

Circle diffeomorphisms forced by expanding circle maps

Ale Jan Homburg^{1,2}

¹KdV Institute for Mathematics, University of Amsterdam, Science park 904, 1098 XH
Amsterdam, Netherlands

²Department of Mathematics, VU University Amsterdam, De Boelelaan 1081, 1081 HV
Amsterdam, Netherlands

October 10, 2012

Abstract

We discuss dynamics of skew product maps defined by circle diffeomorphisms forced by expanding circle maps. We construct an open class of such systems that are robust topologically mixing and for which almost all points in the same fiber converge under iteration. This property follows from the construction of an invariant attracting graph in the natural extension, a skew product of circle diffeomorphisms forced by a solenoid homeomorphism.

MSC 37C05, 37D30, 37C70, 37E10

1 Introduction

We will treat the dynamics of a class of circle diffeomorphisms that are forced by expanding circle maps. We start with a numerical experiment on the skew product map

$$(y, x) \mapsto (3y, x + \frac{1}{8} \sin(2\pi x) + y) \pmod{1} \quad (1)$$

on the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, the results of which are presented in Figure 1. Note that this map is given by a circle diffeomorphism $x \mapsto x + \frac{1}{8} \sin(2\pi x) + y \pmod{1}$ in the fiber forced by an expanding circle map $y \mapsto 3y \pmod{1}$ in the base. The left panel of Figure 1 shows ten thousand points of an orbit, appearing to lie dense in the torus. The right panel shows a time series of the second coordinate of twenty different orbits, for equidistant distributed initial points in the same fiber (i.e. with identical first coordinate). There appears to be a fast contraction inside the fiber.

Numerical experiments as described above may be explained by the following result. Endow the space of smooth skew product systems $(y, x) \mapsto F(y, x) = (g(y), f_y(x))$ on \mathbb{T}^2 , considered as a subset of smooth endomorphisms, with the C^k topology, $k \geq 2$. A smooth endomorphism g on the circle is called expanding if $|g'| > 1$. Recall that F is topologically mixing if for each nonempty open $U, V \subset \mathbb{T}^2$, $F^n(U)$ intersects V for all large enough positive integers n ; this implies the existence of dense positive orbits.

Theorem 1.1. *There is an open class of forced circle diffeomorphisms $(y, x) \mapsto F(y, x) = (g(y), f_y(x))$, forced by expanding circle maps $y \mapsto g(y)$, with the following properties:*

1. each map F is topologically mixing,
2. there is a subset $\Lambda \subset \mathbb{T}^2$ of full Lebesgue measure, so that for any $(y, x_1), (y, x_2) \in \Lambda$,

$$\lim_{n \rightarrow \infty} |F^n(y, x_1) - F^n(y, x_2)| = 0. \quad (2)$$

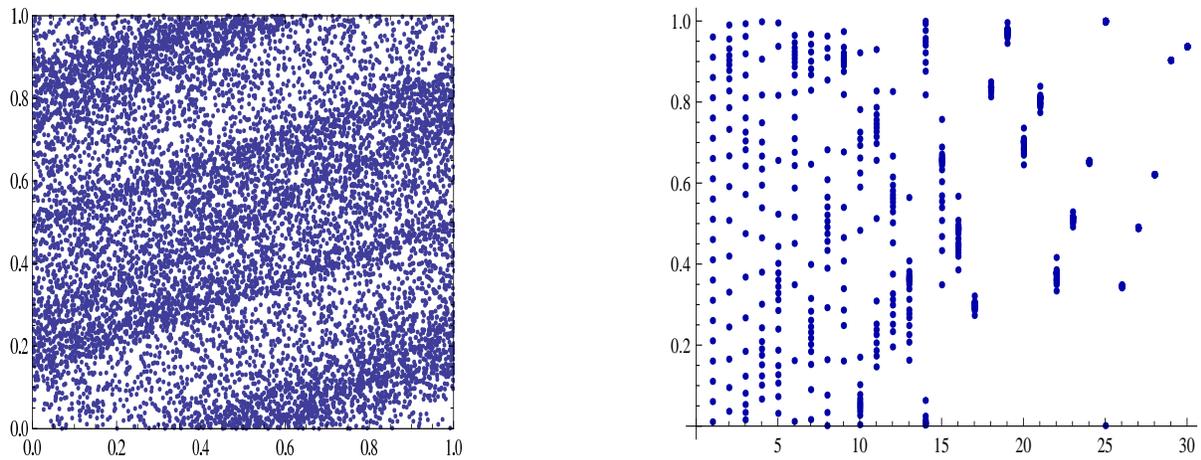


Figure 1: Numerical experiments on $(y, x) \mapsto (3y, x + \frac{1}{8} \sin(2\pi x) + y) \bmod 1$. The left frame shows ten thousand points of an orbit. The right frame shows time series for the x -coordinate starting from twenty different initial conditions with identical y -coordinates.

To prove the convergence property in Theorem 1.1, we apply the natural extension of the endomorphism F to a homeomorphism on the product of a solenoid and a circle (this construction is described in Section 2). This homeomorphism is likewise a skew product map formed by a circle diffeomorphism forced by a solenoid map. It is shown to admit an attracting invariant graph (Theorem 5.1 below), from which the result follows. In the physics literature a convergence phenomenon as in (2) falls under the study of synchronization, see [4] for a review. Forced circle maps appear in various contexts where some sort of convergence of orbits features. We give pointers to the literature for these different contexts and discuss the relation to our result.

