Computing Invariant Sets

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Computing Invariant Sets

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Robin de Vilder would like to dedicate this paper to the loving memory of his wife Annemiek.

Abstract

We describe algorithms for computing hyperbolic invariant sets of diffeomorphisms and their stable and unstable manifolds. This includes the calculation of Smale horseshoes and the stable and unstable manifolds of periodic points in any finite dimension.

Introduction

In understanding the dynamics of a dynamical system, governed by some diffeomorphism, stable and unstable manifolds play a crucial role. If the stable and unstable manifolds of a hyperbolic periodic point have transverse intersections, this implies the existence of nontrivial hyperbolic sets ('horseshoes'). For a family of diffeomorphisms, the dynamics can alter profoundly if the family goes through a homoclinic tangency (where the stable and unstable manifolds of some hyperbolic periodic point are tangent). However stable and unstable manifolds are not easily available analytically. An algorithm implemented on a computer computing stable and unstable manifolds of hyperbolic periodic points and detecting tangencies can therefore be of great help in comprehending the dynamics. In this paper we describe and discuss such an algorithm. Additionally we describe an algorithm to compute orbits in a hyperbolic invariant set, e.g. a horseshoe, as well as their stable and unstable manifolds.

These algorithms are currently being implemented in the program DUNRO [Sands & de Vilder,1995]. The authors hope to make DUNRO publically available in the near future. Please contact them for further information.

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1 Stable and unstable manifolds of hyperbolic fixed points

In this section we explain the algorithm to compute stable and unstable manifolds of a hyperbolic fixed point. The algorithm is based on a variant of an existence proof by Perron [Perron,1929], see also [Irwin,1980], [Shub,1980], [Homburg,1993].

First we recall some definitions, mostly to fix the notation, see [Shub,1980] for details. Let f be a diffeomorphism of $\mathbb{R}^n$. The stable set $W^s(x)$ of $x \in \mathbb{R}^n$ is the set of points y such that the distance between $f^n(y)$ and $f^n(x)$ goes to zero for $n \to \infty$:

$$W^s(x) \quad = \quad \{y \in \mathbb{R}^n, \qquad \|f^n(y) - f^n(x)\| \to 0, \;\; n \to \infty\}. \tag{1}$$

The local stable set $W^s_{\rm loc}(x)$ consists of the points y in $W^s(x)$ such that $f^n(y)$ is in a small neighborhood of $f^n(x)$ for all $n \geq 0$. Observe that

$$W^s(x) = \bigcup_{n \leq 0} f^n(W^s_{\rm loc}(x)). \tag{2}$$

The unstable set $W^u(x)$ and the local unstable set $W^u_{\rm loc}(x)$ are defined as the (local) stable set for f^{-1}:

$$W^u(x) \quad = \quad \{y \in \mathbb{R}^n, \qquad \|f^n(y) - f^n(x)\| \to 0, \;\; n \to -\infty\}. \tag{3}$$

and

$$W^u(x) = \bigcup_{n \geq 0} f^n(W^u_{\rm loc}(x)). \tag{4}$$

If we want to stress the dependence of stable sets on f, we write $W^s_f(x)$ for the stable set of x and $W^s_{f,\rm loc}$ for the local stable set of x. Similarly for unstable sets.

Suppose f has a hyperbolic fixed point at p, that is, $f(p) = p$ and the eigenvalues of $Df(p)$ all have magnitude different from 1. Let E^s be the sum of the generalized eigenspaces of those eigenvalues of $Df(p)$ that have magnitude smaller than one. Let E^u be the sum of the generalized eigenspaces of those eigenvalues of $Df(p)$ that have magnitude larger than one. For simplicity we work in stable and unstable coordinates. By this we mean that we will work in $E^s \times E^u$, and identify $E^s \times E^u$ with $\mathbb{R}^n$ via the map $(x^s, x^u) \in E^s \times E^u \mapsto p + x^s + x^u$, taking $(0,0)$ to p. We continue to write the diffeomorphism as f. In these coordinates the derivative $Df(0,0)$ is block diagonal of the form

$$Df(0,0) \quad = \quad \begin{pmatrix} S & 0 \\ 0 & U \end{pmatrix}$$

where $S : E^s \to E^s$ and $U : E^u \to E^u$. For some $m > 0$, S^m contracts vectors and U^m expands vectors. Define $\lambda_s = \max\{|\lambda|,\ \lambda \text{ is an eigenvalue of } S\} < 1$ and $\lambda_u = \min\{|\lambda|,\ \lambda \text{ is an eigenvalue of } U\} > 1$.

We recall the stable manifold theorem, saying that stable and unstable sets of hyperbolic fixed points are smooth manifolds, see [Shub,1980], [Irwin,1980].

