CRITICAL INTERMITTENCY IN RATIONAL MAPS

ALE JAN HOMBURG, HAN PETERS, AND VAHATRA RABODONANDRIANANDRAINA

ABSTRACT. Intermittent dynamics is characterized by long periods of different types of dynamical characteristics, for instance almost periodic dynamics alternated by chaotic dynamics. Critical intermittency is intermittent dynamics that can occur in iterated function systems, and involves a superattracting periodic orbit.

This paper will provide and study examples of iterated function systems by two rational maps on the Riemann sphere that give rise to critical intermittency. The main ingredient for this is a superattracting fixed point for one map that is mapped onto a common repelling fixed point by the other map. We include a study of topological properties such as topological transitivity.

1. INTRODUCTION

This paper will provide and study examples of iterated function systems by two rational maps on the Riemann sphere that give rise to intermittent time series. The central example of the paper is intermittency of a type that we call critical intermittency, where the main ingredient is a superattracting fixed point for one map that is mapped by the other map onto a common repelling fixed point. We consider a topological description of the dynamics for which we study density of orbits of the semi group generated by the iterated function system. And we consider a metrical description by looking at statistical properties of the intermittent time series.

In dynamical systems theory, intermittency stands for time series that alternate between different characteristics. It in particular indicates times series that appear stationary over long periods of time and are interrupted by bursts of nonstationary dynamics. These are called the laminar phase and relaminarization. Explanations for the occurrence of intermittent time series were given by Pomeau and Manneville [13], see also [3]. They offered explanations using bifurcation theory, and distinguished different types of intermittency caused by different local bifurcations. Later research added to the list of mechanisms giving intermittency, including crisis induced intermittency, homoclinic intermittency, on-off intermittency and in-out intermittency.

1.1. **Critical intermittency in iterated function systems of logistic maps.** The type of dynamics we consider in this paper is related to the following example from interval dynamics.

Denote $g_a(x) = ax(1 - x)$ for the logistic map on the interval $[0, 1]$, with $0 < a \le 4$. Consider the iterated function system generated by the two maps $f_0 = g_2$ and $f_1 = g_4$. This defines a semi-group $\langle f_0, f_1 \rangle$ of compositions of f_0 and f_1. For each iterate we pick $i \in \{0, 1\}$ at random, i.i.d., with probabilities p_0 and $p_1 = 1 - p_0$, and then iterate using

f_i. Note that this defines a Markov process and recall that a stationary measure for the Markov process is a measure m satisfying

$$m = p_0 f_0 m + p_1 f_1 m.$$

Here $f_i m$, $i = 0, 1$, stands for the push forward $f_i m(A) = m(f_i^{-1}(A))$. Carlsson [4] observed that the only stationary measure for this iterated function system is the delta measure at 0 if $p_0 > 1/2$. This was further investigated in [1] where σ-finite stationary measures were constructed for $p_0 > 1/2$, and in [9] that studied stationary measures for all values of p_0.

We summarize results on this iterated function system in the theorem below. Write $\Sigma = \{0, 1\}^{\mathbb{N}}$ and endow Σ with the product topology and the Borel σ-algebra. Further write $\omega = (\omega_i)_{i \in \mathbb{N}}$ for elements of Σ. We denote

$$f_\omega^n(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x).$$

On Σ we consider the Bernoulli measure ν_{p_0, p_1} given the probabilities p_0, p_1.

Theorem 1.1 (see [1]). *Consider the iterated function system $\langle f_0, f_1 \rangle$ on $[0, 1]$ given with probabilities p_0, p_1.*

Assume $p_1 \geq 1/2$. Then the delta measure δ_0 at 0 is the unique stationary probability measure. There is an absolutely continuous σ-finite stationary measure with support equal to $[0, 1]$. For any $\varepsilon > 0$ and for Lebesgue almost any $x \in [0, 1]$,

   (1) $f_\omega^n(x) \notin (0, \varepsilon)$ for infinitely many n;
   (2) $\lim_{N \to \infty} \frac{1}{N} |\{0 \leq n < N : f_\omega^n(x) \in (0, \varepsilon)\}| = 1$,

for almost all $\omega \in \Sigma$.

The theorem expresses the occurrence of intermittent time series; orbits spend most time near 0 but there are infrequent but repeated bursts away from 0. Under the conditions of the theorem, the only stationary probability measure is δ_0, even though 0 is repelling for both maps f_0, f_1. The explanation lies in the existence of a superattracting fixed point $1/2$ for f_0, which will be mapped onto the repelling fixed point 0 after iterating under f_1. Iterates of f_0 bring a point superexponentially close to $1/2$, after iterating f_1 the point will get superexponentially close to 0, after which many iterates are needed to escape neighborhoods of 0.

The bound $p_1 \geq 1/2$ is optimal: it is shown in [9] that for $p_1 < 1/2$ there does exist an absolutely continuous stationary probability measure.

Other examples of this phenomenon, which we call critical intermittency, are possible [9]. In section 2 we will introduce conditions on pairs of 1-parameter families of rational functions that guarantee critical intermittency. Two key assumptions are the existence of a joint fixed point, which is attracting for the first map, and repelling for the second. In order for critical intermittency to occur, we assume the common fixed point is repelling on average. The first map will moreover have a superattracting fixed point, which is mapped back to the common fixed point by the second map. The proof of intermittency is aided by proving the density of semigroup orbits, which is shown for parameter values for which the common fixed point is not resonant.

In section 3 we introduce an explicit pair of 1-parameter rational maps of degree 2 that satisfy the conditions that imply critical intermittency for non-resonant parameter values. We moreover show that density of orbits still frequently occurs for parameter values where the common fixed point is resonant. In section 4 we will treat another explicit example, where the common fixed point is on average neutral instead of repelling. We show that intermittency can still occur, just as for real interval maps.