1. Quasiperiodically forced circle diffeomorphisms; the circle diffeomorphisms $f_y(x)$ are forced by $g(y) = y + \alpha \bmod 1$ with α irrational. A large body of work is available in this area of research, related to the existence of strange non-chaotic attractors, see [17] and references therein. The notable difference with the context here is that the forcing consist of ergodic, but not mixing, dynamics.
2. Randomly perturbed circle diffeomorphisms, including iterated function systems [1, 11, 19] and circle diffeomorphisms with absolutely continuous noise [20, 34]. Such systems allow a formulation as a skew product system. The mentioned references give precise classifications of dynamics in the fibers both for iterated function systems and circle diffeomorphisms with absolutely continuous i.i.d. noise. For an iterated function system consisting of m circle diffeomorphisms f_1, \dots, f_m this yields circle diffeomorphisms forced by a shift on m symbols: consider $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$ endowed with the product topology and, for $\omega = (\omega_0, \omega_1, \dots) \in \Sigma$, the left shift $\sigma\omega = (\omega_1, \omega_2, \dots)$. The skew product system F acting on $\Sigma \times \mathbb{T}$ is then given by

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

The left shift is a topologically mixing map. The dependence of the circle diffeomorphisms on ω is of a restricted form: they depend only on ω_0 and not on $\omega_i, i > 0$ (in [14] the term “step skew product” is used). Our result can be seen as an extension where this restriction is removed and also as an extension to more general topologically mixing base dynamics.

3. (Volume preserving) skew products over hyperbolic torus automorphisms [27, 28, 30]. One may think of small perturbations from $F : \mathbb{T}^2 \times \mathbb{T} \rightarrow \mathbb{T}^2 \times \mathbb{T}$,

$$F(y, x) = (Ay, x),$$

where A is a hyperbolic torus automorphism. This research relates to the phenomenon of stable ergodicity. It also relates to work on partially hyperbolic systems with mostly contracting central

directions [7, 25]. The above references contain results on delta measures in fibers, which go in the direction of the convergence result in Theorem 1.1. As reviewed in Section 3, one may embed the solenoid from the natural extension of the expanding circle map as a hyperbolic attractor for a smooth diffeomorphism on a manifold, so that the natural extension of the skew product system is partially hyperbolic on $\mathcal{S} \times \mathbb{T}$.

Finally, skew product systems of circle diffeomorphisms over horseshoes and solenoids are also treated in [14] with a different emphasis, proving the robust occurrence of dense sets of hyperbolic periodic orbits with different index (attracting or repelling in the fiber). In [15], and continued in [6, 12], the existence of ergodic measures with zero Lyapounov exponent is investigated in related contexts of skew product systems and partially hyperbolic systems.

I acknowledge discussions with Pablo Barrientos that helped me to clean up some of the arguments.

2 Natural extensions

In this section we collect mostly known facts on extensions of skew product torus endomorphisms to skew product homeomorphisms. These facts are used in the arguments in the following sections.

Consider a smooth expanding endomorphism $g : \mathbb{T} \rightarrow \mathbb{T}$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We note that g possesses an absolutely continuous invariant measure ν^+ , equivalent to Lebesgue measure, see e.g. [21, Section III.1]. In fact, g is topologically conjugate to a linear expanding circle map for which Lebesgue measure is invariant [29]. The measure ν^+ has density that is bounded and bounded away from zero. We will consider skew product systems of circle diffeomorphisms $x \mapsto f_y(x)$ forced by the expanding circle map $y \mapsto g(y)$. Write

$$F(y, x) = (g(y), f_y(x)) \tag{3}$$

for the skew product map on the torus \mathbb{T}^2 . For iterates of F , we denote

$$F^n(y, x) = (g^n(y), f_{g^{n-1}(y)} \circ \cdots \circ f_y(x)) = (g^n(y), f_y^n(x)).$$

The inverse limit construction [33] extends g to a homeomorphism on the solenoid, i.e. the space

$$\mathcal{S} = \{(\dots, y_{-1}, y_0) \in \mathbb{T}^{-\mathbb{N}} \mid y_{-i} = g(y_{-i-1})\}$$

endowed with the product topology. We will also write g for the extended map, where the context makes clear whether g acts on \mathbb{T} or \mathcal{S} . So, for $\mathbf{y} = (\dots, y_{-1}, y_0)$,

$$g(\mathbf{y}) = (\dots, y_{-1}, y_0, g(y_0)).$$

The induced skew product map on $\mathcal{S} \times \mathbb{T}$ will likewise be denoted by F , and we write

$$F(\mathbf{y}, x) = (g(\mathbf{y}), f_{\mathbf{y}}(x)).$$

The inverse map is given by

$$F^{-1}(\mathbf{y}, x) = (\dots, y_{-2}, y_{-1}, (f_{y_{-1}})^{-1}(x)).$$

On \mathbb{T} and \mathcal{S} we use Borel σ -algebras \mathcal{F}^+ and \mathcal{F} respectively. Define the projection $\psi : \mathcal{S} \rightarrow \mathbb{T}$; $\psi(\mathbf{y}) = y_0$. Then with $\mathcal{G} = \psi^{-1}(\mathcal{F}^+)$ we have $g^n \mathcal{G} \uparrow \mathcal{F}$ and $g : \mathcal{S} \rightarrow \mathcal{S}$ is a natural extension of $g : \mathbb{T} \rightarrow \mathbb{T}$ [2, Appendix A]. The solenoid \mathcal{S} has an invariant measure ν inherited from the invariant measure ν^+ for g on the circle;

$$\nu(\{y_{-r} \in I_r, \dots, y_0 \in I_0\}) = \nu^+(g^{-r}(I_0) \cap g^{-r+1}(I_1) \cap \cdots \cap I_r).$$

We will write λ for Lebesgue measure on \mathbb{T} . We also write $|I|$ for the length of an interval $I \subset \mathbb{T}$.