Theorem 1.1 *Let f be a diffeomorphism of $E^s \times E^u$ with a hyperbolic fixed point at $(0,0)$ as above. Then $W^s_{\text{loc}}(0,0)$ is diffeomorphic to a disk tangent to $E^s \times \{0\}$ at $(0,0)$, $W^u_{\text{loc}}(0,0)$ is diffeomorphic to a disk tangent to $\{0\} \times E^u$ at $(0,0)$. $W^s(0,0)$ and $W^u(0,0)$ are smooth injectively immersed manifolds.*

1.1 Algorithms

In this section we describe iterative algorithms for calculating the local stable manifold of the hyperbolic fixed point $(0,0)$. In order to use the basic algorithm one must know both f and its inverse f^{-1}. A variant is provided in case only f is known. We also propose an alternative algorithm which has a theoretically faster convergence rate. The corresponding algorithms for the local unstable manifold are given at the end of the section.

Let $\pi_s : E^s \times E^u \to E^s$, $\pi_u : E^s \times E^u \to E^u$ be the coordinate projections $\pi_s(x^s, x^u) = x^s$, $\pi_u(x^s, x^u) = x^u$.

The projection π_s restricted to $W^s(0,0)$ is one-to-one near $(0,0)$ because the stable manifold $W^s(0,0)$ is tangent at $(0,0)$ to $E^s \times \{0\}$. Therefore there is an inverse map σ from a neighborhood of 0 in E^s onto a neighborhood of $(0,0)$ in $W^s(0,0)$. The algorithms calculate $\sigma(a)$ given $a \in E^s$ close enough to 0; in fact they calculate the entire forward orbit of $\sigma(a)$ under f as a fixed point of a contraction mapping $\mathcal{S}_a$ defined on an appropriately chosen sequence space $C_B(\mathbb{N}, E^s \times E^u)$, see below. Starting with an approximation to the forward orbit of $\sigma(a)$, one simply iterates $\mathcal{S}_a$. The iterates will converge to the forward orbit of $\sigma(a)$ as long as the initial approximation was good enough. In practice, of course, one only iterates finite sequences, as discussed in the section on implementation below.

Let $C_B(\mathbb{N}, E^s \times E^u)$ denote the set of bounded sequences $\mathbb{N} \to E^s \times E^u$, equipped with the supnorm. An element $\mathbf{x}$ of $C_B(\mathbb{N}, E^s \times E^u)$ will be written $\mathbf{x} = \{\mathbf{x}_k,\ k \in \mathbb{N}\} = \{(\mathbf{x}^s_k, \mathbf{x}^u_k),\ k \in \mathbb{N}\}$.

The basic algorithm. Consider the following map $\mathcal{S}_a$ on $C_B(\mathbb{N}, E^s \times E^u)$, depending on a parameter $a \in E^s$:

$$\begin{array}{lcll} \mathcal{S}_a & : & \mathbf{x} \mapsto \mathbf{x}', & \\ \mathbf{x}'_0 & = & (a & , \quad \pi_u f^{-1}(\mathbf{x}_1)), \\ \mathbf{x}'_k & = & (\pi_s f(\mathbf{x}_{k-1}) & , \quad \pi_u f^{-1}(\mathbf{x}_{k+1})) \text{ if } k > 0. \end{array}$$

For a small enough, $\mathcal{S}_a$ is a contraction in a neighborhood of $(0,0)$. If f is linear then this is clear. Since f is differentiable at $(0,0)$ it is approximately linear near $(0,0)$, and thus $\mathcal{S}_a$ is a contraction on a small enough neighborhood of $(0,0)$. The contraction rate can be made arbitrarily close to $\max\{\lambda_s, 1/\lambda_u\}$ by taking sufficiently small neighborhoods of $(0,0)$, see [Homburg et al,1995].

The unique fixed point of $\mathcal{S}_a$ in a small neighborhood of $(0,0)$ is the sequence $(\sigma(a), f(\sigma(a)), \ldots)$, the forward orbit of $\sigma(a)$, so we get $\sigma(a)$ as the first coordinate. By varying a, one obtains the local stable manifold.

The algorithm when f^{-1} is unknown. For the above construction, the inverse f^{-1} of f has to be explicitly known. A variant of the construction does not require knowledge of f^{-1}:

$$\begin{array}{lcll} \hat{\mathcal{S}}_a & : & \mathbf{x} \mapsto \mathbf{x}', & \\ \mathbf{x}'_0 & = & (a & , \quad \pi_u \mathbf{x}_0 - U^{-1}(\pi_u \mathbf{x}_1 - \pi_u f(\mathbf{x}_0))), \\ \mathbf{x}'_k & = & (\pi_s f(\mathbf{x}_{k-1}) & , \quad \pi_u \mathbf{x}_k - U^{-1}(\pi_u \mathbf{x}_{k+1} - \pi_u f(\mathbf{x}_k))) \text{ if } k > 0. \end{array}$$

The contraction rate is the same as for the basic algorithm; namely, arbitrarily close to $\max\{\lambda_s, 1/\lambda_u\}$.

