Essentially asymptotically stable homoclinic networks

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April 24, 2009

Abstract

In [15] Melbourne discusses an example of a robust heteroclinic network that is not asymptotically stable but which has the strong attracting property called essential asymptotic stability. We establish that this phenomenon is possible for homoclinic networks, where all heteroclinic trajectories are symmetry related. Moreover, we study a transverse bifurcation from an asymptotically stable to an essentially asymptotically stable homoclinic network. The essentially asymptotically stable homoclinic network turns out to attract all nearby points except those on codimension-one stable manifolds of equilibria outside the homoclinic network.

1 Introduction

In symmetric (i.e. equivariant) ordinary differential equations heteroclinic cycles can, in contrast to differential equations without symmetry, occur robustly. The heteroclinic networks given as group orbits of such heteroclinic cycles can moreover be asymptotically stable. For differential equations with discrete symmetry, an example of Melbourne [15] shows that a more intricate stability property, called essential asymptotic stability, is possible for heteroclinic networks. Here, for sufficiently small neighborhoods of the heteroclinic network, trajectories for an arbitrary large portion of initial points in it converge to the heteroclinic network but trajectories for the remainder of initial points escape. Additional research has shown a number of contexts in which heteroclinic networks arise that are essentially asymptotically stable, see for example [3, 9, 12]. These examples are all for heteroclinic networks containing equilibria with different index (dimension of the unstable manifold) which are therefore not symmetry related. Postlethwaite and Dawes [16] study heteroclinic networks in $\mathbb{Z}_6 \ltimes \mathbb{Z}_2^6$-equivariant differential equations in $\mathbb{R}^6$, for which the equilibria lie in a single group orbit. They establish trajectories that follow heteroclinic trajectories in an irregular order while converging to the heteroclinic network.

The possibility of essentially asymptotically stable homoclinic cycles is mentioned in a specific setting and without proof in [5]. The purpose of this paper is to prove, through the construction of an elementary example, the occurrence of essentially asymptotically stable homoclinic cycles. Furthermore we study a transverse bifurcation [4] through which an asymptotically stable homoclinic network loses its asymptotic stability and becomes essentially asymptotically stable. We prove that after the bifurcation there is an attractor and that for initial points outside a codimension-one manifold, their trajectories converge to the original network. (Here, an attractor is a compact invariant set that is Lyapunov stable so that the ω-limit set of each nearby point is in it.)

We now briefly introduce concepts relevant to our study, thus refraining from a treatment in full generality, referring to [10] for further information. Consider a differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{1.1}$$

in $\mathbb{R}^n$. Let G be a finite group with a representation on $\mathbb{R}^n$. A system of differential equations (1.1) is G-equivariant if $\mathbf{x}(t)$ is a solution to (1.1) if and only if $g\mathbf{x}(t)$ is a solution to (1.1), for all $g \in G$. Equivalently, if

$$gf(\mathbf{x}) = f(g\mathbf{x}),$$

for all $g \in G$.

We adopt the following definitions of heteroclinic and homoclinic cycles and networks. Note that different definitions occur in the literature.

Definition. A *heteroclinic network* H is an invariant set consisting of finitely many equilibria and of a set of heteroclinic trajectories connecting these equilibria, that is G-symmetric, i.e. $GH = H$ (without real loss of generality, one may in addition demand connectedness).

A *homoclinic cycle* Γ is an invariant set that is equal to a group orbit $\langle h\rangle\bar{\gamma}$, for a heteroclinic trajectory γ connecting p to hp for some $h \in G$. Here $\bar{\gamma}$ is the closure of γ and $\langle h\rangle$ denotes the cyclic subgroup generated by h. The element h is called the *twist* for the homoclinic cycle. A *homoclinic network* is the group orbit $G\Gamma$ of a homoclinic cycle Γ.

Let $\gamma(t)$ be a heteroclinic trajectory satisfying $\lim_{t\to-\infty}\gamma(t) = p$, for a hyperbolic equilibrium p, and $\lim_{t\to\infty}\gamma(t) = hp$ for some $h \in G$. The isotropy groups $\Sigma_{\gamma(t)} = \{g \in G\ ;\ g\gamma(t) = \gamma(t)\}$ do not depend on t, so that one can speak of the isotropy group Σ_γ of the heteroclinic trajectory γ. Let P denote the fixed point space $\{x \in \mathbb{R}^n\ ;\ gx = x, g \in \Sigma_\gamma\}$. Assume that $hp \in P$ is a sink when restricting the differential equations to P. The system then possesses a robust homoclinic network $G\overline{\gamma}$.

Definition ([3, 15])**.** A flow-invariant compact set H is *essentially asymptotically stable* if for any open neighborhood U of H there is a set C so that for any given number $a \in (0,1)$ there is an open ε-neighborhood $V \subset U$ of H such that all trajectories starting in $V - C$ remain in U and are asymptotic to H and $\mu(V - C)/\mu(V) > a$, where μ is the Lebesgue measure.

Our example of an essentially asymptotically stable robust homoclinic network will be for differential equations (1.1) in $\mathbb{R}^5$. It is constructed by taking a robust homoclinic network in $\mathbb{R}^3$ (reminiscent of the example in [1, 17]), adding two transverse directions and extending the symmetries in such a way that the old three-dimensional space is invariant.

