}(z w^{-T+\Omega}) \end{pmatrix},$$

for some $\Omega > 0$.

Remark 5. As in Remark 4 the existence of codimension one fixed point spaces implies, for a twist h as in (20), that $\gamma_1 > 0$ and $\gamma_2 > 0$.

Consider trajectories from Σ^{in} to $\Sigma^{\text{out}} = \{w = 1\}$, starting at a point $x_0 = (u_0, v_0, w_0, z_0)$. Write $\tau = -\frac{1}{\lambda_e} \ln w_0$ for the transition time. The solution for w is given by

$$w(t) = e^{-\lambda_e(\tau - t)}. \tag{24}$$

Lemma 11. *There exists $\omega > 0$ so that*

$$u(\tau) = \mathcal{O}\big(e^{-\omega\tau}\big),$$
$$v(\tau) = v_0 e^{\lambda_c t} + \mathcal{O}\big(e^{(\lambda_c - \omega)\tau}\big) + z_0 \mathcal{O}\big(e^{(\lambda_t - \omega)\tau}\big),$$
$$z(\tau) = z_0 e^{\lambda_t \tau} + z_0 \mathcal{O}\big(e^{(\lambda_t - \omega)\tau}\big).$$

Here u, v, z depend smoothly on u_0, v_0, z_0 and τ; derivatives of the higher order terms yield terms of the same order:

$$D^k_{u_0, v_0, z_0} D^l_\tau u(\tau) = \mathcal{O}\big(e^{-\omega\tau}\big),$$
$$D^k_{u_0, v_0, z_0} D^l_\tau v(\tau) = \mathcal{O}\big(e^{\lambda_c \tau}\big) + \mathcal{O}\big(e^{\lambda_t t}\big),$$
$$D^k_{u_0, v_0, z_0} D^l_\tau z(\tau) = \mathcal{O}\big(e^{\lambda_t \tau}\big).$$

Remark 6. Filling in $\tau = -\frac{1}{\lambda_e} \ln w_0$, the expansions yield

$$u(\tau) = \mathcal{O}\big(w_0^{-\omega/\lambda_e}\big),$$
$$v(\tau) = v_0 w_0^{-\lambda_c/\lambda_e} + \mathcal{O}\big(w_0^{(-\lambda_c + \omega)/\lambda_e}\big) + z_0 \mathcal{O}\big(w_0^{(-\lambda_t + \omega)/\lambda_e}\big),$$
$$z(\tau) = z_0 w_0^{-\lambda_t/\lambda_e} + z_0 \mathcal{O}\big(w_0^{(-\lambda_t + \omega)/\lambda_e}\big).$$

Proof. Consider a trajectory $x(t) = (u(t), v(t), e^{-\lambda_e(\tau-t)}, z(t))$ for $0 \leqslant t \leqslant \tau$, with $x(0) \in \Sigma^{\text{in}}$ and $x(\tau) \in \Sigma^{\text{out}}$. Note that u, v and z depend on t, τ, u_0, v_0 and z_0. We suppress the dependence on τ, u_0, v_0, z_0 from the notation and write $u(t)$, $v(t)$, $z(t)$. We write down expressions for $u(t)$, $v(t)$, $z(t)$, with $0 \leqslant t \leqslant \tau$, from the variation of constants formula:

$$u(t) = e^{\lambda_r t} u_0 + \int_0^t e^{\lambda_r(t-s)} P(x(s)) \, ds,$$

$$v(t) = e^{\lambda_c t} v_0 + \int_0^t e^{\lambda_c(t-s)} \big[v(s) w(s) Q(x(s)) + z(s) R(x(s)) \big] \, ds,$$

$$z(t) = e^{\lambda_t t} z_0 + \int_0^t e^{\lambda_t(t-s)} v(s) w(s) S(x(s)) \, ds.$$