Let μ^+ be an invariant measure for $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ with marginal ν^+ ; existence is guaranteed by [10, Lemma 2.3]. Write μ_y^+ for the disintegrations of μ^+ . Occasionally we also write μ_y^+ with the understanding that μ_y^+ depends only on the coordinate y_0 in $\mathbf{y} = (\dots, y_{-1}, y_0)$. Invariance of μ^+ means

$$\int_{g^{-1}(A)} f_y \mu_y^+ d\nu^+(y) = \int_{g^{-1}(A)} \mu_{g(y)}^+ d\nu^+(y) = \int_A \mu_y^+ d\nu^+(y)$$

for $A \in \mathcal{F}^+$ (the second equality by invariance of ν^+ under g), see [10], [2, Theorem 1.4.5].

The following lemma originating from [10] relates invariant measures for the skew product system with one and two sided time.

Lemma 2.1. *Given the invariant measure μ^+ for F acting on \mathbb{T}^2 , with marginal ν^+ on \mathbb{T} , there is an invariant measure μ for F acting on $\mathcal{S} \times \mathbb{T}$, with marginal ν on \mathcal{S} . For ν -almost all $\mathbf{y} = (\dots, y_{-1}, y_0) \in \mathcal{S}$, the limit*

$$\mu_{\mathbf{y}} = \lim_{n \rightarrow \infty} f_{y_{-n}}^n \mu_{y_{-n}}^+ \quad (4)$$

gives its disintegrations.

Proof. The lemma is implied by [2, Theorem 1.7.2]. We include the line of reasoning. To avoid confusion we write \mathcal{B} (and not again \mathcal{F}^+) for the Borel σ -algebra on the circle of x -coordinates. For fixed $B \in \mathcal{B}$, and for $\mathbf{y} = (\dots, y_{-n}, \dots, y_0) \in \mathcal{S}$, define

$$\nu_{\mathbf{y}}^n(B) = f_{g^{-n}(\mathbf{y})}^n \mu_{y_{-n}}^+(B)$$

as the push-forward by $f_{g^{-n}(\mathbf{y})}^n$ of $\mu_{y_{-n}}^+$, evaluated in B . Recall $\mathcal{G} = \psi^{-1}\mathcal{F}^+$, with $\mathcal{G}_n = g^n\mathcal{G}$ we have $\mathcal{G}_n \uparrow \mathcal{F}$ as $n \rightarrow \infty$. One computes that $\mathbb{E}(\nu_{\mathbf{y}}^n(B)|\mathcal{G}_m) = \nu_{\mathbf{y}}^m(B)$, i.e. $y \mapsto \nu_{\mathbf{y}}^n(B)$ is a martingale with respect to the filtration \mathcal{G}_n . As this holds for all fixed B , $\mu_{\mathbf{y}}(B) = \lim_{n \rightarrow \infty} f_{y_{-n}}^n \mu_{y_{-n}}^+(B)$ defines a probability measure for ν -almost all \mathbf{y} . \square

Vice versa, given an invariant measure μ for F on $\mathcal{S} \times \mathbb{T}$,

$$\mu_y^+ = \mathbb{E}(\mu|\mathcal{F}^+)_y \quad (5)$$

is an invariant measure for F on \mathbb{T}^2 [2, Theorem 1.7.2]. Moreover, the correspondence maps ergodic measures to ergodic measures in either direction [10, Section 3].

We will also need to study iterates of the inverse map F^{-1} on $\mathcal{S} \times \mathbb{T}$. Noting that this interchanges stable and unstable directions, one obtains a convergence result similar to Lemma 2.1. To state it, it is convenient to think of g as acting on $[0, 1]$; one can identify 0 with 1 to obtain the expanding circle map. The inverse limit construction extends g to a map, also denoted by g , on

$$\mathcal{I} = \{(\dots, y_{-1}, y_0) \in [0, 1]^{-\mathbb{N}} \mid y_{-i} = g(y_{-i-1})\}.$$

We may think of g as acting on $\Sigma \times [0, 1]$ for a Cantor set $\Sigma = \{0, \dots, m-1\}^{\mathbb{N}}$. The solenoid \mathcal{S} is then given as a quotient

$$\mathcal{S} = \Sigma \times [0, 1] / \sim, \quad (6)$$

identifying points

$$\begin{aligned} (0 \ \omega_1 \ \omega_2 \ \dots, 1) &\sim (1 \ \omega_1 \ \omega_2 \ \dots, 0), \\ (1^{n-1} \ 0 \ \omega_n \ \omega_{n+1} \ \dots, 1) &\sim (0^{n-1} \ 1 \ \omega_n \ \omega_{n+1} \ \dots, 0). \end{aligned}$$

