Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

DYNAMICS AND BIFURCATIONS OF RANDOM CIRCLE DIFFEOMORPHISMS

Hicham Zmarrou

KdV Institute for Mathematics
University of Amsterdam, Plantage Muidergracht 24, 1018TV Amsterdam, The Netherlands

Ale Jan Homburg

KdV Institute for Mathematics
University of Amsterdam, Plantage Muidergracht 24, 1018TV Amsterdam, The Netherlands

(Communicated by the associate editor name)

Abstract. We discuss iterates of random circle diffeomorphisms with identically distributed noise, where the noise is bounded and absolutely continuous. Using arguments of B. Deroin, V.A. Kleptsyn and A. Navas, we provide precise conditions under which random attracting fixed points or random attracting periodic orbits exist. Bifurcations leading to an explosion of the support of a stationary measure from a union of intervals to the circle are treated. We show that this typically involves a transition from a unique random attracting periodic orbit to a unique random attracting fixed point.

1. **Introduction.** We study bifurcations of randomly perturbed diffeomorphisms on the circle. The randomness of the diffeomorphisms $x \mapsto f_\omega(x)$ is expressed through the dependence on a parameter ω chosen independently from an absolutely continuous distribution ν on a compact interval Ω. Bifurcations are studied in the context of families $f_{a;\omega}$ of random diffeomorphisms parameterized by a deterministic parameter a.

Iterates of a random diffeomorphism f_ω can be studied with time either in $\mathbb{N}$ or $\mathbb{Z}$. With one sided time in $\mathbb{N}$ it is natural to consider the discrete Markov process defined by the random diffeomorphisms. In this context it makes sense to study bifurcations of stationary measures [28]. A particularly interesting bifurcation from this perspective is the explosion of the support of a stationary measure. In families $f_{a;\omega}$ of random circle diffeomorphisms the following scenario occurs:

- for a smaller than a bifurcation value a_0, $f_{a;\omega}$ admits a stationary measure supported on q disjoint intervals that are mapped cyclically under iteration of $f_{a;\omega}$,
- for $a > a_0$, $f_{a;\omega}$ admits a single stationary measure supported on all of $\mathbb{S}^1$.

A description with two sided time in $\mathbb{Z}$ is better suited for a study of dynamical properties like the occurrence of random attracting periodic orbits. The explosion of stationary measures typically manifests itself in the following scenario:

- for $a < a_0$ there is a single random attracting periodic orbit of period q,
- for $a > a_0$ there is a single random attracting fixed point.

2000 *Mathematics Subject Classification.* Primary: 37E10, 37H20; Secondary: 37G35.
Key words and phrases. Random dynamical systems, circle diffeomorphisms, bifurcations.

It is the purpose of this paper to make precise and prove this scenario.

Let us present the set-up. We consider random diffeomorphisms $f_\omega : \mathbb{S}^1 \to \mathbb{S}^1$, depending on a random parameter ω from an interval Ω (without loss of generality, we may take $\Omega = [-1, 1]$) drawn from a measure ν with density function g. We assume sufficient regularity:

$$(x, \omega) \mapsto f_\omega(x) \in C^\infty(\Omega \times \mathbb{S}^1, \mathbb{S}^1) \tag{1}$$

and

$$g \in C^\infty(\Omega) \tag{2}$$

(differentiability on Ω is always understood in the sense of differentiability on an open neighborhood of Ω). Note that uniform bounded noise gives $g = \frac{1}{2}$ in $C^\infty(\Omega)$. At various places in the manuscript we can do with less regularity, and it is often straightforward to adapt results correspondingly. For instance arguments as in § 4 on random fixed points work for random homeomorphisms. It is however interesting to observe the consequences of regularity assumptions of the noise distribution on the dynamics. Finally, we assume that

$$\omega \mapsto f_\omega(x) \text{ is an injective map for each } x. \tag{3}$$

This condition occurs naturally in the context of representations of Markov processes, see [28].

Write $\mathcal{R}^\infty(\mathbb{S}^1)$ for the space of random diffeomorphisms $x \mapsto f_\omega(x)$ on $\mathbb{S}^1$, depending on a random parameter $\omega \in \Omega$ drawn from the fixed distribution ν, with $f_\omega(x)$ smooth jointly in (x, ω). Iterates of f_ω are written as

$$f^k_{\omega_1,\ldots,\omega_k}(x) \quad = \quad f_{\omega_k}(f^{k-1}_{\omega_1,\ldots,\omega_{k-1}}(x)),$$

for $k > 0$. More generally, write $\Omega^\mathbb{Z}$ for all infinite sequences $\boldsymbol{\omega} = \{\omega_i\}_{i\in\mathbb{Z}}$ with each $\omega_i \in \Omega$. Denote $f^k_{\boldsymbol{\omega}}(x) = f^k_{\omega_1,\ldots,\omega_k}(x)$. Let $\vartheta : \Omega^\mathbb{Z} \to \Omega^\mathbb{Z}$ be the left shift operator; $\vartheta\{\omega_i\}_{i\in\mathbb{Z}} = \{\omega_{i+1}\}_{i\in\mathbb{Z}}$ and define the skew product system $S : \Omega^\mathbb{Z} \times \mathbb{S}^1 \to \Omega^\mathbb{Z} \times \mathbb{S}^1$ by

$$S(\boldsymbol{\omega}, x) = (\vartheta\boldsymbol{\omega}, f_{\omega_0}(x)). \tag{4}$$

On $\Omega^\mathbb{Z}$ one considers a measure ν^∞ which is the product of the measure ν over each Ω. Restricting time to $\mathbb{N}$ yields the left shift operator $\vartheta_+ : \Omega^\mathbb{N} \to \Omega^\mathbb{N}$ and the skew product system $S_+ : \Omega^\mathbb{N} \times \mathbb{S}^1 \to \Omega^\mathbb{N} \times \mathbb{S}^1$. Write ν^∞_+ for the measure on $\Omega^\mathbb{N}$ induced by ν.

A stationary measure m for the smooth random diffeomorphism f is a probability measure on $\mathbb{S}^1$ for which $\mu_+ = \nu^\infty_+ \times m$ is an S_+-invariant measure. Equivalently, see [18, 2],

$$m(A) = \int_\Omega (f_\omega)_* \, m(A) d\nu(\omega)$$

for Borel sets $A \subset \mathbb{S}^1$. A random circle diffeomorphism in $\mathcal{R}^\infty(\mathbb{S}^1)$ admits finitely many stationary measures [8]. In § 3 we collect some facts on stationary measures and invariant measures for the skew product systems with one and two sided time.