A faster algorithm. Define $\bar{\mathcal{S}}_a$ by

$$\begin{array}{lcll} \bar{\mathcal{S}}_a & : & \mathbf{x} \mapsto \mathbf{x}', & \\ \mathbf{x}'_0 & = & (a & , \quad \pi_u \mathbf{x}_0 - U^{-1}(\pi_u f(\mathbf{x}_0) - \pi_u \mathbf{x}'_1)), \\ \mathbf{x}'_k & = & (\pi_s f(\mathbf{x}_{k-1}) + S(\pi_s \mathbf{x}'_{k-1} - \pi_s \mathbf{x}_{k-1}) & , \quad \pi_u \mathbf{x}_k - U^{-1}(\pi_u f(\mathbf{x}_k) - \pi_u \mathbf{x}'_{k+1})) \text{ if } k > 0. \end{array}$$

This map is well defined in spite of $\mathbf{x}'$ appearing on both sides of the equations. Indeed, the stable coordinates $\pi_s \mathbf{x}'_k$ can calculated inductively from $\pi_s \mathbf{x}'_0 = a$. The unstable coordinates $\pi_u \mathbf{x}'_k$ are more problematic since one needs to know $\pi_u \mathbf{x}'_{k+1}$ in order to calculate $\pi_u \mathbf{x}'_k$. If one chooses $\pi_u \mathbf{x}'_N = 0$ then this determines values for $\pi_u \mathbf{x}'_0, \ldots, \pi_u \mathbf{x}'_N$ in an N-dependent way. These values converge in the limit $N \to \infty$, and we take the limit values as defining $\mathbf{x}'$.

The contraction rate can be made arbitrarily close to

$$\left(\frac{\lambda_u}{\lambda_u - 1} + \frac{\lambda_s}{1 - \lambda_s}\right) \cdot \|a\| \cdot \|D^2 f(0,0)\|$$

by taking small enough neighborhoods of $(0,0)$. This formula can be derived by expanding $f(x)$ in a Taylor series around $(0,0)$ up to second order terms. Thus the contraction rate can be made arbitrarily close to zero by taking a close enough to zero.

Algorithms for the unstable manifold. The local unstable manifold is the local stable manifold for f^{-1} and can thus be computed analogously to the local stable manifold. If one does not explicitly know f^{-1}, use the following contraction $\mathcal{U}_b$ on $C_B(-\mathbb{N}, E^s \times E^u)$, where $b \in E^u$:

$$\begin{aligned}
\hat{\mathcal{U}}_b &: \mathbf{x} \mapsto \mathbf{x}', \\
\mathbf{x}'_0 &= (\pi_s f(\mathbf{x}_{-1}) \quad , \quad b), \\
\mathbf{x}'_k &= (\pi_s f(\mathbf{x}_{k-1}) \quad , \quad \pi_u \mathbf{x}_k - U^{-1}(\pi_u \mathbf{x}_{k+1} - \pi_u f(\mathbf{x}_k))) \text{ if } k < 0.
\end{aligned}$$

The unique fixed point of $\mathcal{U}_b$ in a small neighborhood of $(0,0)$ is the sequence $(\ldots, f^{-1}(\mu(b)), \mu(b))$, the backward orbit of $\mu(b)$, where $\mu : E^u \to W^u(0,0)$ is the local inverse of $\pi_u : W^u(0,0) \to E^u$. We get $\mu(b)$ as the first coordinate. By varying b, one obtains the local unstable manifold. The contraction rate is the same as for the basic algorithm for the stable manifold; namely, arbitrarily close to $\max\{\lambda_s, 1/\lambda_u\}$.

The analogous fast algorithm for the unstable manifold is

$$\begin{aligned}
\bar{\mathcal{U}}_b &: \mathbf{x} \mapsto \mathbf{x}', \\
\mathbf{x}'_0 &= (\pi_s f(\mathbf{x}_{-1}) + S(\pi_s \mathbf{x}'_{-1} - \pi_s \mathbf{x}_{-1}) \quad , \quad b), \\
\mathbf{x}'_k &= (\pi_s f(\mathbf{x}_{k-1}) + S(\pi_s \mathbf{x}'_{k-1} - \pi_s \mathbf{x}_{k-1}) \quad , \quad \pi_u \mathbf{x}_k - U^{-1}(\pi_u f(\mathbf{x}_k) - \pi_u \mathbf{x}'_{k+1})) \text{ if } k < 0.
\end{aligned}$$

1.2 Implementation

In this section we discuss some aspects of the implementation of the above algorithms on a computer.

Stability. Since we are iterating a *contraction* $\mathcal{S}_a$, the method is unaffected by small perturbations, eg: rounding errors; errors in the evaluation of f; errors in the calculation of the fixed point p or of the splitting into stable and unstable eigenspaces; the truncation of sequences to a finite number of terms, etc. A small perturbation of $\mathcal{S}_a$ is still a contraction with approximately the same fixed point and contraction rate. Thus the method is numerically stable.