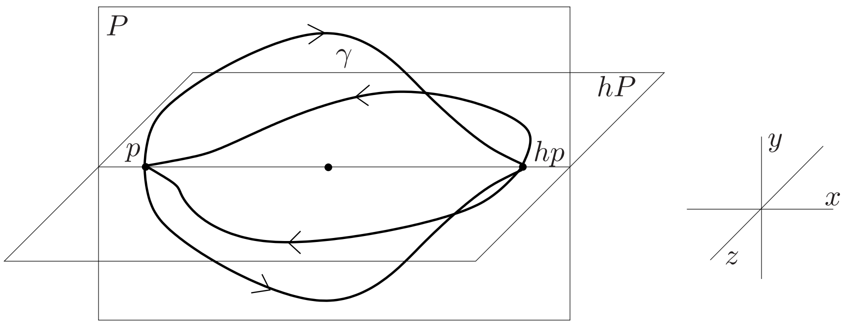

Figure 1: A sketch of the three-dimensional homoclinic network $G\Gamma$ in R.

With $\mathbf{x} = (x, y, z, u, v)$, we will assume that the differential equations are G-equivariant under the representation of $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ generated by

$$\begin{aligned} g_1 &: (x, y, z, u, v) \to (x, y, -z, u, v), \\ g_2 &: (x, y, z, u, v) \to (x, -y, z, u, v), \\ g_3 &: (x, y, z, u, v) \to (x, y, z, u, -v), \\ h &: (x, y, z, u, v) \to (-x, z, y, -v, u). \end{aligned}$$

We make the following assumptions on the existence of a homoclinic cycle. Write $\Sigma_\gamma = \mathbb{Z}_2^2(h^2, g_1)$.

Assumption. *Suppose $p = (-\bar{x}, 0, 0, 0, 0)$ and $hp = (\bar{x}, 0, 0, 0, 0)$ in $\mathrm{Fix}(\mathbb{Z}_2^3(g_1, g_2, h^2))$ are equilibria, such that restricted to the fixed point space*

$$P = \mathrm{Fix}(\Sigma_\gamma) = \{z = u = v = 0\},$$

p is a saddle and hp is a sink. Assume there is a heteroclinic trajectory $\gamma = \{\gamma(t), t \in \mathbb{R}\}$ connecting p to hp, with isotropy subgroup Σ_γ and hence contained in the fixed point space P. Let Γ be the homoclinic cycle $\langle h \rangle \overline{\gamma}$.

Note that $G\Gamma$ is a robust homoclinic network that consists of four heteroclinic trajectories contained in

$$R = \mathrm{Fix}(\mathbb{Z}_2(h^2)) = \{u = v = 0\},$$

see Figure 1. The stability of $G\Gamma$ depends on the eigenvalues of Df at p. From the G-equivariance it follows that the eigenvalues are real. Choose local coordinates by shifting the origin to p. It follows that $Df(p) = \Lambda$ is diagonal. Write λ_x, λ_y, λ_z, λ_u and λ_v for the eigenvalues which are the radial, expanding, contracting and two transverse eigenvalues respectively, see [11]. By assumption, $\lambda_x < 0$, $\lambda_z < 0$ and $\lambda_y > 0$. From [11] it follows that $G\Gamma$ is asymptotically stable within the subspace R if $-\lambda_z > \lambda_y$. Hence, $G\Gamma$ is asymptotically stable if $-\lambda_z > \lambda_y$ and $\lambda_u, \lambda_v < 0$.

When one transverse eigenvalue is positive, say $\lambda_v > 0$, the homoclinic network $G\Gamma$ is not asymptotically stable but may be essentially asymptotically stable. Key to this stability property is the swapping of the u and v directions by h. The instability when passing the equilibrium p, caused by the positive eigenvalue λ_v in the direction of the

v-axis, can be compensated by a contraction in the same direction when passing near hp. The theorem below is proved in Section 2. We need the following definition.

Definition. A set of eigenvalues $\{\lambda_1, \ldots, \lambda_j\}$ satisfies the *nonresonance condition up to order N* if for every $i_1, \ldots, i_j \in \mathbb{N}$, $2 \leq \sum_{k=1}^{j} i_k \leq N$ and $\lambda \in \{\lambda_1, \ldots, \lambda_j\}$

$$\sum_{k=1}^{j} i_k \lambda_k \neq \lambda. \tag{1.2}$$

Theorem 1.1. *Let $\dot{\mathbf{x}} = f(\mathbf{x})$ be a G-equivariant differential equation on $\mathbb{R}^5$, $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ generated by g_1, g_2, g_3 and h, possessing a robust homoclinic network $G\Gamma$ contained in the fixed point space R as above. Assume nonresonance conditions on $\{\lambda_x, \lambda_y, \lambda_z, \lambda_u, \lambda_v\}$ up to a sufficiently high order N. Assume that $0 < \lambda_v/\lambda_y < \min\{\lambda_u/\lambda_z, 1\}$, $\lambda_u < 0$ and $-\lambda_z/\lambda_y > 1$. Then $G\Gamma$ is essentially asymptotically stable.*

We note that the stability results in [12], consisting of checkable conditions for asymptotic stability and essential asymptotic stability of heteroclinic networks, do not apply in the context of the above theorem (from conditions $(S1)$–$(S4)$ in [12], condition $(S2)$ fails).