A random fixed point is a measurable map $X : \Omega^\mathbb{Z} \to \mathbb{S}^1$ providing an invariant graph

$$S(\boldsymbol{\omega}, X(\boldsymbol{\omega})) = (\vartheta\boldsymbol{\omega}, X(\vartheta(\boldsymbol{\omega})))$$

for ν^∞-almost all $\boldsymbol{\omega} \in \Omega^\mathbb{Z}$. A random periodic orbit of period q is likewise an invariant set with cardinality q in fibers $\{\boldsymbol{\omega}\} \times \mathbb{S}^1$ for ν^∞ almost every $\boldsymbol{\omega}$, see [21] for this notion and more restrictive variants. A random fixed point $X(\boldsymbol{\omega})$ is attracting if for a set of initial conditions $(\boldsymbol{\omega}, x_0) \in \Omega^{\mathbb{Z}} \times \mathbb{S}^1$ with positive $\nu^\infty \times \lambda$-measure (here λ is Lebesgue measure),

$$\lim_{n\to\infty} d\left(f^n_{\boldsymbol{\omega}}(x_0), f^n_{\boldsymbol{\omega}}(X(\boldsymbol{\omega}))\right) = 0 \tag{5}$$

(d being the distance on the circle). Similarly for random periodic orbits $\cup_{i=1}^k X^i(\boldsymbol{\omega})$,

$$\lim_{n\to\infty} d\left(f^n_{\boldsymbol{\omega}}(x_0), f^n_{\boldsymbol{\omega}}(\cup_{i=1}^k X^i(\boldsymbol{\omega}))\right) = 0. \tag{6}$$

In the cases we consider, there is a single random attracting and a single random repelling fixed or periodic orbit, while every point away from the random repelling orbit converges to the the random attracting orbit.

Results by Y. Le Jan [22] imply the existence of a random attracting periodic orbit of unspecified period for a class of randomly perturbed circle diffeomorphisms acting transitively on the circle (with an exceptional case in which an absolutely continuous measure is invariant under each diffeomorphism f_ω, $\omega \in \Omega$). The existence of a random attracting fixed point for a class of transitively acting iterated function systems on the circle was obtained earlier by V.A. Antonov [1]. In § 4 a precise result is presented (see Theorems 4.2 and 4.5) determining in particular the period of the random attracting periodic orbit. Main parts of this result, in slightly different contexts, are contained in the just mentioned reference [1] and [17, 19, 7]. We follow arguments from [7] where random fixed points for certain iterated function systems on the circle are treated. In § 5 we extend the results to families of random diffeomorphisms, depending on a deterministic parameter, discussing the bifurcation from a random period q orbit to a random fixed point. This is the typical bifurcation that appears in an explosion of the support of the stationary measure from q intervals to the circle.

2. **Random standard circle maps.** To illustrate the issues considered in this paper, we present some numerical computations involving the standard circle map with additive noise. The standard circle map acting on $x \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and depending on parameters a, ε is given by

$$s_a(x) = x + a + \frac{\varepsilon}{2\pi}\sin(2\pi x) \mod 1.$$

Consider s_a for a fixed value of $\varepsilon \in (0, 1)$ for which s_a is a diffeomorphism. With uniform noise added, we get

$$s_{a;\omega}(x) = x + \frac{\varepsilon}{2\pi}\sin(2\pi x) + a + \sigma\omega \mod 1 \tag{7}$$

for $x \in \mathbb{S}^1$ and a random parameter ω chosen from a uniform distribution on $\Omega = [-1, 1]$. The value of σ determines the amplitude of the noise; we take it fixed.

It is well known that random circle diffeomorphism have finitely many stationary measures with mutually disjoint support on intervals or the circle. Considering the action of the transfer operator on the space of smooth densities, one checks smoothness of densities of stationary measures [28].

Proposition 2.1. *A random circle diffeomorphism f_ω has finitely many stationary measures. The support E of a stationary measure m consists either of q mutually disjoint intervals, or the circle $\mathbb{S}^1$. The density function ϕ of m is in $C^\infty(\mathbb{S}^1)$.*

The randomly perturbed standard circle diffeomorphism has a unique stationary measure [28]. For families of random circle diffeomorphisms with unique stationary

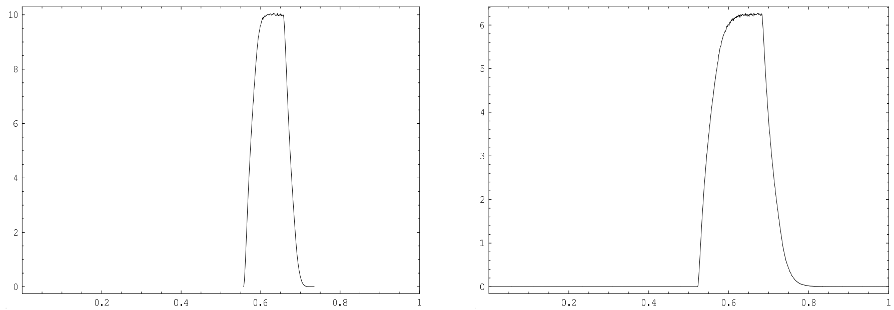

Figure 1. Numerically computed stationary densities of the random standard circle map (drawn are positive values of the density functions $\mathbb{S}^1 \to [0, \infty)$). On the left for $|a| + \sigma < \varepsilon/2\pi$, supported on an interval. On the right for $|a| + \sigma > \varepsilon/2\pi$, with the circle as support.

measures, such as the random standard circle diffeomorphism, the stationary densities are smooth functions that depend smoothly on the parameter as well. This is shown by an application of the implicit function theorem [28].

Proposition 2.2. *Let $f_{a;\omega}$ be a family of random circle diffeomorphisms with a unique stationary measure m_a for each value of a. The support of m_a consists either of q mutually disjoint intervals, or the circle $\mathbb{S}^1$. The density function ϕ_a of m_a is in $C^\infty(\mathbb{S}^1)$ and depends C^∞ on a.*

Observe that the deterministic standard circle diffeomorphism s_a has a hyperbolic fixed point for $a \in (-\frac{\varepsilon}{2\pi}, \frac{\varepsilon}{2\pi})$. Hence, $s_{a;\omega}$ has a stationary measure supported on a single interval precisely if both $a - \sigma > -\frac{\varepsilon}{2\pi}$ and $a + \sigma < \frac{\varepsilon}{2\pi}$. This occurs for a nonempty interval of values for a if $\sigma < \frac{\varepsilon}{2\pi}$.

