Lorenz attractors in unfoldings of homoclinic flip bifurcations

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Abstract

Lorenz like attractors are known to appear in unfoldings from certain codimension two homoclinic bifurcations for differential equations in $\mathbb{R}^3$ that possess a reflectional symmetry. This includes homoclinic loops under a resonance condition and the inclination flip homoclinic loops. We show that Lorenz like attractors also appear in the third possible codimension two homoclinic bifurcation (for homoclinic loops to equilibria with real different eigenvalues); the orbit flip homoclinic bifurcation. We moreover provide a bifurcation analysis, computing the bifurcation curves of bifurcations from periodic orbits and discussing the creation and destruction of the Lorenz like attractors. Known results for the inclination-flip are extended to include a bifurcation analysis.

1 Introduction

The Lorenz equations, the system of ordinary differential equations on $\mathbb{R}^3$ given by

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= \rho x - y - xz,\\ \dot{z} &= -\beta z + xy,\end{aligned}$$

was coined in the 1960s by Lorenz [16] as a much simplified model for Rayleigh-Bénard convection. Numerical computations done by Lorenz revealed the existence of strange attractors for parameter values $\sigma = 10, b = 8/3, \rho = 28$. These numerical observations of strange attractors were first explained in geometric Lorenz models [1, 10, 35]. A fundamental property of the strange attractor is its robust occurrence under variations of the parameters (or perturbations of the system, details of the dynamics may change). The occurrence of a robust strange attractor in the actual Lorenz system was proven by Tucker [33, 34]. A general theory, for three dimensional flows, of robust partially hyperbolic attractors containing one or more equilibria has been developed, see [19]. These attractors are called singular hyperbolic attractors. We refer to [2, 4, 5, 17, 18] for further results, mostly discussing ergodic properties, on Lorenz like attractors and other singular hyperbolic attractors. We use the phrase Lorenz like attractor for a singular hyperbolic attractor in three dimensional differential equations with a geometry as in the Lorenz system (that is, containing a single equilibrium).

There has been substantial interest in the question how Lorenz like attractors may appear through bifurcations from ODEs with simpler dynamics. For instance miniature Lorenz like attractors were shown to occur in unfoldings of certain nilpotent singularities [9]. Shilnikov [32] suggested that Lorenz-like attractors may occur in bifurcations from ODEs with two homoclinic loops for three different codimension two (in the context of $\mathbb{Z}_2$-equivariant systems) bifurcations: resonant leading eigenvalues, an inclination-flip condition or an orbit-flip condition. These are all codimension two homoclinic bifurcations involving a hyperbolic equilibrium with real distinct eigenvalues. The first possibility was worked out by Robinson [26–28] (see also [20, 21]), the second by Rychlik [29]. Both Robinson and Rychlik provided cubic differential equations containing Lorenz like attractors. We treat the third type of bifurcation, the orbit-flip bifurcation, and establish the existence of Lorenz like attractors in its unfolding. We present a more detailed bifurcation analysis going beyond the observation that Lorenz like attractors occur in the unfolding, thus giving more insight in the creation of the attractors.

The word flip in both orbit flip and inclination flip indicates a change in geometry of the two dimensional stable manifold along the homoclinic orbit; both bifurcations trigger a change in orientation where the stable manifold near the homoclinic orbit is either an annulus or a Möbius band. Both bifurcations show many similarities and in fact a bifurcation analysis for the inclination flip can be given along the lines of this paper. This is further discussed in § 5.

The subject of our study will be two parameter families

$$\dot{u} = f(u, \mu) \tag{1}$$

of differential equations with $u \in \mathbb{R}^3$ and $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$. The differential equations we consider are symmetric with respect to a reflection against an axis and possess a symmetric hyperbolic equilibrium at the origin. Symmetry is given by a linear involution $\mathcal{R}$ (a linear map with $\mathcal{R}^2 = \mathrm{id}$) which maps orbits to orbits. More formally this means that

$$\mathcal{R} f(u, \mu) = f(\mathcal{R} u, \mu). \tag{2}$$

We assume that $Df(0,0)$ has three real eigenvalues $\lambda_{ss}, \lambda_s, \lambda_u$ with $\lambda_{ss} < \lambda_s < 0 < \lambda_u$. Write

$$\alpha = -\lambda_{ss}/\lambda_u, \qquad \beta = -\lambda_s/\lambda_u.$$

Note that α, β are the stable eigenvalues one obtains by a time rescaling that brings the unstable eigenvalue to 1. The eigenspaces corresponding to λ_{ss}, λ_s are called the strong and leading stable directions. The equilibrium at the origin possesses thus a one dimensional unstable manifold $W^u(0)$ consisting of points whose negative orbit converges to the origin and a two dimensional stable manifold $W^s(0)$ consisting of points whose positive orbit converges to the origin. Within the stable manifold there is a one dimensional strong stable manifold $W^{ss}(0)$ consisting of points whose positive orbit converges at a rate $\sim e^{-\alpha t}$ to the origin. Locally near the origin one can construct center unstable manifolds $W^{s,u}(0)$, tangent to the sum of the leading stable direction and the unstable direction at the origin. Such manifolds are not unique, but are at least C^1 [11] and possess a unique tangent bundle along the unstable manifold [12].

We assume that for $\mu = 0$, a homoclinic orbit

$$\gamma = \{\gamma(t) \mid \lim_{t \to \pm\infty} \gamma(t) = 0\} \tag{3}$$

to the equilibrium at the origin exists. We assume that $\mathcal{R}\gamma$ is a second homoclinic orbit different from γ.

We treat orbit flip bifurcations characterized by the following hypothesis.

Hypothesis 1.1 (Orbit flip). *The homoclinic orbit γ is a generically unfolding orbit flip homoclinic orbit:*

1. *Orbit flip: the homoclinic orbit γ is contained in the strong stable manifold $W^{ss}(0)$.*
2. *No inclination flip: along the homoclinic orbit γ, $W^{s,u}(0)$ is transverse to $W^s(0)$.*
3. *Generic unfolding: the traces of the parameter dependent manifolds $W^{ss}(0)$ and $W^u(0)$ in the product of state space and parameter space $\mathbb{R}^3 \times \mathbb{R}^2$, intersect transversally along $(\gamma, 0)$.*

Let Σ be a smooth cross section transverse to γ. The generic unfolding condition can also be formulated as follows: the map assigning the difference vector in Σ between the two points $W^u(0) \cap \Sigma$ and $W^{ss}(0) \cap \Sigma$