An explicit example of differential equations admitting an essentially asymptotically stable homoclinic network can be obtained by extending differential equations in $\mathbb{R}^3$ from [18] to G-equivariant differential equations in $\mathbb{R}^5$. Consider the family of differential equations

$$\begin{aligned}
\dot{x} &= \nu x + z^2 - y^2 - x^3 + \beta x(y^2 + z^2),\\
\dot{y} &= y(\lambda + ay^2 + bz^2 + cx^2) + yx,\\
\dot{z} &= z(\lambda + az^2 + by^2 + cx^2) - zx,\\
\dot{u} &= \mu u + dux,\\
\dot{v} &= \mu v - dvx.
\end{aligned}$$

A homoclinic cycle in the (x, y, z)-space exists for $\nu > 0$ small, $\lambda \in (\lambda_H(\nu), \sqrt{\nu} + c\nu)$ for some $\lambda_H(\nu) = -\frac{1}{2}\nu + \mathcal{O}(v)$ [18]. It connects the equilibria $p_{\pm}{=}(\pm\nu, 0, 0, 0, 0)$. The contracting, expanding, radial and transverse eigenvalues are given by

$$\begin{aligned}
&\lambda_x = -2\nu, \quad \lambda_y = \lambda + c\nu + \sqrt{\nu}, \quad \lambda_z = \lambda + c\nu - \sqrt{\nu},\\
&\lambda_u = \mu + d\sqrt{\nu}, \quad \lambda_v = \mu - d\sqrt{\nu}.
\end{aligned}$$

The eigenvalue conditions for asymptotic stability, i.e. $-\lambda_z > \lambda_y$ and $\lambda_u, \lambda_v < 0$, are equivalent to $c < -\lambda/\nu$, $\mu < 0$ and $|d| < |\mu|/\sqrt{\nu}$. For $c < -\lambda/\nu$, $\mu < 0$ and $|d| > |\mu|/\sqrt{\nu}$, the homoclinic network is essentially asymptotically stable.

We include a bifurcation study of a transition from an asymptotically stable to an essentially asymptotically stable homoclinic network. Consider G-equivariant differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon) \tag{1.3}$$

as above, now depending on a parameter ε. Assume Γ is a robust homoclinic cycle as before. We make the following assumptions.

Assumption. *Assume there is a transverse bifurcation at $\varepsilon = 0$, in which the transverse eigenvalue λ_v is zero [4]. We assume the unfolding condition $\frac{\partial}{\partial\varepsilon}\lambda_v(\varepsilon)|_{\varepsilon=0} > 0$ and also that the transverse bifurcation results from a supercritical pitchfork bifurcation.*

The homoclinic network $G\Gamma$ is then asymptotically stable for small negative ε and unstable for small positive ε. Near the equilibrium p two new equilibria p', g_3p' are created in the subspace $\mathrm{Fix}(\mathbb{Z}_2^2(g_1g_3h^2, g_2g_3h^2)) = \{y = z = u = 0\}$. For small positive ε a heteroclinic network $G\Gamma'$ containing the equilibria p, p', g_3p' and the images hp, hp', hg_3p' exists. Indeed, the one-dimensional unstable manifold of p' lies inside the invariant subspace

$$Q = \mathrm{Fix}(\mathbb{Z}_2(g_1g_3h^2)) = \{z = u = 0\}$$

in which hp is a sink. For $\varepsilon > 0$ small enough, the unstable manifold of p' is close to the unstable manifold of p for $\varepsilon = 0$ and therefore there is a heteroclinic connection between p' and hp, see Figure 2. Clearly a heteroclinic connection exists between the nearby equilibria p and p'. The heteroclinic network $G\Gamma'$ arises as the group orbit under G of these heteroclinic connections. For $\varepsilon > 0$ small enough we not only have essential

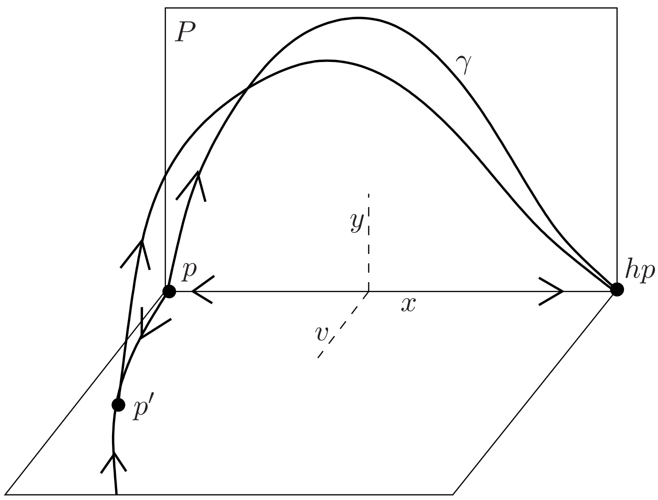

Figure 2: Inside Q one finds connecting heteroclinic trajectories from p to p' and from p, p' to hp. Note that hp is a sink within Q.

asymptotic stability of $G\Gamma$ but also the stability property that all points near $G\Gamma$ outside the stable manifolds of Gp', are attracted to $G\Gamma$. The following theorem is proved in Section 3.

Theorem 1.2. *Let $\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon)$ be a G-equivariant system of differential equations depending on a parameter ε. Assume that it possesses a robust homoclinic network $G\Gamma$ as above, asymptotically stable for $\varepsilon < 0$, with at $\varepsilon = 0$ a supercritical transverse bifurcation and no other bifurcation simultaneously, i.e. $-\lambda_z(0) > \lambda_y(0) > 0$, $\lambda_x(0) < 0$, $\lambda_u(0) < 0$ and $a(0) > 0$ (here $a(\varepsilon)$ is a coefficient in the local normal form close to the equilibrium p, see* (3.7)*). Assume the generic unfolding condition*

$$\frac{d\lambda_v(\varepsilon)}{d\varepsilon}|_{\varepsilon=0} > 0$$

and nonresonance conditions up to a sufficiently high order N for the set of eigenvalues $\{\lambda_x, \lambda_y, \lambda_z, \lambda_u\}$. Then for $\varepsilon > 0$ small enough the heteroclinic network $\mathcal{A} = G(W^u(p') \cup W^u(p))$ is an attractor. Furthermore there is an open invariant neighborhood U of $G\Gamma$, so that for $\varepsilon > 0$ small enough, trajectories starting at $\mathbf{x} \in U - GW^s(p')$ converge for positive time to $G\Gamma$.