2. Iterated function systems on the Riemann sphere

In this section we will cover key assumptions that cause critical intermittency for rational iterated function systems on the Riemann sphere. We will work with a pair of 1-parameter families of rational functions, and show that critical intermittency holds for almost every parameter value. In later sections we will discuss explicit pairs of rational functions where it is possible to decide more precise when critical intermittency holds.

Key assumptions. Throughout this section let $f_0 = f_{0,\lambda}$ and $f_1 = f_{1,\lambda}$ be rational functions, both depending analytically on a parameter $\lambda \in \mathbb{D}$. We will stipulate the behavior of f_0 and f_1 on two marked points in $\widehat{\mathbb{C}}$. Without loss of generality we may assume that these two points are 0 and ∞.

We assume that 0 is a fixed point for each f_0 and f_1. We assume it to be repelling for each f_0 and attracting for each f_1. The point ∞ is assumed to be a superattracting fixed point of f_0, and mapped to the fixed point 0 by f_1. We will write $d \geq 2$ for the local degree of f_0 at infinity, so that f_0 is locally conjugate to $z \mapsto z^d$. Finally, we assume that each function f_0 is hyperbolic, and that all the other attracting fixed or periodic points of f_0 are mapped to the immediate basin of ∞ for f_0 by some iterate f_1^k. Note that by hyperbolicity, the attracting periodic points vary holomorphically with λ, hence the iterate k only depends on the choice of attracting periodic point, not on the value of λ.

For each fixed λ we write $\langle f_0, f_1 \rangle$ for the iterated function system generated by the functions f_0 and f_1. We iterate by picking the maps at random, i.i.d., with probabilities p_0, p_1 for f_0, f_1. We write $\mu(\lambda) = f_0'(0)$ and $\nu(\lambda) = f_1'(0)$, where $|\mu(\lambda)| > 1$ and $|\nu(\lambda)| < 1$ for each λ. The Lyapunov exponent at 0 is the average $p_0 \ln |f_0'(0)| + p_1 \ln |f_1'(0)|$ and we assume this to be positive:

$$p_0 \ln |\mu(\lambda)| + p_1 \ln |\nu(\lambda)| > 0. \tag{2.1}$$

This makes the common fixed point 0 repelling on average.

2.1. **Dense semigroup orbits.** Recall that the action of the semi-group $G = G_\lambda = \langle f_0, f_1 \rangle$ is said to be *topologically transitive* if for every non-empty and open $U, V \subset \widehat{\mathbb{C}}$ there exists $g \in G$ with $g(U) \cap V \neq \emptyset$.

Given an invariant set $S = f_0(S) \cup f_1(S)$, the action of G is said to be *minimal* on S if for all $z \in S$ the G-orbit of z is dense in S. We are interested in G-orbits that are dense in $\widehat{\mathbb{C}}$. As this can not hold for all points, it can for instance not hold for $z = 0, \infty$, the best that we can hope for is that G-orbits of all but finitely many points are dense.

We start with a result on topological transitivity.

Lemma 2.1. *For each $\lambda \in \mathbb{D}$ the action of the semi-group G is topologically transitive.*

Proof. Consider an open set $U \subset \widehat{\mathbb{C}}$. By hyperbolicity of f_0 the set U must intersect the attracting basin of some attracting fixed point or periodic cycle. Let $V \subset U$ be an open connected subset contained in this basin. Then for some large n the set $f_0^n(V)$ is contained in a small neighborhood of an attracting periodic point x of f_0. Let k be such that $f_1^k(x)$ is in a small neighborhood of ∞. Then $f_1^k \circ f_0^n(V)$ is contained in a small neighborhood of the point ∞.

Recall that near ∞ the map f_0 is holomorphically conjugate to a map $z \mapsto z^d$, for $d \geq 2$. It follows that for large ℓ the set $f_0^\ell \circ f_1^k \circ f_0^n(U)$ contains an annulus around the point ∞. Moreover, by increasing ℓ if necessary we can guarantee that the modulus of this annulus is arbitrarily large.

It follows that $f_1 \circ f_0^\ell \circ f_1^k \circ f_0^n(U)$ contains a small annulus of arbitrarily large modulus around the point 0. Since the repelling fixed point 0 is a non-isolated point of the Julia set of f_0, and since f_0 acts in local coordinates as multiplication by the multiplier at the fixed point, it follows that a small annulus around 0 of sufficiently large modulus must contain a point on the Julia set of f_0, and thus also an open neighborhood of such a point. It follows that

$$\bigcup_{m \in \mathbb{N}} f_0^m \circ f_1 \circ f_0^\ell \circ f_1^k \circ f_0^n(U) \supset \widehat{\mathbb{C}} \setminus \mathcal{E}_{f_0},$$

where $\mathcal{E}_{f_0}$ is the exceptional set of f_0, which contains at most 2 points. This completes the proof. $\square$

Remark 2.2. One can verify that the exceptional set of f_0 is in fact empty. By compactness of $\widehat{\mathbb{C}}$ it follows that there exist integers n, k, ℓ, and m such that

$$f_0^m \circ f_1 \circ f_0^\ell \circ f_1^k \circ f_0^n(U) = \widehat{\mathbb{C}}.$$

Theorem 2.3. *Let λ be such that the set*

$$S = S_\lambda := \overline{\{\mu(\lambda)^n \cdot \nu(\lambda)^m \ : \ n, m \in \mathbb{N}\}}$$

coincides with $\mathbb{C}$. Then there exists an $r > 0$ such that for all $z \in B_r(0) \setminus \{0\}$ the G_λ-orbit of z is dense in $\widehat{\mathbb{C}}$.

Remark 2.4. For $\nu(\lambda) \in B(0,1)$ one of the following must be satisfied:

(1) $S = \mathbb{C}$.
(2) S is a finite union of real lines passing through the origin.
(3) S is a discrete union of concentric circles.
(4) S is discrete.