Finite sequences. Only finite pieces of orbits can be stored in a computer. Denote by π_N the projection $\pi_N(\mathbf{x}) = (\mathbf{x}_0, \ldots, \mathbf{x}_N, 0, 0, \ldots)$. Effectively we are replacing $\mathcal{S}_a$ by $\mathcal{S}_{a,N} = \pi_N \circ \mathcal{S}_a \circ \pi_N$ for some sufficiently large N. This map is a small perturbation of $\mathcal{S}_a$ if N is large enough, so will be a contraction on the space of sequences $\{0, \ldots, N\} \to \mathbb{R}^n$, with fixed point close to the fixed point of $\mathcal{S}_a$.

To get points on $W^s_{\text{loc}}(p)$ with precision Δ, compute the fixed point ν of $\mathcal{S}_{a,N}$ — for some parameter value a — with N large enough that $\|\nu_N\| \leq \Delta(1-\kappa)$, where $0 < \kappa < 1$ is the contraction rate of $\mathcal{S}_a$, see above. One easily checks that then $\|\eta - \nu\|_{\sup} \leq \Delta$, where η is the fixed point of $\mathcal{S}_a$ for the same value of a. There is another reason for checking that $\|\nu_N\|$ is small: if N is not large enough then the algorithm may nonetheless converge, but to the wrong sequence, with ν_N close to some other fixed point of f.

Global manifolds. The above algorithms enable one to calculate $W^u_{\text{loc}}(0,0)$ to any desired accuracy. Iterating $W^u_{\text{loc}}(0,0)$ by f (or more exactly, iterating the points approximating $W^u_{\text{loc}}(0,0)$ by f) gives the global unstable manifold $W^u(0,0)$. One has to take care that the distance between calculated points on the manifold does not grow too large when iterating. Keeping track of this distance and if necessary computing additional points, one ensures a correct approximation of a piece of the unstable manifold by a sequence of computed points.

If f^{-1} is known then $W^s(0,0)$ can be calculated by iterating $W^s_{\mathrm{loc}}(0,0)$ by f^{-1}. If not, then continuation methods can be used (these may work even if f is not invertible): the edge of the piece of the stable manifold calculated so far is sent by f closer to the origin, into a part of $W^s(0,0)$ already calculated (for example $W^s_{\mathrm{loc}}(0,0)$). This leads to the continuation problem of extending the known piece of stable manifold beyond the current edge, using the fact that the extension must be sent by f into a known curve (the stable manifold closer to $(0,0)$). Standard methods allow one to extend $W^s(0,0)$ until a singularity of Df is encountered.

2 Computing jets of manifolds

In the introduction we mentioned the importance of being able to detect tangencies of invariant manifolds. Suppose f_μ is a family of diffeomorphisms on $\mathbb{R}^n$, such that f_μ has a hyperbolic fixed point p_μ, depending continuously on μ. Assume further that at $\mu = \mu_0$, the stable and unstable manifolds of p_μ are tangent. To be able to give information on the bifurcations occurring when μ is varied, one has to know the order of tangency and the change in relative position of the stable and unstable manifolds as μ varies. In this section we indicate how the relevant derivatives can be computed.

First we discuss the computation of higher order jets of stable and unstable manifolds of a single map f with a hyperbolic fixed point at the origin 0, as before. Let n_s and n_u be the dimensions of E^s and E^u respectively. Write $\mathcal{G}$ for the Grassmanian manifold of n_s-dimensional linear subspaces L of $\mathbb{R}^n$. Define $F : \mathbb{R}^n \times \mathcal{G} \to \mathbb{R}^n \times \mathcal{G}$ by $F(x, L) = (f(x), Df(x)L)$. It can be shown that $(0, E^s)$ is a hyperbolic fixed point of F, its eigenvalues being those of $Df(0)$ supplemented by eigenvalues of the form λ_u/λ_s where λ_u and λ_s are eigenvalues of $Df(0)$ with $|\lambda_u| > 1$ and $|\lambda_s| < 1$. The stable manifold of F at $(0, E^s)$ consists of all points (x, L) where $x \in W^s(0)$ and L is the tangent space to $W^s(0)$ at x. That is, one can calculate the tangent space to $W^s(0)$ at x by calculating the corresponding point of $W^s_F(0, E^s)$. Doing this for both $W^s(0)$ and $W^u(0)$ lets one detect, for example, homoclinic tangencies.