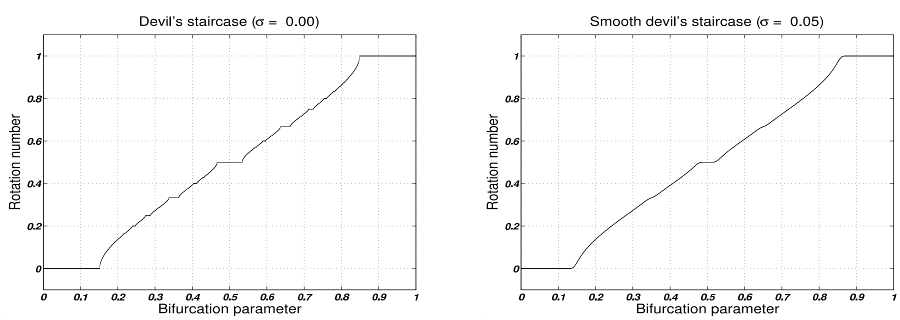

Figure 2. The function $a \mapsto \rho_a$. On the left the devil's staircase; the rotation number of the deterministic standard family. On the right the rotation number of the random standard family.

Central in the study of deterministic circle diffeomorphisms is the notion of rotation number, see e.g. [6]. Given $f_{a;\omega} \in \mathcal{R}^\infty(\mathbb{S}^1)$, write $F_{a;\omega} : \mathbb{R} \to \mathbb{R}$ for its lift. We define the rotation number of $f_{a;\omega}$, when it exists, by

$$\rho_{a;\boldsymbol{\omega}}(x) = \lim_{k\to\infty} \frac{F^k_{a;\boldsymbol{\omega}}(x) - x}{k} \tag{8}$$

(see [25]). The rotation number measures the average rotation per iterate of the circle diffeomorphisms. Note that $\rho_{a;\boldsymbol{\omega}}$ is a random variable, depending also on the starting point x. By Birkhoff's ergodic theorem and Proposition 2.2, we get that the rotation number is a smooth function of the parameter.

Proposition 2.3. *For fixed a, the rotation number $\rho_{a;\boldsymbol{\omega}}$ exists for ν_+^∞ almost all $\boldsymbol{\omega}$ and is a constant ρ_a independent of x and $\boldsymbol{\omega}$. For a family of random circle diffeomorphisms $f_{a;\omega}$ with a unique stationary measure m_a for each value of a, $a \mapsto \rho_a$ is C^∞.*

The rotation number is constant in a when the support of the stationary measure is not the circle but a union of intervals; in Figure 2 this is visible for the standard circle diffeomorphisms. The rotation number is a flat function of a at values $\rho_a = 0$ and $\rho_a = 1/2$. Bifurcations of the stationary measure (discontinuous changes of the support) occur at the end values of these intervals.

3. **Invariant and stationary measures.**

A stationary measure gives rise to an invariant measure for the skew product system S. The following results discuss this relation and properties of the invariant measure.

Proposition 3.1. *Let m be a stationary measure for the random diffeomorphism f_ω. Then there exists a measurable map $\mathcal{L} : \Omega^{\mathbb{Z}} \to \mathcal{M}(\mathbb{S}^1)$ (the space of probability measures on $\mathbb{S}^1$), such that*

$$\left(f^n_{\omega_{-n},\ldots,\omega_{-1}}\right)_* m \to \mathcal{L}(\boldsymbol{\omega})$$

in the weak-$*$ topology as $n \to \infty$, ν_+-almost surely. The measure μ on $\Omega^{\mathbb{Z}} \times \mathbb{S}^1$ with marginal ν^∞ and disintegration $\mu_{\boldsymbol{\omega}} = \mathcal{L}(\boldsymbol{\omega})$ is an S-invariant measure.

Proof. See [10, Theorem 7.5], [22, Lemme 1] or [3, Theorem 1.7.2]. □

Recall that μ is mixing if

$$\lim_{n\to\infty} \mu(S^{-n}(V) \cap W) = \mu(V)\mu(W)$$

(similarly for $\mu_+ = \nu_+^\infty \times m$ and S_+), see e.g. [27]. A stationary measure m for the discrete Markov process is called mixing if the averaged correlations decay to zero:

$$\lim_{n\to\infty} \int_{\Omega^n} m(\left(f^n_{\omega_1,\ldots,\omega_n}\right)^{-1}(A) \cap B) d\nu(\omega_1)\cdots d\nu(\omega_n) = m(A)m(B). \tag{9}$$

Proposition 3.2. *Suppose m is a unique stationary measure. The measures μ_+, μ are ergodic. If the support of the stationary measure m is connected, then the stationary measure m is mixing and the invariant measures μ_+, μ are mixing.*

Proof. We will establish equivalence between the different mixing properties of the stationary measure m for the Markov process and the measures μ_+, μ for the skew product systems with one or two sided time. That is, the following statements are equivalent.

(a) $\lim_{n\to\infty} \int_{\Omega^n} m(\left(f^n_{\omega_1,\ldots,\omega_n}\right)^{-1}(A) \cap B) d\nu(\omega_1)\cdots d\nu(\omega_n) = m(A)m(B)$ for Borel sets $A, B \subset \mathbb{S}^1$.
(b) $\lim_{n\to\infty} \mu(S_+^{-n}(V) \cap W) = \mu_+(V)\mu_+(W)$ for Borel sets $V, W \subset \Omega^{\mathbb{N}} \times \mathbb{S}^1$.
(c) $\lim_{n\to\infty} \mu(S^{-n}(V) \cap W) = \mu(V)\mu(W)$ for Borel sets $V, W \subset \Omega^{\mathbb{Z}} \times \mathbb{S}^1$.

The proposition follows since, in our context with a unique stationary measure, the stationary measure is ergodic and mixing in case its support is connected [8, 28].

To prove that (a) implies (b), it suffices to consider product sets $V = A_1 \times B_1$, $W = A_2 \times B_2$ (compare [27, Theorem 1.17]). We have

$$\mu_+(S^{-n}(V)) = \mu_+\left(\bigcup_{\omega_1,\ldots,\omega_n\in\Omega} \{(\omega_1,\ldots,\omega_n)\} \times A_1 \times \left(f^n_{\omega_1,\ldots,\omega_n}\right)^{-1}(B_1)\right).$$

Hence

$$\begin{aligned} &\nu_+^\infty \times m(S^{-n}(V) \cap W) \\ &= \nu_+^\infty(\vartheta^{-n}(A_1) \cap A_2) \int_{\Omega^n} m\left(\left(f^n_{\omega_1,\ldots,\omega_n}\right)^{-1}(B_1) \cap B_2\right) d\nu(\omega_1)\cdots d\nu(\omega_n) \\ &\to \nu_+^\infty(A_1)\nu_+^\infty(A_2) m(B_1) m(B_2) \\ &= \nu_+^\infty \times m(V) \nu_+^\infty \times m(W), \end{aligned}$$

as $n \to \infty$. That (b) implies (a) follows from the above computation with the fact that ν_+^∞ is mixing and μ_+ is mixing.