It is possible that case (2), (3), or (4) occurs persistently for all λ, in which case we say that G is *persistently resonant*. If G_λ is not persistently resonant, the equality $S = \mathbb{C}$ will hold for almost every $\lambda \in \mathbb{D}$. Indeed, the parameters λ for which cases (2), (3), and (4) hold are given by countably many real analytic equations in λ. Each such equation either holds throughout, in which case G_λ is persistently resonant, or is satisfied in a real analytic subvariety of real dimension 1.

Proof of Theorem 2.3. We will consider semi-group orbits that remain in $B_r(0)$, and accumulate on a small annulus around the origin of arbitrarily large modulus. The argument that concludes the proof of 2.1 can then be used to determine density of the orbit in all of $\widehat{\mathbb{C}}$.

Since we will remain in $B_r(0)$, we may assume that $r > 0$ is sufficiently small so that we can use linearizing coordinates for the map f_0, i.e. $f_0(z) = \mu(\lambda)z$. Since the multiplier is preserved under conjugation we obtain $f_1(z) = \nu(\lambda)z + O(z^2)$. Let us denote the local linearization map of this function by $\varphi = \varphi_\lambda$, which is unique once we demand that $\varphi(z) = z + O(z^2)$. Thus $\varphi \circ f_1 \circ \varphi^{-1} : z \mapsto \nu(\lambda)z$ on $B_r(0)$, by shrinking r if necessary.

Consider diverging sequences (m_j) and (n_j) for which $\mu(\lambda)^{m_j} \cdot \nu(\lambda)^{n_j}$ converges to $w \in B_r(0) \setminus \{0\}$. Then

$$f_0^{m_j} \circ f_1^{n_j} = \mu^{m_j} \varphi^{-1}(\nu^{n_j} \varphi(z_0)) \to w \cdot \varphi(z_0).$$

It follows that the semi-group orbit of z accumulates on a small annulus around 0 of arbitrarily large modulus, which completes the proof. $\qquad\square$

Remark 2.5. Theorem 2.3 implies the density of the G-orbit $G(z_0)$ of z_0 for an arbitrary initial value z_0 whenever some element in $G(z_0)$ lies in a small punctured neighborhood of the origin, which is of course a necessary condition. It is also clear that this condition is not always satisfied. For example, if z equals one of the attracting periodic points of f_0 and is mapped exactly onto ∞ by f_1, then the orbit $G(z)$ is finite. In general there could be different subsets in the Julia set of f_0 that are invariant under both f_0 and f_1, which we exclude in the Lemma below. We will later discuss explicit examples of the functions f_0 and f_1 for which we can deduce that there are no non-trivial invariant subsets, and as a result we obtain density for all but finitely many initial values $z_0 \in \widehat{\mathbb{C}}$. In those examples we can moreover deduce that density also occurs for most resonant parameters λ, where S satisfies case (ii), (iii), or (iv). In the resonant setting more precise knowledge of the higher order terms is required to deduce density of local orbits near the origin.

Theorem 2.3 and Remark 2.5 yield the following result.

Lemma 2.6. *Suppose that the hypotheses of Theorem 2.3 are satisfied. Moreover, assume that none of the attracting fixed or periodic points of f_0 are mapped exactly to ∞ under iteration by f_1. In addition, assume that for any $z_0 \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$ there is $g \in G_\lambda$ so that $g(z_0)$ lies in the immediate basin of attraction of ∞ for f_0. Then for $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$ the G_λ-orbit of z is dense in $\widehat{\mathbb{C}}$.*

2.2. **Critical Intermittency.** As in the real setting we write $\Sigma = \{0,1\}^{\mathbb{N}}$ and endow Σ with the product topology and the Borel σ-algebra. We write $\omega = (\omega_i)_{i \in \mathbb{N}}$ for elements of Σ. The iterated function system $\langle f_0, f_1 \rangle$ defines a skew product system $F : \Sigma \times \widehat{\mathbb{C}} \to \Sigma \times \widehat{\mathbb{C}}$ given by

$$F(\omega, z) = (\sigma\omega, f_{\omega_0}(z)).$$

Here σ is the left shift operator. Denote

$$F^n(\omega, z) = (\sigma^n \omega, f_\omega^n(z)).$$

We write $[i_0 \ldots i_k]$ for the cylinder $\{\omega \in \Sigma \ : \ \omega_0 = i_0, \ldots, \omega_k = i_k\}$. On Σ we consider the Bernoulli measure ν_{p_0,p_1} for the probabilities p_0, p_1, which is invariant under σ. A stationary measure m for the iterated function system defines an invariant measure $\nu_{p_0,p_1} \times m$ for F.

Theorem 2.7. *Consider the iterated function system $\langle f_0, f_1 \rangle$ given with probabilities p_0, p_1. Suppose that the key assumptions and the additional hypotheses of Lemma 2.6 hold. Assume*

$$p_0 > \frac{1}{d}.$$

Then the delta measure δ_0 is the only finite stationary measure. Moreover, for any $\varepsilon > 0$, and for Lebesgue almost any $z \in \widehat{\mathbb{C}}$,

(1) $f_\omega^n(z) \notin B(0, \varepsilon)$ *for infinitely many n;*
(2) $\lim_{N \to \infty} \frac{1}{N} |\{0 \le n < N \ : \ f_\omega^n(z) \in B(0, \varepsilon)\}| = 1$,

for almost all $\omega \in \Sigma$.

In the proof we use Kač theorem. We recall a version we use. Consider a measurable map $f : X \to X$ with a finite invariant measure μ. Let $E \subset X$ with $\mu(E) > 0$. Define the first return time $R : E \to \mathbb{N} \cup \{\infty\}$ by

$$R(x) = \min\{i > 0 \ : \ f^i(x) \in E\}.$$

We use the statement that the average first return time to E is finite.

