ATOMIC DISINTEGRATIONS FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

ALE JAN HOMBURG

Abstract. Shub & Wilkinson and Ruelle & Wilkinson studied a class of volume preserving diffeomorphisms on the three dimensional torus that are stably ergodic. The diffeomorphisms are partially hyperbolic and admit an invariant central foliation of circles. The foliation is not absolutely continuous, in fact, Ruelle & Wilkinson established that the disintegration of volume along central leaves is atomic. We show that in such a class of volume preserving diffeomorphisms the disintegration of volume along central leaves is a single delta measure. We also formulate a general result for conservative three dimensional skew product like diffeomorphisms on circle bundles, providing conditions for delta measures as disintegrations of the smooth invariant measure.

MSC 37C05, 37D30

1. Introduction

We consider volume preserving perturbations of the following diffeomorphisms on the three dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$:

$$(x, y, z) \mapsto (A(x, y), z),$$

where $A \in \text{GL}(2, \mathbb{Z})$ is a hyperbolic torus automorphism.

The interest in these systems stems from their role in the study of stable ergodicity. Indeed, Shub & Wilkinson [24] show the existence, arbitrarily close to (1), of a $C^1$ open set of $C^2$ volume preserving diffeomorphisms that are ergodic with respect to volume. Stable ergodicity has since been shown to occur abundantly in conservative partially hyperbolic diffeomorphisms [12, 18].

Our interest comes from the phenomenon of Fubini’s nightmare [16] that appears in these diffeomorphisms and is related to non absolutely continuous foliations. By classical work on normal hyperbolicity [15], perturbations of (1) admit an invariant center foliation with leaves that are circles close to $\{(x, y) = \text{constant}\}$ (which is the invariant center foliation for (1)). The diffeomorphisms studied in [24] are shown by Ruelle & Wilkinson [23] to possess a set of full Lebesgue measure that intersects almost every circle from the center foliation in $k$ points for some finite integer $k$. The number $k$ remained unspecified in their result.

We will show that the result in [23] is true with $k = 1$. We thus get robust examples of conservative diffeomorphisms on $\mathbb{T}^3$ with a center foliation of circles and an invariant set of full Lebesgue measure that intersects almost every center leaf in a single point.

The theorem below recalls the results of [23, 24]. Note that the central Lyapunov exponent in the formulation of the theorem is negative, the inverse diffeomorphisms possess a
positive central Lyapunov exponent as in [24]. Also, [24] takes $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; the extension to arbitrary hyperbolic torus automorphisms is in [8, Section 7.3.1].

**Theorem 1.1** ([23, 24]). In any neighborhood of $(1)$ there is a $C^1$ open set $U$ of $C^2$ volume preserving diffeomorphisms on $\mathbb{T}^3$, so that for each $F \in U$,

(i) $F$ is ergodic with respect to Lebesgue measure;

(ii) there is an invariant center foliation of $C^2$ circles $W^c(p)$, $p \in \mathbb{T}^3$, so that for Lebesgue almost all $p$, if $v \in T_p W^c(p)$, then

$$\lim_{n \to \infty} \frac{1}{n} \ln |DF^n(p)v| = \lambda_c$$

for some $\lambda_c < 0$;

(iii) for some positive integer $k$, the disintegrations of Lebesgue measure along center leaves are point measures consisting of $k$ points with mass $\frac{1}{k}$ (in particular, there is an invariant set of full Lebesgue measure in $\mathbb{T}^3$ that intersects almost every center leaf in $k$ points).

The arguments followed by Ruelle & Wilkinson involve Pesin theory, in particular the construction of local unstable manifolds in nonuniformly hyperbolic systems. With such methods it is not clear how to obtain further information on the number of atoms $k$. As mentioned above, we show that Theorem 1.1 holds with $k = 1$.

**Theorem 1.2.** In any neighborhood of $(0, 0)$ there is a $C^1$ open set $U$ of $C^2$ volume preserving diffeomorphisms on $\mathbb{T}^3$, so that each $F \in U$ satisfies properties (i), (ii) of Theorem 1.1 and furthermore

(iii) the disintegrations of Lebesgue measure along center leaves are delta measures (in particular, there is an invariant set of full Lebesgue measure in $\mathbb{T}^3$ for $F$ that intersects almost every center leaf in a single point).

The study in [23] provides a specific two parameter family of diffeomorphisms for which Theorem 1.1 is shown to hold. Define $F_{a,b} = (j \circ h)^{-1}$ with

$$h(x, y, z) = (2x + y, x + y, z + x + y + b \sin(2\pi y)),$$
$$j(x, y, z) = (x + (1 + \sqrt{5})a \cos(2\pi z), y + 2a \cos(2\pi z), z).$$

(2)

For $a, b = 0$, $F_{0,0}$ can be brought to a form (1) by a linear coordinate change. By [23], $F_{a,b}$ for small nonzero values of $a, b$ satisfies the conclusions of Theorem 1.1. We show that atomic disintegrations with $k = 1$ occur within this family.

**Theorem 1.3.** In any neighborhood of $(0, 0)$ there is a set $\Phi$ of positive measure so that for $(a, b) \in \Phi$, $F_{a,b}$ satisfies properties (i), (ii) of Theorem 1.1 and furthermore

(iii) the disintegrations of Lebesgue measure along center leaves of $F_{a,b}$ are delta measures (in particular, there is an invariant set of full Lebesgue measure in $\mathbb{T}^3$ for $F_{a,b}$ that intersects almost every center leaf in a single point).
It should be noted that disintegrations with $k > 1$ points do occur for specific diffeomorphisms in any neighborhood of \([1]\). Namely, if \(j\) in \([2]\) is replaced by \((x, y, z) \mapsto (x + (1 + \sqrt{5})a \cos(2\pi qz), y + 2a \cos(2\pi qz), z)\) for an integer \(q\) with \(q \geq 2\), then \(F_{a,b}\) satisfies the \(\mathbb{Z}_q\)-symmetry relation \(F_{a,b}(x, y, z + 1/q) = F_{a,b}(x, y, z) + (0, 0, 1/q)\). By a remark due to Katok and contained in Ruelle and Wilkinson’s paper, this forces \(k\) to be a multiple of \(q\).