The right-hand side defines a map $\mathscr{S}$ on a space of continuous functions (u, v, z) defined for $0 \leqslant t \leqslant \tau$. We first obtain exponential bounds for the solutions, before determining more precise exponential expansions. We claim that for some $\omega > 0$, $0 \leqslant t \leqslant \tau$,

$$u(t) = \mathcal{O}(e^{-\omega t}),$$

$$v(t) = \mathcal{O}(e^{\lambda_c t}) + z_0 \mathcal{O}(e^{\lambda_t t}),$$

$$z(t) = z_0 \mathcal{O}(e^{\lambda_t t}).$$

This follows from the observation that $\mathscr{S}$ maps a space of functions with the prescribed exponential bounds into itself. More in detail, write

$$B_C^{\omega, \lambda_c, \lambda_t} = \big\{ (u, v, w) : [0, \tau] \to \mathbb{R}^3 \ \big| \ |u(t)| < Ce^{-\omega t}, \ |v(t)| < Ce^{\lambda_c t} + Cz_0 e^{\lambda_t t}, \ |z(t)| < Cz_0 e^{\lambda_t t} \big\}.$$

Then, as direct estimates show, for $0 < \omega < \min\{-\lambda_r, -\lambda_c, -\lambda_t\}$ and large enough $C > 0$, $\mathscr{S}$ maps $B_C^{\omega, \lambda_c, \lambda_t}$ into itself.

Further estimates bound the integrals in the variation of constants formulas:

$$\int_0^\tau e^{\lambda_c(\tau-s)} v(s) w(s) Q(x(s)) \, ds = \mathcal{O}(e^{(\lambda_c - \omega)\tau}),$$

$$\int_0^\tau e^{\lambda_c(\tau-s)} z(s) R(x(s)) \, ds = z_0 \mathcal{O}(e^{(\lambda_t - \omega)\tau}),$$

$$\int_0^\tau e^{\lambda_t(\tau-s)} v(s) w(s) S(x(s)) \, ds = z_0 \mathcal{O}(e^{(\lambda_t - \omega)\tau}).$$

This proves the given expansions. It remains to consider estimates for derivatives of (u, v, w) with respect to τ and the initial data u_0, v_0, z_0. We claim that for some $\omega > 0$, $0 \leqslant t \leqslant \tau$,

$$D^k_{u_0,v_0,z_0} u(t) = \mathcal{O}(e^{-\omega t}),$$
$$D^k_{u_0,v_0,z_0} v(t) = \mathcal{O}(e^{\lambda_c t}) + \mathcal{O}(e^{\lambda_t t}),$$
$$D^k_{u_0,v_0,z_0} z(t) = \mathcal{O}(e^{\lambda_t t}).$$

Derivatives are treated by differentiating the variation of constants formulas. In a similar way one treats derivatives with respect to τ, providing bounds

$$D^k_{u_0,v_0,z_0} D^l_\tau u(t) = \mathcal{O}(e^{-\omega t} e^{-\lambda_e(\tau-t)}),$$
$$D^k_{u_0,v_0,z_0} D^l_\tau v(t) = \mathcal{O}(e^{\lambda_c t} e^{-\lambda_e(\tau-t)}) + \mathcal{O}(e^{\lambda_t t} e^{-\lambda_e(\tau-t)}),$$
$$D^k_{u_0,v_0,z_0} D^l_\tau z(t) = \mathcal{O}(e^{\lambda_t t} e^{-\lambda_e(\tau-t)})$$

and thus proving the statement if we fill in $t = \tau$. $\quad\square$

Proposition 10 follows by composing the local and global transition maps.

3.3. Bifurcation analysis

Proposition 12. *Assume we have a family of differential equations in $\mathbb{R}^4$ depending smoothly on μ with robust homoclinic cycles Γ of type C which are stable for $\mu > 0$ and unstable for $\mu < 0$. Assume the unfolding condition (4) and the nondegeneracy condition $\Phi = \gamma_1(0)\gamma_2(0)v^*(0) \neq 1$. If $\Phi < 1$ then there exists an attracting periodic trajectory close to the homoclinic cycle for $\mu > 0$ small enough. If $\Phi > 1$ then there exists a repelling periodic trajectory for $\mu < 0$ small enough.*