Theorem 2.8 (Kač theorem, see [14]). *Let $f : X \to X$ be a measurable map with a finite invariant measure μ. Let $E \subset X$ with $\mu(E) > 0$.*
Then

$$\int_E R(x) \, d\mu(x) < \infty.$$

If μ is an ergodic invariant measure, then $\int_E R(x) \, d\mu(x) = \mu(X)/\mu(E)$.

Proof of Theorem 2.7. Assume there is a finite stationary measure m that assigns zero measure to $\{0, \infty\}$. By Lemma 2.6 the support of m is all of $\widehat{\mathbb{C}}$. Then $\nu_{p_0,p_1} \times m$ is a finite invariant measure for F. (We may assume that $\nu_{p_0,p_1} \times m$ is ergodic.) Given a set $A \subset \Sigma \times \widehat{\mathbb{C}}$ of positive measure $\nu_{p_0,p_1} \times m(A) > 0$, Kač theorem yields finite average return time. We will derive a contradiction by providing a set A of positive measure and with infinite average return time.

For A we take the product set $A = [0] \times \mathcal{A}$ where $\mathcal{A}$ is an annulus around 0 between a small circle $C(0, \delta)$ around 0 and $f_0\big(C(0, \delta)\big)$. Take A so that it includes $[0] \times C(0, \delta)$ but excludes $[0] \times f_0\big(C(0, \delta)\big)$. Since the support of m is all of $\widehat{\mathbb{C}}$ and $\nu_{p_0,p_1} \times m$ is a product measure, we find $\nu_{p_0,p_1} \times m(A) > 0$. Then Kač theorem yields

$$\int_A R(\omega, z) \, d\nu_{p_0,p_1} \times m(\omega, z) < \infty$$

for the first return time R to A.

Since ∞ is a superattracting fixed point for f_0 and $f_1(\infty) = 0$, it follows that for every $z \in \widehat{\mathbb{C}}$ with $|z| \geq R$ we have $f_1 \circ f_0^N(z) \in B(0, \delta)$ for large enough R and N larger than some N_0.

A calculation in the spirit of [1, 4] establishes infinite average return time to $\mathcal{A}$ for $p_0 > 1/d$. Recall that the local degree of f_0 at infinity is d. Thus for given $z \in \mathbb{C}$ with $|z| \geq R$ we have $|f_1 \circ f_0^N(z)| \leq c/|z|^{d^N}$ for some $c > 0$. Each iterate maps a point in $B(0, \delta)$ at most a constant factor further away from 0. Therefore for a given $\omega \in \Sigma$, $f_\omega^M \circ f_1 \circ f_0^N(z)$ may enter $\mathcal{A}$ only if $M > Cd^N$ for some $C > 0$ and

$$\int_A R \, d\nu_{p_0,p_1} \times m \geq \sum_{i \geq N_0} \int_{[0^i 1] \times \mathcal{A}} R \, d\nu_{p_0,p_1} \times m \geq C \sum_{n \geq N_0} p_0^n d^n$$

for some $C > 0$. Hence $\int_A R \, d\nu_{p_0,p_1} \times m = \infty$ if $p_0 > 1/d$.

Consequently the only stationary probability measure is the delta measure δ_0. Item (1) follows from (2.1), which implies that for $z \in B(0, \varepsilon)$, with probability one the orbit $f_\omega^n(z)$ leaves $B(0, \varepsilon)$.

We continue with item (2). We follow reasoning from [2] which is also used in [1]. Instead of using $B(0, \varepsilon)$ we find it convenient to replace it with the union W of $B(0, \varepsilon)$ and a small disc $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ around ∞. We will establish that for almost all ω, $f_\omega^n(z)$ spends on average a bounded number of iterates between leaving and re-entering W. Item (2) in the formulation of the theorem will be deduced from this, and it gives in fact additional information on the duration of the relaminarization.

Given $\varepsilon > 0$, there is a finite partition $\{Q_i\}$ of $\widehat{\mathbb{C}} \setminus W$ so that for Q_i there is a cylinder B_i of uniformly bounded depth $K_i \leq K$, so that $f_\omega^{K_i}(z) \in W$ for $(\omega, z) \in B_i \times Q_i$. As $\nu_{p_0,p_1}(B_i)$ is bounded from below, it follows that the expected first hitting time for $z \in \widehat{\mathbb{C}} \setminus W$ to enter W is finite: if $U(\omega, z) = \min\{i > 0 : f_\omega^i(z) \in W\}$, then $U(\omega, z) < \infty$ almost surely and

$$\int_\Sigma U(\omega, z) \, d\nu_{p_0,p_1} < \infty$$

uniformly in z.

Take an orbit $z_n = f_\omega^n(z_0)$ with for definiteness $z_0 \in W$. Also assume that z_0 is not in the inverse orbit of 0. Define subsequent escape times from W and $\widehat{\mathbb{C}} \setminus W$: $T_0 = 0$ and

$$T_{2k+1} = \inf\{n \in \mathbb{N} \mid n > T_{2k} \text{ and } z_n \notin W\},$$
$$T_{2k} = \inf\{n \in \mathbb{N} \mid n > T_{2k-1} \text{ and } z_n \in W\}.$$

Note that such a sequence of escape times exists almost surely. Write $\eta_k = T_{2k-1} - T_{2k-2}$ and $\xi_k = T_{2k} - T_{2k-1}$ for the duration of the orbit pieces in W and $\widehat{\mathbb{C}} \setminus W$. Define for $n \in [T_{2k}, T_{2k+1})$,

$$N_\eta(n) = k, \, N_\xi(n) = k$$

and $\tilde{\eta}(n) = n + 1 - T_{2k}$, $\tilde{\xi}(n) = 0$, so that $\tilde{\eta}$ counts the number of iterates from T_{2k} on where $z_n \in W$. Likewise define for $n \in [T_{2k+1}, T_{2k+2})$,

$$N_\eta(n) = k + 1, \, N_\xi(n) = k$$

and $\tilde{\eta}(n) = 0$, $\tilde{\xi}(n) = n + 1 - T_{2k+1}$, so that $\tilde{\xi}$ counts the number of iterates from T_{2k+1} on where $z_n \notin W$.

