CRITICAL INTERMITTENCY IN RATIONAL MAPS

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Abstract. This paper will provide and study examples of iterated function systems by two rational maps on the Riemann sphere that give rise to critical intermittency. The main ingredient for this is a superattracting fixed point for one map that is mapped onto a common repelling fixed point by the other map.

1. Introduction

This paper will provide and study examples of iterated function systems by two rational maps on the Riemann sphere that give rise to intermittent time series. The central example of the paper is intermittency of a type that we call critical intermittency, where the main ingredient is a superattracting fixed point for one map that is mapped by the other map onto a common repelling fixed point. We consider a topological description of the dynamics for which we study density of orbits of the semi group generated by the iterated function system. And we consider a metrical description by looking at properties of the intermittent time series.

In dynamical systems theory, intermittency stands for time series that alternate between different characteristics. It in particular indicates times series that appear stationary over long periods of time and are interrupted by bursts of nonstationary dynamics. These are called the laminar phase and relaminarization. Explanations for the occurrence of intermittent time series were given by Pomeau and Manneville [11], see also [3]. They offered explanations using bifurcation theory, and distinguished different types of intermittency caused by different local bifurcations. Later research added to the list of mechanisms giving intermittency, including crisis induced intermittency, homoclinic intermittency, on-off intermittency and in-out intermittency.

1.1. **Critical intermittency in iterated function systems of logistic maps.**

The type of dynamics we consider in this paper is related to the following example from interval dynamics.

Denote $g_a(x) = ax(1-x)$ for the logistic map on the interval $[0,1]$, with $0 < a \le 4$. Consider the iterated function system generated by the two maps $f_0 = g_2$ and $f_1 = g_4$. This defines a semi-group $\langle f_0, f_1 \rangle$ of compositions of f_0 and f_1. For each iterate we pick $i \in \{0,1\}$ at random, i.i.d., with probabilities p_0 and $p_1 = 1 - p_0$, and then iterate using f_i. Note that this a Markov process and recall that a stationary measure for the Markov process is a measure m satisfying

$$m = p_0 f_0 m + p_1 f_1 m.$$

Here $f_i m$, $i = 0, 1$, stands for the push forward $f_i m(A) = m(f_i^{-1}(A))$. Carlsson [4] observed that the only stationary measure for this iterated function system is the delta measure at 0 if $p_0 > 1/2$. This was further investigated in [1] where σ-finite stationary measures were constructed for $p_0 > 1/2$.

We summarize results on this iterated function system in the theorem below. Write $\Sigma = \{0,1\}^{\mathbb{N}}$ and endow Σ with the product topology and the Borel σ-algebra. Further write $\omega = (\omega_i)_{i\in\mathbb{N}}$ for elements of Σ. We denote

$$f_\omega^n(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x).$$

On Σ we consider the Bernoulli measure $\nu = \nu_{p_0,p_1}$ given the probabilities p_0, p_1.

Theorem 1.2 (see [1]). *Consider the iterated function system $\langle f_0, f_1\rangle$ on $[0,1]$ given with probabilities p_0, p_1.*

For all $x \in (0,1)$ the G-orbit of x is dense in $[0,1]$.

Assume $p_1 > 1/2$. Then the delta measure δ_0 at 0 is the unique stationary probability measure. There is an absolutely continuous σ-finite stationary measure assigning zero measure to the points $\{0,1\}$. For any $\varepsilon > 0$ and for Lebesgue almost any $x \in [0,1]$,

(1) *$f_\omega^n(x) \notin (0,\varepsilon)$ for infinitely many n;*
(2) *$\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \ : \ f_\omega^n(x) \in (0,\varepsilon)\}| = 1$,*

for almost all $\omega \in \Sigma$.

The last part of the theorem expresses the occurrence of intermittent time series; orbits spend most time near 0 but there are infrequent but repeated bursts away from 0. Under the conditions of the theorem, the only stationary probability measure is δ_0, even though 0 is repelling for both maps f_0, f_1. The explanation lies in the existence of a superattracting fixed point $1/2$ for f_0, which will be mapped onto the repelling fixed point 0 after iterating under f_1. Iterates of f_0 bring a point superexponentially close to $1/2$, after iterating f_1 the point will get superexponentially close to 0, after which may iterates are needed to escape neighborhoods of 0.

Other examples of this phenomenon, which we call critical intermittency, are possible. In the next section we introduce and study an example of an iterated function system of rational maps on the Riemann sphere.