Recall that a compact invariant set A is weakly asymptotically stable if the ω-limit set of each nearby point is contained in A, but A is not Lyapunov stable (trajectories starting near A may leave neighborhoods of A before converging to A). The set $G\Gamma$ has stability properties akin to weak asymptotic stability, except for the existence of some codimension-one manifolds accumulating onto $G\Gamma$ whose points have ω-limit sets outside $G\Gamma$.

This research was supported by a grant from the Netherlands Scientific Organization (NWO). We are grateful for the detailed comments by the referees.

2 Essential asymptotic stability

To prove the stability result of Theorem 1.1 we compute a first return map Π as the composition of a local first hit map Π^{loc} and a connecting diffeomorphism Π^{far}. Consider two cross sections close to p, in local coordinates given by

$$\begin{aligned}\Sigma^{\text{in}} &= \{x^2 + z^2 = d^2; |y|, |u|, |v| < d\},\\ \Sigma^{\text{out}} &= \{|y| = d; |x|, |z|, |u|, |v| < d\}.\end{aligned}$$

Scale the coordinates so that $d = 1$. Define Π^{loc} as the first hit map from Σ^{in} to Σ^{out} and Π^{far} as the first hit map from Σ^{out} to $h\Sigma^{\text{in}}$. By identifying Σ^{in} with $h\Sigma^{\text{in}}$ through the twist map h, the transition map from Σ^{in} to $h\Sigma^{\text{in}}$ yields a "first return map" $\Pi : \Sigma^{\text{in}} \to \Sigma^{\text{in}}$;

$$\Pi = h \circ \Pi^{\text{far}} \circ \Pi^{\text{loc}}.$$

Lemma 2.1. *For any k there is a function $N = N(\lambda_x, \lambda_y, \lambda_z, \lambda_u, \lambda_v)$ such that if the set of eigenvalues $\{\lambda_x, \lambda_y, \lambda_z, \lambda_u, \lambda_v\}$ satisfies the nonresonance condition up to order N, then* (1.1) *is C^k-equivalent to a G-equivariant system that around p equals $\dot{\mathbf{x}} = \Lambda \mathbf{x}$ and outside a neighborhood of Gp equals* (1.1)*. Here $\Lambda = \mathrm{Diag}(\lambda_x, \lambda_y, \lambda_z, \lambda_u, \lambda_v)$.*

Proof. Note that the linear part of the equation is already diagonal by the G-equivariance. The nonresonance conditions give a C^k-equivalence to linear differential equations on a neighborhood of p, see e.g. [7]. By using a test function there is a C^k-equivalence globally that leaves the system invariant outside a small neighborhood of p and is linear and diagonal inside a smaller neighborhood. This can be done so that the reflectional symmetries are respected. We do the same around hp (by using h composed with this function) and this yields a system that is linear and diagonal locally around p and hp and it is G-equivariant. ∎

We will refer to this C^k-equivalent system as the locally linearized system. Define the local transition time $\tau : \Sigma^{\text{in}} \to \mathbb{R}$ as the time it takes to flow from Σ^{in} to Σ^{out}. By solving the linearized equation $\dot{y}(t) = \lambda_y y(t)$ and using $y(\tau) = 1$ we get $\tau = \frac{-1}{\lambda_y} \log |y(0)|$. The following lemma yields asymptotic expansions for Π.

Lemma 2.2. *The "first return map" $\Pi : \Sigma^{\rm in} \to \Sigma^{\rm in}$ of the locally linearized system is at lowest order*

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \overset{\Pi}{\to} \begin{pmatrix} x^* + A_1 x|y|^{-\lambda_x/\lambda_y} + B_1 z|y|^{-\lambda_z/\lambda_y} + C_1 u|y|^{-\lambda_u/\lambda_y} + D_1 v|y|^{-\lambda_v/\lambda_y} \\ B_2 z|y|^{-\lambda_z/\lambda_y} \\ z^* + A_3 x|y|^{-\lambda_x/\lambda_y} + B_3 z|y|^{-\lambda_z/\lambda_y} + C_3 u|y|^{-\lambda_u/\lambda_y} + D_3 v|y|^{-\lambda_v/\lambda_y} \\ D_4 v|y|^{-\lambda_v/\lambda_y} \\ C_5 u|y|^{-\lambda_u/\lambda_y} \end{pmatrix},$$

where $(x^*, 0, z^*, 0, 0) \in G\Gamma \cap \Sigma^{\rm in}$, and $A_1, \dots, D_4$ are constants. The constants B_2, C_5, D_4 as well as z^* are nonzero (note that z^* does depend on the sign of y).

Proof. Solving the linear system with initial condition (x, y, z, u, v) and evaluating at time τ yields the local first hit map $\Pi^{\rm loc} : \Sigma^{\rm in} \to \Sigma^{\rm out}$,

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \overset{\Pi^{\rm loc}}{\to} \begin{pmatrix} x|y|^{-\lambda_x/\lambda_y} \\ \pm 1 \\ z|y|^{-\lambda_z/\lambda_y} \\ u|y|^{-\lambda_u/\lambda_y} \\ v|y|^{-\lambda_v/\lambda_y} \end{pmatrix}.$$