For the equivalence of (b) and (c) we note that the skew product S with the invariant measure μ is the natural extension of S_+ with the invariant measure $\mu_+ = \nu_+^\infty \times m$ [3, Appendix A]. A natural extension inherits ergodicity and mixing properties. Clearly (c) implies (b) as the system with time $\mathbb{N}$ is a factor of the system with time $\mathbb{Z}$. To see that (b) implies (c), we need to show

$$\lim_{n\to\infty} \mu(S^{-n}(V) \cap W) = \mu(V)\mu(W), \tag{10}$$

for Borel sets V, W in $\Omega^{\mathbb{Z}} \times \mathbb{S}^1$. Write O_1, O_2 for the coordinate projections of V, W onto $\Omega^{\mathbb{Z}}$. For $\varepsilon > 0$, take two sets V', W' with $\mu(V \bigtriangleup V') < \varepsilon$ and $\mu(W \bigtriangleup W') < \varepsilon$, such that the coordinate projections $O_1', O_2' \subset \Omega^{\mathbb{Z}}$ are cylinder sets. Then for some $n > 0$, $\vartheta^{-n}(O_1')$ defines a cylinder set in $\Omega^{\mathbb{N}}$. By the mixing property of μ_+, (10) holds for V', W'. By approximation it is true for all Borel sets. □

The disintegrations $\mu_{\boldsymbol{\omega}}$ are called fiber mixing if

$$\lim_{n\to\infty} \mu_{\boldsymbol{\omega}}(\left(f^n_{\omega_1,\ldots,\omega_n}\right)^{-1}(A) \cap B) = \mu_{\vartheta^n\boldsymbol{\omega}}(A)\mu_{\boldsymbol{\omega}}(B), \tag{11}$$

see [4]. Even when μ is mixing, the $\mu_{\boldsymbol{\omega}}$'s need not be fiber mixing. This follows from Theorem 4.2; random circle diffeomorphisms that are equivariant under the action of a cyclic group can be expected to have multiple random attracting fixed points in each fiber.

4. **Random periodic and fixed points.** In this section we consider dynamics of skew product systems $S : \Omega^{\mathbb{Z}} \times \mathbb{S}^1$ defined by random circle diffeomorphisms f_ω. Initially we work under the assumption of a stationary measure m whose support equals $\mathbb{S}^1$, so that m is equivalent to Lebesgue measure λ. Note that this occurs if one of the diffeomorphisms f_ω, $\omega \in \Omega$, has irrational rotation number. Recall that Proposition 3.1 constructs the invariant measure μ, with disintegrations $\mu_{\boldsymbol{\omega}}$, for the skew product system S on $\Omega^{\mathbb{Z}} \times \mathbb{S}^1$. Theorem 4.2 below gives a precise characterization of $\mu_{\boldsymbol{\omega}}$. Theorem 4.5 similarly treats random circle diffeomorphisms f_ω for which the stationary measure is supported on a union of intervals.

Although we state the results in the context of random diffeomorphisms, differentiability of f_ω is not used in the proof; one only needs to assume that f_ω are homeomorphisms depending continuously on ω and the existence of stationary measures equivalent to Lebesgue for the semigroups generated by the maps f_ω and the maps f_ω^{-1}. The proof owes much to arguments in [7, Proposition 5.7], where random fixed points for a class of iterated function systems of circle homeomorphisms are considered. References [1, 22, 17, 19, 20] contain earlier results with a similar flavor. It is interesting to compare the result with characterizations for groups of homeomorphisms, as in [11, 23, 14].

Skew products with different dynamics in the base have also been frequently studied. See [12], compare also [13], for skew products over horseshoes, generalizing iterated function systems. For a result on random fixed points in skew products over a hyperbolic torus automorphism, see [24]. In the context of circle diffeomorphisms or homeomorphisms with quasiperiodic forcing (giving rise to skew product systems with a minimal system in the base), invariant measures and invariant graphs have been studied by many authors starting from [9], see e.g. [5, 26, 15, 16].

Definition 4.1. Consider one of the following properties, where the second property is the opposite of the first and the third property is a special stronger case of the second property.

1. The random diffeomorphisms are equicontinuous: for each $\varepsilon > 0$, there exists $\delta > 0$ so that for all $\boldsymbol{\omega} \in \Omega^{\mathbb{N}}$ and each interval $I \subset \mathbb{S}^1$ with $\lambda(I) < \delta$, we have $(f^n_{\boldsymbol{\omega}})_* \lambda(I) < \varepsilon$ for all $n \in \mathbb{N}$.
2. The random diffeomorphisms are contractive: there exists $\varepsilon_0 > 0$ so that for all $\delta > 0$, there is $\boldsymbol{\omega} \in \Omega^{\mathbb{N}}$ and an interval $I \subset \mathbb{S}^1$ with $\lambda(I) < \delta$ so that $(f^n_{\boldsymbol{\omega}})_* \lambda(I) > \varepsilon_0$ for some $n \geq 0$.
3. The random diffeomorphisms are strongly contractive: for all $\varepsilon > 0$, $\delta > 0$, there is $\boldsymbol{\omega} \in \Omega^{\mathbb{N}}$ and an interval $I \subset \mathbb{S}^1$ with $\lambda(I) < \delta$ so that $(f^n_{\boldsymbol{\omega}})_* \lambda(I) > 1-\varepsilon$ for some $n \geq 0$.

The next result gives precise conditions for the existence of random attracting fixed points, and shows in particular that unique random attracting fixed points are a common feature in random circle diffeomorphisms.