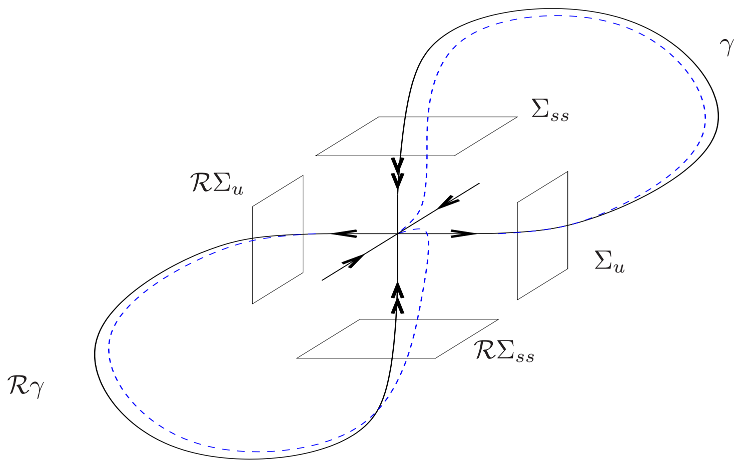

Figure 1: *A double homoclinic loop at an orbit flip bifurcation. The dashed curves indicate a double homoclinic loop in a butterfly configuration bifurcating from the orbit flip. Cross sections as introduced in § 2.2 are also indicated.*

(assuming that Σ is contained in a linear subspace) is locally injective at $\mu = 0$. We will use the unfolding condition to reparameterize the parameter plane so that, given suitable coordinates on Σ, in new parameters $\mu = (\mu_1, \mu_2)$ we can write $W^{ss}(0) \cap \Sigma = (0, 0)$, $W^u(0) \cap \Sigma = (\mu_2, \mu_1)$ and a homoclinic orbit exists for $\{\mu_2 = 0\}$.

Depending on the dimension of the fixed point space of the involution $\mathcal{R}$ (which is either 0 if $\mathcal{R}$ is a reflection against the origin or 1 if $\mathcal{R}$ is a reflection against a line), there are two different unfoldings of the symmetric orbit flip homoclinic loop. We restrict our study to the case of one dimensional fixed point space $\mathrm{Fix}(\mathcal{R}) = \{u \mid \mathcal{R}u = u\}$, this is the case where Lorenz like attractors occur.

Hypothesis 1.2 (Symmetry)**.** *We assume equivariance with respect to a linear involution $\mathcal{R}$ for which*

$$\dim \mathrm{Fix}(\mathcal{R}) = 1.$$

Note that γ and $\mathcal{R}\gamma$ form a figure eight, meaning

$$\mathcal{R} \lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|} = - \lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|}, \qquad \mathcal{R} \lim_{t\to-\infty} \frac{\gamma(t)}{\|\gamma(t)\|} = - \lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|}.$$

It is a consequence of Hypothesis 1.2 that perturbed homoclinic orbits that are no longer forming an orbit flip, are in a butterfly configuration (see Figure 1); still writing γ and $\mathcal{R}\gamma$ for the perturbed homoclinic orbits this means

$$\mathcal{R} \lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|} = \lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|}, \qquad \mathcal{R} \lim_{t\to-\infty} \frac{\gamma(t)}{\|\gamma(t)\|} = - \lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|}.$$

In contrast, if $\dim \mathrm{Fix}(\mathcal{R}) = 0$, then perturbed homoclinic orbits are still forming a figure eight.

We consider eigenvalue conditions for which the unfolding of a single orbit flip homoclinic orbit gives rise to a homoclinic doubling bifurcation [14, 30].

Hypothesis 1.3 (Eigenvalue conditions)**.** *Consider the following eigenvalue conditions:*

$$\alpha > 1, \qquad \frac{1}{2} < \beta < 1.$$

The following result is our main theorem, discussing bifurcations from a codimension two orbit flip bifurcation of two symmetric homoclinic orbits and the occurrence of Lorenz like attractors. The following sections contain further discussions, elaborating on the bifurcation result.

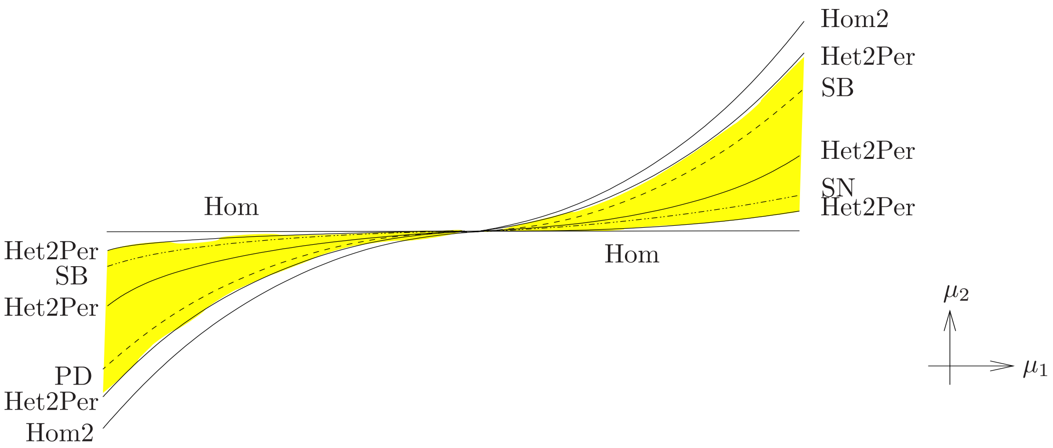

Figure 2: *Bifurcation diagram for Theorem 1.1.*

Theorem 1.1. *Let $\dot{x} = f(x, \mu)$, $\mu \in \mathbb{R}^2$, be a smooth two parameter family of ODEs on $\mathbb{R}^3$ for which Hypotheses 1.1, 1.2, 1.3 are met. Up to a reparameterization of the parameter plane, the bifurcation diagram is as in the following description referring to Figure 2.*

The continuation of the homoclinic orbits γ and $\mathcal{R}\gamma$ exists along $\mu_2 = 0$. Both in $\mu_1 < 0$ and $\mu_1 > 0$ there are curves Hom2 *of homoclinic orbits branching from the codimension two bifurcation point: in $\mu_1 < 0$ with double homoclinic loops following γ (or $\mathcal{R}\gamma$ by symmetry) twice before closing, in $\mu_1 > 0$ with homoclinic loops following near $\gamma \cup \mathcal{R}\gamma$ before closing. Further bifurcation curves are indicated by* SN *(Saddle-node bifurcations of periodic orbits),* SB *(Symmetry-breaking bifurcations of periodic orbits),* PD *(Period-doubling bifurcations of periodic orbits) and* Het2Per *(Heteroclinic connections from the origin to periodic orbits). The bifurcation curves are contained in a wedge*

$$|\mu_2| \leqslant C|\mu_1|^{\frac{1}{1-\beta}}$$

for some $C > 0$.

A Lorenz like attractor exists inside a wedge that is contained in the depicted filled-in region.

We conjecture that Lorenz like attractors exist inside the entire region that is filled-in in the figure (i.e. between two heteroclinic bifurcation curves of heteroclinic connections from the equilibrium to periodic orbits). Lacunae (see [1]) in the Lorenz like attractors are formed along the heteroclinic bifurcation curve Het2Per inside the filled-in region. Possibly other similar bifurcation curves exist, depending on the value of β. This will be discussed in terms of reduced interval maps below.