Finally calculate

$$
\begin{aligned}
\frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_W(f_\omega^i(x_0)) &= \frac{1}{n} \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1) \right) \\
&= \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1) \right) \bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1) + \sum_{k=1}^{N_\xi(n-1)} \xi_k + \tilde{\xi}(n-1) \right) \\
&= \left(1 + \left(\sum_{k=1}^{N_\xi(n-1)} \xi_k + \tilde{\xi}(n-1) \right) \bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1) \right) \right)^{-1} \\
&\geq \left(1 + \left(\sum_{k=1}^{N_\xi(n-1)+1} \xi_k \right) \bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k \right) \right)^{-1}.
\end{aligned}
\tag{2.2}
$$

Construct independent stochastic variables $\sigma_k \geq \xi_k$ that have uniformly bounded expectation and variance. This can be done as follows: by shrinking B_i one constructs cylinders B_i of constant depth K and with constant measure $\nu_{p_0,p_1}(B_i)$, still satisfying $f_\omega^K(Q_i) \subset W$ for $\omega \in B_i$. Define $G : \Sigma \times \widehat{\mathbb{C}} \to \Sigma \times \widehat{\mathbb{C}}$ by $G \equiv F$ on $\Sigma \times W$ and $G(\omega, z) = (\sigma\omega, 0)$ for $z \in W$. Take a cylinder B in Σ of depth K and measure $\nu_{p_0,p_1}(B) = \nu_{p_0,p_1}(B_i)$ and add (B, W) to the collection $\{(B_i, Q_i)\}$. Consider the stochastic variable σ that gives the first time to enter a $B_i \times Q_i$ by iterating G^K. Take independent copies $\sigma_k \geq \xi_k$ of σ.

An application of the strong law of large numbers (see for instance [11]) gives that $\frac{1}{m} \sum_{k=1}^m \xi_k \leq \frac{1}{m} \sum_{k=1}^m \sigma_k$ stays bounded for large m, almost surely. For $z \in W$, let $V(\omega, z) = \min\{i > 0 : F^i(\omega, z) \in \widehat{\mathbb{C}} \setminus W\}$. Let ρ be the minimal escape time to $\widehat{\mathbb{C}} \setminus W$, minimized over initial points $z \in W$; $\rho = \min_{z \in W} V(\omega, z)$. There are independent copies ρ_n of ρ with $\rho_n \leq \eta_n$. Now $\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \rho_i = \infty$ almost surely and hence $\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \eta_i = \infty$ almost surely. We conclude that the last term in (2.2) goes to 1 for almost all ω, as $n \to \infty$ (note that $N_\eta(n-1) - N_\xi(n-1) \leq 1$). Item (2) follows since the average escape time out of $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$ is finite. $\qquad\square$

Remark 2.9. We get that for any $z \in \widehat{\mathbb{C}}$,

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_\omega^i(z)} = \delta_0,
$$

where the convergence is in the weak star topology, for ν_{p_0,p_1} almost all ω.

3. Degree 2 example

In this section we discuss an explicit pair of 1-parameter rational functions for which critical intermittency can be shown. Given a parameter $\lambda \in \mathbb{C}$, consider the two maps

$$f_0(z) = 2z + z^2,$$
$$f_1(z) = \lambda \frac{z}{(z+1)^2} \tag{3.1}$$

on the Riemann sphere $\widehat{\mathbb{C}}$. The first map, f_0, is conjugate to $z \mapsto z^2$ through a translation by 1. Its Julia set equals the circle $\{|z+1| = 1\}$. The maps f_0 and f_1 have 0 as common fixed point. Moreover, the set of three points $\{0, \infty, -1\}$ is forward invariant by both maps. The points are mapped under f_0, f_1 in the following way:

$$f_0(0) = 0, \qquad\qquad f_1(0) = 0,$$
$$f_0(\infty) = \infty, \qquad\qquad f_1(\infty) = 0,$$
$$f_0(-1) = -1, \qquad\qquad f_1(-1) = \infty.$$

As in the general setting we write $\langle f_0, f_1 \rangle$ for the iterated function system generated by f_0, f_1, and work with associated probabilities p_0, p_1. Again we assume that the Lyapunov exponent at 0 is positive:

$$p_0 \ln |2| + p_1 \ln |\lambda| > 0. \tag{3.2}$$

We will use the skew product notation

$$F(\omega, z) = (\sigma\omega, f_{\omega_0}(z))$$

introduced in the previous section.

3.1. **Dense semi-group orbits.** Since our explicit semigroup satisfies the general assumptions from the previous section, we immediately obtain topological transitivity from Lemma 2.1. Our next goal is to prove the density of orbits. Lemma 2.6 implies that density occurs for almost all values of λ, and for initial values sufficiently close to 0. We will see that in our explicit setting density also occurs frequently for parameter values for which the semigroup G_λ is resonant.

Of course, for $\lambda \in \mathbb{R}$ the invariance of $\mathbb{R}$ implies that density does not occur for real initial values. We will focus on proving density when $\lambda \in B(0,1) \setminus \mathbb{R}$. We need a lemma on linearizing coordinates.