If one chooses a specific parametrization for $W^s_{\mathrm{loc}}(0)$, for example that given by σ in section 1, then it is possible to calculate the derivative of σ at a point $a \in E^s$. Let $\mathcal{L}$ be the space of linear maps from E^s to $E^u \subseteq \mathbb{R}^n$ and define $\bar{F} : \mathbb{R}^n \times \mathcal{L} \to \mathbb{R}^n \times \mathcal{L}$ by $\bar{F}(x, A) = (f(x), \pi_u Df(x)(I + A)(\pi_s Df(x)(I + A))^{-1})$. This has $(0,0)$ as a hyperbolic fixed point. The local stable manifold of $\bar{F}$ at $(0,0)$ consists of all points (x, A) where $x \in W^s(0)$ and A gives a parametrization of the tangent space to $W^s(0)$ at x via the map $D\sigma : E^s \to \mathbb{R}^n$, $v \mapsto v + Av$. The formula for $\bar{F}$ represents composing $v \mapsto v + Av$ with $Df(x)$, writing the composition in the form $v \mapsto v + A'v$, and putting $\bar{F}(x, A) = (f(x), A')$.

Higher dimensional jets of $W^s(0)$ can be calculated similarly, for example quadratic approximations to $W^s(0)$. One works with the space J^k of k-jets from E^s to $\mathbb{R}^n$ (see e.g. [Golubitsky & Guillemin,1973] for definitions and properties of jet bundles). One defines $\mathcal{F} : J^k \to J^k$ by $\mathcal{F}(j) = f_*(j)$ where f_* is the action induced by f on k-jets. For example, if $k = 1$ then $J^k = \mathbb{R}^n \times G$ and $\mathcal{F} = F$ is as above. The k-jet of $W^s(0)$ at 0, which we write as $j_0^k(W^s(0))$, is a hyperbolic fixed point of $\mathcal{F}$. The stable manifold of $\mathcal{F}$ at $j_0^k(W^s(0))$ consists of all k-jets of $W^s(0)$. That is, one can calculate the k-jet of $W^s(0)$ at a point $x \in W^s(0)$ by calculating the corresponding point of $W^s_{\mathcal{F}}(j_0^k(W^s(0)))$. One can calculate for example whether a homoclinic tangency is generic by checking that the second order jets of $W^s(0)$ and $W^u(0)$ are different at the point of tangency.

Now we discuss calculating derivatives with respect to a parameter. Suppose the parameter μ belongs to $\mathbb{R}^m$. The set $W^s = \{(x, \mu),\, x \in W^s_{f_\mu}(p_\mu),\, \mu \text{ near } \mu_0\}$ is an injectively immersed $(n_s + m)$-dimensional submanifold of $\mathbb{R}^{n+m}$. One can calculate its tangent planes as follows (similarly, given a parametrization, for the derivative of $W^s_{f_\mu}(p_\mu)$ with respect to μ). Write $\mathcal{G}$ for the Grassmanian manifold of all $(n_s + m)$-dimensional linear subspaces of $\mathbb{R}^{n+m}$. Define $\hat{f} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by $\hat{f}(x, \mu) = (f_\mu(x), \mu)$. Clearly $\hat{f}$ has fixed points at (p_μ, μ). Let $\hat{E}^s$ be the tangent space to W^s at (p_{μ_0}, μ_0). Define $\hat{F} : \mathbb{R}^n \times \mathcal{G} \to \mathbb{R}^n \times \mathcal{G}$ by $\hat{F}(x, L) = (f(x, \mu_0), D\hat{f}(x, \mu)L)$. It can be shown that $(p_{\mu_0}, \hat{E}^s)$ is a hyperbolic fixed point of $\hat{F}$. The supplementary eigenvalues of $D\hat{F}(p_{\mu_0}, \hat{E}^s)$ are of the form λ_u/λ_s and $\lambda_u/1$. The stable manifold of $\hat{F}$ at $(p_{\mu_0}, \hat{E}^s)$ consists of all points (x, L) where $x \in W^s_{f_{\mu_0}}(p_{\mu_0})$ and L is the tangent space to W^s at (x, μ_0). Using this, one can calculate, for example, whether a tangency of $W^s(0)$ and $W^u(0)$ unfolds with non-zero speed as μ crosses μ_0.

2.1 Implementation

The special structure of the extended function F greatly simplifies the stable manifold algorithms given in section 1. Indeed, suppose the forward orbit $\mathbf{x} = (x_0, x_1 = f(x_0), \ldots)$ of some point $x_0 \in W^s(0)$ with $\pi_s(x_0) = a$ has already been calculated. If $\mathbf{L} = (L_0, L_1, \ldots)$ is a bounded sequence of points in $\mathcal{G}$ then the stable manifold algorithm gives $\mathcal{S}_a(\mathbf{x}, \mathbf{L}) = (\mathbf{x}', \mathbf{L}')$ where $\mathbf{x}' = \mathbf{x}$ and $\mathbf{L}' = (Df(x_1)^{-1}L_1, Df(x_2)^{-1}L_2, \ldots)$. The calculation of the tangent plane to $W^s(0)$ at x_0 reduces to evaluating $\lim_{N\to\infty} \left(Df^N(x_N)\right)^{-1} E^s$. Similar simplifications occur for the other algorithms.