By singular rescalings, a first return map on a cross section contains small perturbations from interval maps. The interval maps are of the form

$$x \mapsto \operatorname{sign}(x)(\bar{\mu}_2 + q_1|x|^\beta),$$

for varying $\bar{\mu}_2$ (related to μ_2 through a rescaling). The coefficient q_1 has the same sign as μ_1, so that reductions to interval maps both with positive and negative slope occur.

This paper, and in particular the proof of Theorem 1.1, is organized as follows. Finding asymptotic expressions for a first return map on a cross section, is facilitated by a local normal form. The derivation of the local normal form is given in § 2. This section also gives the asymptotic expressions for a first return map, see § 2.2. In § 3 we study renormalizations to small perturbations of interval maps. The construction of continuously differentiable stable foliations for the first return map provides a rigorous dimension reduction to interval maps and enables an existence proof of Lorenz like attractors. Finally, we study bifurcations of periodic orbits to arrive at the bifurcation diagram. For this one cannot rely on the reduction to interval maps using a stable foliation, as the stable foliation is only continuously differentiable and higher order differentiability is required for the bifurcation study. One can however use a Lyapunov-Schmidt reduction to obtain reduced bifurcation equations. The construction and analysis of the reduced bifurcation equations is in § 4.

2 Normal forms and return maps

We arrive at the main theorems, yielding bifurcation curves and the existence of Lorenz like attractors in the unfolding of the symmetric orbit flip, through the derivation and analysis of a return map on a cross section. A crucial role lies in the derivation of asymptotic expansions for such a return map. We remark that we do not assume conditions on eigenvalues other than Hypothesis 1.3. In particular we do not apply linearization results which hold only generically (see e.g. [15]). As a consequence there is no need to check for nonresonance conditions on the eigenvalues in order to apply our results.

In the following section, § 2.1, we construct a local normal form, close enough to a linear vector field to enable the derivation of suitable asymptotic expansions for a local transition map. This, as well as asymptotic expansions for a first return map, are considered in § 2.2 below.

2.1 Local normal forms

Take coordinates $u = (x_u, x_s, x_{ss})$ so that $\dot{u} = f(u, \mu)$ is given by a set of ordinary differential equations of the following form:

$$\left\{ \begin{array}{rcl} \dot{x}_u & = & x_u + F_u(x_u, x_s, x_{ss}; \mu), \\ \dot{x}_s & = & -\beta x_s + F_s(x_u, x_s, x_{ss}; \mu), \\ \dot{x}_{ss} & = & -\alpha x_{ss} + F_{ss}(x_u, x_s, x_{ss}; \mu) \end{array} \right. \tag{4}$$

where $D^i F_j(0; \mu) = 0$ for $i = 0, 1$ and $j = u, s, ss$. We may assume

$$\mathcal{R}(x_u, x_s, x_{ss}) = (-x_u, x_s, -x_{ss}). \tag{5}$$

A natural approach to studying bifurcations of homoclinic orbits is through the construction of Poincaré return maps, given as compositions of local transition maps (using the flow near the origin) and global transition maps. The main technical obstacle lies in finding expansions for the flow near the equilibrium at the origin. The computation of a local transition map, through estimates of integrals appearing in variation of constants formulae, is facilitated by the change to a normal form.

Proposition 2.1. *The system of differential equations* (4)*, equivariant under the action of* (5)*, is smoothly equivalent to a system of the same expression with*

$$\begin{array}{rcl} F_u & = & 0, \\ F_s & = & \mathcal{O}(|x_u||x_{ss}, x_s|), \\ F_{ss} & = & \mathcal{O}(|x_u||x_{ss}, x_s|). \end{array}$$

The equivalence is by multiplication by a function and a conjugacy by a diffeomorphism depending smoothly on μ. The equivalence varies smoothly with parameters and leaves the action of the symmetry unaltered.

Proof. By a smooth coordinate change near the origin we may assume that local stable and unstable manifolds are linear:

$$W^s_{\mathrm{loc}}(0) \subset \{x_u = 0\} \tag{6}$$

and

$$W^u_{\mathrm{loc}}(0) \subset \{x_s, x_{ss} = 0\}. \tag{7}$$

By (6), $F_u = \mathcal{O}(x_u)$, so that multiplying the vector field with the positive function

$$\frac{x_u}{x_u + F_u(x_u, x_s, x_{ss}; \mu)}$$

brings the equation for $\dot{x}_u$ to the given form.

Note that (7) ensures that F_s and F_{ss} are of order $\mathcal{O}(|x_s, x_{ss}|)$. Consider the differential equations restricted to the stable manifold $\{x_u = 0\}$. Choosing coordinates in which the strong stable foliation is affine, makes that the equation for $\dot{x}_s$ depends only on x_s. Since one-dimensional vector fields near a sink can always be smoothly linearized, a smooth coordinate change gives $F_s(0, x_s, x_{ss}; \mu) = 0$. From equivariance of the differential equations, the obtained equations are of the form

$$\dot{x}_s = -\beta x_s, \qquad \dot{x}_{ss} = -\alpha x_{ss} + x_{ss} f(x_s, x_{ss}; \mu)$$

We go on to remove the term $x_{ss} f(x_s, x_{ss}; \mu)$. Consider for this a smooth change of variables

$$y_s = x_s, \qquad y_{ss} = x_{ss} + x_{ss} q(x_s, x_{ss}; \mu).$$

for a function q which satisfies $q(x_s, -x_{ss}; \mu) = q(x_s, x_{ss}; \mu)$. Calculating derivatives with respect to time t, we have

$$\dot{y}_s = -\beta y_s, \qquad \dot{y}_{ss} = -\alpha y_{ss} + (1+q)^{-1} y_{ss}(f + qf + \dot{q}).$$

In order to arrive at our aim, $F_{ss}(0, x_s, x_{ss}; \mu) = 0$, consider the following equations, treating q as a variable,

$$\dot{y}_s = -\beta y_s, \qquad \dot{y}_{ss} = -\alpha y_{ss} \qquad \dot{q} = -qf - f.$$

The eigenvalues of the linearized equations about $(y_s, y_{ss}, q) = (0, 0, 0)$, are $-\beta, -\alpha, 0$. Thus q can be obtained from the two-dimensional stable manifold of the above system. The expressions for F_s and F_{ss} from the statement of the proposition follow. □

Remark 2.1. *Note that the normal form restricted to the local stable manifold is linear. Symmetry is essential for this normal form. Formal computations show that the proposition is not valid in the following example when* $\alpha = 2\beta$*:*

$$\dot{x}_u = x_u, \qquad \dot{x}_s = -\beta x_s, \qquad \dot{x}_{ss} = -\alpha x_{ss} + x_s^2.$$

This system of differential equations does in particular not possess a smooth weak stable manifold; any weak stable manifold is only C^1*. Symmetry however forces the existence of a smooth weak stable manifold, namely the symmetry axis.*

2.2 Transition maps

Following a discussion of a local transition map for the flow near the origin, we give asymptotic expansions for a first return map in Theorem 2.1 below.