We write $\mathbf{y} = (\boldsymbol{\omega}, y_0) \in \Sigma \times \mathbb{T}$. Consider the projection $\psi : \Sigma \times [0, 1] \rightarrow \Sigma$, $\psi(\boldsymbol{\omega}, y_0) = \boldsymbol{\omega}$. The Borel σ -algebra on $\Sigma \times [0, 1]$ is $\mathcal{F} = \mathcal{F}^- \otimes \mathcal{F}^+$. The inverse map g^{-1} on $\Sigma \times [0, 1]$ induces an expanding map on Σ with an invariant measure ν^- (with ν the invariant measure for g on $\Sigma \times [0, 1]$). Write $\mathcal{G} = \psi^{-1}\mathcal{F}^-$.

The measure ν^- is computable from ν^+ : for a cylinder $C = C_{\nu_1 \dots \nu_k} = \{\omega \mid \omega_i = \nu_i \text{ for } i = 1, \dots, k\}$, it satisfies $\nu^-(C) = \nu(F^{-k}(C \times [0, 1])) = \nu^+(J)$ with $F^{-k}(C \times [0, 1]) = \Sigma \times J$. Now $g^{-n}\mathcal{G} \uparrow \mathcal{F}$ and $g^{-1} : \mathcal{I} \rightarrow \mathcal{I}$ is the natural extension of $g^{-1} : [0, 1] \rightarrow [0, 1]$. By a continuously differentiable coordinate change, the strong unstable lamination \mathcal{F}^{uu} is affine; $\mathcal{F}^{uu} = \{(\omega, y, x) \mid \omega, x \text{ constant}\}$. This makes F^{-1} like F up to interchanging strong stable and strong unstable directions. In the resulting coordinates, write $F^{-1}(\mathbf{y}, x) = (g^{-1}(\mathbf{y}), k_{\mathbf{y}}^{-1}(x))$ (where $k_{\mathbf{y}}^{-1}(x)$ depends only on ω and x). Suppose ζ^- is an invariant measure for F^{-1} on $\Sigma \times [0, 1] \times \mathbb{T}$ with σ -algebra $\mathcal{G} \otimes \mathcal{B}$ and with marginal ν^- . We write ζ_{ω}^- , $\omega \in \Sigma$, or also $\zeta_{\mathbf{y}}^-$, for its disintegrations.

Lemma 2.2. *Given the invariant measure ζ^- for F^{-1} acting on $\Sigma \times \mathbb{T}$, with marginal ν^- on Σ , there is an invariant measure ζ for F^{-1} acting on $\Sigma \times [0, 1] \times \mathbb{T}$ with marginal ν on $\Sigma \times [0, 1]$. For ν -almost all $\mathbf{y} \in \mathcal{S}$, the limit*

$$\zeta_{\mathbf{y}} = \lim_{n \rightarrow \infty} k_{g^n(\mathbf{y})}^{-n} \zeta_{g^n(\mathbf{y})}^- \quad (7)$$

gives its disintegrations.

Proof. As for Lemma 2.1 one can apply [2, Theorem 1.7.2] to prove the lemma. \square

3 Partial hyperbolicity

See e.g. [18, Section 17.1] for the standard construction of the solenoid as an attractor for a diffeomorphism on $(-1, 1)^2 \times \mathbb{T}$. Likewise the solenoid can appear as an attractor for a diffeomorphism on $(-1, 1)^d \times \mathbb{T}$, $d \geq 2$. Under an assumption

$$m = \max_{y,x} f'_y(x) < \min_y g'(y) = M, \quad (8)$$

one may embed the solenoid as a hyperbolic attractor, so that the class of skew product systems is partially hyperbolic [7] on $\mathcal{S} \times \mathbb{T}$. This results in a partially hyperbolic splitting in one-dimensional strong unstable directions, one-dimensional center directions (the fibers) and the remaining d -dimensional strong stable directions. Write $N = (-1, 1)^d \times \mathbb{T}^2$ for the (open neighborhood in the) manifold that contains $\mathcal{S} \times \mathbb{T}$ as hyperbolic attractor; the map F on $\mathcal{S} \times \mathbb{T}$ is extended to a diffeomorphism F on N .

Write $W^{ss}(\mathbf{y}, x)$ for the strong stable manifold of (\mathbf{y}, x) and $W^{uu}(\mathbf{y}, x)$ for the strong unstable manifold of (\mathbf{y}, x) . The strong stable and strong unstable manifolds form laminations \mathcal{F}^{ss} and \mathcal{F}^{uu} .

Lemma 3.1. *Assuming (8), there exists an embedding of \mathcal{S} as a hyperbolic attractor for a smooth diffeomorphism on a manifold, so that the class of skew product systems is partially hyperbolic on $\mathcal{S} \times \mathbb{T}$. For $\max_{y,x} \{f'_y(x), 1/f'_y(x)\}$ sufficiently close to 1 and $M > 2$, such an embedding exists for which \mathcal{F}^{ss} and \mathcal{F}^{uu} are continuously differentiable laminations.*

Proof. In the strong stable directions, taking the dimension d sufficiently large (depending on the degree of the expanding circle map g), distances can be assumed to be contracted by a factor close to $\frac{1}{2}$.