The method to prove Theorem 1.1 is sufficiently general to treat other conservative partially hyperbolic systems. Main ingredients, apart from ergodicity with respect to a smooth measure, are a minimal strong unstable foliation with one dimensional leaves and a center foliation of circles with a fixed (or periodic) center leaf with Morse-Smale dynamics. We make use of a Markov partition on the leaf space of the center foliation.

Let \(M\) be a compact three dimensional manifold \(M\), for which there exists a circle bundle (a fiber bundle with circles as fibers) \(\pi : M \to \mathbb{T}^2\) over the two dimensional torus. Let \(A\) be an Anosov diffeomorphism on \(\mathbb{T}^2\). We say that a diffeomorphism \(G\) on \(M\) is a partially hyperbolic skew product over \(A\) if \(G\) preserves the fibration of the circle bundle, which is the center foliation, and \(G\) projects to \(A\). The relevance of this definition is underlined by \([10, \text{Theorem 1}]\) and \([5, \text{Theorem 1}]\). We refer to \([8, 17, 22]\) for background and additional information on partially hyperbolic diffeomorphisms.

**Theorem 1.4.** Let \(F\) be a partially hyperbolic diffeomorphism, preserving a smooth measure \(m\), that is topologically conjugate to a partially hyperbolic skew product over \(A\). Assume the following properties.

(i) \(F\) is ergodic with respect to \(m\);
(ii) \(F\) has a central Lyapunov exponent \(\lambda_c < 0\);
(iii) \(F\) has a minimal strong unstable foliation;
(iv) \(F\) admits a periodic center leaf \(F^kW^c(p) = W^c(p)\), so that
   (a) \(F^k\) restricted to \(W^c(p)\) is Morse-Smale with a unique attracting fixed point \(P\) and unique repelling fixed point \(Q\);
   (b) \(\lambda^u(Q)\lambda^c(Q) > \lambda^u(P)\) (where \(\lambda^u(Q)\) is the strong unstable eigenvalue of \(DF^k(Q)\), \(\lambda^c(Q)\) is the central eigenvalue of \(DF^k(Q)\) and likewise at \(P\)).

Then the disintegrations of \(m\) along center manifolds are delta measures.

We illustrate Theorem 1.4 with an example of a partially hyperbolic skew product system from [10]. Start with the map
\[
A_\theta(x, y, z) = (A(x, y), z + \theta(x, y))
\]
on \(\mathbb{T}^3\), where \(A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\) and \(\theta : \mathbb{T}^2 \to \mathbb{R}\) is a smooth map satisfying \(\theta(x, y + \frac{1}{2}) = -\theta(x, y)\). Consider the action of \(\mathbb{Z}_2\) on \(\mathbb{T}^3\) induced by \(\varphi(x, y, z) = (x, y + \frac{1}{2}, -z)\). The quotient of \(\mathbb{T}^3\) by this \(\mathbb{Z}_2\)-action is a smooth manifold \(M\). By [10, Proposition 4.1], \(A_\theta\) projects to a partially hyperbolic skew product diffeomorphism \(F_\theta : M \to M\). The center foliation is a nonorientable circle bundle.

**Proposition 1.1.** In any neighborhood of \(F_\theta\) there is a \(C^1\) open set \(U\) of \(C^2\) conservative diffeomorphisms on \(M\), so that for each \(F \in U\),
(i) $F$ is ergodic with respect to $m$;
(ii) there is an invariant center foliation of $C^2$ circles $W^c(p), p \in M$, with center Lyapunov exponent $\lambda_c \neq 0$;
(iii) the disintegrations of $m$ along center leaves are delta measures.

**Sketch of proof.** Consider the family $A_{a,b} = j \circ h$ on $\mathbb{T}^3$ with

$$h(x, y, z) = (3x + 2y, x + y, z + b \sin(2\pi y)),$$
$$j(x, y, z) = (x + (1 + \sqrt{3})a \cos(2\pi z), y + a \cos(2\pi z), z).$$

Note that we recover $A_0$ if $a, b = 0$. A direct calculation shows that $A_{a,b}$ is volume preserving as well as equivariant with respect to the given $\mathbb{Z}_2$-symmetry, and hence projects to a diffeomorphism on $M$. For small values of $(a, b)$, $A_{a,b}$ possesses a fixed center leaf near $\{(x, y) = (0, 0)\}$. For $a, b \neq 0$, there are precisely two hyperbolic fixed points $(0, 0, 1/4)$ and $(0, 0, 3/4)$ on this leaf. The additional eigenvalue conditions from item (iv) in Theorem 1.4 hold since $A_{a,b}$ has a smooth center unstable foliation, compare Lemma 2.4 and the proof of Lemma 2.8.

Let $F_{a,b}$ denote the projected diffeomorphism on $M$. By Hirsch, Pugh & Shub \[15\], or \[10\, Proposition 4.1\], $F_{a,b}$ and small perturbations thereof are topologically conjugate to a partially hyperbolic skew product over $A$. By \[21\] the set of ergodic partially hyperbolic diffeomorphisms is $C^1$ open and dense. Baraviera and Bonatti \[2\] show how a nonzero center Lyapunov exponent is created through small local perturbations (if needed). A minimal strong unstable foliation is created through an arbitrarily small perturbation with a blender as in Lemma 2.1 below. All these properties are robust, so that an open set of diffeomorphisms is created for which the conditions of Theorem 1.4 hold (take the inverse diffeomorphism in case of a positive center Lyapunov exponent). Apply Theorem 1.4. □

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## 2. Proofs of the results on delta measures as disintegrations

In order to avoid too much jumping between cases, we will prove Theorems 1.2 and 1.3 and deal with Theorem 1.4 by noting that, apart from notation, it follows from the same arguments. Write $W^i(p), i = s, c, u$, for the strong stable, center or strong unstable manifold containing $p$. Further, $W^{sc}(p)$ is the center stable manifold and $W^{cu}(p)$ is the center unstable manifold containing $p$.

Recall that a foliation on a manifold is minimal if all its leaves lie dense in the manifold. Minimal strong stable or strong unstable foliations are abundant in the context of partially hyperbolic diffeomorphisms \[7\].