For some small positive δ, let

$$\Sigma_u = \{(\delta, x_s, x_{ss}) \mid |x_s|, |x_{ss}| < \delta\}, \qquad \Sigma_{ss} = \{(x_u, x_s, \delta) \mid |x_u|, |x_s| < \delta\}$$

be cross sections which intersect the homoclinic orbit γ transversally. By a linear rescaling we may assume that $\delta = 1$. The rescaling makes the higher order terms in the differential equations of order δ:

$$|F_{ss}(u)|, |F_s(u)| \leqslant C\delta |u|^2, \tag{8}$$

for some $C > 0$. Put

$$\Sigma^{in} = \Sigma_{ss} \cup \mathcal{R}\Sigma_{ss}, \qquad \Sigma^{out} = \Sigma_u \cup \mathcal{R}\Sigma_u,$$

see Figure 1 for an illustration.

Denote by $\Pi_{\text{loc}} : \Sigma^{\text{in}} \to \Sigma^{\text{out}}$ the local transition map. The following proposition provides exponential expansions for Π_{loc} in coordinate charts from the normal form given in Proposition 2.1. Note that the leading terms of the expansions equal the exact formulae for a locally linear vector field.

Proposition 2.2. *For $(x_s^{out}, x_{ss}^{out}) = \Pi_{loc}(x_u^{in}, x_s^{in})$ and μ near 0, the following asymptotic formulae apply. If $(x_u^{in}, x_s^{in}) \in \Sigma_{ss}$, then*

$$
\begin{aligned}
x_s^{out} &= |x_u^{in}|^\beta x_s^{in} + |x_u^{in}|^{\beta+\omega} x_s^{in} \varphi_s^1(x_u^{in}, x_s^{in}, \mu) + |x_u^{in}|^{1+\omega} \varphi_s^2(x_u^{in}, x_s^{in}, \mu), \\
x_{ss}^{out} &= |x_u^{in}|^\alpha + |x_u^{in}|^{\alpha+\omega} \varphi_{ss}(x_u^{in}, x_s^{in}, \mu),
\end{aligned}
$$

for some $\omega > 0$. Here $\varphi_s^1, \varphi_s^2, \varphi_{ss}$ are smooth functions for $x_u^{in} \neq 0$. Similar asymptotics hold for $(x_u^{in}, x_s^{in}) \in \mathcal{R}\Sigma_{ss}$ given by symmetry.

Proof. Write $(x_u(v), x_s(v), x_{ss}(v))$ for the solutions to (4) (with higher order terms given by Proposition 2.1). We have, for $0 \leqslant v \leqslant \tau$,

$$x_u(v) = \operatorname{sign}(x_u) e^{-(\tau - v)}.$$

With $x_{ss}(0) = \pm 1$ and writing, for this proof, $x_s(0) = \xi_s$, the variation of constants formula gives, for $0 \leqslant v \leqslant \tau$,

$$
\begin{aligned}
x_s(v) &= e^{-\beta v} \xi_s + \int_0^v e^{-\beta(v-\zeta)} F_s(x_u(\zeta), x_s(\zeta), x_{ss}(\zeta); \mu) d\zeta, \\
x_{ss}(v) &= e^{-\alpha v} + \int_0^v e^{-\alpha(v-\zeta)} F_{ss}(x_u(\zeta), x_s(\zeta), x_{ss}(\zeta); \mu) d\zeta.
\end{aligned}
$$

Write this as $(x_{ss}, x_s) = \Gamma(x_{ss}, x_s)$. For $\omega > 0$, consider a set B_K of continuous functions $(x_{ss}, x_s) : [0, \tau] \to \mathbb{R}^2$ such that $|x_{ss}(v)| e^{(\beta+\omega)v} \leqslant K$, $|x_s(v)| e^{\beta v} \leqslant K \xi_s$.

Direct estimates, making use of (8), show that for $\omega < \min(\beta, \alpha - \beta, 1)$ and some $K > 0$, Γ maps B_K into itself. This implies

$$
\begin{aligned}
x_s(v) &= e^{-\beta v} \xi_s + \mathcal{O}(e^{-(\beta+\omega)v}), \\
x_{ss}(v) &= e^{-\alpha v} + \mathcal{O}(e^{-(\beta+\omega)v}),
\end{aligned}
$$

Plug in $\tau = -\ln |x_u|$ to obtain the end points in Σ^{out}.

One treats derivatives with respect to τ, ξ_s and parameters μ by differentiating the integral formulas and studying them as above. This leads to the following statement. For $k \geqslant 0$, there are positive constants C_k such that, for $0 \leqslant t \leqslant \tau$ and μ near μ_0,

$$
\begin{aligned}
\left|\frac{\partial^k}{\partial(t, \xi_s; \mu)^k} x_s(t, \tau, \xi_s; \mu)\right| &\leqslant C_k e^{-\beta t}, \\
\left|\frac{\partial^k}{\partial(t, \xi_s; \mu)^k} x_{ss}(t, \tau, \xi_s; \mu)\right| &\leqslant C_k e^{-\alpha t}, \\
\left|\frac{\partial^k}{\partial(t, \xi_s; \mu)^k} \frac{\partial}{\partial \tau} x_s(t, \tau, \xi_s; \mu)\right| &\leqslant C_k e^{-\beta t + (t - \tau)}, \\
\left|\frac{\partial^k}{\partial(t, \xi_s; \mu)^k} \frac{\partial}{\partial \tau} x_{ss}(t, \tau, \xi_s; \mu)\right| &\leqslant C_k e^{-\alpha t + (t - \tau)}.
\end{aligned}
$$

Similar estimates yield expansions in case $\xi_s = 0$, proving the proposition. □

The global transition map $\Pi_{\text{far}} : \Sigma_u \to \Sigma_{ss}$ is a local diffeomorphism by the flow box theorem. By the generic unfolding condition in Hypothesis 1.1, we may by applying a reparameterization assume that $\Pi_{\text{far}}(0, 0) = (\mu_2, \mu_1)$. Write

$$
\begin{aligned}
\Pi_{\text{far}}^u(x_s, x_{ss}; \mu) &= \mu_2 + q_1(\mu) x_s + r_1(\mu) x_{ss} + f_1(x_s, x_{ss}; \mu), \\
\Pi_{\text{far}}^s(x_s, x_{ss}; \mu) &= \mu_1 + q_2(\mu) x_s + r_2(\mu) x_{ss} + f_2(x_s, x_{ss}; \mu),
\end{aligned}
$$

where f_1 and f_2 are quadratic and higher order terms. In the sequel we suppress the dependence of q_1, q_2, r_1 and r_2 on μ. Also note that $\mu_2 = 0$ if and only if there are homoclinic orbits to the origin and $(\mu_1, \mu_2) = (0, 0)$

if and only if there are orbit-flip homoclinic orbits to the origin. Symmetry induces a global transition map $\mathcal{R}\Pi_{\text{far}}$ defined on $\mathcal{R}\Sigma_u$ with values in $\mathcal{R}\Sigma_{ss}$.