The connecting diffeomorphism $\Pi^{\rm far} : \Sigma^{\rm out} \to h\Sigma^{\rm in}$ is of the form

$$\begin{pmatrix} x \\ 1 \\ z \\ u \\ v \end{pmatrix} \overset{\Pi^{\rm far}}{\to} \begin{pmatrix} x^* + a_1 x + b_1 z + c_1 u + d_1 v + \mathcal{O}(|(x,z,u,v)|^2) \\ z^* + a_3 x + b_3 z + c_3 u + d_3 v + \mathcal{O}(|(x,z,u,v)|^2) \\ b_2 z + \mathcal{O}(|z||(x,z,u,v)|) \\ c_5 u + \mathcal{O}(|u||(x,z,u,v)|) \\ d_4 v + \mathcal{O}(|v||(x,z,u,v)|) \end{pmatrix},$$

for some constants $a_1, \dots, d_4$ and with coordinates on $h\Sigma^{\rm in}$ obtained from the (x, y, z, u, v) coordinates on $\mathbb{R}^5$ with hp shifted to the origin. This form follows from the G-equivariance (here for C^k with $k \geq 2$, for $k = 1$ the higher order terms are of smaller order then linear). For instance, the existence of the fixed point space $\mathrm{Fix}(\mathbb{Z}_2(g_1)) = \{z = 0\}$ forces the z-coordinate of $\Pi^{\rm far}$ to be of order $|z|$. Likewise $\mathrm{Fix}(\mathbb{Z}_2(g_3 h^2)) = \{u = 0\}$ implies that the u-coordinate of $\Pi^{\rm far}$ is of order $|u|$ and $\mathrm{Fix}(\mathbb{Z}_2(g_3)) = \{v = 0\}$ implies that the v-coordinate of $\Pi^{\rm far}$ is of order $|v|$. Note also that b_2, c_5, d_4 are necessarily nonzero as $\Pi^{\rm far}$ is a local diffeomorphism. Also, $z^* \neq 0$ by invariance of $P = \{z = u = v = 0\}$.

Composition of these maps, and then composing with the twist h (to identify $h\Sigma^{\rm in}$ with $\Sigma^{\rm in}$ through h), gives the first return map with asymptotics as stated in the lemma (where we note that coefficients stay the same or change sign and also that there are other invariant subspaces whose consequences we do not consider). ■

When one of the transverse eigenvalues, say λ_v, is positive the homoclinic network $G\Gamma$ is not asymptotically stable. However, for $-\lambda_v$ not too large $G\Gamma$ is essentially asymptotically stable.

Proof of Theorem 1.1. By the nonresonance conditions, the system is C^k-equivalent, $k \geq 1$, to the locally linearized system and Lemma 2.1 applies. For each δ define $N(\delta)$, a neighborhood of $G\Gamma \cap \Sigma^{\rm in} = (x^*, 0, z^*, 0, 0)$ in $\Sigma^{\rm in}$, as

$$N(\delta) = \{|x - x^*| < \delta^m, |z - z^*| < \delta^m, |y| < \delta, |u| < \delta, |v| < \delta, \mathbf{x} \in \Sigma^{\rm in}\}, \tag{2.4}$$

where $m = \frac{1}{2}\min\{1, -\lambda_x/\lambda_y\}$. We fix a small open neighborhood U of $G\Gamma \cap \Sigma^{\mathrm{in}}$ inside Σ^{in} and $\alpha \in (0,1)$. Choose $\tilde{\delta} > 0$ so that $N(\tilde{\delta}) \subset U$. Define a cusp shaped region $C \subset N(\delta)$,

$$C = \{|v| > \frac{\tilde{\delta}}{K}|y|^{\lambda_v/\lambda_y}\},$$

where $K = \max\{8|D_1|, 8|D_3|, 2|D_4|\}$. This region depends on $\tilde{\delta}$, which depends on U. Note that for $\lambda_v/\lambda_y < 1$ we have

$$\lim_{\delta \to 0} \frac{\mu(N(\delta) - C)}{\mu(N(\delta))} = 1.$$

We claim that

1. for every $\tilde{\delta}$ there is a δ so that $\Pi(N(\delta) - C) \subset N(\tilde{\delta}) - C$,
2. for all δ small enough $\Pi^2(N(\delta) - C) \subset N(\delta/2) - C$,

implying essential asymptotic stability by choosing $\tilde{\delta}$ small enough. Write $\Pi = (\Pi_x, \ldots, \Pi_v)$. We get the following estimates for points in $N(\delta) - C$:

$$|\Pi_y| < 2|B_2||z^*||y|^{-\lambda_z/\lambda_y} < \delta/2 < \tilde{\delta},$$

for δ small enough, using $|z - z^*| < \delta^m$ with $z^* \neq 0$, $-\lambda_z/\lambda_y > 1$ and $-\lambda_u/\lambda_y > 0$. Likewise

$$|\Pi_v| < 2|C_5||u||y|^{-\lambda_u/\lambda_y} < \delta/2 < \tilde{\delta}$$

for δ small enough. The u component Π_u is estimated by

$$|\Pi_u| < 2|D_4||v||y|^{-\lambda_v/\lambda_y} < 2|D_4|\tilde{\delta}/K \leq \tilde{\delta}.$$