Observe that, forced by the form of the map F , the local strong stable manifold $W_{\text{loc}}^{ss}(\mathbf{y}, a_0)$ for any $\mathbf{y} \in \psi^{-1}(y_0)$ equals $\psi^{-1}(y_0) \times \{a_0\}$. The strong stable lamination is therefore continuously differentiable. If $f'_y(x)$ is near 1 for all x, y , then with $M > 2$ (the expanding map g has to be of degree three or higher) spectral gap conditions are satisfied that imply that the strong unstable lamination is continuously differentiable. This is checked by going through the construction of the strong unstable lamination by graph transform techniques [16], as we will indicate.

One obtains the strong unstable lamination by integrating the line field formed by the strong unstable directions. Write $TN = N \times E^{uu} \times E^{ss,c}$ so that the strong unstable directions at a point $x \in \mathcal{S} \times \mathbb{T}$ are given as the graph of a linear map in $\mathcal{L}(E^{uu}, E^{ss,c})$. The strong unstable directions are then given

by the graph of a section $\mathcal{S} \times \mathbb{T} \mapsto \mathcal{L}(E^{uu}, E^{ss,c})$ that is invariant under the induced diffeomorphism $\hat{F} : \mathcal{S} \times \mathbb{T} \times \mathcal{L}(E^{uu}, E^{ss,c}) \rightarrow \mathcal{S} \times \mathbb{T} \times \mathcal{L}(E^{uu}, E^{ss,c})$;

$$\hat{F}(\mathbf{y}, x, \alpha) = (F(\mathbf{y}, x), \beta), \quad \text{graph } \beta = DF(\mathbf{y}, x)\text{graph } \alpha. \quad (9)$$

It is possible to construct strong unstable directions on N that extend those on $\mathcal{S} \times \mathbb{T}$ by choosing a lamination on a fundamental domain in its basin of attraction and iterating under the graph transform [23, Appendix 1]. This produces a graph V^{uu} of a section $N \mapsto \mathcal{L}(E^{uu}, E^{ss,c})$ that is invariant under \hat{F} .

If λ^{ss} is the strongest rate of contraction, i.e. for some $C > 0$ and $i \in \mathbb{N}$,

$$|DF^i(n)v| \geq C(\lambda^{ss})^i|v|,$$

for each $n \in N$, $v \in T_x N$, then such a graph V^{uu} is normally hyperbolic for $m/M < \lambda^{ss}$. Indeed, the contraction of \hat{F} along the fibers $\mathcal{L}(E^{uu}, E^{ss,c})$ is estimated by

$$D\hat{F}^i(n, \alpha)(0, w) \leq C(m/M)^i|w|, \quad (10)$$

compare [23, Appendix 1]. Normal hyperbolicity holds for λ^{ss} near $\frac{1}{2}$, m near 1 and $M > 2$. Normal hyperbolicity implies that V^{uu} is continuously differentiable and this in turn implies that the strong unstable lamination is continuously differentiable [26]. \square

4 Robust transitivity

We record that

$$F_{i,j}(y, x) = (iy, x + jy) \pmod{1},$$

with $i > 1, j$ integers, is not topologically transitive; it leaves all circles parallel to $jx = (i-1)y \pmod{1}$ invariant. Note that $F_{i,j}$ induces a homeomorphism on $\mathcal{S} \times \mathbb{T}$; $F_{i,j}(\mathbf{y}, x) = (i\mathbf{y}, x + iy_0)$, with inverse $F_{i,j}^{-1}(\mathbf{y}, x) = (\dots, y_{-2}, y_{-1}, x - jy_{-1})$.

The following result provides a class of robust topologically mixing skew product maps. We use ad hoc arguments, relying on the skew product structure with topologically mixing base dynamics, to prove it, but the arguments bear a resemblance to the technique of blenders introduced in [5].

Theorem 4.1. *There exist arbitrarily small smooth perturbations F , $F(y, x) = (g(y), f_y(x))$, of $F_{i,0}$, $i > 1$, that are robustly topologically mixing skew product maps (considered on either \mathbb{T}^2 or $\mathcal{S} \times \mathbb{T}$).*

Moreover,

1. there are $k \in \mathbb{N}$, $\hat{y} \in \mathbb{T}$, with $g^k(\hat{y}) = \hat{y}$ and $f_{\hat{y}}^k$ possessing a unique hyperbolic attracting and hyperbolic repelling fixed point,
2. for any $(\mathbf{y}, x) \in \mathcal{S} \times \mathbb{T}$, the strong stable and strong unstable manifolds $W^{ss}(\mathbf{y}, x)$, $W^{uu}(\mathbf{y}, x)$ are dense in $\mathcal{S} \times \mathbb{T}$.

Proof. Consider $\Sigma_n^+ = \{0, \dots, n\}^{\mathbb{N}}$ endowed with the product topology and let $\sigma : \Sigma_n^+ \rightarrow \Sigma_n^+$ be the left shift. The base map g (or some iterate thereof) admits invariant Cantor sets on which the dynamics is topologically conjugate to $\sigma : \Sigma_n^+ \mapsto \Sigma_n^+$. This observation and the following lemma imply the existence of robust topologically mixing maps F acting on \mathbb{T}^2 as stated in the theorem.