The return map Π on Σ^{in} is obtained by composing the global with the local transition maps. The following result is obtained, being deliberately unprecise in the notation by making no difference between Σ_{ss} and $\mathcal{R}\Sigma_{ss}$.

Theorem 2.1. *There are smooth coordinates (x_u, x_s) on Σ^{in}, such that with a smooth reparametrization of the parameter plane, the Poincaré return map has the following asymptotic expansions:*

$$\Pi(x_u, x_s) = \begin{pmatrix} \operatorname{sign}(x_u)\mu_2 + \operatorname{sign}(x_u) q_1 |x_u|^\beta x_s + \mathcal{O}(|x_u|^{\beta+\omega} x_s) + \mathcal{O}(|x_u|^{1+\omega}) \\ \mu_1 + q_2 |x_u|^\beta x_s + \mathcal{O}(|x_u|^{\beta+\omega} x_s) + \mathcal{O}(|x_u|^{1+\omega}) \end{pmatrix},$$

where ω is some positive number.

3 Lorenz like attractors

A renormalization shows how Lorenz like return maps, small perturbations of interval maps embedded in planar maps, arise in the study of the orbit-flip. As in [26, 29] the occurrence of Lorenz like attractors can be studied by constructing a stable foliation and examining the reduced interval maps.

Consider rescaled coordinates $\bar{x} = (\bar{x}_s, \bar{x}_u)$ given by

$$x_u = |\mu_1|^{\frac{1}{1-\beta}} \bar{x}_u, \qquad x_s = |\mu_1|^d \bar{x}_s + \mu_1, \tag{9}$$

with $1 < d < \frac{\beta}{1-\beta}$. Note that this requires $\beta > 1/2$. This coordinate change on Σ^{in} becomes singular for $\mu_1 = 0$. Define a rescaled parameter $\bar{\mu}_2$ by

$$\bar{\mu}_2 = \mu_2 |\mu_1|^{\frac{1}{\beta-1}}.$$

Let $\bar{\Sigma}^{in} \subset \Sigma^{in}$ be regions, depending on μ_2 and μ_1, on which $\bar{x} = (\bar{x}_u, \bar{x}_s)$ and $\bar{\mu}_2$ are bounded. We denote the return map in rescaled coordinates (the renormalized return map) by $(\bar{x}_u, \bar{x}_s) \to \bar{\Pi}(\bar{x}_u, \bar{x}_s, \bar{\mu}_2, \mu_1)$. Write also $\bar{\Pi} = (\bar{\Pi}_s, \bar{\Pi}_u)$. From Theorem 2.1 we obtain

$$\bar{\Pi}(\bar{x}_u, \bar{x}_s, \mu_2, \mu_1) = \begin{pmatrix} \operatorname{sign}(\bar{x}_u)(\bar{\mu}_2 + q_1 |\bar{x}_u|^\beta) + \mu_1^\kappa k_1(\bar{x}_u, \bar{x}_s, \bar{\mu}_2, \mu_1) \\ \mu_1^\kappa k_2(\bar{x}_u, \bar{x}_s, \bar{\mu}_2, \mu_1) \end{pmatrix} \tag{10}$$

where κ is a positive constant. We obtain convergence to a one-dimensional map when $\mu_1 \to 0$.

Proposition 3.1. *For $\mu_1 \to 0$, the Poincaré return map $\bar{\Pi}$ converges to*

$$\bar{\Pi}_0(\bar{x}_u, \bar{x}_s, \bar{\mu}_2, \mu_1) = \begin{pmatrix} \operatorname{sign}(\bar{x}_u)(\bar{\mu}_2 + q_1 |\bar{x}_u|^\beta) \\ 0 \end{pmatrix}.$$

Note that an additional rescaling of the $\bar{x}_u$ variable brings q_1 to ± 1. The reduction to an interval map is completed by the construction of C^1 invariant stable foliations for $\bar{\Pi}$. Here, a C^1 foliation is a foliation whose (in our case one dimensional) leaves can be mapped into straight lines by a C^1 coordinate change. Existence of C^1 stable foliations in related contexts is studied in [12, 25, 26, 31]. We follow [31].

Proposition 3.2. *The Poincaré return map Π admits a C^1 foliation $\mathcal{F}^s$.*

Proof. This essentially follows from [31]. For applying [31] there are two issues to be resolved. In our setting there are two cross sections, instead of a single cross section. It is easily seen that this does not pose a problem; one can think of the two cross sections being connected to form a single cross section again.

The main result in [31] assumes a form for the map in which the coefficients of the lowest order terms $\bar{x}_u^\beta$ are constant, i.e. not depending on $\bar{x}_s$. In our case the coefficients do depend on $\bar{x}_s$ (through terms in the functions k_1, k_2). Additional coordinate changes remove this dependence (compare also [23]). Indeed, a coordinate change of the form $\bar{x}_s \mapsto \bar{x}_s + \bar{x}_u a(\bar{x}_s), \bar{x}_u \mapsto \bar{x}_u$ for some function a removes terms with a factor $\bar{x}_u^\beta$ from the equation for $\bar{\Pi}_s$. A further coordinate change of the form $\bar{x}_s \mapsto \bar{x}_s, \bar{x}_u \mapsto \bar{x}_u b(\bar{x}_s)$ for some function b removes terms with a factor $\bar{x}_u^\beta \bar{x}_s$ from the equation for $\bar{\Pi}_u$. □

Identifying points on leaves of $\mathcal{F}^s$ induces a one dimensional map π which is C^1 close to

$$f : [-1,1] \setminus \{0\} \to \mathbb{R}, \qquad f(\bar{x}) = \operatorname{sign}(\bar{x})(\bar{\mu}_2 + q_1|\bar{x}|^\beta) \tag{11}$$

(the projection along leaves of $\mathcal{F}^s$ is C^1 close to the coordinate projection $(\bar{x}_{ss}, \bar{x}_u) \mapsto \bar{x}_u$ if μ_1 is small, compare [23]). The existence of an invariant interval I on which $|\pi'| > 1$ implies the existence of a Lorenz like attractor [3, 6, 7, 22]. It suffices to consider f as one can take μ_1 as small as required. Suppose $q_1 > 0$. A direct computation shows that $|f'| > 1$ on the invariant interval $[\bar{\mu}_2, -\bar{\mu}_2]$ for $-(q_1\beta)^{\frac{1}{1-\beta}} < \bar{\mu}_2 < -(\frac{q_1}{2})^{\frac{1}{1-\beta}}$. Likewise, if $q_1 < 0$, then for $(-\frac{q_1}{2})^{\frac{1}{1-\beta}} < \bar{\mu}_2 < (-q_1\beta)^{\frac{1}{1-\beta}}$ we find $|f'| > 1$ on $[-\bar{\mu}_2, \bar{\mu}_2]$.