This method of calculating jets has both advantages and disadvantages compared to the numerical differentiation of $W^s(0)$. It seems most suitable for calculating first and second order derivatives of one-dimensional manifolds. The main advantage is in being fairly well-behaved numerically, since the eigenvalues of the manifold problem become more rather than less hyperbolic as additional derivatives are taken. The main disadvantage, particularly when E^s has dimension > 1, or when taking more than one or two derivatives, is in the greatly increased dimension of the space and the number of calculations required. A further problem is that of representing jets in a computer. In general it is necessary to keep them in a normal form or regularly rescale them.

3 Stable and unstable manifolds of hyperbolic orbits

The previously described algorithms can be generalized to compute stable and unstable manifolds of hyperbolic periodic orbits. For this, the splitting in stable and unstable directions along the periodic orbit has to be known. In fact, we will deal with the more general situation of computing stable and unstable manifolds of points contained in some hyperbolic invariant set. We first describe the algorithm that computes such invariant manifolds. In the next section we examine the special case of a diffeomorphism of $\mathbb{R}^2$ possessing a horseshoe.

Suppose $\{x_m\}_{m\in\mathbb{Z}}$ is an orbit of f contained in a hyperbolic invariant set $\mathfrak{D}$. This means that at each point $x_m \in \mathfrak{D}$ a splitting $\mathbb{R}^n = E^s(x_m) \times E^u(x_m)$ exists, depending continuously on x_m, such that

$$Df(x_m)E^s(x_m) = E^s(x_{m+1}), \tag{5}$$

$$Df(x_m)E^u(x_m) = E^u(x_{m+1}), \tag{6}$$

and further, for some $N \in \mathbb{N}$, $Df^N(x_m)\big|_{E^s(x_m)}$ contracts vectors and $Df^N(x_m)\big|_{E^u(x_m)}$ expands vectors. The orbit $\{x_m\}_{m\in\mathbb{Z}}$ can in particular be a hyperbolic periodic orbit where $x_m = x_{m+p}$ for all $m \in \mathbb{Z}$, where p is the period.

Let $\pi_{s,m} : \mathbb{R}^n \to E^s(x_m)$ and $\pi_{u,m} : \mathbb{R}^n \to E^u(x_m)$ be the linear projections of $\mathbb{R}^n$ onto $E^s(x_m)$ and $E^u(x_m)$ respectively. A point $x \in \mathbb{R}^n$ can be decomposed as $x = \pi_{s,m}\, x + \pi_{u,m}\, x$. Consider the following map $\mathcal{S}_a$, depending on a parameter $a \in E^s(x_0)$, on the space of those sequences $\mathbf{x} : \mathbb{N} \to \mathbb{R}^n$ such that $\sup_{m\in\mathbb{N}} \|\mathbf{x}_m - x_m\|$ is bounded:

$$\begin{aligned}
\mathcal{S}_a &: \mathbf{x} \mapsto \mathbf{x}', \\
\mathbf{x}'_0 &= a + \pi_{s,0}\, x_0 + \pi_{u,0}\, f^{-1}(\mathbf{x}_1), \\
\mathbf{x}'_k &= \pi_{s,k}\, f(\mathbf{x}_{k-1}) + \pi_{u,k}\, f^{-1}(\mathbf{x}_{k+1}) \text{ if } k > 0.
\end{aligned}$$

The map $\mathcal{S}_a$ is a contraction if a is small. Its fixed point is the positive orbit $\{\nu_m\}_{m\in\mathbb{N}}$ on $W^s_{\mathrm{loc}}(x_0)$ with $\pi_{u,0}(\nu_0 - x_0) = a$. By varying a, $W^s_{\mathrm{loc}}(x_0)$ is obtained. The local unstable manifold $W^u_{\mathrm{loc}}(x_0)$ is computed analogously.

3.1 Implementation

In order to apply these algorithms, one needs to know the hyperbolic orbit $\{x_m\}_{m\in\mathbb{Z}}$. Here we make some comments on the calculation of periodic orbits. The calculation of general orbits is discussed in the next section.

A point x_0 of period p should not in general be calculated as a fixed point of f^p. The reason is that the evaluation of f^p can be highly inaccurate especially if p is large, due to the possible sensitive dependence to initial conditions of f.[1] An alternative approach is to calculate the orbit $(x_0, f(x_0), \ldots, f^{p-1}(x_0))$ as

[1] None of our algorithms involve iterating f many times on a point; they involve iterating associated contraction mappings many times. This is numerically stable.

a fixed point of the mapping $F:\mathbb{R}^{np}\to\mathbb{R}^{np}$ taking $(y_0,\dots,y_{p-1})$ to $(f(y_{p-1}),f(y_0),\dots,f(y_{p-2}))$. This results in an important gain in numerical accuracy at the price of dealing with large vectors.