**Lemma 2.1.** In any neighborhood of \((1)\) there is a diffeomorphism $F$ with the following properties:
(i) there is a center leaf, fixed for $F$, with a unique hyperbolic attracting fixed point $P$ and a unique hyperbolic repelling fixed point $Q$;

(ii) the strong unstable and strong stable foliations are minimal.

Moreover, these properties are robust.

Proof. A calculation shows that $F_{a,b}$, for nonzero $a$, has hyperbolic fixed points $(0, 0, \frac{1}{4})$ and $(0, 0, \frac{3}{4})$. So in, and near, the family $F_{a,b}$ with $a, b$ small we find examples of diffeomorphisms for which the first item holds. The fixed center leaf can be replaced by a periodic center leaf with only notational changes in the following.

For the second item, [7] discusses minimal strong unstable and strong stable foliations in general, not necessarily conservative (volume preserving) diffeomorphisms. But as the basic tool of blenders [6, 8] is available for conservative diffeomorphisms [20], their construction can be followed and thus the second item holds.

For convenience of the reader we spend a few words on clarifying the use of blenders. Start with a diffeomorphism possessing a fixed point $P$ with one dimensional unstable manifold and a fixed point with two dimensional unstable manifold $Q$, such as (2). A blender associated with $P$ is an open set $V$ near $P$ so that $W^u(P)$ intersects each center stable strip that stretches through $V$ (see the references mentioned above). In [20] it is established that there are arbitrarily small perturbations of such diffeomorphisms that admit a heterodimensional cycle. Blenders are found in further arbitrarily small perturbations from here, and hence blenders occur arbitrarily close to $F_{a,b}$.

We note that a blender associated with $P$ gives a hyperbolic set, containing a dense set of periodic points with one dimensional unstable manifold, close to $P$. Again resorting to [20], an arbitrarily small perturbation ensures that $W^s(Q)$ intersects $V$. Then $W^{cu}(Q) \subset W^u(P)$: high iterates of a small neighborhood $O$ of a point in $W^{cu}(Q)$ under $F^{-1}$ accumulate onto $W^s(Q)$ by the $\lambda$-lemma and hence contain points accumulating onto $W^u(P)$ due to the blender associated with $P$.

Since center unstable leaves are dense in $\mathbb{T}^3$ and hence $W^{cu}(Q)$ is dense in $\mathbb{T}^3$, we get that $W^u(P)$ is dense in $\mathbb{T}^3$. Since strong unstable manifolds accumulate onto $W^u(P)$, all strong unstable manifolds are dense in $\mathbb{T}^3$, that is, the strong unstable foliation is minimal. Similarly one obtains a minimal strong stable foliation.

For the concrete family $F_{a,b} = (j \circ h)^{-1}$, see (2), item (i) holds for small $(a, b)$ with $a$ nonzero; fixed points are $(0, 0, \frac{1}{4})$ and $(0, 0, \frac{3}{4})$. We can further show that minimal strong stable and strong unstable foliations occur for many values of $a, b$.

**Lemma 2.2.** In any neighborhood of $(0,0)$ there is a set $\Phi$ of positive measure so that for $(a, b) \in \Phi$, the strong unstable and strong stable foliations are minimal.

**Proof.** Consider the skew product system $F_{a,b}$ for small $b$ and $a = 0$. Calculate

$$F_{0,b}^{-4}(x, y, z) = (34x + 21y, 21x + 13y, z + 33x + 21y + bR_4(x, y))$$

with

$$R_4(x, y) = \sin(2\pi y) + \sin(2\pi(x + y)) + \sin(2\pi(3x + 2y)) + \sin(2\pi(8x + 5y)).$$
Note that $F_{0,b}$ has a period four fiber $(\frac{1}{15}, \frac{2}{15}, T)$:

$$F_{0,b}^{-4} \left( \frac{1}{15}, \frac{2}{15}, z \right) = \left( \frac{1}{15}, \frac{2}{15}, z + bR_4 \left( \frac{1}{15}, \frac{2}{15} \right) \right)$$

with

$$R_4 \left( \frac{1}{15}, \frac{2}{15} \right) = \sin \left( \frac{4}{15} \pi \right) + \sin \left( \frac{6}{15} \pi \right) + \sin \left( \frac{14}{15} \pi \right) + \sin \left( \frac{6}{15} \pi \right).$$

Since $R_4 \left( \frac{1}{15}, \frac{2}{15} \right) > 0$ (all four terms are positive), we find that for a full measure set of values of $b$, $F_{0,b}^{-4}$ has irrational rotation on the period four fiber.

Treat $F_{a,b}$ as a small perturbation of the family $F_{0,b}$ with $b \neq 0$. By normal hyperbolicity this family possesses a smooth normally hyperbolic period four fiber $V_{a,b}^c$ near $(\frac{1}{15}, \frac{2}{15}, T)$. For a positive measure set $\Phi$ of parameter values, the rotation number of $F_{a,b}^4$ on $V_{a,b}^c$ is irrational, see e.g. [14].

So the strong unstable manifold of a point in $V_{a,b}^c$, $(a, b) \in \Phi$, is dense in $\mathbb{T}^3$. The same reasoning applies to strong stable manifolds. Compare also [11, Theorem 12]. □

In the following, when working with a diffeomorphism $F_{a,b}$ with a fixed choice of $a, b$, we frequently suppress the dependence on $a, b$ from the notation and write $F$ for $F_{a,b}$.

We use a partition of $\mathbb{T}^3$ which is perhaps easiest explained by making use of a topological conjugacy to a skew product system, as in the following result from Hirsch, Pugh & Shub [15].

**Proposition 2.1.** There is a homeomorphism $h$ on $\mathbb{T}^3$ with $h \circ F = G \circ h$, where $G$ is a skew product diffeomorphism

$$G(x, y, z) = (A(x, y), G_{x,y}(z)),$$

for the hyperbolic torus automorphism $A$ and with $z \mapsto G_{x,y}(z)$ a diffeomorphism depending continuously on $(x, y)$.