The estimates on the x and z components are similar to each other. We give the estimate for the x component, assuming for simplicity that the constants are nonzero (otherwise replace them by the maximum of the constant and 1):

$$\begin{aligned}
|\Pi_x - x^*| &< 2|A_1||x^*||y|^{\frac{-\lambda_x}{\lambda_y}} + 2|B_1||z^*||y|^{\frac{-\lambda_z}{\lambda_y}} + 2|C_1||u||y|^{\frac{-\lambda_u}{\lambda_y}} + 2|D_1||v||y|^{\frac{-\lambda_v}{\lambda_y}} \\
&< 2|A_1||x^*|\delta^{\frac{-\lambda_x}{\lambda_y}} + \delta/4 + \delta/4 + \tilde{\delta}/4 \\
&< 2|A_1||x^*|\delta^{\frac{-\lambda_x}{2\lambda_y}}\delta^{\frac{-\lambda_x}{2\lambda_y}} + \frac{1}{2}\tilde{\delta}^m \\
&< \frac{1}{2}\tilde{\delta}^{\frac{-\lambda_x}{2\lambda_y}} + \frac{1}{2}\tilde{\delta}^m < \tilde{\delta}^m,
\end{aligned}$$

where we used that $\delta < \tilde{\delta}/2$, $\tilde{\delta} \leq \tilde{\delta}^m$ and $\tilde{\delta}^{-\lambda_x/2\lambda_y} \leq \tilde{\delta}^m$. We show that $\Pi(N(\delta) - C) \cap C = \emptyset$ which concludes the proof for the first part of the claim. We have to show that

$$|\Pi_v| < \frac{\tilde{\delta}}{K}|\Pi_y|^{\lambda_v/\lambda_y}. \tag{2.5}$$

We estimate $|\Pi_y|$ from below and $|\Pi_v|$ from above as follows:

$$|\Pi_y| > \tfrac{1}{4}|B_2||z^*||y|^{-\lambda_z/\lambda_y}, \quad |\Pi_v| < 2|C_5||u||y|^{-\lambda_u/\lambda_y}.$$

It follows that if the following estimate holds then so does (2.5),

$$|y|^{-\frac{\lambda_u}{\lambda_y}+\frac{\lambda_z}{\lambda_y}\frac{\lambda_v}{\lambda_y}} < \frac{(|B_2||z^*|/4)^{\lambda_v/\lambda_y}}{2|C_5|K}.$$

For $\lambda_v/\lambda_y < \lambda_u/\lambda_z$ there is a δ so that this estimate holds for all $\mathbf{x} \in N(\delta) - C$.

Next we prove the second part of the claim, i.e for δ small enough $\Pi^2(N(\delta) - C) \subset N(\delta/2) - C$. We first apply Π two times and estimate

$$\begin{aligned}
|(\Pi^2)_x - x^*| &< 2x^*|A_1||\Pi_y|^{\frac{-\lambda_x}{\lambda_y}} + 2z^*|B_1||\Pi_y|^{\frac{-\lambda_z}{\lambda_y}} + 2|C_1||\Pi_u||\Pi_y|^{\frac{-\lambda_u}{\lambda_y}} + 2|D_1||\Pi_v||\Pi_y|^{\frac{-\lambda_v}{\lambda_y}} \\
&< M|y|^{\frac{-\lambda_x}{\lambda_y}} + M|y|^{\frac{-\lambda_z}{\lambda_y}} + M|v||y|^{\frac{-\lambda_v}{\lambda_y}+\frac{\lambda_z}{\lambda_y}\frac{\lambda_u}{\lambda_y}} + M|u||y|^{\frac{-\lambda_u}{\lambda_y}+\frac{\lambda_z}{\lambda_y}\frac{\lambda_v}{\lambda_y}} \\
&< (\delta/2)^m/4 + \delta/4 + \delta/4 + \delta/4 < (\delta/2)^m,
\end{aligned}$$

where $M > 0$ is some constant. In this estimate we used $-\lambda_z/\lambda_y > 1$ and $\lambda_v/\lambda_y < \lambda_u/\lambda_z$, ensuring $\frac{-\lambda_v}{\lambda_y}+\frac{\lambda_z}{\lambda_y}\frac{\lambda_u}{\lambda_y} > 0$ and $\frac{-\lambda_u}{\lambda_y}+\frac{\lambda_z}{\lambda_y}\frac{\lambda_v}{\lambda_y} > 0$. In the same way we have $|\Pi_z^2 - z^*| < (\delta/2)^m$. The other coordinates are estimated by

$$\begin{aligned}
&|(\Pi^2)_y| < 2|z^*||B_2||\Pi_y|^{-\lambda_z/\lambda_y} < M|y|^{\lambda_z^2/\lambda_y^2} < \delta/2, \\
&|(\Pi^2)_u| < 2|D_4||C_5||u||y|^{-\lambda_u/\lambda_y}(|B_2||z^*||y|^{-\lambda_z/\lambda_y})^{-\lambda_v/\lambda_y} < M|u||y|^{\frac{-\lambda_u}{\lambda_y}+\frac{-\lambda_z}{\lambda_y}\frac{-\lambda_v}{\lambda_y}} < \delta/2, \\
&|(\Pi^2)_v| < |C_5||D_4||v||y|^{-\lambda_v/\lambda_y}(|B_2|z^*||y|^{-\lambda_z/\lambda_y})^{-\lambda_u/\lambda_y} < M|v||y|^{\frac{-\lambda_v}{\lambda_y}+\frac{-\lambda_z}{\lambda_y}\frac{-\lambda_u}{\lambda_y}} < \delta/2.
\end{aligned}$$

Note that these estimates only hold for points that are close to $G\Gamma$ after one return, so the estimates hold for points in $N(\delta) - C$. To complete the proof of the claim we need to show that

$$\Pi^2(N(\delta) - C) \cap C = \emptyset. \tag{2.6}$$

There are $K_1 > 0$, $K_2 > 0$ so that:

$$\begin{aligned}
|(\Pi^2)_v| &< K_1\tilde{\delta}|y|^{-\lambda_v/\lambda_y+\lambda_z\lambda_u/\lambda_y^2}, \\
\tilde{\delta}/K|(\Pi^2)_y|^{\lambda_v/\lambda_y} &> \tilde{\delta}/K(\frac{1}{2}|B_2||z^*||\Pi_y|^{-\lambda_z/\lambda_y})^{\lambda_v/\lambda_y} > K_2\tilde{\delta}|y|^{\lambda_v\lambda_z^2/\lambda_y^3}.
\end{aligned}$$