Theorem 4.2. *Consider random circle diffeomorphisms for which supp*$(m) = \mathbb{S}^1$*. Exactly one of the following possibilities occurs:*

1. *The random diffeomorphisms are equicontinuous. The stationary measure m is invariant under each f_ω.*
2. *The random diffeomorphisms are contractive but not strongly contractive. Then there exists a smooth nontrivial periodic diffeomorphism θ, $\theta^k = id$, on $\mathbb{S}^1$ that commutes with every f_ω. Moreover, for almost all $\boldsymbol{\omega}$, $\mu_{\boldsymbol{\omega}}$ is a union $\frac{1}{k}\sum_{i=1}^k \delta_{X^i(\boldsymbol{\omega})}$ of k delta-measures of mass $1/k$.*
3. *The random diffeomorphisms are strongly contractive and $\mu_{\boldsymbol{\omega}}$ is a delta-measure $\delta_{X(\boldsymbol{\omega})}$ for almost all $\boldsymbol{\omega}$.*

A contractive random diffeomorphism has a unique random attracting set and a unique random repelling set both consisting of k points ($k = 1$ for a strongly contractive random diffeomorphism).

Proof. We begin the proof with a separate lemma.

Lemma 4.3. *Assume the random diffeomorphisms are contractive. Then there exists $0 < \varepsilon_0 < 1$, so that for all $\varepsilon > 0$ the following holds: for almost all $\boldsymbol{\omega} \in \Omega^{\mathbb{Z}}$, there is an interval I with $|I| < \varepsilon$ so that $\left(f^j_{\omega_{-j},\ldots,\omega_{-1}}\right)_* m(I) > \varepsilon_0$ for some j.*

Proof. Take $\varepsilon > 0$. We provide $\delta > 0$, $l \in \mathbb{N}$, $0 < \varepsilon_0 < 1$, so that the following holds. For each $\omega_{-n}, \ldots, \omega_{-1} \in \Omega^n$ we construct an interval I with length $|I| < \varepsilon$ and a subset $\Sigma \subset \Omega^{l'}$ for some $l' \leq l$ with $\nu^{l'}(\Sigma) > \delta$, so that for each $(\omega_{-n-l'}, \ldots, \omega_{-n-1}) \in \Sigma$,

$$\left(f^{n+l'}_{\omega_{-n-l'},\ldots,\omega_{-1}}\right)_* m(I) > \varepsilon_0.$$

The lemma will be shown to follow from this.

Take $q > 1/\varepsilon$ and an orbit piece $a_i = f^i_{\boldsymbol{\eta}}(a_0)$, $0 \leq i < q$, consisting of q different points. Take an open interval U containing a_0 so that $U_i = f^i_{\boldsymbol{\eta}}(U)$, $0 \leq i < q$, are mutually disjoint. By Definition 4.1(2) and $\mathrm{supp}(m) = \mathbb{S}^1$, there exists an interval V with $m(V) > \varepsilon_0$ for some positive ε_0 and $(\beta_1, \ldots, \beta_j) \in \Omega^j$, so that

$$f^j_{\beta_1,\ldots,\beta_j}(V) \subset U. \tag{12}$$

There is further $0 \leq i_* < q$ with

$$\left| f^n_{\omega_{-n},\ldots,\omega_{-1}}(U_{i_*}) \right| < \varepsilon. \tag{13}$$

This is clear as the q disjoint intervals $f^n_{\omega_{-n},\ldots,\omega_{-1}}(U_i)$, $0 \leq i < q$, can not all have length $\geq \varepsilon$ by $q > 1/\varepsilon$.

Write

$$I = f^n_{\omega_{-n},\ldots,\omega_{-1}}(U_{i_*}).$$

Define $(\omega_{-n-l'}, \ldots, \omega_{-n-1})$ as the sequence consisting of $\beta_1, \ldots, \beta_j$ followed by $\eta_1, \ldots, \eta_{i_*}$. Then

$$\begin{aligned} \left(f^{n+l'}_{\omega_{-n-l'},\ldots,\omega_{-1}}\right)_* m(I) &= \left(f^n_{\omega_{-n},\ldots,\omega_{-1}} \circ f^{i_*+j}_{\omega_{-n-l'},\ldots,\omega_{-n-1}}\right)_* m(I) \\ &= m\left(\left(f^j_{\beta_1,\ldots,\beta_j}\right)^{-1}(U)\right) \geq m(V) > \varepsilon_0. \end{aligned} \tag{14}$$

Note that i_* depends on $\omega_{-n}, \ldots, \omega_{-1}$. By the continuous dependence of f_ω on ω, there is for each i_* an open set of sequences $\omega_{-n-l'}, \ldots, \omega_{-n-1}$ in Ω^{i_*+j} for which (14) holds. Let δ be the minimum of the measures of the sets of sequences in Ω^{i_*+j} thus constructed. Define

$$\begin{aligned} \Delta_N = \{&\boldsymbol{\omega} \in \Omega^{\mathbb{Z}} \mid \text{ for each interval } I \text{ with } |I| < \varepsilon \text{ and each } i \leq N, \\ &\left(f^i_{\omega_{-i},\ldots,\omega_{-1}}\right)_* m(I) \leq \varepsilon_0\}. \end{aligned}$$

It follows from the above that $\nu^\infty(\Delta_{tl}) \leq (1-\delta)^t$. Thus $\nu^\infty(\Delta) = 0$, where

$$\begin{aligned} \Delta = \{&\boldsymbol{\omega} \in \Omega^{\mathbb{Z}} \mid \text{ for each interval } I \text{ with } |I| < \varepsilon \text{ and each } i, \\ &\left(f^i_{\omega_{-i},\ldots,\omega_{-1}}\right)_* m(I) \leq \varepsilon_0\}. \end{aligned}$$

This proves the lemma. □

Now let ε in the statement of the lemma go to 0. Then j in the lemma goes to ∞ and by Proposition 3.1, $\mu_{\boldsymbol{\omega}}$ contains a union of delta measures. As the set of points with positive measure (or with measure lying in some interval) in the fiber is invariant under S, by ergodicity (recall Proposition 3.2) we can write

$$\mu_{\boldsymbol{\omega}} = \frac{1}{k} \sum_{i=1}^{k} \delta_{x_i(\boldsymbol{\omega})}$$

as a sum of delta measures at points $x_i(\boldsymbol{\omega})$ with mass $1/k$ each (for some $k \leq 1$).