This interval map f, as well as the related continuous map

$$\tilde{f}(\bar{x}) = \begin{cases} -f(\bar{x}), & \bar{x} < 0, \\ f(\bar{x}), & \bar{x} > 0, \end{cases}$$

occur in other bifurcation problems as well, see e.g. [23, 24]. Note that a periodic point of period k for f is a periodic point of period k or $2k$ for $\tilde{f}$, and vice versa. In [23] an open subinterval of $(\frac{1}{2}, 1)$ of values for β is given for which it is proved that a Lorenz like attractor exists for all parameter in the filled-in region in Theorem 1.1 (see also Figure 3).

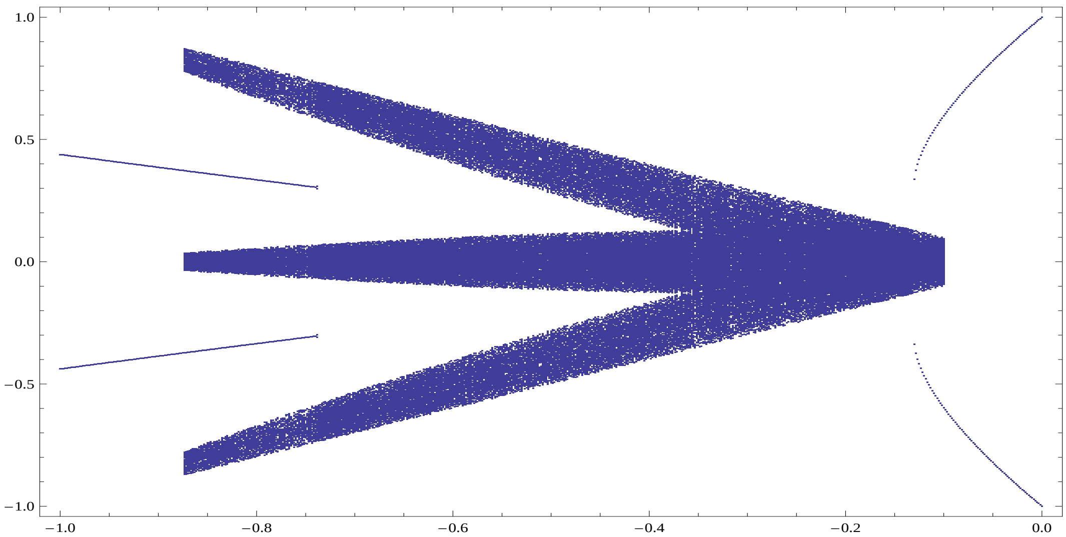

Figure 3: *Numerical computation of the attractors inside* $[-,1,1]$ *for the interval map* f *as in* (11) *for varying parameter* $\bar{\mu}_2$ *on the horizontal axis. Here* $\beta = 0.7$*, details of the bifurcation diagram depend on* β*.*

4 Bifurcation curves

The analysis of bifurcations of periodic orbits such as saddle-node bifurcations or symmetry-breaking bifurcations, requires more differentiability than C^1 and can thus not be done using reduced interval maps. Note further that we have not included parameter dependence in the discussion of stable foliations. The asymptotic expansions for the (rescaled) return map allow one to set-up the bifurcation equations and derive reduced bifurcation equations from a Lyapunov-Schmidt procedure. These are then easily studied, following an analysis along the lines of e.g. [8, 13, 23, 30]. Here we indicate the construction of reduced bifurcation equations, in § 4.1 below we consider the bifurcation curves from the bifurcation theorem. As a similar analysis can be found in other papers, like the ones just mentioned, we will not include full details.

For $0 \leqslant j < N$, let

$$\Psi_j = \begin{pmatrix} x_u^{in,j+1} \\ x_s^{in,j+1} \end{pmatrix} - \Pi(x_u^{in,j}, x_s^{in,j}; \mu).$$

The existence of N-periodic and N-homoclinic orbits is reduced to solving a set of equations

$$\Psi(\mathbf{x}_u, \mathbf{x}_s; \mu) = 0, \tag{12}$$

where $\Psi = (\Psi_0, \ldots, \Psi_{N-1})$, $\mathbf{x}_u = (x_u^{in,0}, \ldots, x_u^{in,N-1})$, $\mathbf{x}_s = (x_s^{in,0}, \ldots, x_s^{in,N-1})$ and the indices are taken modulo N. Note that for an N-periodic orbit $x_u^{in,j} \neq 0$ for all j. Also $x_u^{in,N} = x_u^{in,0} = 0$ and $x_u^{in,j} \neq 0$ for $0 < j \leqslant N-1$ yields an N-homoclinic orbit.

It is easy to see that with $\mathbf{x}_u = (x_u^{in,0}, \ldots, x_u^{in,N-1})$, $\mathbf{x}_s = (x_s^{in,0}, \ldots, x_s^{in,N-1})$,

$$rank(L := D_{\mathbf{x}_s}\Psi|_{\mathbf{x}_u=0}) = N.$$

Following the setup of of Lyapunov-Schmidt reduction method, split (12) into an equivalent pair of equations

$$P\Psi(\mathbf{x}_u, \mathbf{x}_s; \mu) = 0, \qquad (I-P)\Psi(\mathbf{x}_u, \mathbf{x}_s; \mu) = 0, \tag{13}$$

where P is the orthogonal projection onto the image Im $D_{\mathbf{x}_s}\Psi|_{\mathbf{x}_u=0}$; i.e.

$$\begin{aligned} P : \mathbb{R}^{2N} &\rightarrow \text{Im } D_{\mathbf{x}_s}\Psi|_{\mathbf{x}_u=0} \subset \mathbb{R}^N \\ (\mathbf{x}_u, \mathbf{x}_s) &\mapsto (0, \mathbf{x}_s). \end{aligned}$$

Applying the Implicit Function Theorem to solve the first part of 13 for $\mathbf{x}_s$ as a function of $\mathbf{x}_u$ and μ leads to the following lemma:

Lemma 4.1. *The equation $P\Psi = 0$ can be solved for $\boldsymbol{x}_s$ as a function of $\boldsymbol{x}_u$ and μ. Here $\boldsymbol{x}_s(\mathbf{0}, \mu) = \mu_1$ and the following estimate holds for $k, l \geqslant 0$:*

$$\left\| \frac{\partial^{k+l}}{\partial^k \mu \partial^l \boldsymbol{x}_u} \boldsymbol{x}_s(\boldsymbol{x}_u, \mu) - \mu_1 \right\| \leqslant C_{k+l} \|\boldsymbol{x}_u\|^{\beta - l} \tag{14}$$

Proof. The remarks preceding the lemma proves the existence, uniqueness and smoothness of $\mathbf{x}_s(\mathbf{x}_u, \mu)$. Estimate 14 follows from the asymptotic expansion for the rescaled return map in (10). □

We substitute $\mathbf{x}_s(\mathbf{x}_u, \mu)$ into the second part of (13) to obtain the reduced bifurcation equation

$$\Phi(\mathbf{x}_u, \mu) = (I-P)\Psi(\mathbf{x}_u, \mathbf{x}_s(\mathbf{x}_u, \mu); \mu) = 0, \tag{15}$$

with $\Phi : \ker L \times \mathbb{R}^2 \rightarrow \mathbb{R}^N$. This gives the following result.