Lemma 4.1. *There exists a skew product map*

$$H(\omega, x) = (\sigma\omega, h_\omega(x))$$

on $\Sigma_n^+ \times \mathbb{T}$, $n \geq 4$, that is robustly topologically mixing under continuous perturbations of $\omega \mapsto h_\omega$ in the C^1 topology.

Proof. Following [13], take circle diffeomorphisms h_0, h_1, h_2 so that

1. h_i has a unique hyperbolic attracting fixed point p_i and a unique hyperbolic repelling fixed point q_i , $i = 0, 1, 2$; the fixed points are mutually disjoint,
2. p_0, p_1 are close to each other and h_0, h_1 are affine on $[p_0, p_1]$,
3. $p_2 \in (p_0, p_1)$,
4. $\frac{1}{2} < (h_0)'(p_0), (h_1)'(p_1) < 1$.

The iterated function system generated by $h_0, h_1, h_2, h_3 = h_2^{-1}$ is robustly minimal under C^1 small perturbations of h_0, \dots, h_3 . We give the main steps in the reasoning, referring to [13] for details. Consider the iterated function system generated by h_0, h_1 . For a compact subset $S \subset \mathbb{T}$, write $\mathcal{L}(S) = h_0(S) \cup h_1(S)$. Let $E_{in} \subset [p_0, p_1] \subset E_{out}$ be intervals close to $[p_0, p_1]$ on which h_0, h_1 are affine. Then

$$E_{in} \subset \mathcal{L}(E_{in}) \subset [p_0, p_1] \subset \mathcal{L}(E_{out}) \subset E_{out} \quad (11)$$

and $\mathcal{L}^i(E_{in}), \mathcal{L}^i(E_{out})$ converge to $[p_0, p_1]$ in the Hausdorff topology as $i \rightarrow \infty$. Since h_0 and h_1 are contractions, this shows that the iterated function system generated by h_0, h_1 is minimal on $[p_0, p_1]$. From the properties of h_2, h_3 it is easily concluded that the iterated function system generated by h_0, h_1, h_2, h_3 is minimal on \mathbb{T} .

The skew product system $H(\omega, x) = (\sigma\omega, h_{\omega_0}(x))$ is topologically mixing. Indeed, write $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}} \subset \Sigma_n^+$ and take an open set $U \subset \Sigma_2^+ \times [p_0, p_1]$. A high iterate $H^n(U)$ contains a strip $\Sigma_2^+ \times J$ in $\Sigma_2^+ \times [p_0, p_1]$. Observe that $H^k(\Sigma_2^+ \times [p_0, p_1])$ consists of 2^k strips with width going to 0 as $k \rightarrow \infty$ and together covering $[p_0, p_1]$. It follows that $H^{n+k}(U)$ contains 2^k strips that lie increasingly dense in $\Sigma_2^+ \times [p_0, p_1]$, as $k \rightarrow \infty$. Iterates of $\Sigma_n^+ \times [p_0, p_1]$ under H lie dense in $\Sigma_n^+ \times \mathbb{T}$ since the repelling fixed point q_2 of h_2 lies inside $[p_0, p_1]$. This shows that H is topologically mixing.

This reasoning also applies to small perturbations of H , where also the fiber maps may depend on all of ω instead of just ω_0 . We note the following changes in the reasoning. The inclusions (11) get replaced by

$$\Sigma_2^+ \times E_{in} \subset H(\Sigma_2^+ \times E_{in}), \quad H(\Sigma_2^+ \times E_{out}) \subset \Sigma_2^+ \times E_{out}$$

The map H acting on $\Sigma_2^+ \times E_{out}$ acts by contractions in the fibers $\omega \times E_{out}$. A high iterate $H^n(U)$ may not contain a product $\Sigma_2^+ \times J$ but contains a strip lying between the graphs of two maps $\Sigma_2^+ \rightarrow \mathbb{T}$. Again $H^{n+k}(U)$ contains 2^k strips, lying increasingly dense in a region that contains $\Sigma_2^+ \times E_{in}$, as $k \rightarrow \infty$.

If ω starts with a sequence of i symbols 2, then $h_{\sigma^i \omega} \circ \dots \circ h_{\omega}$ maps an interval $I \subset \mathbb{T}$ that contains q_2 to an interval with length approaching 1 as $i \rightarrow \infty$. Also, any point in \mathbb{T} can be mapped into $[\tilde{p}_0, \tilde{p}_1]$ by an iterate that involves ω with a long sequence of symbols 3.

The lemma follows. □

As a consequence, F acting on \mathbb{T}^2 is topologically mixing. Indeed, take an open set U in $\Sigma_n^+ \times \mathbb{T}$. The construction in Lemma 4.1 gives that $\cup_{n \in \mathbb{N}} F^n(U)$ is open and dense in $\Sigma_n^+ \times \mathbb{T}$. Now take open sets $U, V \subset \mathbb{T}^2$. As g is expanding, some iterate of U under F intersects $\Sigma_n^+ \times \mathbb{T}$. Again as g is expanding, a higher iterate will intersect V , establishing topological mixing of $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$.