Take a Markov partition $\mathcal{R} = \{R_1, \ldots, R_n\}$ for the base dynamics $(x, y) \mapsto A(x, y)$. Recall that a partition element $R_i$ is a rectangle, bounded by segments in local stable and local unstable manifolds. One can bound the diameter of the rectangles by any given $d > 0$. Consider the partition of $\mathbb{T}^3$ with partition elements $R_i \times \mathbb{T}$. The image under the topological conjugacy $h^{-1}$ is a partition $\{S_1, \ldots, S_n\}$ of $\mathbb{T}^3$. The conjugacy $h^{-1}$ maps boundaries of $R_i \times \mathbb{T}$ into center stable and center unstable manifolds of $F$, so that the boundaries of $S_i$ lie in center stable and center unstable manifolds of $F$. A partition element $S_i$ is therefore diffeomorphic to a product of a rectangle and a circle. Note that the boundaries of the partition elements (and their forward and backward orbits) are of zero Lebesgue measure. For $p$ in the interior of $S_i$, we write $W^s_{loc}(p)$ for the local strong stable manifold containing $p$ with boundary points in the boundary of a partition element $S_i$. Likewise other local invariant manifolds such as $W^c_{loc}(p)$ have their boundary inside the boundary of a partition element $S_i$. 
As mentioned in the introduction, Ruelle & Wilkinson [23] prove the existence of a set of full Lebesgue measure for which the Lyapunov exponents exist and that intersects almost every circle from the center foliation in k points for some finite integer k.

**Proposition 2.2.** Let k be as above. There are \( R > 0 \) and a set \( \Lambda \subset T^3 \) that is of positive Lebesgue measure, so that

(i) For \( p \in \Lambda \), \( \Lambda \cap W^c(p) \) consists of k intervals \( B^i(p) \subset W^c(p) \), \( 1 \leq i \leq k \), with a length uniformly bounded from below by \( R \);

(ii) There are \( C > 0, \nu < 1 \) so that

\[
|F^n(q) - F^n(r)| \leq C \nu^n
\]

for \( q, r \) from the same interval \( B^i(p) \).

Moreover, there is a set \( \Lambda \) with these properties that is an s-saturated set:

\[
\Lambda = \cup_{p \in \Lambda} W^s_{loc}(p)
\]

**Proof.** The statements on the existence of a set \( \Lambda \) of positive Lebesgue measure so that items (i), (ii) hold can be found in [23]. The bound (3) (possibly with a different constant \( C \)) also holds when one replaces \( \Lambda \) by its s-saturation \( \cup_{p \in \Lambda} W^s_{loc}(p) \). This is true since the stable holonomy map \( h_{p,q} : W^c(p) \to W^c(q) \), defined for \( q \in W^c_{loc}(p) \) by \( h_{p,q}(x) = W^s_{loc}(x) \cap W^c(q) \), is uniformly \( C^1 \) [12, 19]. This shows that we may take \( \Lambda \) to be an s-saturated set. \( \square \)

In fact one can find \( \Lambda \) as above with Lebesgue measure arbitrarily close to one. The following lemma contains a key argument for the proof of Theorems 1.2 and 1.3. Its proof uses the above proposition and also relies on minimality of the strong unstable foliation.

We denote Lebesgue measure by \( \lambda \), and also the leaf measure (Lebesgue measure) on center leaves by \( \lambda \).

**Lemma 2.3.** Let \((a,b) \in \Phi\). For Lebesgue almost all \( p \in T^3 \), \( \{ F^n|_{W^c(F^{-n}(p))} \lambda \} \) contains a delta measure in its limit points in the weak star topology.

**Proof.** For intervals \( I \) we write \( |I| \) to denote their length, so for intervals \( I \) inside center leaves we also write \( |I| = \lambda(I) \). Fix \( \varepsilon > 0 \).

**Step 1.** Recall from Lemma 2.2 the existence of a center leaf, fixed by \( F \), containing an attracting fixed point \( P \) and a repelling fixed point \( Q \). Note that any closed interval in \( W^c(P) \setminus Q \) is contracted under iteration by \( F \). The existence of strong stable and strong unstable foliations near \( W^c(P) \) shows that a similar contraction occurs on center leaves near \( W^c(P) \) as long as iterates remain near \( W^c(P) \). Let \( K_0 \subset W^u_{loc}(P) \) be a fundamental interval with endpoints \( k_0, k_1 = F^{-1}(k_0) \). Write \( K_n = F^{-n}(K_0) \). Note that the intervals \( K_n \) converge to \( P \) as \( n \to \infty \). Now there is \( N \in \mathbb{N} \) so that for \( q_{-N} \in W^u_{loc}(K_N) \), there is \( V \subset W^c(q_{-N}) \) with both

\[
|V| > 1 - \varepsilon \text{ and } |F^N(V)| < \varepsilon.
\]
Larger values of $N$ are needed for smaller values of $\varepsilon$. For use in the following step we note that a stronger contraction is obtained (the image $F^N(V)$ can be made smaller) when taking $N$ larger.

**Step 2.** The second step in the proof leads to the following statement. For any $\varepsilon > 0$ there exists a set $\Lambda_\varepsilon$ of positive Lebesgue measure and an integer $L$, so that for $r \in \Lambda_\varepsilon$ there is an interval $V \subset W^c(r)$ of length at least $1 - \varepsilon$ so that for any integer $n \geq L$, $f^n(V)$ has length smaller than $\varepsilon$. The lemma will follow easily from this.

The following steps are illustrated in Figure 1. Let $\Lambda$ be the set of positive Lebesgue measure provided by Proposition 2.2. For simplicity we assume $C = 1$ in (3). For $p \in \Lambda$, let $D^1(p)$ be a closed subinterval of $B^1(p)$ some distance, say $R/10$, away from the boundary of $B^1(p)$. The strong unstable manifold of $P$ lies dense and in fact iterates of a fundamental interval $K_0$ lie dense in $T^3$. We therefore get that for all $p \in \Lambda$, there is a positive integer $M = M(p)$ and a point $q_0 = q_0(p) \in K_0$ with $F^M(q_0) \in W^s_{loc}(D^1(p)) \subset W^s_{loc}(p)$. By replacing $\Lambda$ with a smaller set we get $M$ to be constant. Namely, write $\tilde{\Lambda}_j \subset \Lambda$ for the set of points $p \in \Lambda$ with $M(p) = j$. At least one of the sets $\tilde{\Lambda}_j$ has positive Lebesgue measure. Now replace $\Lambda$ by this $\tilde{\Lambda}$ and $M$ will be constant. Let

$$\Lambda_0 = \bigcup_{p \in \Lambda} W^s_{loc}(q_0(p))$$  (4)

denote the union of the local center stable manifolds of the points $q_0(p) \in K_0$.