Proof. By the unfolding condition (4), after a reparametrization we may assume $C - T = 1 - \mu$ for μ close to zero. By the implicit function theorem we can find u and v as functions of w and z and then the fixed point problem reduces to finding a solution of

$$w = \gamma_1(\mu)zw^{-T} + \mathcal{O}(zw^{-T+\Omega}),$$
$$z = \gamma_2(\mu)v^*(\mu)w^C + \mathcal{O}(w^{C+\Omega}) + \mathcal{O}(zw^{-T+\Omega}).$$

First we solve z in terms of w from the second equation

$$z = \gamma_2(\mu)v^*(\mu)w^C(1 + \mathcal{O}(w^\Omega))$$

and then put this into the first equation

$$w = \gamma_1(\mu)\gamma_2(\mu)v^*(\mu)w^{C-T} + \text{h.o.t.}$$

If $\gamma_1(0)\gamma_2(0)v^*(0) < 1$, there is a solution for $\mu > 0$ small enough. We prove stability in the same way as in Proposition 8. The fixed point is given by

$$(\bar{u}, \bar{v}, \bar{w}, \bar{z}) \approx (u^*, v^*, B(\mu)^{1/\mu}, \gamma_2(\mu)v^*(\mu)B(\mu)^{C/\mu}),$$

where $B(\mu) = \gamma_1(\mu)\gamma_2(\mu)v^*(\mu) < 1$. From the linearization of the first return map about the fixed point, compute that the characteristic polynomial is given by

$$k(\lambda, \mu) = \lambda^2(\lambda^2 + T\lambda - C) + \text{h.o.t.},$$

where the higher order terms are flat terms in μ. It follows by continuity of solutions of this equation that there are two eigenvalues close to zero. The other two eigenvalues are close to the nonzero solutions of $k(\lambda, 0) = 0$. We solve

$$\lambda_\pm(0) = \frac{-T \pm \sqrt{T^2 + 4C}}{2}.$$

While $C - T = 1$ for $\mu = 0$, $\lambda_+(0) = 1$ and $\lambda_-(0) = -C$. We assumed that $T \neq 0$ and thus it follows that $|\lambda_-(\mu)| < 1$ for μ small. We will now show that $|\lambda_+(\mu)| < 1$ for $\mu > 0$ small enough. Compute

$$\frac{d}{d\lambda} k(\lambda, \mu)|_{\lambda=1, \mu=0} = 2 + T > 1.$$

The implicit function theorem implies that we can express λ as a smooth function of μ for μ small with $\lambda(0) = 1$. Derivation of $k(\lambda, \mu) = 0$ with respect to μ in the point $(\lambda, \mu) = (1, 0)$ yields

$$\lambda'(0) = \frac{C'(0) - T'(0)}{4 + 3T(0) - 2C(0)} = \frac{-1}{2 + T(0)} < -1$$

and thus $|\lambda_+(\mu)| < 1$ for μ small and positive. This concludes the proof that the periodic trajectory is stable.

If $\gamma_1(0)\gamma_2(0)v^*(0) > 1$ and $\mu < 0$ small enough there also exists a periodic trajectory. Note that $B(\mu)^{1/\mu}$ is flat in $\mu < 0$ for $B(\mu) > 1$. It follows directly from the proof above that this periodic trajectory is unstable. $\quad\square$

4. Simple homoclinic cycles of type A

Groups that admit homoclinic cycles of type A do not possess three-dimensional fixed point spaces. As a consequence, an appropriate normal form near equilibria in homoclinic cycles of type A is more sophisticated then near equilibria in homoclinic cycles of type B or C. The techniques used in the derivation, and following that in the bifurcation analysis, are not different. We will therefore at some points be brief and refer to similar computations in the earlier sections.

4.1. Normal forms

Write the differential equation near p, in coordinates (5), as

$$\dot{u} = \lambda_r u + F^u(u, v, w, z),$$

$$\dot{v} = \lambda_c v + F^v(u, v, w, z),$$

$$\dot{w} = \lambda_e w + F^w(u, v, w, z),$$

$$\dot{z} = \lambda_t z + F^z(u, v, w, z). \tag{25}$$

We consider a homoclinic cycle of type A. We will assume that we are close to the resonance $-\lambda_c = \lambda_e$. In particular this assumption guarantees that

$$\lambda_t - \lambda_e < \lambda_c. \tag{26}$$

This estimate will be assumed throughout this section.