Proposition 4.1. *With the above notations, the reduced bifurcation equations have the expansion*

$$x_u^{in,j+1} = \text{sign}(x_u^{in,j})(\mu_2 + q_1\mu_1|x_u^{in,j}|^\beta) + U_j(\boldsymbol{x}_u, \mu), \tag{16}$$

for $0 \leqslant j < N$. The function U_j is smooth for $x_u^{in,j} \neq 0$, $0 \leqslant j < N$ and

$$\left\| \frac{\partial^{k+l}}{\partial^k \mu \partial^l \boldsymbol{x}_u} U_j(\boldsymbol{x}_u, \mu) \right\| \leqslant C_{k+l} \|\boldsymbol{x}_u\|^{\upsilon - l},$$

for some $\upsilon > \beta$

4.1 Bifurcation analysis

We proceed with an analysis of bifurcation curves. Bifurcations of periodic orbits are treated using the reduced bifurcation equation 16. We restrict to an analysis of bifurcation curves indicated in Theorem 1.1 and do for instance not prove that no additional bifurcations occur outside the wedge around the μ_1 axis indicated in Theorem 1.1. Compare [13, Appendix].

Homoclinic bifurcations: Obviously, at $\mu_2 = 0$ the reduced bifurcation equation

$$x_u = \mathrm{sign}(x_u)(\mu_2 + q_1\mu_1|x_u|^\beta) + O(|x_u|^\upsilon)$$

has the solution $x_u = 0$ for all μ_1, so along the line $\mu_2 = 0$ a couple of symmetrical 1-homoclinic orbits exists. Suppose for simplicity that $q_1 < 0$ and $\mu_1 > 0$. For 2-homoclinic orbits we solve the equations

$$\begin{aligned} x_u^{in,1} &= \mathrm{sign}(x_u^{in,0})(\mu_2 + q_1\mu_1|x_u^{in,0}|^\beta) + U_0(x_u^{in,0}, x_u^{in,1};\mu), \\ x_u^{in,0} &= \mathrm{sign}(x_u^{in,1})(\mu_2 + q_1\mu_1|x_u^{in,1}|^\beta) + U_1(x_u^{in,0}, x_u^{in,1};\mu), \end{aligned}$$

with $x_u^{in,0} = 0$ and $x_u^{in,1} \neq 0$. Then, for say $x_u^{in,1} > 0$,

$$\begin{aligned} x_u^{in,1} &= \mu_2 \\ 0 &= \mu_2 + q_1\mu_1|\mu_2|^\beta + U_1(0,\mu_2;\mu), \end{aligned}$$

and therefore

$$\mu_2 = |q_1\mu_1|^{\frac{1}{1-\beta}} + o((q_1\mu_1)^{\frac{1}{1-\beta}}).$$

Saddle node bifurcations: Suppose for simplicity that $q_1 < 0$ and $\mu_1 > 0$. To compute the curve of parameter values with a saddle-node bifurcation of a periodic orbit, consider the bifurcation equations

$$\begin{aligned} x_u &= \mathrm{sign}(x_u)(\mu_2 + q_1\mu_1|x_u|^\beta) + U_0(x_u,\mu), \\ 1 &= \mathrm{sign}(x_u)\beta q_1\mu_1|x_u|^{\beta-1} + U_0'(x_u,\mu). \end{aligned}$$

Solving these equations, see [8], we obtain

$$\mu_2 = |q_1\mu_1|^{\frac{1}{1-\beta}}(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}) + o((q_1\mu_1)^{\frac{1}{1-\beta}}).$$

Symmetry breaking and period doubling bifurcations: Consider $q_1 > 0$. A symmetry breaking bifurcation occurs if

$$\begin{aligned} x_u &= -\mu_2 - q_1\mu_1|x_u|^\beta + O(|x_u|^\upsilon), \\ -1 &= -\beta q_1\mu_1|x_u|^{\beta-1} + O(|x_u|^{\upsilon-1}). \end{aligned}$$

Solving these equations, we get

$$\mu_2 = -(q_1\mu_1)^{\frac{1}{1-\beta}}(\beta^{\frac{1}{1-\beta}} + \beta^{\frac{\beta}{1-\beta}}) + o((q_1\mu_1)^{\frac{1}{1-\beta}}).$$

Note that this gives the bifurcation curve, and does not entail an analysis of the unfolding (compare [8, 13, 30]). In a similar way one treats the curve of periodic doubling bifurcations when $q_1 < 0$.

Heteroclinic connections to periodic orbits: For the analysis of bifurcation curves with heteroclinic connections between the origin and hyperbolic periodic orbits, we refer to [24]. Note that here estimates on first order derivatives suffice and knowledge on higher order derivatives is not required. One can study the connections for the limit map given in Proposition 3.1 and conclude their existence for small values of μ_1.

5 Lorenz like attractors from inclination flip bifurcations

Inclination-flip homoclinic orbits in $\mathbb{Z}_2$ symmetric differential equations can be studied in much the same way, using the normal form and renormalization as in [13, Appendix] (where inclination-flips without symmetry are considered). This gives an approach to Rychliks results including a bifurcation analysis. We merely state the results.