Lemma 3.1. *Let $\lambda \neq 0$. Then the rational functions f_0 and f_1 are not simultaneously linearizable at 0.*

Proof. Recall that the linearization map is unique up to a multiplicative constant. It is therefore sufficient to consider the linearization map φ for f_0 of the form $\varphi(z) = z + a_2 z^2 + a_3 z^3 + O(z^4)$, and to show that φ does not also linearize f_1. Since $f_0(z) = 2z + z^2$ we have that $a_2 = -1/2$ and $a_3 = 1/6$. Observe that

$$f_1(z) = \frac{\lambda z}{1 + 2z + z^2} = \lambda z - 2\lambda z^2 + 3\lambda z^3 + O(z^3).$$

It follows that

$$\varphi \circ f_1 - \lambda\varphi = \frac{1}{6}\lambda z^2 \left(-3(3+\lambda) + (17 + 12\lambda + \lambda^2)z + O(z^2)\right).$$

Therefore the second order part of $\varphi \circ f_1 - \lambda\varphi$ only vanishes when $\lambda = -3$, in which case

$$\varphi \circ f_1 - \lambda\varphi = 5z^3 + O(z^4).$$

Hence regardless of the value of $\lambda \neq 0$ the maps f_0 and f_1 cannot be simultaneously linearizable. $\square$

Next we exclude nontrivial forward invariant sets for the semi-group outside $\{-1, 0, \infty\}$.

Lemma 3.2. *Let $\lambda = \mathbb{C} \setminus \mathbb{R}$. If the semi-group orbit of $z \in \widehat{\mathbb{C}}$ is contained in $\{|z + 1| = 1\} \cup \{-1, \infty\}$, then $z \in \{0, -1, \infty\}$.*

Proof. For simplicity of notation we work with $w = z + 1$, which gives $f_0 : w \mapsto w^2$ and

$$f_1 : w \mapsto \frac{\lambda(w-1)}{w^2} + 1.$$

Assume that both w and $f_1(w)$ are contained in $\partial\mathbb{D}$. Analysis of the image of $\partial\mathbb{D}$ under $w \mapsto \frac{w-1}{w^2}$ shows that for each fixed λ there are at most 4 points $w \in \partial\mathbb{D}$ whose image can lie in $\partial\mathbb{D}$, including the fixed point $w = 1$. Therefore it is sufficient to only consider
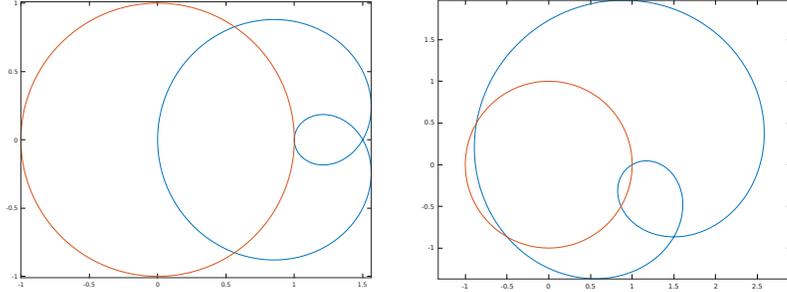
FIGURE 1.   Left panel: The curve $f_1(\partial\mathbb{D})$ for $\lambda = 1/2$, together with $\partial\mathbb{D}$. Right panel: $f_1(\partial\mathbb{D})$ for $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{3}$.

$w \in \partial\mathbb{D}$ whose forward f_0-orbit contains at most 3 points, possibly with 1 added. Up to complex conjugation we therefore obtain the following eight candidates for a forward invariant sets.

$$\{1, -1\}, \qquad\qquad\qquad\qquad \{1, i, -1\},$$
$$\{1, e^{2\pi i/3}, e^{4\pi i/3}\}, \qquad\qquad\qquad \{1, -1, -i, i\}$$
$$\{1, e^{\pi i/4}, i, -1\}, \qquad\qquad\qquad \{1, i, -1, e^{5\pi i/4}\},$$
$$\{1, e^{2\pi i/7}, e^{4\pi/7}, e^{8\pi i/7}\}, \qquad\qquad \{1, e^{\pi i/3}, e^{2\pi i/3}, e^{4\pi i/3}\}.$$

A case by case analysis shows that for any of these sets the f_1-image does not leave the set invariant for $\lambda \notin \mathbb{R}$. □

Theorem 3.3. *Let* $\lambda \in B(0,1) \setminus \mathbb{R}$ *and let* $z_0 \in \mathbb{C} \setminus \{-1, 0, \infty\}$. *Then the G-orbit of z_0 is dense in* $\widehat{\mathbb{C}}$.

Proof. We use the notation introduced in the proof of the previous proposition. Since $z_0 \notin \{-1, 0, \infty\}$ Lemma 3.2 implies that its G-orbit intersects $B(0,r) \setminus \{0\}$, hence we may assume that $z_0 \in B(0,r) \setminus \{0\}$. By topological transitivity of the G-action it is sufficient to prove that the G-orbit is dense in some open $U \subset B(0,r)$. We will prove this using only the local orbits $f_\omega^n(z_0)$ with $\omega \in \Omega$ (recall that Ω is the set of all sequences $\omega \in \{0,1\}$ for which $f_\omega^n(z_0) \in B(0,r)$).

Define
$$S := \overline{\{2^m \cdot \lambda^n\}}.$$

As in the general setting we distinguish four cases:

(i) S is dense in $\mathbb{C}$.
(ii) S is a union of real lines passing through the origin.
(iii) S is a union of concentric circles.
(iv) S is discrete.

The simplest case (i) was treated in the general setting, in Lemma 2.1. It is clear that in the other cases it is not sufficient to consider the linear part only, and one needs to take into account the higher order terms. An idea that will be used in several different places is the following: if the G-orbit of a accumulates on a point b, and the G-orbit of b accumulates on c, then the G-orbit of a accumulates on c as well.

Case (ii). Observe that there exists a minimal k such that $\lambda^k > 0$. Use linearization coordinates as in case (i). Since by Lemma 3.1 f_0 and f_1 are not simultaneously linearizable at 0 it follows that φ is not linear.