Using the first step, we find $N$ large and $V \subset W^c(q_{-N})$, with $q_{-N} = F^{-N}(q_0)$, so that the iterate $F^M$ maps $F^N(V) \subset W^c(q_0)$ into $W^s_{loc}(B^1(p))$. By the last sentence of Step 1, we may take an $N$ that works for all $p \in \Lambda$. Write $L = N + M$. Observe that $F^L$ maps $V \subset W^c(q_{-N})$ into $W^s_{loc}(B^1(p))$.

Now

$$\Lambda_N = \bigcup_{p \in \Lambda} W^s_{loc}(q_{-N})$$

is the required set. Note that $\Lambda_N$ is defined as a union of local center stable manifolds. It remains to prove that $\Lambda_N$ has positive Lebesgue measure. Its measure equals the measure of $F^L(\bigcup_{p \in \Lambda} W^s_{loc}(q_{-N}))$. For fixed $p$, $F^L(W^s_{loc}(q_{-N}))$ is a cylinder inside $W^s_{loc}(F^M(q_0)) = W^s_{loc}(p)$ and hence it intersects $W^s_{loc}(p)$ in a subinterval. Since $\Lambda$ is $s$-saturated, see Proposition 2.2, it follows that $F^L(\Lambda_N) \cap \Lambda$ consists of a subinterval in each local strong stable leaf inside $\Lambda$. Since $L$ is fixed, there exists $c > 0$ so that for each $r \in \Lambda_N$,

$$\left| F^L(W^s_{loc}(r)) \right| > c.$$  (5)

We finish the argument by employing absolute continuity of the strong stable foliation. We may write, for a Borel set $A$ contained in a partition element $S_i$ and for a choice of $r \in S_i$,

$$\lambda(A) = \int_{W^c_{loc}(r)} \lambda^p_s(A \cap W^s_{loc}(p)) \, d\nu^{cu}(p),$$

where $\nu^{cu}$ is projected measure of local strong stable manifolds; $\nu^{cu}(B) = \lambda(\bigcup_{p \in B} W^s_{loc}(p))$. By e.g. [3] Section 8.6], $\nu^{cu}$ is equivalent to leaf measure on $W^c_{loc}(r)$ and the conditional
Figure 1. An illustration of the proof of Lemma 2.3: for $\varepsilon > 0$ and a given strip $W_{loc}^{s}(B^{1}(p))$, we find a uniformly bounded $L$ so that $F^{L}$ maps a large interval $V \subset W^{s}(q_{-N})$ of length at least $1 - \varepsilon$, for a $q_{-N} \in K_{N}$, into $W_{loc}^{s}(B^{1}(p))$. This uses minimality of the strong unstable foliation.

By ergodicity, $F^{-n}(p)$ intersects $\Lambda_{N}$ infinitely often for almost all $p \in \mathbb{T}^{3}$. The lemma follows by taking a sequence of values for $\varepsilon$ going to zero. $\Box$

The previous lemma is used to prove Theorems 1.2 and 1.3. We start with Theorem 1.3 which follows from Proposition 2.3 below. Specific to the two parameter family of diffeomorphisms $F_{a,b}$ is the existence of a smooth center stable foliation. This makes the argument more straightforward.

Lemma 2.4. The center stable foliation of $F_{a,b}$ is the affine foliation with leaves tangent to the planes spanned by $v_{0} = (1 + \sqrt{5}, 2, 0)$ and $(0, 0, 1)$.

Proof. Observe that $(1 + \sqrt{5}, 2)$ is the unstable eigenvector of the torus automorphism given by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The lemma is clear from the observations that $h$ is a skew product diffeomorphism and that $j$ leaves the given affine foliation invariant. $\Box$

Proposition 2.3. For small nonzero values of $(a, b) \in \Phi$, the disintegrations of Lebesgue measure along center leaves of $F_{a,b}$ are delta measures.
Proof. Recall the partition \( \{ S_1, \ldots, S_n \} \) of \( \mathbb{T}^3 \) and consider \( F \) acting on the union \( S = \bigcup_i S_i \) of partition elements. Note that \( F \) acting on \( \mathbb{T}^3 \) is obtained by gluing partition elements along boundaries.

The lemma is proved by applying [13 Proposition 3.1] (see also [1 Theorem 1.7.2]) that treats relations between invariant measures for endomorphisms and their natural extensions. These results are formulated for skew product diffeomorphisms and translate to our setting by Proposition 2.1.

For a point \( p \) from a partition element \( S_i \), write \( \pi^s(p) \) for its projection along the leaf \( W^s_{loc}(p) \) onto a center unstable side, which we denote by \( T_i \), of \( S_i \). Write \( F^+ \) for the dynamical system on \( T = \bigcup_i T_i \), obtained by composing \( F \) with \( \pi^s \). Write \( \mu^+ = \pi^s \lambda \).

Lemma 2.5. The measure \( \mu^+ \) is \( F^+ \)-invariant.