The construction also implies that strong unstable manifolds are dense in $\mathcal{S} \times \mathbb{T}$. To see this, consider the periodic points P and Q for $F : \mathcal{S} \times \mathbb{T} \rightarrow \mathcal{S} \times \mathbb{T}$, where P corresponds to p_0 in the proof of the lemma and Q corresponds to q_2 . The two-dimensional unstable manifold of Q lies dense in $\mathcal{S} \times \mathbb{T}$ since unstable manifolds for g lie dense in \mathcal{S} . Note that the stable manifold of Q contains points arbitrarily close to P . We claim that $W^u(Q) \subset \overline{W^{uu}(P)}$ (compare [5, Lemma 1.9]): take a point $x \in W^u(Q)$ and a neighborhood V of it, iterate backwards and note that $F^{-m}(V)$ intersects $W^{uu}(P)$. Thus $W^{uu}(P)$, and therefore each strong unstable manifold, lies dense in $\mathcal{S} \times \mathbb{T}$.

Topological mixing is a direct consequence of minimality of the strong unstable lamination, since iterates of an open set accumulate onto $W^{uu}(P)$.

Finally use these arguments for inverse diffeomorphisms to see that there are skew product maps for which also strong stable manifolds lie dense in $\mathcal{S} \times \mathbb{T}$. \square

Definition 4.1. *The skew-product map F is called strongly contractive if for all $\varepsilon > 0$, there are $\hat{y} \in \mathbb{T}$, an interval $V \subset \mathbb{S}^1$, $n \in \mathbb{N}$, so that $|V| > 1 - \varepsilon$ and $|f_{\hat{y}}^n(I)| < \varepsilon$.*

The following lemma, that provides a robust condition for F being strongly contractive, is immediate.

Lemma 4.2. *Suppose there exist $k \in \mathbb{N}$, $\hat{y} \in \mathbb{T}$, with $g^k(\hat{y}) = \hat{y}$ and $f_{\hat{y}}^k$ possessing a unique hyperbolic attracting and hyperbolic repelling fixed point. Then F is strongly contractive.*

5 Attracting invariant graphs

Contraction of positive orbits starting in the same fiber, is explained by the following result. Theorem 1.1 follows from it. The arguments that establish random fixed points in iterated function systems, see [11, Proposition 5.7] (compare also [22, Section 2.3]) and [34], are based on pushing forward a stationary measure by the circle diffeomorphisms and identifying limit measures. Although there is no stationary measure in our context, our proof of Theorem 5.1 is inspired by this approach. Different approaches using the theory of nonuniform hyperbolic systems to provide invariant delta-measures in forced circle diffeomorphisms, are followed in [10, 20, 30]. Such approaches do not determine the number of points in each fiber and would therefore not allow to explain Theorem 1.1.

Theorem 5.1. *Let $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be robust topologically mixing as in Theorem 4.1, so that F is also strongly contractive. Then F acting on $\mathcal{S} \times \mathbb{T}^2$ admits an invariant graph $\{(\mathbf{y}, \omega^+(\mathbf{y})) \mid \mathbf{y} \in \mathcal{S}\}$ for a measurable function ω^+ , that attracts the positive orbits of $\nu \times \lambda$ -almost all initial points.*

Proof. The main steps in the proof are the following. We show that for ν -almost all \mathbf{y} , the push-forwards $f_{g^{-n}(\mathbf{y})}^n \lambda$ of Lebesgue measure λ contain delta measures in the fiber over \mathbf{y} as accumulation points in the weak star topology. Invoking Lemma 2.1, we establish that $f_{g^{-n}(\mathbf{y})}^n \lambda$ in fact converges to a delta measure, thus proving the existence of an invariant graph for F acting on $\mathcal{S} \times \mathbb{T}$. For the attraction properties, we must likewise consider F^{-1} and construct an invariant graph for F^{-1} .

We start with a lemma.

Lemma 5.1. *Given $\varepsilon > 0$, for ν -almost all $\mathbf{y} \in \mathcal{S}$ there are an interval $I \subset \mathbb{T}$ with $|I| > 1 - \varepsilon$ and $n \in \mathbb{N}$, so that $|f_{g^{-n}(\mathbf{y})}^n(I)| < \varepsilon$.*

Proof. The lemma will be a consequence of a construction in which we provide $\delta > 0$, $L \in \mathbb{N}$ so that the following holds. Given an interval $J \subset \mathbb{T}$ and $n \in \mathbb{N}$, we construct for each $y \in J$, an interval $I \subset \mathbb{T}$ with $|I| > 1 - \varepsilon$, an open subset $J' \subset \psi^{-1}(J) \subset \mathcal{S}$ with $\nu(J')/\nu^+(J) = \nu(J')/\nu(\psi^{-1}(J)) > \delta$ and a positive integer $l \leq L$, so that for $\mathbf{y} \in J'$, $|f_{g^{-n-l}(\mathbf{y})}^{n+l}(I)| < \varepsilon$.

Fix $\varepsilon > 0$.

Step 1. There are intervals $K \subset \mathbb{T}$, $V \subset \mathbb{T}$ with $|V| > 1 - \varepsilon$ and $N \in \mathbb{N}$, so that $|f_y^N(V)| < \varepsilon$ for $y \in K$. We make this more explicit. By Theorem 4.1, $f_{\hat{y}}^k$ has a hyperbolic attracting fixed point p .

By taking suitable smooth coordinates near the forward orbit of (\hat{y}, p) , we may assume that the local unstable manifold $W_{\text{loc}}^u(\hat{y}, p)$ of (\hat{y}, p) is contained in $\mathbb{T} \times \{p\}$.