It follows that (2.6) holds for δ small and $\lambda_v/\lambda_y < \lambda_u/\lambda_z$. The claim implies that

$$\Pi^{2n}(N(\delta) - C) \subset N(\delta/2^n) - C \subset N(\tilde{\delta}/2^n) - C$$

and

$$\Pi^{2n+1}(N(\delta) - C) \subset \Pi^{2n}(N(\tilde{\delta}) - C) \subset N(\tilde{\delta}/2^n) - C$$

and thus all trajectories converge to $G\Gamma$. ■

3 Transverse bifurcation

In this section we prove Theorem 1.2 dealing with a transverse bifurcation from an asymptotically stable to an essentially asymptotically stable homoclinic network. The unfolding condition $\frac{\partial}{\partial\varepsilon}\lambda_v(\varepsilon)|_{\varepsilon=0} > 0$ enables a reparametrization $\lambda_v = \varepsilon$ for ε close to zero. Assuming that the bifurcation is supercritical, there are two equilibria p', $g_3 p'$ bifurcating from p for $\varepsilon > 0$.

The homoclinic network $G\Gamma$ is asymptotically stable for $\varepsilon < 0$, but unstable for $\varepsilon > 0$. For $\varepsilon > 0$ small it follows from Theorem 1.1 that the homoclinic network $G\Gamma$ is essentially asymptotically stable. To study the dynamics in an open neighborhood of $G\Gamma$ there is need of a normal form containing higher order terms in the equation for v.

Lemma 3.1. *Suppose* (1.3) *unfolds a supercritical pitchfork bifurcation at* $\varepsilon = 0$. *For any* $k \in \mathbb{N}$ *there is a* $N = N(\lambda_x, \lambda_y, \lambda_z, \lambda_u) \in \mathbb{N}$ *so that under the condition of nonresonance up to order* N *for the set of eigenvalues* $\{\lambda_x, \lambda_y, \lambda_z, \lambda_u\}$, *the system* (1.1) *is* C^k*-equivalent to a system that is locally around* p *given by*

$$\begin{aligned}
\dot{x} &= \lambda_x(v,\varepsilon)x,\\
\dot{y} &= \lambda_y(v,\varepsilon)y,\\
\dot{z} &= \lambda_z(v,\varepsilon)z,\\
\dot{u} &= \lambda_u(v,\varepsilon)u,\\
\dot{v} &= \lambda_v(\varepsilon)v - a(\varepsilon)v^3,
\end{aligned} \tag{3.7}$$

where the occurring functions are smooth, $a(\varepsilon) > 0$, *and this system is* G*-equivariant.*

Proof. Takens [20] proved that under these nonresonance conditions there are $C^{l(N)}$ coordinates, $l(N) \to \infty$ for $N \to \infty$, so that the system is locally in standard form, i.e.

$$\begin{aligned}
\dot{x} &= a_{11}(v,\varepsilon)x + a_{12}(v,\varepsilon)z + a_{13}(v,\varepsilon)u,\\
\dot{y} &= \lambda_y(v,\varepsilon)y,\\
\dot{z} &= a_{21}(v,\varepsilon)x + a_{22}(v,\varepsilon)z + a_{23}(v,\varepsilon)u,\\
\dot{u} &= a_{31}(v,\varepsilon)x + a_{32}(v,\varepsilon)z + a_{33}(v,\varepsilon)u,\\
\dot{v} &= f(v,\varepsilon)
\end{aligned}$$

for smooth functions $a_{11}, \dots, a_{33}$, λ_y, see also Il'yashenko and Yakovenko [8]. Furthermore Bonckaert [2] shows that we can do this so that the system respects a symmetry S. In the proof in [2] it is used that S is similar to an orthonormal matrix T, i.e. there is an invertible matrix M and an orthonormal matrix T so that $MSM^{-1} = T$. Then a suitable cut-off function is chosen of the form $\phi(v) = \psi(|Mv|)$. In our case we can choose $M = Id$, for all $S \subset G$ and it follows that there exists a coordinate transformation which respects the entire symmetry group G.

The symmetries of G force the matrix $A = (a_{ij})$ to be diagonal, i.e.

$$a_{12}(v,\varepsilon) = a_{13}(v,\varepsilon) = a_{21}(v,\varepsilon) = a_{23}(v,\varepsilon) = a_{31}(v,\varepsilon) = a_{32}(v,\varepsilon) = 0.$$

The equivariance implies $f(v,\varepsilon) = -f(-v,\varepsilon)$ and thus $f(0,0) = f_{vv}(0,0) = 0$. Because there is a pitchfork bifurcation at $\varepsilon = 0$ in the v direction, we have $f_v(0,0) = 0$. Furthermore we assume the nondegeneracy condition $f_{vvv}(0,0) \neq 0$. From the finitely smooth case of the Malgrange preparation theorem, see [6, 14], we can write

$$f(v,\varepsilon) = H(v,\varepsilon)P(v,\varepsilon)$$

where $P(v,\varepsilon) = v^3 + \sum_{i=0}^2 b_i(\varepsilon)v^i$ is a polynomial with C^h coefficients and $H(v,\varepsilon)$ is a C^l invertible function. Here $h = [l(N)/3] - 1$ and $l = [(l(N)-1)/3] - 1$. Because we assumed the bifurcation to be supercritical, $H(v,\varepsilon) < 0$ on a small neighborhood of $(0,0)$. Simple calculations show that $b_0(\varepsilon) = b_2(\varepsilon) = 0$ and $\lambda_v(\varepsilon) = H(0,\varepsilon)b_1(\varepsilon)$. Multiply the vectorfield by $H(0,\varepsilon)/H(v,\varepsilon)$ and write $a(\varepsilon) = -H(0,\varepsilon)$ so that the equation for v reads

$$\dot{v} = \lambda_v(\varepsilon)v - a(\varepsilon)v^3.$$

Thus for each k there is a N so that (1.1) is C^k-equivalent to (3.7) in some neighborhood of p. ∎