Proposition 13. *For homoclinic cycles of type A the differential equations* (25) *are smoothly equivalent to differential equations of the form*

$$\dot{u} = \lambda_r(\mu)u + P(u, v, w, z, \mu),$$

$$\dot{v} = \lambda_c(\mu)v + vwQ(u, v, w, z, \mu) + zR(u, v, w, z, \mu),$$

$$\dot{w} = \lambda_e(\mu)w,$$

$$\dot{z} = \lambda_t(\mu)z + zwS(u, v, w, z, \mu) + vwT(u, v, w, z, \mu),$$

for smooth functions P, Q, R, S, T *satisfying*

$$P(x), Q(x), S(x), T(x) = \mathcal{O}(|(u, v, z)|).$$

In case $\lambda_c \leqslant \lambda_t < 0$, $R(x) = \mathcal{O}(|(u, v, z)|)$. *In case* $\lambda_t < \lambda_c < 0$, $R(x) = \mathcal{O}(|w|)$. *The differential equations are G-equivariant and depend smoothly on* μ.

Proof. Take coordinates in which local stable and unstable manifolds are linear. By a reparametrization of time we may assume that $\dot{w} = \lambda_e w$. From invariance of $\{w, z = 0\}$, $\{v, z = 0\}$, it follows that the differential equations are given by a set of equations

$$\dot{u} = \lambda_r u + \tilde{P}(u, v, w, z),$$

$$\dot{v} = \lambda_c v + v\tilde{Q}(u, v, w, z) + z\tilde{R}(u, v, w, z),$$

$$\dot{w} = \lambda_e w,$$

$$\dot{z} = \lambda_t z + z\tilde{S}(u, v, w, z) + vw\tilde{T}(u, v, w, z).$$

The statements on $\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}$ are proved as before. For the statement on $\tilde{T}$ consider a coordinate change of the form $\bar{z} = z + vwH(w)$. Then

$$\dot{\bar{z}} = \dot{z} + (\lambda_c + \lambda_e)vwH + vw\dot{H} + \text{h.o.t.}$$

$$= \lambda_t\bar{z} + vw(\tilde{T} + (\lambda_c + \lambda_e - \lambda_t)H + \dot{H}) + \text{h.o.t.}$$

Along the unstable manifold $\{(u, v, z) = 0\}$, we add to the differential equation $\dot{w} = \lambda_e w$ the differential equation

$$\dot{H} = (\lambda_t - \lambda_e - \lambda_c)H + \tilde{T}(0, 0, w, 0).$$

Under the nonresonance condition $\lambda_t - \lambda_e \neq \lambda_c$ (which is met by (26)) terms vw from the equation for $\dot{z}$ can be removed. This guarantees $T(0, 0, w, 0) = \mathcal{O}(|w|)$. Note that $\lambda_t - \lambda_e < \lambda_c$. We therefore obtain H as a smooth function of w as the unstable manifold. Finally, G-equivariance follows since the coordinate changes can be made to be G-equivariant. $\quad\square$

4.2. Asymptotic expansions

Lemma 14. *Suppose $\lambda_t < \lambda_c < 0$. There exists $\omega > 0$ and a smooth function $\phi(x_0)$ so that*

$$u(\tau) = \mathcal{O}(e^{-\omega\tau}),$$

$$v(\tau) = \phi(x_0)e^{\lambda_c\tau} + \mathcal{O}(e^{(\lambda_c-\omega)\tau}),$$

$$z(\tau) = \mathcal{O}(e^{(\lambda_c-\omega)\tau}).$$

Remark 7. Filling in $\tau = -\frac{1}{\lambda_e}\ln w_0$, the expansions yield

$$u(\tau) = \mathcal{O}(w_0^{\omega/\lambda_e}),$$

$$v(\tau) = \phi(x_0)w_0^{-\lambda_c/\lambda_e} + \mathcal{O}(w_0^{(-\lambda_c+\omega)/\lambda_e}),$$