Denote by H a union of k radial half-lines, invariant under the linear map $z \mapsto \lambda z$. Then it follows that g_1 maps $\varphi^{-1}(H)$ into itself. Observe that $\varphi^{-1}(H)$ consists of k real analytic curves, each tangent at the origin to a half-line. It follows that the g_1-orbit of z_0 is contained in $\varphi^{-1}(H_0)$ for some choice of half-lines H_0. As in case (i), $g_0^m \circ g_1^n(z_0)$ converges to $2^m \lambda^n \varphi(z_0)$ as $2^m \lambda^n$ converges along a sequence with $m, n \to \infty$. It follows that the set of accumulation points of $g_0^m \circ g_1^n(z_0)$ contains a set of half-lines containing $\varphi(z_0)$.

Since this description of the accumulation points holds for any base point z_0 in $B(0,r)$, we can apply it also to each of the points z in one of the k radial half-lines in H_0. Thus we obtain a union of k-half lines $H(z)$ that varies continuously with z. It remains to be shown that these k-half lines vary with z, and is not constant. But this follows since $\varphi(H)$ is *not* a union of half-lines, since φ is not linear. It follows that we obtain an open set of accumulation points, which completes the proof.

Case (iii). We have that $2^m \cdot \lambda^n = e^{2\pi i \theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. It follows that
$$f_1^m \circ f_0^n(z) = e^{2\pi i \theta} z + h.o.t..$$

Note that the rationality of f_0 and f_1 implies that the higher order terms are non-zero. Since f_0 and f_1 are not simultaneously linearizable, it follows that

$$f_0^n \circ f_1^m \neq f_1^m \circ f_0^n,$$

and hence

$$f_1^{2m} \circ f_0^{2n} \neq (f_1^m \circ f_0^n)^2.$$

Therefore the semi-group $H = \langle h_0, h_1 \rangle$ induced by the two distinct neutral maps $h_0 := f_1^{2m} \circ f_0^{2n}$ and $h_1 := (f_1^m \circ f_0^n)^2$ with the same rotation number, cannot be normal in a neighborhood of 0, see [6]. In particular the action of H on any closed curve winding around 0 is unbounded.

Observe that $\{f_0^m \circ f_1^n(z_0)\}_{m,n \in \mathbb{N}}$ accumulates on closed curves winding around 0 of arbitrarily small radius. Hence there are points on those curves whose H-orbits must be unbounded. But since the generators of H are both neutral at the origin, it follows that the unbounded orbit under H starting at such a point must be arbitrarily dense on an annulus enclosed by two of such closed curves winding around 0. It follows that by considering the H-action on smaller and smaller scales, and repeatedly composing with f_0 to get back to a given scale, we obtain density on an open set which completes the proof of case (iii).

Case (iv). Let us again work in linearization coordinates for f_0, such that we can write $g_0(z) = 2z$. By the assumption in case (iv) it follow that $2^m \lambda^n = 1$ for certain minimal n and m, from which it follows that

$$h(z) := g_0^m \circ g_1^n(z) = z + h.o.t..$$

Straightforward computation shows that the second order term of g_1 does not vanish when $|\lambda| < 1$ (Lemma 3.1), from which it follows that the second order term of h also does not vanish. Thus h has a parabolic fixed point at the origin with a single parabolic basin. In order to simplify the discussion we can apply a linear coordinate transformation to give $h(z) = z + z^2 + O(z^3)$, so that all orbits in the parabolic basin of h converge to zero along the negative real axis.

Consider base points $z_n = g_1^n(z_0)$ for n large with $\arg(z_n)$ bounded away from 0. Then z_n lies in the parabolic basin of h at 0. Write $z_{n,j} = h^j(z_n)$ where j runs from 0 to a large $k = k(n)$ satisfying $|z_{n,k}| << |z_n|$. Recall that the points $z_{n,j}$ converge to zero as $j \to \infty$, along a real analytic curve tangent to the negative real axis. Moreover, the ratios between consecutive points satisfy

$$\frac{z_{n,j}}{z_{n,j+1}} \to 1$$

as $n \to \infty$. Write $w_{n,j} = g_1(z_{n,j})$, so that the points $w_{n,j}$ converge to zero along the half line through $-\lambda$, which is different from the negative real axis, and since $\lambda \notin \mathbb{R}$ also different from the positive real axis. Choose $J > 0$ such that $\arg(w_{n,j}) \sim \arg(-\lambda)$ for $j \geq J$. It follows that the points $w_{n,j}$ still lie in the parabolic basin for $j \geq J$. Now define $w_{n,j,\ell} = h^\ell(w_{n,j})$ for $j \geq J$ and $\ell \geq 0$.

Recall the existence of the Fatou coordinate on the parabolic basin: a change of coordinates, again denoted by φ, conjugating h to $z \mapsto z + 1$. Recall that φ is of the form

$z \mapsto -\frac{1}{z} + b \log(z) + o(1)$ for some $b \in \mathbb{C}$. It follows that each of the orbits $\{w_{n,j,\ell}\}_{\ell \in \mathbb{N}}$ lies on a real analytic curve, and these real analytic curves are all transverse to the half line through $-\lambda$, with angles bounded away from zero. After scaling by an iterate g_0^s to bring $w_{j,J,\ell}$ back to fixed scale, we obtain an arbitrarily dense set of points lying in an open set of uniform size. By increasing n and taking a convergent subsequence of $g_0^s(w_{j,J,\ell})$ we obtain a dense set of accumulation points in an open subset, completing the proof. $\square$

Remark 3.4. It is not clear to the authors whether Theorem 3.3 also holds for nonreal λ with $|\lambda| > 1$. However, it does hold for generic λ. Assume that the set

$$S' := \overline{\{2^{-m} \cdot \lambda^n\}}$$

is dense in $\mathbb{C}$. Using the attraction under f_0 to the point ∞, we can consider a starting value z_0 that is unequal to but arbitrarily close to 0. We may therefore assume that, for $k \in \mathbb{N}$ large, the point $2^k z_0$ is still close to zero. For $j \leq k$ we obtain that

$$f_0^j f_1^n(z_0) \sim 2^j \lambda^n z_0 = 2^{-(k-j)} \lambda^n (2^m z_0)$$

when $f_0^j f_1^n(z_0)$ is still sufficiently close to the origin. The assumption that S' is dense implies that by starting with smaller and smaller values of z_0, the set of points $f_0^j f_1^n(z_0)$ becomes more and more dense in a round annulus centered at 0 of arbitrarily large modulus. As in the proof of Theorem 2.3 it follows that the semi-group orbit is dense in $\widehat{\mathbb{C}}$.