Proof. By the Markov property of the partition we have

\[
F^{-1} (\pi^s)^{-1} (A) = (\pi^s)^{-1} (F^+)^{-1} (A),
\]

for Borel sets \( A \subset T \). Hence

\[
\mu^+ (A) = \lambda ( (\pi^s)^{-1} (A)) = \lambda (F^{-1} (\pi^s)^{-1} (A)) = \lambda ( (\pi^s)^{-1} (F^+)^{-1} (A)) = \mu^+ ( (F^+)^{-1} (A)),
\]

which expresses \( F^+ \) invariance of \( \mu^+ \). \( \square \)

We have the following properties, implying that \( F \) is the natural extension of \( F^+ \), see [1 Appendix A]:

(i) \( F^+ \) is a factor of \( F \);

(ii) With \( F \) the Borel \( \sigma \)-algebra on \( S \), \( F^+ \) the Borel \( \sigma \)-algebra on \( T \), and \( \mathcal{G} = (\pi^s)^{-1} (F^+) \), we have \( \sigma (F^n (\mathcal{G}), n \in \mathbb{N}) = F \mod 0 \). Here \( \sigma (F^n (\mathcal{G}), n \in \mathbb{N}) \) is the \( \sigma \)-algebra generated by \( F^n (\mathcal{G}) \).

In this context we obtain the following convergence. Let \( \mu_p \) denote the disintegrations of Lebesgue measure along center leaves \( W^c(p) \). So, if \( \nu^c \) is the measure \( \nu^c (A) = \lambda (\cup_{p \in A} W^c(p)) \) on the leaf space \( \mathbb{T}^2 \), we have

\[
\lambda (A) = \int \mu_p (A \cap W^c(p)) \, d\nu^c (p)
\]

for Borel sets \( A \subset \mathbb{T}^3 \). Considering \( \mu^+ \) as a measure on \( \sigma \)-algebra \( \mathcal{G} \), we also get disintegrations \( \mu_p^+ \) on \( W^c(p) \) satisfying \( \mu_p^+ (B) = \mu_{\pi^s_1 (p)}^+ ((\pi^s (B)) \) for Borel sets \( B \subset W^c(p) \).

Lemma 2.6. For Lebesgue almost all \( p \in \mathbb{T}^3 \),

\[
F^n \big|_{W^c(F^{-n}(p))} \mu_{F^{-n}(p)}^+ \to \mu_p
\]

as \( n \to \infty \), with convergence in the weak star topology.

Proof. Under the homeomorphism \( h \) that provides the topological conjugacy \( h \circ F = G \circ h \) from Proposition 2.1 Lebesgue measure \( \lambda \) on \( \mathbb{T}^3 \) is pushed forward to the measure \( h \lambda \) with a marginal \( \Omega \) on \( \mathbb{T}^2 \). Let \( G^+: H(T) \to h(T) \) be given by \( G^+ = h \circ F^+ \circ h^{-1} \). Note that
\( \nu^+ = \pi^\ast \lambda \) is an invariant measure for \( G^+ \). Interpret \( \nu^+ \) as a measure on \( h(S) = \cup_i R_i \) with \( \sigma \)-algebra \( h(G) \). Now [13, Proposition 3.1] provides convergence of measures

\[
G^n_{A^{-n}(x,y)} \nu^+_A \longrightarrow \nu(x,y),
\]

in the weak star topology, for \( \Omega \)-almost all \( x, y \).

If \( C \subset T^2 \) is a set of full \( \Omega \) measure, then \( h \lambda(C \times T) = 1 \), and therefore \( \lambda(h^{-1}(C \times T)) = 1 \).

We hence obtain the following statement. Take Lebesgue measure \( \lambda \) and consider the corresponding invariant measure \( \mu^+ = \pi^\ast \lambda \) for \( F^+ \). While \( \lambda \) is ergodic, by [13] also \( \mu^+ \) is ergodic. One finds convergence

\[
F^n|_{W^c(F^{-n}(p))} \lambda_F \longrightarrow \lambda_p,
\]

for Lebesgue almost all \( p \in T^2 \), with convergence in the weak star topology. By [1 Theorem 1.7.2] the measures \( \mu^+ \) and \( \mu \) are in one-to-one correspondence so that \( \mu \) equals Lebesgue measure.

By [23, 24], \( F^n|_{W^c(F^{-n}(p))} \lambda_F \) converges to \( k \) point measures of mass \( \frac{1}{k} \) each, for Lebesgue almost all \( p \in T^2 \). We wish to mimic the argument in the proof of Lemma 2.3 [23] with Lebesgue measure on center leaves \( W^c(q) \) replaced by \( \mu^+_q \).

Let \( S_1 \) be the partition element of the Markov partition containing the fixed center leaf with the attracting fixed point \( P \) and the repelling fixed point \( Q \). For the fundamental interval \( K_0 \subset W^s_{loc}(P) \) introduced in the proof of Lemma 2.3 [23] consider the region

\[
V_0 = W^s_{loc}(K_0).
\]

For \( n \geq 1 \), write \( V_n = F^{-1}(V_{n-1}) \cap S_1 \). Denote by \( B_d(q) \subset W^c(q) \) the interval of diameter \( d \) around \( q \) inside \( W^c(q) \). Consider the union of segments \( B_d(q) \) over \( q \in K_0 \) and let \( W_0 \) be the local strong stable manifolds of this union;

\[
W_0 = W^s_{loc}(\cup_{q \in K_0} B_d(q)),
\]

see Figure 2. Recall from the proof of Lemma 2.3 the regular set \( \Lambda \) of positive Lebesgue measure and the integers \( N, M \). By taking \( d \) depending on \( \varepsilon \) small enough, we get

\[
W_0 \cap W^s_{loc}(q) \subset F^{-M}(W^s_{loc}(B^1(p)))
\]

for \( F^M(q) \in W^s_{loc}(D^1(p)) \) and \( p \in \Lambda \). Write \( W_N = F^{-1}(W_{N-1}) \cap S_1 \) for the images under \( F^{-1} \) inside the partition element \( S_1 \). Observe that for large \( N \), \( V_N \cap W_N \) is a box inside \( V_N \), very thin in the center direction. See again Figure 2.

**Lemma 2.7.** For \( N \) depending on \( \varepsilon \) large enough, we get for \( q \in K_N, \mu^+_q(W^c(q) \cap W_N) > \frac{1}{2} \).