We continue by showing that $k > 1$ occurs only if there is a nontrivial periodic diffeomorphism that commutes with each f_ω. If J is a small interval around $x_i(\boldsymbol{\omega})$, disjoint from $x_j(\boldsymbol{\omega})$ for $j \neq i$, then from Proposition 3.1 one deduces that $m(f_{\boldsymbol{\omega}}^{-n}(J))$ converges to $1/k$ as $n \to \infty$. Consider the skew product system $S_-(\boldsymbol{\omega}, x) = (\vartheta^{-1}\boldsymbol{\omega}, (f_{\boldsymbol{\omega}})^{-1}(x))$ on $\Omega^{-\mathbb{N}} \times \mathbb{S}^1$. Write ν_-^∞ for the product measure on $\Omega^{-\mathbb{N}}$. There is an invariant measure $\nu_-^\infty \times m_-$ for S_-. As this invariant measure for S_- is mixing and m_- is equivalent to Lebesgue measure, $S_-(\boldsymbol{\omega}, x)$ has a dense forward orbit for $\nu_-^\infty \times \lambda$ almost every point $(\boldsymbol{\omega}, x)$. Then also for almost all fibers $\{\boldsymbol{\omega}\} \times \mathbb{S}^1$, the forward orbit of all points from these fibers under S_- is dense. We may therefore assume that this holds for the boundary points of $f_{\boldsymbol{\omega}}^{-n}(J)$. Define $\theta : \mathbb{S}^1 \to \mathbb{S}^1$ by $\theta(x) = y$ with $m(x, y) = 1/k$. Smoothness of the density function of m implies that θ is a diffeomorphism. We claim that θ commutes with each f_ω. Writing $J = (q, p)$ and $(q_{-n}, p_{-n}) = f_{\boldsymbol{\omega}}^{-n}(J)$, if q_{-n} converges to x then p_{-n} converges to $\theta(x)$. If θ does not commute with some f_ω, then by continuity $\theta f_{\boldsymbol{\omega}}(x) \neq f_{\boldsymbol{\omega}}\theta(x)$ for $(\boldsymbol{\omega}, x)$ from some open set U in $\Omega^{\mathbb{Z}} \times \mathbb{S}^1$. There are arbitrary large values of n for which $(f_{\boldsymbol{\omega}}^{-n}(x), \vartheta^{-n}\boldsymbol{\omega})$ lies in U. Proposition 3.1 however implies that $f_{\omega_{-n}} \circ \theta(q_{-n})$ is close to $\theta \circ f_{\omega_{-n}}(q_{-n})$ for n large, leading to a contradiction. Note that by dividing out the action of θ, one obtains a random diffeomorphism that acts strongly contractive.

Next we show, following [19], that $\cup_{i=1}^k X^i(\boldsymbol{\omega})$ yields an attractor in the sense (6). Suppose for simplicity that the random diffeomorphism acts strongly contractive. By the same construction a random point measure at $Y^1(\boldsymbol{\omega})$ for the inverse maps is obtained. Thus for almost all $\boldsymbol{\omega}$ one has for $y, z \neq Y^1(\boldsymbol{\omega})$,

$$|f_{\boldsymbol{\omega}}^i(y) - f_{\boldsymbol{\omega}}^i(z)| \to 0$$

as $i \to \infty$. The distribution of the points $Y^1(\boldsymbol{\omega})$ is absolutely continuous. Also $X^i(\boldsymbol{\omega})$ and $Y^i(\boldsymbol{\omega})$ are independent; $X^i(\boldsymbol{\omega})$ depends only on the past of $\boldsymbol{\omega}$ while $Y^i(\boldsymbol{\omega})$ depends only on the future of $\boldsymbol{\omega}$. Therefore $Y^1(\boldsymbol{\omega}) \neq X^1(\boldsymbol{\omega})$ for almost all $\boldsymbol{\omega}$. It follows that for almost all points x_0, $f_{\boldsymbol{\omega}}^n(x_0)$ converges to one of the points $X^i(\vartheta^n\boldsymbol{\omega})$.

Finally, statement (i) is proved in [7, Lemme 5.4]. Here the existence of a measure invariant under each f_ω is proved. This measure equals the stationary measure. □

The random attracting fixed points form a dense subset of $\Omega^{\mathbb{Z}} \times \mathbb{S}^1$.

Lemma 4.4. *Suppose supp$(m) = \mathbb{S}^1$. For almost all $\boldsymbol{\omega} \in \Omega^{\mathbb{Z}}$, the orbit of $(\boldsymbol{\omega}, X^i(\boldsymbol{\omega}))$ under S lies dense in $\Omega^{\mathbb{Z}} \times \mathbb{S}^1$.*

Proof. The statement is a consequence of the following observation: the set of points $(\boldsymbol{\omega}, x) \in \Omega^{\mathbb{Z}} \times \mathbb{S}^1$ with dense orbits under the action of the skew product S, has full $\nu^\infty \times \lambda$-measure.

For this, recall first that with one sided time, S_+ has a mixing invariant measure $\nu_+^\infty \times m$. Hence, $\nu_+^\infty \times m$ almost every point has a dense forward orbit. Note that m is equivalent to Lebesgue measure, so that $\nu_+^\infty \times \lambda$ almost every point has a dense forward orbit under S_+.

One can likewise consider the one sided time skew product defined by iterating the inverse diffeomorphisms f_ω^{-1}, for which the same statement holds.

Together this implies the observation and thus the lemma. □

If the support of the stationary measure is not the entire circle, it consists of a finite union of intervals E.

Theorem 4.5. *Consider a random diffeomorphism f_ω in $R^k(\mathbb{S}^1)$, with a stationary measure m supported on a union E of q closed intervals. For almost all $\boldsymbol{\omega}$, $\mu_{\boldsymbol{\omega}}$ is a union $\frac{1}{q}\sum_{i=1}^q \delta_{X^i(\boldsymbol{\omega})}$ of q delta measures of mass $1/q$, with one point in each interval in E. The orbit of the points $X^i(\boldsymbol{\omega})$ give a random attracting periodic orbit.*

Proof. The type of argument used to prove Theorem 4.2 can be applied to show the existence of a random point measure $\frac{1}{q}\sum_{i=1}^q \delta(X^i(\boldsymbol{\omega}))$, with one point in each connected component of E. The existence of a random point measure follows alternatively by reasoning as in [3, Theorem 1.8.4 (iv)].

To show that $\cup_{i=1}^q X^i(\boldsymbol{\omega})$ is a random attractor in the sense of (6), we consider the reasoning used for Theorem 4.2. Assume for simplicity that m is a unique stationary measure and that the inverse diffeomorphisms have a unique stationary measure supported on q intervals F. The inverse circle diffeomorphisms give rise to a random point measure $\frac{1}{q}\sum_{i=1}^q \delta(Y^i(\boldsymbol{\omega}))$, with one point in each connected component of F.