2. Iterated function systems on the Riemann sphere

Given a parameter $\lambda \in \mathbb{C}$, consider the two maps

$$\begin{aligned} f_0(z) &= 2z + z^2, \\ f_1(z) &= \lambda \frac{z}{(z+1)^2} \end{aligned} \tag{2.1}$$

on the Riemann sphere $\widehat{\mathbb{C}}$. The first map, f_0, is conjugate to $z \mapsto z^2$ through a translation by 1. Its Julia set equals the circle $\{|z+1| = 1\}$. The maps f_0 and f_1 have 0 as common fixed point. Moreover, the set of three points $\{0, \infty, -1\}$ is forward invariant by both maps. The points are mapped under f_0 , f_1 in the following way:

$$f_0(0) = 0, \qquad f_1(0) = 0,$$
$$f_0(\infty) = \infty, \qquad f_1(\infty) = 0,$$
$$f_0(-1) = -1, \qquad f_1(-1) = \infty.$$

Write $\langle f_0, f_1 \rangle$ for the iterated function system generated by f_0, f_1. We iterate by picking the maps at random, i.i.d., with probabilities p_0, p_1 for f_0, f_1. The Lyapunov exponent at 0 is the average $p_0 \ln(f_0'(0)) + p_1 \ln(f_1'(0)) = p_0 \ln(2) + p_1 \ln(\lambda)$ and we assume this to be positive:

$$p_0 \ln(2) + p_1 \ln(\lambda) > 0. \tag{2.2}$$

This makes the common fixed point 0 repelling on average.

As above we write $\Sigma = \{0,1\}^{\mathbb{N}}$ and endow Σ with the product topology and the Borel σ-algebra. We write $\omega = (\omega_i)_{i \in \mathbb{N}}$ for elements of Σ. The iterated function system defines a skew product system $F : \Sigma \times \widehat{\mathbb{C}} \to \Sigma \times \widehat{\mathbb{C}}$ given by

$$F(\omega, z) = (\sigma\omega, f_{\omega_0}(z)).$$

Here σ is the left shift operator. Denote

$$F^n(\omega, z) = (\sigma^n \omega, f_\omega^n(z)).$$

We write $[i_0 \dots i_k]$ for the cylinder $\{\omega \in \Sigma \ : \ \omega_0 = i_0, \dots, \omega_k = i_k\}$.

2.1. **Dense semi-group orbits.** Recall that the iterated function system given by the action of the semi-group $G = \langle f_0, f_1 \rangle$ is said to be *topologically transitive* if for every non-empty and open $U, V \subset \widehat{\mathbb{C}}$ there exists $g \in G$ with $g(U) \cap V \neq \emptyset$. Given a closed invariant set $S = f_0(S) \cup f_1(S)$, the action of G is said to be *minimal* on S if for all $z \in S$ the G-orbit of z is dense in S. We are interested in G-orbits that are dense in $\widehat{\mathbb{C}}$. As this can not hold for points $z = -1, 0, \infty$, the best that we can hope for is that G-orbits of points $z \in \widehat{\mathbb{C}} \setminus \{-1, 0, \infty\}$ are dense. We will prove that for $\lambda \in B(0,1) = \{\lambda \in \mathbb{C} \ : \ |\lambda| < 1\}$, the G-orbit of a point $z \in \widehat{\mathbb{C}} \setminus \{-1, 0, \infty\}$ is dense in $\widehat{\mathbb{C}}$ if and only if $\lambda \notin \mathbb{R}$.

The fact that for $\lambda \in \mathbb{R}$, the semi-group orbits of points $z \neq -1, -0, \infty$ are not all dense follows immediately from the invariance of $\mathbb{R}$. In order to prove dense semi-group orbits when $\lambda \in B(0,1) \setminus \mathbb{R}$ we first prove topological transitivity. We need a lemma on linearizing coordinates.

Lemma 2.2. *Let $\lambda \neq 0$. Then the rational functions f_0 and f_1 are not simultaneously linearizable at* 0.

Proof. Recall that the linearization map is unique up to a multiplicative constant. It is therefore sufficient to consider the linearization map φ for f_0 of the form $\varphi(z) = z + a_2 z^2 + a_3 z^3 + O(z^4)$, and to show that φ does not also linearize f_1. Since $f_0(z) = 2z + z^2$ we have that $a_2 = -1/2$ and $a_3 = 1/6$. Observe that

$$f_1(z) = \frac{\lambda z}{1 + 2z + z^2} = \lambda z - 2\lambda z^2 + 3\lambda z^3 + O(z^3).$$

It follows that

$$\varphi \circ f_1 - \lambda\varphi = \frac{1}{6}\lambda z^2 \left(-3(3+\lambda) + (17 + 12\lambda + \lambda^2)z + O(z^2)\right).$$

Therefore the second order part of $\varphi \circ f_1 - \lambda\varphi$ only vanishes when $\lambda = -3$, in which case

$$\varphi \circ f_1 - \lambda\varphi = 5z^3 + O(z^4).$$

Hence regardless of the value of $\lambda \neq 0$ the maps f_0 and f_1 cannot be simultaneously linearizable. $\square$

Proposition 2.3. *For $\lambda \in B(0,1) \setminus \mathbb{R}$ the action of G is topologically transitive.*

Proof. For $\lambda \in B(0,1) \setminus \mathbb{R}$ fixed, we choose $r > 0$ such that $B(0,1)$ is mapped strictly into itself by f_1. Let $z_0 \in B(0,r)$ be a marked point, and let Ω be the set of all sequences $\omega \in \{0,1\}$ for which $f_\omega^n(z_0) \in B(0,r)$. We will prove that the family of maps

$$z \mapsto f_\omega^n(z), \quad \omega \in \Omega,\ n \in \mathbb{N},$$

is not a normal family in any neighborhood of z_0. As a consequence the family cannot avoid three distinct points in $\widehat{\mathbb{C}}$, which implies topological transitivity.