**Proof.** This follows from smoothness of the center stable foliation as stated in Lemma 2.4. Indeed, with \( \lambda \) denoting Lebesgue measure on \( W^s_{loc}(q), \lambda(W^s_{loc}(q) \cap (V_N \setminus W_N)) \) is uniformly small if \( N \) is large. Therefore also the projected measure \( \mu^+_q(W^c(q) \cap (V_N \setminus W_N)) \) is uniformly small if \( N \) is large.
As a consequence, when replacing Lebesgue measure with $\mu^+$ on center leaves $W^c(q)$ in the reasoning of Lemma 2.3, we find that for Lebesgue almost all $p \in \mathbb{T}^3$, the limit points of $F^n(U_\epsilon(F^{-n}(p), \mu^+)_{F^{-n}(p)}^+$ (again writing $F$ for $F_{a,b}$) contain point measures of mass more than $\frac{1}{2}$. Recall that $F^n(U_\epsilon(F^{-n}(p), \mu^+)_{F^{-n}(p)}^+$ converges to $k$ point measures of mass $\frac{1}{k}$ each. So $k$ cannot be 2 or higher and $F^n(U_\epsilon(F^{-n}(p), \mu^+)_{F^{-n}(p)}^+$ converges to a delta measure for Lebesgue almost all $p \in \mathbb{T}^3$. This proves Proposition 2.3 and Theorem 1.3. □

We continue with the proof of Theorem 1.2. We follow the proof above, adjusting for the lack of smoothness of the center stable foliation. The center stable foliation is absolutely continuous by [25].

**Proposition 2.4.** For $F$ as in Theorem 1.2, the disintegrations of Lebesgue measure along center leaves of $F$ are delta measures.

**Proof.** The proof of Proposition 2.3 carries over up to Lemma 2.7. Smoothness of the center stable foliation as expressed by Lemma 2.4 does not hold in general and it is not clear whether Lemma 2.7 applies in general. This lemma will be replaced by a slightly weaker version. We take the proof of Proposition 2.3 up to Lemma 2.7 and start the discussion from there. In particular the sets $W_0 \subset V_0$ are chosen as before, so that holds. The sets $W_N \subset V_N$ are again the inverse images of $W_0$ and $V_0$ inside the partition element $S_1$. Let $\nu^{sc}$ be the projected measure of local center stable manifolds on $W_{loc}^{u}(P)$; $\nu^{sc}(J) = \lambda(\cup_{q \in J}W_{loc}^{sc}(q))$. For a set $A \subset S_1$ we have

$$\lambda(A) = \int \lambda^+_q(A \cap W_{loc}^{sc}(q)) d\nu^{sc}(q).$$
Lemma 2.7 is replaced by the following. The proof of the lemma relies on eigenvalue conditions at the equilibria \( P \) and \( Q \) that hold for \( F_{a,b} \) and perturbations thereof as well as for the diffeomorphisms considered in \[ \text{Section 7.3.1}. \]

**Lemma 2.8.** For any \( \eta > 0 \), there is \( N > 0 \) so that there is a set \( J \subset K_N \) with \( \nu^c(J) > (1 - \eta)\nu^c(K_N) \) and \( \mu_\eta^*(W^c(q) \cap W_N) > \frac{1}{2} \) for \( q \in J \).

**Proof.** Write \( \lambda^s(P) < \lambda^c(P) < \lambda^u(P) \) for the eigenvalues of \( DF(P) \). Write likewise \( \lambda^s(Q) < \lambda^c(Q) < \lambda^u(Q) \) for the eigenvalues of \( DF(Q) \). For the system \( (j \circ h)^{-1} \) with \( j \) and \( h \) as in \[ \text{2} \] we have, because of the affine center stable foliation, \( \lambda^u(Q) = \lambda^u(P) \). The same applies to the diffeomorphisms considered in \[ \text{[8 Section 7.3.1].} \] As \( \lambda^c(Q) > 1 \) we get

\[
\lambda^u(Q)\lambda^c(Q) > \lambda^u(P). \tag{7}
\]

We consider diffeomorphisms close to \( (j \circ h)^{-1} \) so that this inequality holds.

We claim that, thanks to \[ \text{[7]}, \]

\[
\lim_{N \to \infty} \lambda(V_N \setminus W_N)/\lambda(V_N) = 0. \tag{8}
\]

For the computations we use local linearizing coordinates near \( P \) and \( Q \). As \( F \) is a \( C^2 \) diffeomorphism, there are local \( C^1 \) diffeomorphisms defined on neighborhoods \( O_P \) of \( P \) and \( O_Q \) of \( Q \) in \( \mathbb{T}^3 \), that transform \( F \) into its linearization at \( P \) and \( Q \) \[ \text{[4].} \] The required nonresonance conditions \( \lambda_u(Q) \neq \lambda_c(Q) \) and \( \lambda_u(P) \lambda_c(P) \neq \lambda_u(P) \) to apply \[ \text{[4]} \] hold since the diffeomorphism is conservative and the products \( \lambda_u(Q) \lambda_c(Q) \lambda_u(P) \) and \( \lambda_u(P) \lambda_c(P) \lambda_u(P) \) are therefore equal to 1. By iteration under \( F \) we can extend the neighborhoods with linearizing coordinates and we may therefore assume \( W^c(P) \subset O_P \cup O_Q \). There is no loss in assuming that \( S_1 = O_P \cup O_Q \) and \( V_0 \setminus W_0 \subset O_Q \).

In linearizing coordinates in \( O_Q \), distances in the strong unstable direction get contracted by a factor \( 1/\lambda_u(Q) \) each iterate under iteration by \( F^{-1} \). This applies to points starting in \( V_0 \setminus W_0 \) that remain in \( S_1 \) under iteration by \( F^{-1} \). Points in \( V_N \setminus W_N \) moreover satisfy an estimate \( |x_1| \leq C/(\lambda^c(Q))^N \) for some \( C > 0 \). It easily follows from these computations that \( \lambda(V_N \setminus W_N) \sim (\lambda^u(Q)\lambda^c(Q))^{-N} \cdot \) for some \( C > 1 \),

\[
\frac{1}{C}(\lambda^u(Q)\lambda^c(Q))^{-N} \leq \lambda(V_N \setminus W_N) \leq C(\lambda^u(Q)\lambda^c(Q))^{-N}.
\]

Likewise one obtains \( \lambda(O_P \cap V_N) \sim (\lambda^u(P))^{-N} \). By \[ \text{[7]} \] we find that for large \( N \), the volume of \( V_N \setminus W_N \) is much smaller than the volume of \( O_P \cap V_N \) and hence much smaller than the volume of \( V_N \). The claim follows.