Hypothesis 5.1 (Inclination flip)**.** *The homoclinic orbit γ is a generically unfolding inclination flip homoclinic orbit:*

1. *Inclination flip: Along the homoclinic orbit γ for $\mu = 0$, $W^{s,u}(0)$ is tangent to $W^s(0)$.*
2. *No orbit flip: for $\mu = 0$, the homoclinic orbit γ is not contained in the strong stable manifold $W^{ss}(0)$.*
3. *Generic unfolding: the traces of the parameter dependent manifolds $W^{s,u}(0)$ and $W^s(0)$ in the product of state space and parameter space $\mathbb{R}^3 \times \mathbb{R}^2$, intersect transversally along $(\gamma, 0)$.*

Theorem 5.1. *Let $\dot{x} = f(x, \mu)$, $\mu \in \mathbb{R}^2$, be a smooth two parameter family of ODEs on $\mathbb{R}^3$ for which Hypotheses 5.1, 1.2, 1.3 are met. Up to a reparameterization of the parameter plane, the bifurcation diagram is as in the following description referring to Figure 2.*

The continuation of the homoclinic orbits γ and $\mathcal{R}\gamma$ exists along $\mu_2 = 0$. Both in $\mu_1 < 0$ and $\mu_1 > 0$ there are curves Hom2 *of homoclinic orbits branching from the codimension two bifurcation point: in $\mu_1 < 0$ with double homoclinic loops following γ (or $\mathcal{R}\gamma$ by symmetry) twice before closing, in $\mu_1 > 0$ with homoclinic loops following $\gamma \cup \mathcal{R}\gamma$ close before closing. Further bifurcation curves are indicated by* SN *(Saddle-node bifurcations of periodic orbits),* SB *(Symmetry-breaking bifurcations of periodic orbits),* PD *(Period-doubling bifurcations of periodic orbits) and* Het2Per *(Heteroclinic connections from the origin to periodic orbits). The bifurcation curves are contained in a wedge*

$$|\mu_2| \leqslant C|\mu_1|^{\frac{1}{1-\beta}}$$

for some $C > 0$.

A Lorenz like attractor exists inside a wedge that is contained in the depicted filled-in region.

References

[1] V. S. Afraĭmovich, V. V. Bykov, and L. P. Shil′nikov. On structurally unstable attracting limit sets of Lorenz attractor type. *Trans. Mosc. Math. Soc.*, 1983(2):153–216, 1983.

[2] V. S. Afraĭmovich and Ya. B. Pesin. Dimension of Lorenz type attractors. In *Mathematical physics reviews, Vol. 6*, volume 6 of *Soviet Sci. Rev. Sect. C Math. Phys. Rev.*, pages 169–241. Harwood Academic Publ., Chur, 1987.

[3] J. F. Alves, J. L. Fachada, and J. Sousa Ramos. A condition for transitivity of Lorenz maps. In *Proceedings of the Eighth International Conference on Difference Equations and Applications*, pages 7–13. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[4] V. Araújo, M. J. Pacífico, E. R. Pujals, and M. Viana. Singular-hyperbolic attractors are chaotic. *Trans. Amer. Math. Soc.*, 361(5):2431–2485, 2009.

[5] S. Bautista and C. A. Morales. Existence of periodic orbits for singular-hyperbolic sets. *Mosc. Math. J.*, 6(2):265–297, 2006.

[6] Y. Choi. Attractors from one dimensional Lorenz-like maps. *Discrete Contin. Dyn. Syst.*, 11(2-3):715–730, 2004.

[7] Y. Choi. Topology of attractors from two-piece expanding maps. *Dyn. Syst.*, 21(4):385–398, 2006.

[8] S.-N. Chow, B. Deng, and B. Fiedler. Homoclinic bifurcation at resonant eigenvalues. *J. Dynam. Differential Equations*, 2(2):177–244, 1990.

[9] F. Dumortier, H. Kokubu, and H. Oka. A degenerate singularity generating geometric Lorenz attractors. *Ergodic Theory Dynam. Systems*, 15(5):833–856, 1995.

[10] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, 50:59–72, 1979.

[11] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds.* Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 583.

[12] A. J. Homburg. Global aspects of homoclinic bifurcations of vector fields. *Mem. Amer. Math. Soc.*, 121(578):viii+128, 1996.

[13] A. J. Homburg, H. Kokubu, and V. Naudot. Homoclinic-doubling cascades. *Arch. Rational Mech. Anal.*, 160(3):195–243, 2001.

[14] A. J. Homburg and B. Krauskopf. Resonant homoclinic flip bifurcations. *J. Dynam. Differential Equations*, 12(4):807–850, 2000.

[15] Yu. S. Il′yashenko and S. Yu. Yakovenko. Finitely-smooth normal forms of local families of diffeomorphisms and vector fields. *Russ. Math. Surv.*, 46(1):1–43, 1991.

[16] E. N. Lorenz. Deterministic non-periodic flow. *J. Atmos. Sci.*, 20:130–141, 1963.

[17] S. Luzzatto, I. Melbourne, and F. Paccaut. The Lorenz attractor is mixing. *Comm. Math. Phys.*, 260(2):393–401, 2005.

[18] C. A. Morales. The explosion of singular-hyperbolic attractors. *Ergodic Theory Dynam. Systems*, 24(2):577–591, 2004

[19] C. A. Morales, M. J. Pacífico, and E. R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math. (2)*, 160(2):375–432, 2004.

[20] C. A. Morales, M. J. Pacífico, and B. San Martín. Expanding Lorenz attractors through resonant double homoclinic loops. *SIAM J. Math. Anal.*, 36(6):1836–1861, 2005.

[21] C. A. Morales, M. J. Pacífico, and B. San Martín. Contracting Lorenz attractors through resonant double homoclinic loops. *SIAM J. Math. Anal.*, 38(1):309–332, 2006.

[22] C. A. Morales and E. R. Pujals. Singular strange attractors on the boundary of Morse-Smale systems. *Ann. Sci. École Norm. Sup. (4)*, 30(6):693–717, 1997.

[23] H. K. Nguyen and A. J. Homburg. Global bifurcations to strange attractors in a model for skewed varicose instability in thermal convection. *Phys. D*, 211:235–262, 2005.

[24] H. K. Nguyen and A. J. Homburg. Resonant heteroclinic cycles and Lorenz type attractors in models for skewed varicose instability. *Nonlinearity*, 18:155–173, 2005.

[25] R. C. Robinson. Differentiability of the stable foliation for the model Lorenz equations. In *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, volume 898 of *Lecture Notes in Math.*, pages 302–315. Springer, Berlin, 1981.

[26] R. C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. *Nonlinearity*, 2:495–518, 1989.

[27] R. C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. II. *SIAM J. Math. Anal.*, 23:1255–1268, 1992.

[28] R. C. Robinson. Nonsymmetric Lorenz attractors from a homoclinic bifurcation. *SIAM J. Math. Anal.*, 32:119–141, 2000.

[29] M. R. Rychlik. Lorenz attractors through Shil′nikov-type bifurcation. Part I. *Ergod. Th. & Dynam. Syst.*, 10:793–821, 1990.

[30] B. Sandstede. *Verzweigungstheorie homokliner Verdopplungen*. PhD thesis, University of Stuttgart, 1993.

[31] M. V. Shashkov and L. P. Shil′nikov. The existence of a smooth invariant foliation for Lorentz-type maps. *Differ. Equations*, 30(4):536–544, 1994.

[32] L. P. Shil′nikov. Bifurcations and strange attractors. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 349–372, Beijing, 2002. Higher Ed. Press.

[33] W. Tucker. The Lorenz attractor exists. *C. R. Acad. Sci. Paris Sér. I Math.*, 328:1197–1202, 1999.

[34] W. Tucker. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.*, 2(1):53–117, 2002.

[35] R. F. Williams. The structure of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, 50:73–99, 1979.