$$z(\tau) = \mathcal{O}(w_0^{(-\lambda_c+\omega)/\lambda_e}). \tag{27}$$

Proof. The proof mostly follows the reasoning from earlier sections, details are left to the reader. We consider exponential expansions for $v(\tau)$. Following Deng [5], define

$$\bar{v}(\tau) = v(\tau)e^{-\lambda_c\tau}.$$

The lowest order term in the exponential expansion for v (the term $\phi(x_0)e^{-\lambda_c\tau}$ in the statement of the proposition) is obtained as the limit $\lim_{\tau\to\infty}\bar{v}(\tau)$. For existence of this limit, consider

$$\frac{\partial}{\partial\tau}\bar{v}(\tau) = e^{-\lambda_c\tau}F^v(x(\tau)) + \int_0^\tau e^{-\lambda_c s}\frac{\partial}{\partial\tau}F^v(v(s), w(s), e^{-\lambda_e(\tau-s)}, z(s))\,ds,$$

and recall that $F^v(x) = vw\mathcal{O}(|(u,v)|) + z\mathcal{O}(|(u,v,z)|)$. Straightforward estimates show that $\frac{\partial}{\partial\tau}\bar{v}(\tau)$ converges exponentially fast to 0, implying that $\lim_{\tau\to\infty}\bar{v}(\tau)$ exists. Similarly one proves that the limit depends smoothly on the initial conditions. □

Lemma 15. *Suppose that $\lambda_c \leqslant \lambda_t < 0$. There is a smooth nonvanishing function $\phi(x_0)$ and an $\omega > 0$, so that*

$$u(\tau) = \mathcal{O}(e^{-\omega\tau}),$$

$$v(\tau) = \phi(x_0)e^{\lambda_c\tau} + z_0\mathcal{O}(e^{(\lambda_t-\omega)\tau}) + \mathcal{O}(e^{(\lambda_c-\omega)\tau}),$$

$$z(\tau) = z_0 e^{\lambda_t\tau} + z_0\mathcal{O}(e^{(\lambda_t-\omega)\tau}) + \mathcal{O}(e^{(\lambda_c-\omega)\tau}).$$

Remark 8. Plugging in $\tau = -\frac{1}{\lambda_e}\ln w_0$, the expansions read

$$u(\tau) = \mathcal{O}(w_0^{\omega/\lambda_e}),$$

$$v(\tau) = \phi(x_0)w_0^{-\lambda_c/\lambda_e} + \mathcal{O}(w_0^{(-\lambda_c+\omega)/\lambda_e}) + z_0\mathcal{O}(w_0^{-(\lambda_t-\omega)/\lambda_e}),$$

$$z(\tau) = z_0 w_0^{-\lambda_t/\lambda_e} + z_0\mathcal{O}(w_0^{-(\lambda_t-\omega)/\lambda_e}) + \mathcal{O}(w_0^{(-\lambda_c+\omega)/\lambda_e}). \tag{28}$$

Proof. We write down expressions for a trajectory from the variation of constants formula for $0 \leqslant t \leqslant \tau$ and $x(t) = (u(t), v(t), e^{-\lambda_e(\tau-t)}, z(t))$,

$$u(t) = e^{\lambda_r t} u_0 + \int_0^t e^{\lambda_r(t-s)} P(x(s)) \, ds,$$

$$v(t) = e^{\lambda_c t} v_0 + \int_0^t e^{\lambda_c(t-s)} \big[v(s)w(s)Q(x(s)) + z(s)R(x(s)) \big] \, ds,$$

$$z(t) = e^{\lambda_t t} z_0 + \int_0^t e^{\lambda_t(t-s)} \big[z(s)w(s)S(x(s)) + v(s)w(s)T(x(s)) \big] \, ds.$$

The right-hand side defines a map $\mathscr{S}$ on a space of continuous functions (u, v, z) defined for $0 \leqslant t \leqslant \tau$.