Instead of working with f_0 and f_1 we change coordinates and work with $g_0(z) = 2z + z^2 + h.o.t.$ and $g_1(z) = \lambda z$. From the proof of Lemma 2.2 is follows that this is possible.

We prove non-normality by showing that there exist $\omega \in \Omega$ and $n \in \mathbb{N}$ for which $(g_\omega^n)'(z_0)$ is arbitrarily large. Let v_0 be a non-zero tangent vector at z_0, and given $\omega \in \Omega$ and $n \in \mathbb{N}$ write $(z_n, v_n) = dg_\omega^n(z_0; v_0)$. We consider the ratios

$$R_n := \frac{\|v_n\|}{|z_n|}$$

for specific sequences ω. Notice that if $\omega_n = 1$, then $R_{n+1} = R_n$, since g_1 is linear. On the other hand we have that

$$R_{n+1} = \frac{|2 + 2z + O(z^2)|}{|2 + z + O(z^2)|} \cdot R_n$$

when $\omega_n = 0$. It folows that for $|z|$ sufficiently small, which can be guaranteed by choosing $r > 0$ sufficiently small, R_n increases by a multiplicative factor when $\mathrm{Re}(z) > 0$. Moreover, this multiplicative factor is bounded away from 1 on a compact subset bounded away from 0.

Let us define the open conical region

$$C = \{z \ : \ 0 < |z| < \frac{r}{3},\ |\arg(z)| < \frac{2\pi}{5}\}.$$

Observe that when $r > 0$ is chosen sufficiently small $g_0(C) \subset B(0,r)$. On the other hand, the angle $\frac{2\pi}{5}$ is chosen such that for any $z \in B(0,1) \setminus \{0\}$ the orbit of z under the linear map g_1 lands in C in at most a fixed number of steps; a number that depends only on the argument of λ and does not depend on the choice of r.

It is now clear how to construct the sequence ω: first repeatedly apply the linear map g_1 until $z_m \in C$, and then repeatedly apply g_0 until $z_n \notin C$. By choosing r sufficiently small we can guarantee that the orbit from z_m to z_n remains in the right half plane, and thus R_n is at least a fixed multiplicative constant larger than R_m. By repeating the process we obtain that $R_n \to \infty$. As the constructed orbit stays in $B(0,r)$ it follows from $R_n \to \infty$ that $(g_\omega^n)'(z_0) \to \infty$, which implies that the constructed family is not normal, which completes the proof. □

Remark 2.4. We note that topological transitivity of the semigroup $\langle f_0, f_1 \rangle$ also holds when $|\lambda| \geq 1$. Clearly it is in this case insufficient to consider only the behavior near the origin. Instead one uses the global behavior of the maps. Starting with any open set, the semi-group orbit reaches the basin of infinity of f_0. Observe that by applying f_0 repeatedly to an open set in the superattracting basin, one first obtains an annulus, and afterwards the image contains a round annulus of arbitrarily large modulus. By applying f_1 one obtains a round annulus A around 0 of arbitrarily large modulus. Observe that the exceptional set of f_1 is empty, i.e. f_1 does not admit a proper backwards invariant subset. Recall that since 0 lies in the Fatou set of f_1, for any open neighborhood U of 0 we have that

$$\bigcup_{n \in \mathbb{N}} f_1^n(U) = \widehat{\mathbb{C}}.$$

Choose U sufficiently small. Then, assuming that the modulus of A is sufficiently large, the forward orbit of any point in $U \setminus \{0\}$ must pass through A. It follows that

$$\bigcup_{n \in \mathbb{N}} f_1^n(A) = \widehat{\mathbb{C}},$$

which implies the topological transitivity of $\langle f_0, f_1 \rangle$.

Before proceeding with a theorem on dense semi-group orbits we give a lemma which excludes nontrivial forward invariant sets for the semi-group outside $\{-1, 0, \infty\}$.