Remark 3.5. The Fatou set $F(G)$ of the semi-group $G = \langle f_0, f_1 \rangle$ is the set of points where G is normal. The Julia set $J(G)$, defined as the complement $\widehat{\mathbb{C}} \setminus F(G)$, is a closed backward invariant set. Under the assumptions of Theorem 3.3, $J(G)$ equals $\widehat{\mathbb{C}}$. Indeed, the Julia set contains 0 and by backward invariance also ∞. By [7] it contains a neighborhood of ∞ and then using Theorem 3.3 it equals $\widehat{\mathbb{C}}$. By [8], repelling fixed points of elements of G lie dense in $J(G) = \widehat{\mathbb{C}}$.

3.2. **Intermittency.** Our explicit semigroup satisfies the hypotheses of Theorem 2.3. Moreover Lemma 3.2 implies that the only invariant subset is $\{-1, 0, \infty\}$. Thus the result of Theorem 2.7 still holds for our specific example. As the density occurs for parameter values for which the semigroup is resonant, we can state the intermittency with larger values of λ.

Theorem 3.6. *Consider the iterated function system $\langle f_0, f_1 \rangle$ given with probabilities p_0, p_1. Let $\lambda \in B(0, 1) \setminus \mathbb{R}$. Assume (3.2) holds and*

$$p_0 > \frac{1}{2}.$$

Then the only finite stationary measure is the delta measure δ_0. For any $\varepsilon > 0$, for Lebesgue almost any $z \in \widehat{\mathbb{C}}$,

(1) *$f_\omega^n(z) \notin B(0, \varepsilon)$ for infinitely many n;*
(2) *$\lim_{N \to \infty} \frac{1}{N} |\{0 \leq n < N \ : \ f_\omega^n(z) \in B(0, \varepsilon)\}| = 1$,*

for almost all $\omega \in \Sigma$.

4. Vanishing Lyapunov exponents

For iterated function systems of interval maps, Gharaei and the first author [5] showed how intermittent time series occur if there is a common fixed point with a vanishing Lyapunov exponent, so that the common fixed point is neutral on average. We will present an analogous example for iterated function systems of Möbius transformations on the Riemann sphere. Consider the maps

$$f_0(z) = \mu z,$$
$$f_1(z) = \frac{z}{\mu + z}$$

We pick the maps f_0, f_1 with equal probability.

Theorem 4.1. *Consider the iterated function system $G = \langle f_0, f_1 \rangle$ given with probabilities $p_0 = p_1 = 1/2$. Assume that $|\mu| > 1$ and $\mu \notin \mathbb{R}$.*

The G-orbit of any $z_0 \in \widehat{\mathbb{C}} \setminus \{0\}$ is dense.

The only finite stationary measure is the delta measure δ_0. For any $\varepsilon > 0$, for Lebesgue almost any $z \in \widehat{\mathbb{C}}$,

(1) *$f_\omega^n(x) \notin B(0, \varepsilon)$ for infinitely many n;*
(2) *$\lim_{N \to \infty} \frac{1}{N} |\{0 \le n < N : f_\omega^n(x) \in B(0, \varepsilon)\}| = 1$,*

for ν_{p_0, p_1}-almost all $\omega \in \Sigma$.

Proof. That semi-group orbits lie dense is proved as in Theorem 3.3. To prove the remaining statements on intermittency we follow the reasoning of Theorem 3.6. Key statement is again

$$\int_A R \, d\nu_{p_0, p_1} \times m = \infty, \tag{4.1}$$

where

$$R(\omega, z) = \min\{i > 0 : F^i(\omega, z) \in A\}$$

is the return time to A. Here as before $A = [0] \times \mathcal{A}$ with $\mathcal{A}$ an annulus between a small circle S around 0 and its image $f_0(S)$. By Kač theorem this implies there is no finite stationary measure with support intersecting $\mathcal{A}$, so that the only stationary probability measure is δ_0.

To obtain (4.1), fix z inside S and let $H(\omega) = \min\{i > 0 : F^i(\omega, z) \in A\}$ be the first time that $f_\omega^i(z)$ that enters A. We are done if we prove

$$\int_\Sigma H \, d\nu_{p_0, p_1} = \infty. \tag{4.2}$$

A sequence $z_n = |f_\omega^n(z_0)|$ that stays near 0 satisfies $|z_{n+1} - \mu z_n| \le C z_n^2$ (if $\omega_n = 0$) or $|z_{n+1} - \frac{1}{\mu} z_n| \le C z_n^2$ (if $\omega_n = 1$) for some $C > 0$. Now (4.2) follows just as in the proof of [5, Theorem 5.2].

The remaining statements follow as in the proof of Theorem 3.6. □

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KdV Institute for Mathematics, University of Amsterdam, Science park 107, 1098 XH Amsterdam, Netherlands

Department of Mathematics, VU University Amsterdam, De Boelelaan 1081, 1081 HV Amsterdam, Netherlands

E-mail address: `a.j.homburg@uva.nl`

KdV Institute for Mathematics, University of Amsterdam, Science park 107, 1098 XH Amsterdam, Netherlands

E-mail address: `h.peters@uva.nl`

KdV Institute for Mathematics, University of Amsterdam, Science park 107, 1098 XH Amsterdam, Netherlands

E-mail address: `v.f.rabodonandrianandraina@uva.nl`