Proof of Theorem 1.2. Because of the nonresonance conditions up to order N, we may assume that the system is of the form (3.7). The generic unfolding condition allows us to reparametrize so that $\lambda_v = \varepsilon$ for ε small. For ε going through zero there are two new equilibria bifurcating from p given in local coordinates by

$$p' = (0,0,0,0,\sqrt{\varepsilon/a(\varepsilon)}), \quad g_3p' = (0,0,0,0,-\sqrt{\varepsilon/a(\varepsilon)})$$

and by symmetry two equilibria are bifurcating from hp. The unstable manifolds of p', g_3p' are inside the invariant subspace $Q = \{z = u = 0\}$ and are close to $G\Gamma$. Because hp is a sink in Q, there are heteroclinic connections from p' and g_3p' to hp. Note that $\dot{v} = \varepsilon v - av^3$ is solved by

$$v(t) = \mathrm{sign}(v(0))\sqrt{\tfrac{\varepsilon}{a}}\sqrt{\tfrac{1}{1+\frac{\varepsilon - av(0)^2}{av(0)^2}e^{-2\varepsilon t}}}.$$

We restrict to the case where $v(0) > 0$, the case $v(0) < 0$ is handled by symmetry. We suppress some of the dependencies from the notation. Using the formula for the transition time, $\tau = -\log|y|/\lambda_y$, the v-coordinate Π_v^{loc} of the local first hit map is

$$\Pi_v^{\mathrm{loc}}(\mathbf{x},\varepsilon) = \sqrt{\tfrac{\varepsilon}{a}}\sqrt{\tfrac{1}{1+\frac{\varepsilon - av^2}{av^2}|y|^{\frac{2\varepsilon}{\lambda_y}}}}\ ,$$

where again $\mathbf{x} = (x,y,z,u,v) \in \Sigma^{\mathrm{in}}$. The transition map $\Pi : \Sigma^{\mathrm{in}} \to h\Sigma^{\mathrm{in}}$ is at lowest order given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \overset{\Pi}{\to} \begin{pmatrix} x^* + A_1x|y|^{-\lambda_x/\lambda_y} + B_1z|y|^{-\lambda_z/\lambda_y} + C_1u|y|^{-\lambda_u/\lambda_y} + D_1\Pi_v^{\mathrm{loc}} \\ B_2z|y|^{-\lambda_z/\lambda_y} \\ z^* + A_3x|y|^{-\lambda_x/\lambda_y} + B_3z|y|^{-\lambda_z/\lambda_y} + C_3u|y|^{-\lambda_u/\lambda_y} + D_3\Pi_v^{\mathrm{loc}} \\ D_4\Pi_v^{\mathrm{loc}} \\ C_5u|y|^{-\lambda_u/\lambda_y} \end{pmatrix}.$$

We choose a neighborhood $N = N(\mu)$, see (2.4), of $G\Gamma \cap \Sigma^{\mathrm{in}}$ small enough so that for some $K > 0$ the following estimates hold

$$\begin{aligned}
|\Pi_x - x^*| &\le K|y|^{-\lambda_x/\lambda_y} + K|y|^{-\lambda_z/\lambda_y} + K|u||y|^{-\lambda_u/\lambda_y} + K\Pi_v^{\mathrm{loc}}, \\
|\Pi_y| &\le K|y|^{-\lambda_z/\lambda_y}, \\
|\Pi_z - z^*| &\le K|y|^{-\lambda_x/\lambda_y} + K|y|^{-\lambda_z/\lambda_y} + K|u||y|^{-\lambda_u/\lambda_y} + K\Pi_v^{\mathrm{loc}}, \\
|\Pi_u| &\le K\Pi_v^{\mathrm{loc}}, \\
|\Pi_v| &\le K|u||y|^{-\lambda_u/\lambda_y}.
\end{aligned}$$

For $v < \sqrt{\varepsilon/a}$ an estimate on $\Pi_v^{\rm loc}$ is given by

$$\Pi_v^{\rm loc} \le \min\{v y^{-\varepsilon/\lambda_y}, \sqrt{\varepsilon/a}\}.$$

For $\varepsilon > 0$ small enough, the arguments to prove Theorem 1.1 establish that $G\Gamma$ is weakly asymptotically stable. Moreover, following the proof of Theorem 1.1, we get that for μ and ε small, $\Pi^2(N(\mu)) \subset N(\mu/2)$ (recall the definition of N in (2.4)). Thus, the positive trajectory of each point $\mathbf{x}$ in $N(\mu)$ with $y \neq 0$ (so $\mathbf{x} \notin GW^s(p'))$ converges to $G\Gamma$. We claim that the heteroclinic network $\mathcal{A} = G(W^u(p') \cup W^u(p))$ is an attractor. To see this, first note that every open neighborhood of $G\Gamma$ has a nonempty intersection with $GW^s(p')$. The smallest invariant set which has an open neighborhood U so that its forward flow converges to the invariant set is $\mathcal{A}$. Thus $\mathcal{A}$ is an attractor. ■

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