Lemma 2.5. *Let $\lambda = \mathbb{C} \setminus \{0, 1\}$. If the semi-group orbit of $z \in \widehat{\mathbb{C}}$ is contained in $\{|z+1| = 1\} \cup \{-1, \infty\}$ then $z \in \{0, -1, \infty\}$.*

Proof. For simplicity of notation we work with $w = z + 1$, which gives $f_0 : w \mapsto w^2$ and

$$f_1 : w \mapsto \frac{\lambda(w-1) + w^2}{w^2}.$$

Assume that both w and $f_1(w)$ are contained in $\partial\mathbb{D}$. Analysis of the image of $\partial\mathbb{D}$ under $w \mapsto \frac{w-1}{w^2}$ shows that for each fixed λ there are at most 3 solutions to this equation apart from $w = 1$. Therefore it is sufficient to only consider $w \in \partial\mathbb{D}$ whose forward f_0-orbit

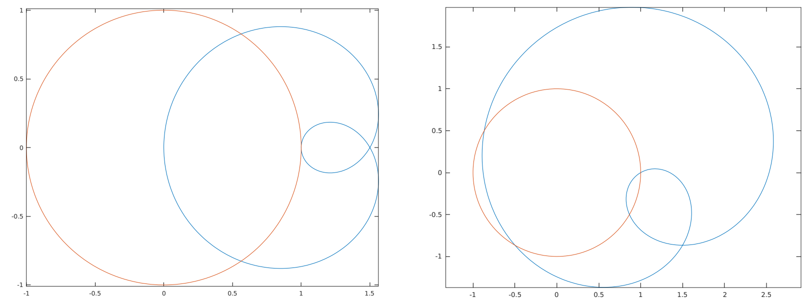

FIGURE 1. Left panel: The curve $f_1(\partial\mathbb{D})$ for $\lambda = 1/2$, together with $\partial\mathbb{D}$. Right panel: $f_1(\partial\mathbb{D})$ for $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{3}$.

contains at most 3 points (possibly with 1 added). That gives the following possible sets.

$$\begin{aligned}
&\{-1\}, && \{-i,-1\},\\
&\{i,-1\}, && \{e^{2\pi i/3}, e^{4\pi i/3}\},\\
&\{i,-1,e^{\pi i/4}\}, && \{i,-1,e^{7\pi i/4}\},\\
&\{-1,-i,i\}, && \{e^{2\pi i/3}, e^{4\pi i/3}, e^{\pi i/3}\},\\
&\{e^{2\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}\}, && \{e^{2\pi i/7}, e^{4\pi/7}, e^{8\pi i/7}\},\\
&\{e^{6\pi i/7}, e^{12\pi i/7}, e^{10\pi i/7}\}.
\end{aligned}$$

A case by case analysis shows that for any of these sets the f_1-image does not leave the set invariant. □

Theorem 2.6. *Let $\lambda \in B(0,1)\setminus\mathbb{R}$ and let $z_0 \in \mathbb{C}\setminus\{-1,0,\infty\}$. Then the G-orbit of z_0 is dense in $\widehat{\mathbb{C}}$.*

Proof. We use the notation introduced in the proof of the previous proposition. Since $z_0 \notin \{-1,0,\infty\}$ Lemma 2.5 implies that its G-orbit intersects $B(0,r)\setminus\{0\}$, hence we may assume that $z_0 \in B(0,r)\setminus\{0\}$. By topological transitivity of the G-action it is sufficient to prove that the G-orbit is dense in some open $U \subset B(0,r)$. We will prove this using only the local orbits $f_\omega^n(z_0)$ with $\omega \in \Omega$ (recall that Ω is the set of all sequences $\omega \in \{0,1\}$ for which $f_\omega^n(z_0) \in B(0,r)$).

Define
$$S := \overline{\{2^m \cdot \lambda^n\}}.$$

We distinguish four cases:

(i) S is dense in $\mathbb{C}$.
(ii) S is a union of real lines passing through the origin.
(iii) S is a union of concentric circles.
(iv) S is discrete.

Case (i). Use linearization coordinates for f_0 such that $g_0(z) = 2z$ and $g_1(z) = \lambda z + O(z^2)$. Recall that $\varphi \circ g_1 \circ \varphi^{-1} : z \mapsto \lambda z$ for a linearization map of the form $\varphi(z) = z + O(z^2)$.

If we take a sequence of values for $m, n \to \infty$ such that $2^m \lambda^n$ converges, then $g_0^m \circ g_1^n = g_0^m \circ \varphi^{-1} \circ \lambda^n \varphi(z_0)$ converges to $2^m \lambda^n \varphi(z_0)$. It follows that orbits are dense in $\mathbb{C}$, which completes the proof.

It is clear that in the other cases it is not sufficient to consider the linear part only, and one needs to take into account the higher order terms. An idea that will be used in several different places is the following: if the G-orbit of a accumulates on a point b, and the G-orbit of b accumulates on c, then the G-orbit of a accumulates on c as well.

Case (ii). Observe that there exists a minimal k such that $\lambda^k > 0$. Use linearization coordinates as in case (i). Since by Lemma 2.2 f_0 and f_1 are not simultaneously linearizable at 0 it follows that φ is not linear.