We first obtain exponential bounds for the solutions, before determining more precise exponential expansions. We claim that for some $\omega > 0$, $0 \leqslant t \leqslant \tau$,

$$u(t) = \mathcal{O}(e^{-\omega t}),$$

$$v(t) = \mathcal{O}(e^{\lambda_c t}) + z_0 \mathcal{O}(e^{(\lambda_t - \omega)t}),$$

$$z(t) = \mathcal{O}(e^{(\lambda_c - \omega)t}) + z_0 \mathcal{O}(e^{\lambda_t t}).$$

This follows from the observation that $\mathscr{S}$ maps a space of functions with the prescribed exponential bounds into itself. A key estimate here uses

$$\int_0^t e^{\lambda_t(t-s)} v(s)w(s)T(x(s)) \, ds = \int_0^t e^{(\lambda_t - \lambda_e)(t-s)} e^{-\lambda_e(\tau-t)} v(s)T(x(s)) \, ds$$

and $\lambda_t - \lambda_e < \lambda_c$. Similarly one shows for derivatives

$$D_{u_0,v_0,z_0}^k u(t) = \mathcal{O}(e^{-\omega t}),$$

$$D_{u_0,v_0,z_0}^k v(t) = \mathcal{O}(e^{\lambda_c t}) + \mathcal{O}(e^{(\lambda_t - \omega)t}),$$

$$D_{u_0,v_0,z_0}^k z(t) = \mathcal{O}(e^{(\lambda_c - \omega)t}) + \mathcal{O}(e^{\lambda_t t})$$

and

$$D_{u_0,v_0,z_0}^k D_\tau^l u(t) = \mathcal{O}(e^{-\omega t} e^{\lambda_e(t-\tau)}),$$

$$D_{u_0,v_0,z_0}^k D_\tau^l v(t) = \mathcal{O}(e^{\lambda_c t} e^{\lambda_e(t-\tau)}) + z_0 \mathcal{O}(e^{\lambda_t t} e^{\lambda_e(t-\tau)}),$$

$$D_{u_0,v_0,z_0}^k D_\tau^l z(t) = z_0 \mathcal{O}(e^{\lambda_t t} e^{\lambda_e(t-\tau)}).$$

To derive an exponential expansion for $v(\tau)$ (i.e. to compute the function $\phi(x_0)$), consider

$$\bar{v}(\tau) = v(\tau) e^{-\lambda_c \tau}.$$

Compute

$$\frac{\partial}{\partial \tau}\bar{v}(\tau) = e^{-\lambda_c \tau}v(\tau)w(\tau)Q\big(x(\tau)\big) + e^{-\lambda_c \tau}z(\tau)R\big(x(\tau)\big)$$

$$+ \int_0^\tau e^{-\lambda_c s}\frac{\partial}{\partial \tau}\big(v(s)w(s)Q\big(u(s), v(s), e^{-\lambda_e(\tau-s)}, z(s)\big)\big)\,ds$$

$$+ \int_0^\tau e^{-\lambda_c s}\frac{\partial}{\partial \tau}\big(z(s)R\big(u(s), v(s), e^{-\lambda_e(\tau-s)}, z(s)\big)\big)\,ds.$$

Straightforward estimates show that $\frac{\partial}{\partial \tau}\bar{v}(\tau)$ converges exponentially fast to 0 as $\tau \to \infty$, implying that $\lim_{\tau\to\infty}\bar{v}(\tau)$ exists. Formulas for derivatives are obtained by taking derivatives of the equations obtained from the variation of constants formula. This shows that $\lim_{\tau\to\infty}\bar{v}(\tau)$ is a smooth function of the initial conditions and the parameters. As $\phi(x_0)$ is within order δ of 1, it is a nonvanishing function. A straightforward estimate shows

$$\int_0^t e^{\lambda_t(t-s)}\big[v(s)w(s)S\big(x(s)\big) + v(s)w(s)T\big(x(s)\big)\big]\,ds = z_0\mathcal{O}\big(e^{(\lambda_t-\omega)\tau}\big) + \mathcal{O}\big(e^{(\lambda_c-\omega)\tau}\big),$$

providing the expansion for $z(\tau)$. $\square$

4.3. Bifurcation analysis

Write x^* for the point of intersection of Γ with Σ, at $\mu = 0$.