By \[ \text{[8]}, \] it is not possible that the conditional measures \( \lambda_q^{sc} \) assign mass \( \frac{1}{2} \), or more, to \( (V_N \setminus W_N) \cap W_{loc}^{sc}(q) \) for a nonzero proportion of points \( q \) in \( K_N \) as \( N \to \infty \). Namely, if \( \lambda_q^{sc}(V_N \setminus W_N) \cap W_{loc}^{sc}(q) \geq \frac{1}{2} \) for \( q \in K_N \setminus J \) and we pose \( \nu^{sc}(K_N \setminus J)/\nu^{sc}(K_N) \geq \rho \) for some \( \rho > 0 \), then

\[
\lambda(V_N \setminus W_N) = \int_{K_N} \lambda_q^{sc}(V_N \setminus W_N) \cap W_{loc}^{sc}(q) \, d\nu^{sc}(q)
\]

\[
\geq \int_{K_N \setminus J} \frac{1}{2} \, d\nu^{sc}(q) = \frac{1}{2} \nu^{sc}(K_N \setminus J) \geq \frac{1}{2} \rho \lambda(V_N),
\]
contradicting (8) for \( N \) large. Because \( \mu_q^x(A) = \lambda_q^{sc}(\cup_{p \in A} W^s_{loc}(p)) \) for Borel sets \( A \subset W^c(q) \), the lemma follows.

Consider \( \Lambda_0 \) as constructed in the proof of Lemma 2.3 see (4), and write \( \Sigma_0 = \Lambda_0 \cap W^u_{loc}(P) \). We may take \( \Lambda_0 \) so that the following statement holds, as follows from Pesin theory. We take the formulation from [25, Lemma 6.6]. There is a homeomorphism \( h : \Sigma_0 \times [-1,1]^2 \to \Lambda_0 \), such that

(i) \( h(\{x_u\} \times [-1,1]^2) \subset W^u_{loc}(h(x_u,0,0)) \);

(ii) there is \( K > 0 \) so that for any transversals \( \tau_1, \tau_2 \) to the center stable foliation, near \( K_0 \), the center stable foliation induces a holonomy map \( h^{sc} \) from \( \tau_1 \cap h(x_u \times [-1,1]^2) \) to \( \tau_2 \cap h(x_u \times [-1,1]^2) \) whose Jacobian is bounded by \( K \) from above and \( 1/K \) from below.

As a consequence, there is \( \gamma > 0 \) so that for each transversal \( \tau \) to the center stable foliation, near \( K_0 \),

\[
\lambda(\Lambda_0 \cap \tau) > \gamma. \tag{9}
\]

We claim that for \( J \subset K_N \) as in Lemma 2.3 and \( N \) large enough, \( F^N(W^{sc}_{loc}(J)) \) intersects \( \Lambda_0 \) in a set of positive Lebesgue measure. This follows by combining Lemma 2.8 and (9). Namely, take a smooth foliation \( \mathcal{G} \) of \( S_1 \) with curves transversal to the local center stable manifolds. For a measurable set \( A \subset S_1 \), we can write \( \lambda(A) = \int_{W^{u}_{loc}(P)} \lambda(\mathcal{G}_q \cap A) \ dm(q) \) for a smooth measure \( m \). By Lemma 2.8 \( \lambda(W^{sc}_{loc}(J))/\lambda(V_N) > t \) for some \( t \) close to one, if \( N \) is large. Write

\[
\frac{\lambda(W^{sc}_{loc}(J))}{\lambda(V_N)} = \frac{\lambda(W^{sc}_{loc}(J) \cap W_N)}{\lambda(V_N)} + \frac{\lambda(W^{sc}_{loc}(J) \cap (V_N \setminus W_N))}{\lambda(V_N)}
\]

and observe that the second term on the right hand side goes to zero as \( N \to \infty \) by (8). Hence also \( \lambda(W^{sc}_{loc}(J) \cap W_N)/\lambda(V_N) > t \) for some \( t \) close to one, if \( N \) is large. From this and

\[
\lambda(W^{sc}_{loc}(J) \cap W_N) = \int_{W^{sc}_{loc}(P)} \lambda(\mathcal{G}_q \cap W^{sc}_{loc}(J) \cap W_N) \ dm(q)
\]

we find that if \( N \) is large, \( \lambda(\mathcal{G}_q \cap W^{sc}_{loc}(J))/\lambda(\mathcal{G}_q \cap V_N) \) is close to one for some \( q \) with \( \mathcal{G}_q \cap V_N \subset W_N \).

By bounded distortion [9, Lemma 3.3], with \( \lambda(\mathcal{G}_q \cap W^{sc}_{loc}(J))/\lambda(\mathcal{G}_q \cap V_N) \) close to one, also \( \lambda(F^N(\mathcal{G}_q \cap W^{sc}_{loc}(J)))/\lambda(F^N(\mathcal{G}_q \cap V_N)) \) is close to one. (Bounded distortion of \( F^N \) on \( \mathcal{G}_q \) means there is \( C > 0 \) so that

\[
\frac{1}{C} \leq \frac{|DF^N(q_1)e^n|}{|DF^N(q_2)e^n|} \leq C,
\]

\( q_1, q_2 \in \mathcal{G}_q \), uniformly in \( N \), where \( e^n \) is a unit tangent vector to \( \mathcal{G}_q \). Consequently, iterating under \( F^N \) does not change too much relative length of sets.) By (9), \( F^N(\mathcal{G}_q \cap W^{sc}_{loc}(J)) \) has nonempty intersection, in fact with positive Lebesgue measure, with \( \Lambda_0 \cap F^N(\mathcal{G}_q) \), for large enough \( N \). By item (ii) above, this shows the claim.
The remainder of the proof again follows the arguments of Proposition 2.3 with a smaller set $\Lambda$ still of positive Lebesgue measure ($\Lambda_0$ being replaced by $F^N(W^c_{loc}(J) \cap \Lambda_0)$). This concludes the proof of Theorem 1.2.

References