Denote by H a union of k radial half-lines, invariant under the linear map $z \mapsto \lambda z$. Then it follows that g_1 maps $\varphi^{-1}(H)$ into itself. Observe that $\varphi^{-1}(H)$ consists of k real analytic curves, each tangent at the origin to a half-line. It follows that the g_1-orbit of z_0 is contained in $\varphi^{-1}(H_0)$ for some choice of half-lines H_0. As in case (i), $g_0^m \circ g_1^n(z_0)$ converges to $2^m \lambda^n \varphi(z_0)$ as $2^m \lambda^n$ converges along a sequence with $m, n \to \infty$. It follows that the set of accumulation points of $g_0^m \circ g_1^n(z_0)$ contains a set of half-lines containing $\varphi(z_0)$.

Since this description of the accumulation points holds for any base point z_0 in $B(0, r)$, we can apply it also to each of the points z in one of the k radial half-lines in H_0. Thus we obtain a union of k-half lines $H(z)$ that varies continuously with z. It remains to be shown that these k-half lines vary with z, and is not constant. But this follows since $\varphi(H)$ is *not* a union of half-lines, since φ is not linear. It follows that we obtain an open set of accumulation points, which completes the proof.

Case (iii). We have that $2^m \cdot \lambda^n = e^{2\pi i \theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. It follows that

$$f_1^m \circ f_0^n(z) = e^{2\pi i \theta} z + h.o.t..$$

Note that the rationality of f_0 and f_1 implies that the higher order terms are non-zero. Since f_0 and f_1 are not simultaneously linearizable, it follows that

$$f_0^n \circ f_1^m \neq f_1^m \circ f_0^n,$$

and hence

$$f_1^{2m} \circ f_0^{2n} \neq (f_1^m \circ f_0^n)^2 .$$

Therefore the semi-group $H = \langle h_0, h_1 \rangle$ induced by the two distinct neutral maps $h_0 := f_1^{2m} \circ f_0^{2n}$ and $h_1 := (f_1^m \circ f_0^n)^2$ with the same rotation number, cannot be normal in a neighborhood of 0, see [6]. In particular the action of H on any closed curve winding around 0 is unbounded.

Observe that $\{f_0^m \circ f_1^n(z_0)\}_{m,n \in \mathbb{N}}$ accumulates on closed curves winding around 0 of arbitrarily small radius. Hence there are points on those curves whose H-orbits must be unbounded. But since the generators of H are both neutral at the origin, it follows that the unbounded orbit under H starting at such a point must be arbitrarily dense on an annulus enclosed by two of such closed curves winding around 0. It follows that by considering the H-action on smaller and smaller scales, and repeatedly composing with f_0 to get back to a given scale, we obtain density on an open set which completes the proof of case (iii).

Case (iv). Let us again work in linearization coordinates for f_0, such that we can write $g_0(z) = 2z$. By the assumption in case (iv) it follow that $2^m\lambda^n = 1$ for certain minimal n and m, from which it follows that

$$h(z) := g_0^m \circ g_1^n(z) = z + h.o.t..$$

Straightforward computation shows that the second order term of g_1 does not vanish when $|\lambda| < 1$ (Lemma 2.2), from which it follows that the second order term of h also does not vanish. Thus h has a parabolic fixed point at the origin with a single parabolic basin. In order to simplify the discussion we can apply a linear coordinate transformation to give $h(z) = z + z^2 + O(z^3)$, so that all orbits in the parabolic basin of h converge to zero along the negative real axis.

Consider base points $z_n = g_1^n(z_0)$ for n large with $\arg(z_n)$ bounded away from 0. Then z_n lies in the parabolic basin of h at 0. Write $z_{n,j} = h^j(z_n)$ where j runs from 0 to a large $k = k(n)$ satisfying $|z_{n,k}| << |z_n|$. Recall that the points $z_{n,j}$ converge to zero as $j \to \infty$, along a real analytic curve tangent to the negative real axis. Moreover, the ratios between consecutive points satisfy

$$\frac{z_{n,j}}{z_{n,j+1}} \to 1$$

as $n \to \infty$. Write $w_{n,j} = g_1(z_{n,j})$, so that the points $w_{n,j}$ converge to zero along the half line through $-\lambda$, which is different from the negative real axis, and since $\lambda \notin \mathbb{R}$ also different from the positive real axis. Choose $J > 0$ such that $\arg(w_{n,j}) \sim \arg(-\lambda)$ for $j \geq J$. It follows that the points $w_{n,j}$ still lie in the parabolic basin for $j \geq J$. Now define $w_{n,j,\ell} = h^\ell(w_{n,j})$ for $j \geq J$ and $\ell \geq 0$.