Proposition 16. *Assume we have a family of differential equations in $\mathbb{R}^4$ depending smoothly on μ with robust homoclinic cycles Γ of type A which are stable for $\mu > 0$ and unstable for $\mu < 0$. Assume the unfolding condition (4) and the nondegeneracy condition $\Phi = \gamma_1(0)\phi(x^*) \neq 1$. If $\Phi < 1$, then there exists an attracting periodic trajectory close to the homoclinic cycle for $\mu > 0$ small enough. If $\Phi > 1$, then there exists a repelling periodic trajectory for $\mu < 0$ small enough.*

Proof. Again we assume that we have a family of differential equations depending on μ for which there is a homoclinic cycle Γ of type C. We assume that it is stable for $\mu > 0$ and unstable for $\mu < 0$. After a reparametrization $C = 1 - \mu$ for μ close to zero.

We can use the normal form from Proposition 13. Depending on the relative magnitude of λ_c and λ_t we can then write the local transition map as (27) or (28). By the implicit function theorem we can write the (u, v) coordinates of the fixed point of the first return map as $(u, v) = (u^* + \mathcal{O}(w^\Omega), v^* + \mathcal{O}(w^\Omega))$. First we consider the case where $\lambda_t < \lambda_c$. Using the expressions for u and v the fixed point satisfies

$$w = \gamma_1\phi(x)w^C + \text{h.o.t.},$$
$$z = \gamma_2\phi(x)w^C + \text{h.o.t.} \tag{29}$$

We solve $\bar{w}$ and $\bar{z}$ from these equations, i.e.

$$\bar{w} \approx B(\mu)^{1/\mu}, \tag{30}$$

$$\bar{z} \approx \gamma_2\phi\big(x^*\big)B(\mu)^C/\mu, \tag{31}$$

where $B(\mu) = \gamma_1\phi(x^*, \mu)$. From the linearization of the first return map about the fixed point, compute that the characteristic polynomial is given by

$$k(\lambda, \mu) = -\lambda^3 (C - \lambda) + \text{h.o.t.,}$$

where the higher order terms are flat in μ. By continuity it follows that there are three eigenvalues close to zero for μ small enough. The fourth eigenvalue is approximately 1 and we must show that it is smaller than 1 for $\mu > 0$. Compute therefore

$$\frac{d}{d\lambda} k(\lambda, \mu)|_{\lambda=1, \mu=0} = 4 - 3C(0) = 1.$$

By the implicit function theorem we can express λ as a smooth function of μ with $\lambda(0) = 1$ for μ small. Taking the derivative at both sides of the equation $k(\lambda(\mu), \mu) = 0$ with respect to μ yields:

$$4\lambda^3 \lambda' - 3C\lambda^2 \lambda' - C'\lambda^3 = 0.$$

When $\mu = 0$ we find $\lambda'(0) = -1$ and thus for μ small and positive, $\lambda(\mu) < 1$. So the periodic trajectory is asymptotically stable for μ small enough. For $\gamma_1(0)\phi(\bar{x}, 0) > 1$ we can find a fixed point for $\mu < 0$ which by the reasoning above is repelling.

In the case $\lambda_t \geqslant \lambda_c$ it follows that the fixed point satisfies

$$w = \gamma_1 \phi(x) w^C + \beta_1 z w^{-T} + \text{h.o.t.,}$$
$$z = \gamma_2 \phi(x) w^C + \beta_2 z w^{-T} + \text{h.o.t.}$$

If we solve $\bar{w}$ and $\bar{z}$ from these equations it follows that they are also given by expressions of the form (31). Further analysis shows that also the characteristic equation is of the same form and thus all results for $\lambda_t < \lambda_c$ also apply for $\lambda_t \geqslant \lambda_c$. $\square$

Acknowledgments

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