Suppose for the sake of argument that $q = 1$. The reasoning in the proof of Theorem 4.2 shows that $\left(f^n_{\vartheta^{-n}\boldsymbol{\omega}}\right)_* m$ has a subsequence converging to $\delta_{X(\boldsymbol{\omega})}$. As Proposition 3.1 gives convergence, $\left(f^n_{\vartheta^{-n}\boldsymbol{\omega}}\right)_* m$ converges to $\delta_{X(\boldsymbol{\omega})}$. In fact, the reasoning shows that $f^n_{\vartheta^{-n}\boldsymbol{\omega}}$ maps E into a small interval. Convergence therefore holds with m replaced by any measure supported on E. Each point outside of F eventually falls into E under iterates of the random diffeomorphism. Note that there is a composition $f^r_{\nu_1,\dots,\nu_r}$ whose iterates map almost all points from F outside of F and into E. This implies, again following the argument in the proof of Theorem 4.2, that for any $\varepsilon > 0$ there is an interval V of size at least $1-\varepsilon$ and an iterate $f^n_{\vartheta^{-n}\boldsymbol{\omega}}$ mapping V into an interval of size at most ε. From the invariance of E, this in turn implies that $\left(f^n_{\vartheta^{-n}\boldsymbol{\omega}}\right)_* \lambda$ converges to $\delta_{X(\boldsymbol{\omega})}$ as $n \to \infty$. The argument is now finished as for Theorem 4.2; $q > 1$ is treated with obvious modifications. □

We remark that in case the diffeomorphisms $f^k_{\omega_1,\dots,\omega_k}$ act by a contraction on E, the random periodic points form a continuous graph as a graph transform technique proves. Note further that V. Araújo [2] proves in a more general framework that random repelling sets have zero $\nu_+^\infty \times \lambda$ measure.

The typical situation for random circle diffeomorphisms with a stationary measure equivalent to Lebesgue measure is to possess a unique random attracting and a unique random repelling fixed point. The next lemma emphasizes this property. It illustrates how Hypothesis 4.1(iii) is satisfied.

Lemma 4.6. *For a generic random diffeomorphism f such that the rotation numbers $\rho(f_{-1})$ and $\rho(f_1)$ are different, there exists a map $f^n_{\omega_1,\dots,\omega_n}$ with precisely one hyperbolic attracting and one hyperbolic repelling fixed point.*

Proof. We can take $\nu_1, \nu_2 \in \Omega$ such that f_{ν_1} has rational rotation number, say p/q, and f_{ν_2} has irrational rotation number. By the genericity assumption, we may assume that $f^q_{\nu_1}$ has a finite number of fixed points, all hyperbolic. In the coordinate in $[0,1)$ on the circle, write $a_1,\dots,a_m$ for the attracting fixed points, in order of increasing angle on the circle. We take indices $\mod m$, so that $a_{m+1} = a_1$. Write $r_1,\dots,r_m$ for the repelling fixed points with $r_i \in [a_i, a_{i+1}]$. Generically the distances between neighboring attracting fixed points a_i, a_{i+1} are different for different i. We assume this to be the case. Similarly we assume that the distances between repelling fixed points are all different. By relabeling the fixed points we may assume that the minimal distance between a_i and r_i is assumed for $i = 1$. Finally, by [6, Theorem

6.1] we may assume that f_{ν_2} is a rotation $x \mapsto x + \theta$ (note that this assumes a diophantine condition on the rotation number of f_{ν_2}, which is no restriction in our context).

We will construct a map $f_{\nu_1}^{qp_1} \circ f_{\nu_2}^{p_2} \circ f_{\nu_1}^{qp_3}$ that has $m-1$ attracting fixed points and $m-1$ repelling fixed points, close to attracting and repelling fixed points of $f_{\nu_1}^{q}$. The proposition then follows by induction.

Write $B_i = (r_{i-1}, r_i)$ for the basin of attraction of a_i. Take compact intervals $I_i \subset B_i$ with small symmetric difference $I_i \triangle B_i$. For p_1 large, $f_{\nu_1}^{qp_1}$ is a contraction on all intervals I_i and maps I_i into a small neighborhood of the point a_i. As f_{ν_2} is an irrational rotation, we can take $p_2 \in \mathbb{N}$ such that $f_{\nu_2}^{p_2}(a_1) \subset B_2$ and $f_{\nu_2}^{p_2}(a_i) \subset B_i$ for all other i. By taking p_1 large enough, the same holds with a_i replaced by $f_{\nu_1}^{qp_1}(I_i)$. Write $I_{1,2}$ for the convex hull of I_1 and I_2 inside the interior of $\overline{B_1 \cup B_2}$. Take p_3 large enough so that $f_{\nu_1}^{p_3}$ is a contraction on $I_{1,2}$. Then $f_{\nu_1}^{qp_1} \circ f_{\nu_2}^{p_2} \circ f_{\nu_1}^{qp_3}$ has $m-1$ hyperbolic attracting fixed points in $I_{1,2}, I_3, \ldots, I_m$.

The proof is finished by establishing that the $m-1$ intervals in $\mathbb{S}^1 \setminus \{I_{1,2} \cup I_3 \cup \cdots \cup I_m\}$ each contain only a hyperbolic repelling fixed point, for p_1, p_3 large enough. Take compact intervals J_i inside (a_i, a_{i+1}) (the basin of attraction of r_i for $f_{\omega_1}^{-q}$) for $i \neq 1$, so that $J_i \triangle (a_i, a_{i+1})$ is small. The intervals $I_{1,2}, I_3, \ldots, I_m$ and $J_2, \ldots, J_m$ cover $\mathbb{S}^1$. For p_3 large enough, $f_{\nu_1}^{-qp_3}$ is a contraction on the intervals $J_2, \ldots, J_m$ and maps them into small neighborhoods of $r_2, \ldots, r_m$. By construction, $f_{\nu_1}^{qp_3} \circ f_{\nu_2}^{-p_2}$ maps J_i inside (a_i, a_{i+1}), $i = 2, \ldots, m$. For p_1 large enough, $f_{\nu_1}^{qp_3} \circ f_{\nu_2}^{-p_2} \circ f_{\nu_1}^{-qp_1}$ is a contraction on $J_2, \ldots, J_m$. This finishes the proof. □

5. **Random saddle node bifurcations.** The material in the previous two sections shows that generically the following picture holds true. A stationary measure supported on q intervals implies the existence of a unique random attracting periodic orbit of period q. A stationary measure supported on the circle implies the existence of a unique random attracting fixed point. We treat here the bifurcation where the support of the stationary measure explodes from q intervals to the circle. One can expect that in such a bifurcation a random periodic orbit of period q bifurcates to a random fixed point. This picture is confirmed in Theorem 5.2 below.