Recall the existence of the Fatou coordinate on the parabolic basin: a change of coordinates, again denoted by φ, conjugating h to $z \mapsto z + 1$. Recall that φ is of the form $z \mapsto -\frac{1}{z} + b\log(z) + o(1)$ for some $b \in \mathbb{C}$. It follows that each of the orbits $\{w_{n,j,\ell}\}_{\ell\in\mathbb{N}}$ lies on a real analytic curve, and these real analytic curves are all transverse to the half line through $-\lambda$, with angles bounded away from zero. After scaling by an iterate g_0^s to bring $w_{j,J,\ell}$ back to fixed scale, we obtain an arbitrarily dense set of points lying in an open set of uniform size. By increasing n and taking a convergent subsequence of $g_0^s(w_{j,J,\ell})$ we obtain a dense set of accumulation points in an open subset, completing the proof. □

Remark 2.7. It is not clear to the authors whether Theorem 2.6 also holds for nonreal λ with $|\lambda| > 1$. However, it does hold for generic λ. Assume that the set

$$S' := \overline{\{2^{-m} \cdot \lambda^n\}}$$

is dense in $\mathbb{C}$. Using the attraction under f_0 to the point ∞, we can consider a starting value z_0 that is unequal to but arbitrarily close to 0. We may therefore assume that, for

$k \in \mathbb{N}$ large, the point $2^k z_0$ is still close to zero. For $j \leq k$ we obtain that

$$f_0^j f_1^n(z_0) \sim 2^j \lambda^n z_0 = 2^{-(k-j)} \lambda^n (2^m z_0)$$

when $f_0^j f_1^n(z_0)$ is still sufficiently close to the origin. The assumption that S' is dense implies that by starting with smaller and smaller values of z_0, the set of points $f_0^j f_1^n(z_0)$ becomes more and more dense in a round annulus centered at 0 of arbitrarily large modulus. As in Remark 2.4 it follows that the semi-group orbit is dense in $\widehat{\mathbb{C}}$.

Remark 2.8. The Fatou set $F(G)$ of the semi-group $G = \langle f_0, f_1 \rangle$ is the set of points where G is normal. The Julia set $J(G)$, defined as the complement $\widehat{\mathbb{C}} \setminus F(G)$, is a closed backward invariant set. Under the assumptions of Theorem 2.6, $J(G)$ equals $\widehat{\mathbb{C}}$. Indeed, the Julia set contains 0 and by backward invariance also ∞. By [7] it contains a neighborhood of ∞ and then using Theorem 2.6 it equals $\widehat{\mathbb{C}}$. By [8], repelling fixed points of elements of G lie dense in $J(G) = \widehat{\mathbb{C}}$.

2.9. **Intermittency.** On Σ we consider the Bernoulli measure $\nu = \nu_{p_0,p_1}$ for the probabilities p_0, p_1, which is invariant under σ. A stationary measure m for the iterated function system defines an invariant measure $\nu \times m$ for F.

Theorem 2.10. *Consider the iterated function system $\langle f_0, f_1 \rangle$ given with probabilities p_0, p_1. Let $\lambda \in B(0,1) \setminus \mathbb{R}$. Assume* (2.2) *holds and*

$$p_0 > \frac{1}{2}.$$

Then the only finite stationary measure is the delta measure δ_0. For any $\varepsilon > 0$, for Lebesgue almost any $z \in \widehat{\mathbb{C}}$,

1. *$f_\omega^n(z) \notin B(0,\varepsilon)$ for infinitely many n;*
2. *$\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \ : \ f_\omega^n(z) \in B(0,\varepsilon)\}| = 1$,*

for almost all $\omega \in \Sigma$.

Proof. Assume there is a finite stationary measure m that assigns positive measure to $\widehat{\mathbb{C}} \setminus \{0, -1, \infty\}$. Then $\nu \times m$ is a finite invariant measure for F. We may assume that $\nu \times m$ is ergodic. Given a set $A \subset \Sigma \times \widehat{\mathbb{C}}$ of positive measure $\nu \times m(A) > 0$, Kac's lemma (see for instance [13]) yields finite average return time. We will derive a contradiction by providing a set A of positive measure and with infinite average return time.

For A we take the product set $A = [0] \times \mathcal{A}$ where $\mathcal{A}$ is an annulus around 0 between a small circle S around 0 and $f_0(S)$. (Take A so that it includes $[0] \times S$ but excludes $[0] \times f_0(S)$.) By Theorem 2.6 m has full support and thus $m(A) > 0$.

For $\omega \in \Sigma, z \in A$, let

$$R(\omega, z) = \min\{i > 0 \ : \ F^i(\omega, z) \in A\}$$

denote its return time to A. Then Kac's lemma yields

$$\int_A R \, d\nu \times m = \frac{1}{\nu \times m(A)}.$$

Since $\nu \times m$ is a product measure, if $m(B) > 0$ then $\nu \times m(C \times B) > 0$ for any cylinder C. Take large $r > 2$. By Lemma 2.5 we can cover $\mathcal{A}$ with finitely many compact sets P_i so that for each P_i there is a cylinder C_i and an integer K_i with

$$F^{K_i}(C_i \times P_i) \subset \{z \in \widehat{\mathbb{C}} \ : \ |z| \geq r\}.$$

At least one of the P_i's has positive measure $m(P_i) > 0$. Consider sets $[C_i 0^N 1] \times P_i$. Since ∞ is a superattracting fixed point for f_0 and $f_1(\infty) = 0$, we have that for N larger then some N_0, $F^{|C_i|+N+1}$ maps (ω, z) to a point with z coordinate inside the circle S (close to 0).