Similarly, it is better to calculate the stable and unstable manifolds of $(x_0,\dots,f^{p-1}(x_0))$ using the methods of this section rather than by considering x_0 as a fixed point of f^p and using the methods of section 1.

4 Calculating hyperbolic orbits

Suppose $\{x_m\}_{m\in\mathbb{Z}}$ is an orbit of f contained in a hyperbolic invariant set $\mathfrak{D}$. The intersection $W^s_{\rm loc}(x_0)\cap W^u_{\rm loc}(x_0)$ equals the point x_0. From this observation one can derive an algorithm to obtain the orbit $\{x_m\}$, if only known approximately. Observe first that, since the splitting $\mathbb{R}^n=E^s(x)\times E^u(x)$ depends continuously on $x\in\mathfrak{D}$, one can extend this splitting to a continuous splitting over a neighborhood of $\mathfrak{D}$. Denote by $\{\xi_m\}$ a sequence of points near the orbit $\{x_m\}$. Let $\mathcal{S}$ be the following map on the space of those sequences $\mathbf{x}:\mathbb{Z}\to\mathbb{R}^n$ such that $\sup_{m\in\mathbb{N}}\|\mathbf{x}_m-x_m\|$ is bounded:

$$\begin{aligned}\mathcal{S} &: \mathbf{x}\mapsto\mathbf{x}',\\ \mathbf{x}'_k &= \pi_{s,\xi_k}f(\mathbf{x}_{k-1})+\pi_{u,\xi_k}f^{-1}(\mathbf{x}_{k+1}).\end{aligned}$$

If $\mathbf{x}_m$ is near x_m for all m, then $\mathcal{S}^n(\mathbf{x})$ will converge to $\{x_m\}$ as $n\to\infty$.

Let us discuss this idea further for a diffeomorphism of the plane possessing a horseshoe. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a diffeomorphism, such that a square S is mapped over itself in a horseshoe shape as indicated in figure 1 below. Within S, f is almost linear, expanding in the vertical direction and contracting in the horizontal direction. We have indicated vertical and horizontal rectangles satisfying $f(H_0)=V_0, f(H_1)=V_1, f^{-1}(V_0)=H_0, f^{-1}(V_0)=H_0$.

The square B contains a maximal invariant set whose dynamics can be described using symbolic dynamics. There is a 1-1 correspondence between sequences $\mathbb{Z}\to\{0,1\}$ and orbits in this invariant set, given by associating to an orbit $\{f^k(x)\}$ the sequence $\{\sigma_k(x)\}_{k\in\mathbb{Z}}$ defined by

$$\sigma_k(p)=j \text{ if } f^k(x)\in V_j.$$

This symbolic sequence is called the *itinerary* of p. The maximal invariant set in B is not attracting: most points in B leave this box after some iterations, since the invariant set has zero measure (assuming f is C^2). Let us explain how we can approximate orbits in this invariant set. Denote by π_x,π_y the projection to the horizontal x-axis resp. the vertical y-axis. Define a map $\mathcal{S}$ on the space of sequences $\mathbb{Z}\mapsto B$ by

$$\begin{aligned}\mathcal{S} &: \mathbf{x}\mapsto\mathbf{x}',\\ \mathbf{x}'_k &= (\pi_x f(\mathbf{x}_{k-1}),\pi_y f^{-1}(\mathbf{x}_{k+1})).\end{aligned}$$

Theorem 4.1 *Let f and $\mathcal{S}$ be as above. Given a symbolic sequence $\{\sigma_m\}_{m\in\mathbb{Z}}$, let the sequence $\mathbf{x}=\{\mathbf{x}_m\}_{m\in\mathbb{Z}}$ satisfy $\mathbf{x}_m\in V_{\sigma_m}\cap H_{\sigma_{m+1}}$ for all $m\in\mathbb{Z}$. Then $\mathcal{S}^n(\mathbf{x})$ converges, as $n\to\infty$, to the unique orbit $\{x_m\}$ of f for which the itinerary of x_0 is equal to $\{\sigma_m\}_{m\in\mathbb{Z}}$.*