Definition 5.1. The smooth one parameter family of random diffeomorphisms f_a on the circle undergoes a random saddle node bifurcation at $a = a_0$, if there exists $\bar{x}$ in the boundary of the support of a stationary measure such that

$$f^k_{a_0;\omega_1,\ldots,\omega_k}(\bar{x}) = \bar{x}, \qquad \frac{d}{dx} f^k_{a_0;\omega_1,\ldots,\omega_k}(\bar{x}) = 1, \tag{15}$$

for some $\omega_1, \ldots, \omega_k \in \partial\Omega$.

The random saddle node bifurcation is said to unfold generically, if

$$\left(\frac{d}{dx}\right)^2 f^k_{a_0;\omega_1,\ldots,\omega_k}(\bar{x}) \neq 0, \qquad \frac{\partial}{\partial a} f^k_{a;\omega_1,\ldots,\omega_k}(\bar{x}) \neq 0 \tag{16}$$

at $a = a_0$.

It is shown in [28] that, in the context of families of random circle diffeomorphisms, a (generically unfolding) random saddle node bifurcation is the only possible codimension one bifurcation of stationary densities.

Theorem 5.2. *Suppose that f_a is a generic family of random diffeomorphisms with a random saddle node bifurcation at $a = a_0$, so that the support E_a of the stationary measure μ_a has q connected components for $a < a_0$ and equals the circle for $a > a_0$.*

For each $a \leq a_0$ and a close to a_0, f_a^q has one random attracting fixed point in each connected component of E_a. For $a > a_0$ and a close to a_0, f_a has one attracting and one repelling random fixed point.

Proof. For each $a > a_0$ sufficiently close to a_0, the random diffeomorphism f_a satisfies Hypothesis 4.1(iii). Apply Theorem 4.2. For $a \leq a_0$, consider f_a^q. Each interval in E_a is mapped into itself by f_a^q. Theorem 4.5 deals with this case. □

Under the conditions of the above theorem, the density function of the stationary measure m_a is smooth and depends smoothly on the parameter a even though the support of m_a varies discontinuously in the Hausdorff topology [28]. In a random saddle-node bifurcation where the support of the stationary measure changes discontinuously while remaining to consist of q intervals, one will for generic families find a single random attracting period q orbit for all parameter values near the bifurcation value.

Acknowledgements. We are particularly grateful to the referees for their detailed comments.

REFERENCES

[1] V. A. Antonov. Modeling of processes of cyclic evolution type. synchronization by a random signal. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 2:67–76, 1984.
[2] V. Araújo. Attractors and time averages for random maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17:307–369, 2000.
[3] L. Arnold. *Random dynamical systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
[4] T. Bogenschütz and Z.S. Kowalski. A condition for mixing of skew products. *Aequationes Math.*, 59:222–234, 2000.
[5] K.M. Campbell and M.E. Davies. The existence of inertial functions in skew product systems. *Nonlinearity*, 9:801–817, 1996.
[6] W. de Melo and S. van Strien. *One-dimensional dynamics*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.
[7] B. Deroin, V. Kleptsyn, and A. Navas. Sur la dynamique unidimensionnelle en régularité intermédiaire. *Acta Math.*, 199:199–262, 2007.
[8] J. L. Doob. *Stochastic processes*. John Wiley & Sons Inc., New York, 1953.
[9] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.*, 83:573–601, 1961.
[10] H. Furstenberg. Boundary theory and stochastic processes on homogeneous spaces. In *Proc. Sympos. Pure Math.*, volume 26, pages 193–229, 1973.
[11] E. Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.
[12] A. S. Gorodetskiĭ and Yu. S. Il′yashenko. Certain properties of skew products over a horseshoe and a solenoid. Grigorchuk, R. I. (ed.), Dynamical systems, automata, and infinite groups. Transl. from the Russian. Moscow: MAIK Nauka/Interperiodica Publishing, Proc. Steklov Inst. Math. 231, 90-112 (2000); translation from Tr. Mat. Inst. Steklova 231, 96-118 (2000)., 2000.
[13] A. S. Gorodetskiĭ, Yu. S. Il′yashenko, V. A. Kleptsyn, and M.B. Nalskii. Nonremovability of zero lyapunov exponents. *Funct. Anal. Appl.*, 39:21–30, 2005.
[14] S. Hurder. Dynamics of expansive group actions on the circle, 2005. To appear in *Geom. Ded.*
[15] T. H. Jäger and G. Keller. The Denjoy type of argument for quasiperiodically forced circle diffeomorphisms. *Ergodic Theory Dynam. Systems*, 26(2):447–465, 2006.
[16] T. H. Jäger and J. Stark. Towards a classification for quasiperiodically forced circle homeomorphisms. *J. London Math. Soc. (2)*, 73(3):727–744, 2006.
[17] T. Kaijser. On stochastic perturbations of iterations of circle maps. *Phys. D*, 68:201–231, 1993.
[18] Y. Kifer. *Random perturbations of dynamical systems*. Birkhäuser, 1988.

[19] V. A. Kleptsyn and M. B. Nalskii. Contraction of orbits in random dynamical systems. *Funct. Anal. Appl.*, 38:267–282, 2004.

[20] V. A. Kleptsyn and M. B. Nalskii. Letter to the editors: "Convergence of orbits in random dynamical systems on a circle" [Funktsional. Anal. i Prilozhen. **38** (2004), no. 4, 36–54, 95–96; mr2117507]. *Funktsional. Anal. i Prilozhen.*, 39(2):95, 2005.

[21] M. Kluenger. Periodicity and sharkovsky's theorem for random dynamical systems. *Stoch. Dyn.*, 1:299–338, 2001.

[22] Y. Le Jan. Equilibre statistique pour les produits de difféomorphismes aléatoires indépendants. *Ann. Inst. H. Poincaré Probab. Statist.*, 23:111–120, 1987.

[23] G. Margulis. Free subgroups of the homeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(9):669–674, 2000.

[24] D. Ruelle and A. Wilkinson. Absolutely singular dynamical foliations. *Comm. Math. Phys.*, 219(3):481–487, 2001.

[25] P. R. C. Ruffino. A sampling theorem for rotation numbers of linear processes in $\mathbf{R}^2$. *Random Oper. Stochastic Equations*, 8(2):175–188, 2000.

[26] J. Stark. Regularity of invariant graphs for forced systems. *Ergodic Theory Dynam. Systems*, 19:155–199, 1999.

[27] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[28] H. Zmarrou and A. J. Homburg. Bifurcations of stationary measures of random diffeomorphisms. *Ergodic Theory Dynam. Systems*, 27(5):1651–1692, 2007.

Received September 2006; revised February 2007.

E-mail address: a.j.homburg@uva.nl