A calculation in the spirit of [1,4] establishes infinite average return time for $p_0 > 1/2$. For given $u \in \mathbb{C}$ with $|u| > 2$, we have $|f_0^N(u)| \geq |u+1|^{2^N}$, so that $|f_1 f_0^N(u)| \leq C/|u+1|^{2^N}$ for some $C > 0$. Each iterate maps a point inside S at most a constant factor further away from 0. We therefore find that $f_\omega^M f_1 f_0^N(u)$ can enter A in M iterates only for $M > C2^N$ for some $C > 0$. As $\nu([0^N 1]) = p_0^N p_1$, we get

$$\int_A R \, d\nu \times m \geq C \sum_{i \geq N_0} p_0^i 2^i$$

for some $C > 0$. Hence $\int_A R \, d\nu \times m = \infty$ if $2p_0 > 1$. Having shown that a stationary probability measure that assigns zero measure to $\{0, -1, \infty\}$ does not exist, it follows that the only stationary probability measure is the delta measure δ_0.

Item (1) follows from (2.2), which implies that for $z \in B(0, \varepsilon)$, with probability one the orbit $f_\omega^n(z)$ leaves $B(0, \varepsilon)$.

We continue with item (2). Instead of using $B(0, \varepsilon)$ we find it convenient to replace it with the union W of $B(0, \varepsilon)$ and a small disc $\{z \in \mathbb{C} \ : \ |z| > r\} \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ around ∞. We will establish that for almost all ω, $f_\omega^n(z)$ spends on average a bounded number of iterates between leaving and re-entering W. Item (2) in the formulation of the theorem will be deduced from this, and it gives in fact additional information on the duration of the relaminarization.

Given $\varepsilon > 0$, there is a finite partition $\{Q_i\}$ of $\widehat{\mathbb{C}} \setminus W$ so that for Q_i there is a cylinder B_i of uniformly bounded depth $K_i \leq K$, so that $F^{K_i}(\omega, z) \in W$ for $(\omega, z) \in B_i \times Q_i$. As $\nu(B_i)$ is bounded from below, it follows that the expected first hitting time for $z \in \widehat{\mathbb{C}} \setminus W$ to enter W is finite: if $U(\omega, z) = \min\{i > 0 \ : \ F^i(\omega, z) \in W\}$, then $U(\omega, z) < \infty$ almost surely and

$$\int_\Sigma U(\omega, z) \, d\nu < \infty$$

uniformly in z.

Take an orbit $z_n = f_\omega^n(z_0)$ with for definiteness $z_0 \in W$. Also assume that z_0 is not in the inverse orbit of 0. Define subsequent escape times from W and $\widehat{\mathbb{C}} \setminus W$: $T_0 = 0$ and

$$T_{2k+1} = \inf\{n \in \mathbb{N} \ | \ n > T_{2k} \text{ and } z_n \notin W\},$$
$$T_{2k} = \inf\{n \in \mathbb{N} \ | \ n > T_{2k-1} \text{ and } z_n \in W\}.$$

Note that such a sequence of escape times exists almost surely. Write $\eta_k = T_{2k-1} - T_{2k-2}$ and $\xi_k = T_{2k} - T_{2k-1}$ for the duration of the orbit pieces in W and $\widehat{\mathbb{C}} \setminus W$. Define for

$n \in [T_{2k}, T_{2k+1})$,

$$N_\eta(n) = k,\, N_\xi(n) = k$$

and $\tilde{\eta}(n) = n + 1 - T_{2k}$, $\tilde{\xi}(n) = 0$, so that $\tilde{\eta}$ counts the number of iterates from T_{2k} on where $z_n \in W$. Likewise define for $n \in [T_{2k+1}, T_{2k+2})$,

$$N_\eta(n) = k + 1,\, N_\xi(n) = k$$

and $\tilde{\eta}(n) = 0$, $\tilde{\xi}(n) = n + 1 - T_{2k+1}$, so that $\tilde{\xi}$ counts the number of iterates from T_{2k+1} on where $z_n \notin W$.