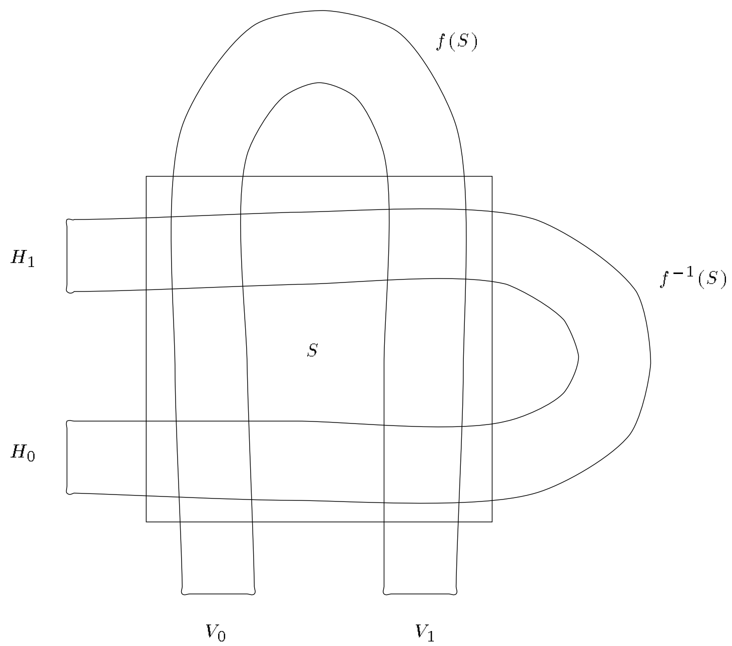

Figure 1: f maps S over itself in a horseshoe shape.

PROOF. One checks that $\mathcal{S}$ contracts the distance between each two sequences in $C_B(\mathbb{Z}, S)$ that have their i^{th} point in the same component of $V \cap H$. The condition imposed on $\mathbf{x}$ implies that $\mathcal{S}(\mathbf{x})_m$ and $\mathbf{x}_m$ are in the same component of $V \cap H$ for each m. ∎

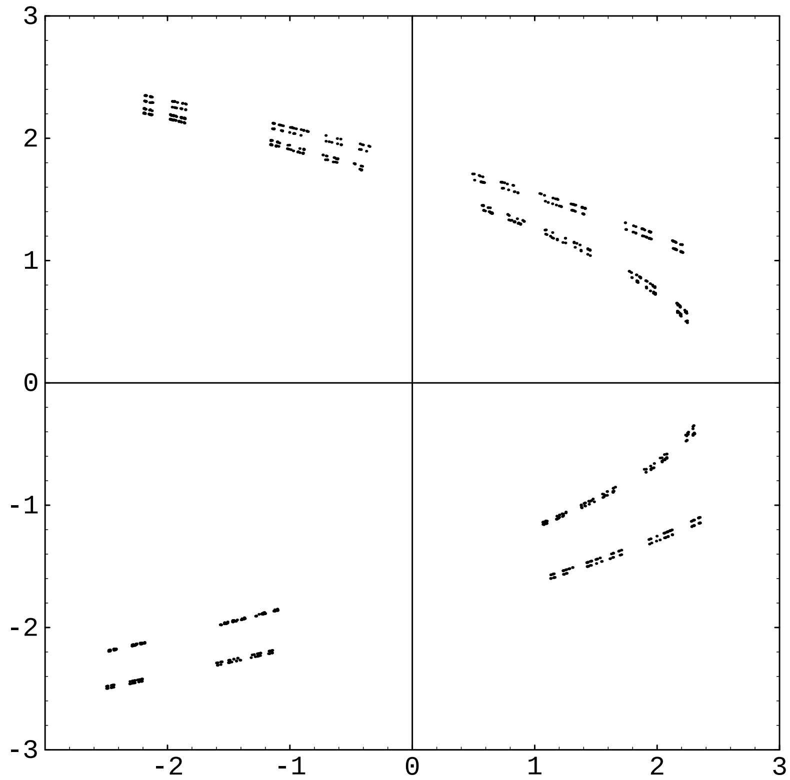

Figure 2: A period 1000 orbit for the Hénon map $(x, y) \mapsto (a - y^2 - bx, x)$ with $a = 3$ and $b = 0.3$, indicating the position of the horseshoe this map possesses. For the splittings $\mathbb{R}^n = E^s(x, y) \times E^u(x, y)$ we let $E^u(x, y)$ be the tangent space of the curve $s \mapsto (a - bt - s^2, s)$ at $(s, t) = h^{-1}(x, y)$ and we let $E^s(x, y)$ be the tangent space of the curve $t \mapsto (t, (a - t^2 - s)/b)$ at $(s, t) = h(x, y)$.

4.1 Implementation

Both the orbit $\{x_n\}$ and the splitting in stable and unstable directions along it need only be known approximately for the above procedures to work. Of course we have to work with finite symbol sequences, finite orbits and finite precision. We can approximate periodic orbits up to any desired precision by working with periodic symbol sequences (we then have only a finite number of points to deal with). We can also approximate long pieces of a non-periodic orbit $\{x_m\}$ with some prescribed symbolic coding $\{\sigma_k\}$ up to any desired precision. Indeed, consider the set $P_N = \{x \in \mathbb{R}^n | f^k(x) \in V_k \text{ for } -N < k < N\}$. It is easily seen that the size of P_N is exponentially small in N. A periodic symbol sequence which matches $\{\sigma_k\}$ for $-N < k < N$ therefore corresponds to a periodic point which is exponentially (in N) close to x_0.

A simple check as to whether the computed sequence actually approximates an orbit is to compare the n^{th} point of the computed sequence to the f-image of the $(n-1)^{\text{th}}$ point, for all n.

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