Finally calculate

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_W(f^i_\omega(x_0)) &= \frac{1}{n}\left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1)\right) \\
&= \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1)\right) \Bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1) + \sum_{k=1}^{N_\xi(n-1)} \xi_k + \tilde{\xi}(n-1)\right) \\
&= \left(1 + \left(\sum_{k=1}^{N_\xi(n-1)} \xi_k + \tilde{\xi}(n-1)\right) \Bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1)\right)\right)^{-1} \\
&\geq \left(1 + \left(\sum_{k=1}^{N_\xi(n-1)+1} \xi_k\right) \Bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k\right)\right)^{-1}. \qquad (2.3)
\end{aligned}$$

Construct independent stochastic variables $\sigma_k \geq \xi_k$ that have uniformly bounded expectation and variance. This can be done as follows: by shrinking B_i one constructs cylinders B_i of constant depth K and with constant measure $\nu(B_i)$, still satisfying $F^K(B_i, Q_i) \subset W$. Define $G : \Sigma \times \widehat{\mathbb{C}} \to \Sigma \times \widehat{\mathbb{C}}$ by $G \equiv F$ on $\Sigma \times \widehat{\mathbb{C}} \setminus W$ and $G(\omega, z) = (\sigma\omega, 0)$ for $z \in W$. Take a cylinder B in Σ of depth K and measure $\nu(B) = \nu(B_i)$ and add (B, W) to the collection $\{(B_i, Q_i)\}$. Consider the stochastic variable σ that gives the first time to enter a $B_i \times Q_i$ by iterating G^K. Take independent copies $\sigma_k \geq \xi_k$ of σ.

An application of the strong law of large numbers (see for instance [12]) gives that $\frac{1}{m}\sum_{k=1}^m \xi_k \leq \frac{1}{m}\sum_{k=1}^m \sigma_k$ stays bounded for large m, almost surely. For $z \in W$, let $V(\omega, z) = \min\{i > 0 \;:\; F^i(\omega, z) \in \widehat{\mathbb{C}} \setminus W\}$. Let ρ be the minimal escape time to $\widehat{\mathbb{C}} \setminus W$, minimized over initial points $z \in W$; $\rho = \min_{z \in W} V(\omega, z)$. There are independent copies ρ_n of ρ with $\rho_n \leq \eta_n$. Now $\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \rho_i = \infty$ almost surely and hence $\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \eta_i = \infty$ almost surely. We conclude that the last term in (2.3) goes to 1 for almost all ω, as $n \to \infty$ (note that $N_\eta(n-1) - N_\xi(n-1) \leq 1$). Item (2) follows since the average escape time out of $\{z \in \mathbb{C} \;:\; |z| > r\} \cup \{\infty\}$ is finite. $\square$

Remark 2.11. We get that for any $z \in \widehat{\mathbb{C}}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_\omega^i(z)} = \delta_0,$$

where the convergence is in the weak star topology, for ν almost all ω.

3. Vanishing Lyapunov exponents

For iterated function systems of interval maps, Gharaei and Homburg [5] show how intermittent time series occur if there is a common fixed point with a vanishing Lyapunov exponent, so that the common fixed point is neutral on average. We will present an analogous example for iterated function systems of Möbius transformations on the Riemann sphere. Consider the maps

$$f_0(z) = \mu z,$$
$$f_1(z) = \frac{z}{\mu + z}$$

We pick the maps f_0, f_1 with equal probability.

Theorem 3.1. *Consider the iterated function system $G = \langle f_0, f_1 \rangle$ given with probabilities $p_0 = p_1 = 1/2$. Assume that $|\mu| > 1$ and $\mu \notin \mathbb{R}$.*

The G-orbit of any $z_0 \in \widehat{\mathbb{C}} \setminus \{0\}$ is dense.

The only finite stationary measure is the delta measure δ_0. For any $\varepsilon > 0$, for Lebesgue almost any $z \in \widehat{\mathbb{C}}$,

1. *$f_\omega^n(x) \notin B(0,\varepsilon)$ for infinitely many n;*
2. *$\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \ : \ f_\omega^n(x) \in B(0,\varepsilon)\}| = 1$,*

for ν-almost all $\omega \in \Sigma$.

Proof. That semi-group orbits lie dense is proved as in Theorem 2.6. To prove the remaining statements on intermittency we follow the reasoning of Theorem 2.10. Key statement is again

$$\int_A R \, d\nu \times m = \infty, \tag{3.1}$$

where

$$R(\omega, z) = \min\{i > 0 \ : \ F^i(\omega, z) \in A\}$$

is the return time to A. Here as before $A = [0] \times \mathcal{A}$ with $\mathcal{A}$ an annulus between a small circle S around 0 and its image $f_0(S)$. By Kac's lemma this implies there is no finite stationary measure with support intersecting $\mathcal{A}$, so that the only stationary probability measure is δ_0.

To obtain (3.1), fix z inside S and let $H(\omega) = \min\{i > 0 \ : \ F^i(\omega, z) \in A\}$ be the first time that $f_\omega^i(z)$ that enters A. We are done if we prove

$$\int_\Sigma H \, d\nu = \infty. \tag{3.2}$$

A sequence $z_n = |f^n_\omega(z_0)|$ that stays near 0 satisfies $|z_{n+1} - \mu z_n| \le C z_n^2$ (if $\omega_n = 0$) or $|z_{n+1} - \frac{1}{\mu} z_n| \le C z_n^2$ (if $\omega_n = 1$) for some $C > 0$. Now (3.2) follows just as in the proof of [5, Theorem 5.2].

The remaining statements follow as in the proof of Theorem 2.10. □

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