The interval K can be taken a fundamental domain (i.e. an interval for which g^k maps one boundary point to the other), so that $g^i(K)$ stays close to the forward orbit of \hat{y} for $0 \leq i \leq N$. We let N be a multiple of k , so that $g^N(K)$ is close to \hat{y} . By replacing K with $g^{-ki}(K)$ for some $i > 0$, and replacing N by $N + ki$, we can

decrease the size of the image $f_y^N(V)$, while keeping $g^N(K)$ fixed. Write $U_y = f_y^N(V)$ and also $V_z = f_y^N(V)$ with $z = g^N(y)$.

Step 2. Take $q > 1/\varepsilon$. Take $\zeta_0 = y_{-n}$ in J . Write \hat{y} for the periodic point of $g : \mathcal{S} \rightarrow \mathcal{S}$ in $\psi^{-1}(\hat{y})$. Iterates of an interval $N \subset W_{\text{loc}}^{uu}(\hat{y}, p)$ lie dense in $\mathcal{S} \times \mathbb{T}$, by Theorem 4.1 and expansion properties of g . One can therefore take a point $\zeta = (\dots, \zeta_0) \in \mathcal{S}$ so that

1. there are positive integers $M_1 < \dots < M_q$ so that $\zeta_{-M_i} \in g^N(K)$, $1 \leq i \leq q$,
2. with a_i, b_i given by $(\zeta_{-M_i}, b_i) \in W_{\text{loc}}^{uu}(\hat{y}, p)$ and $(\zeta_{-M_1}, a_i) = F^{M_i - M_1}(\zeta_{-M_i}, b_i)$, the points a_i , $1 \leq i \leq q$, are disjoint.

Step 3. Take neighborhoods $L_i \subset g^N(K)$ of ζ_{-M_i} so that $f_z^{M_i}(V_z)$, $z \in L_i$, are disjoint for different i . Consider $\cap_{1 \leq i \leq q} g^{M_i}(L_i)$. Since finitely many such intervals (for varying y_0, J) cover the circle \mathbb{T} , the numbers N, M_i are bounded (that is, depend only on ε and the dynamical system F).

Step 4. Let $L = \cup_{1 \leq i \leq q} L_i \subset g^N(K)$. Let $O = \cup_{1 \leq i \leq q} g^{M_i}(L_i)$. For $y \in J \cap O$, there is j , $1 \leq j \leq q$, with $f_y^n(V_z)$ with $z = g^{-M_j}(y) \cap L_j$ that has length smaller than ε (since there are $q > 1/\varepsilon$ such disjoint intervals). This defines a set of y values for which one of $f_y^{N+M_j+n}(U_y)$ is small.

For given $\varepsilon > 0$, there is a bound $\delta > 0$ with $|\cup_{1 \leq i \leq q} L_i|/|J| > \delta$. A similar bound holds with ν^+ replacing the length of intervals, since ν^+ has density that is bounded and bounded away from zero. This ends the construction. Now define

$$\Delta_N = \{\mathbf{y} \in \mathcal{S} \mid \text{for each interval } I \text{ with } |I| > 1 - \varepsilon \text{ and each } i \leq N, |f_{g^{-i}(\mathbf{y})}^i(I)| > \varepsilon\}.$$

The above construction yields the estimate $\nu(\Delta_{tl}) \leq (1 - \delta)^t$. Thus $\nu(\Delta) = 0$, where

$$\Delta = \{\mathbf{y} \in \mathcal{S} \mid \text{for each interval } I \text{ with } |I| > 1 - \varepsilon \text{ and each } i, |f_{g^{-i}(\mathbf{y})}^i(I)| > \varepsilon\}. \quad \square$$

Lemma 5.1 implies that the push-forwards $f_{g^{-n}(\mathbf{y})}^n \lambda$ contain a delta-measure $\delta_{\omega^+(\mathbf{y})}$, concentrated at $\omega^+(\mathbf{y})$, as accumulation point. This yields an invariant graph $\{(\mathbf{y}, \omega^+(\mathbf{y})) \mid \mathbf{y} \in \mathcal{S}\}$ for $F : \mathcal{S} \times \mathbb{T} \rightarrow \mathcal{S} \times \mathbb{T}$. Let $\mu_{y_0}^+$ be obtained from $\delta_{\omega^+(\mathbf{y})}$, $\mathbf{y} = (\dots, y_0)$, as in (5). The following lemma will be applied to find that $f_{g^{-n}(\mathbf{y})}^n \lambda$ and $f_{g^{-n}(\mathbf{y})}^n \mu_{y_0}^+$ converge to the delta measure $\delta_{\omega^+(\mathbf{y})}$ for ν -almost all $\mathbf{y} \in \mathcal{S}$. We refer to [31, 32] for general results on invariant measures for partially hyperbolic endomorphisms. Recall that a measure is diffuse if it has no atoms.

Lemma 5.2. *For each $y_0 \in \mathbb{T}$,*

$$\text{supp } \mu_{y_0}^+ = \mathbb{T}. \quad (12)$$

Moreover, $\mu_{y_0}^+$ is diffuse and depends continuously on y_0 in the weak star topology.

Proof. We use an estimate $m = \max_{y,x} \{f'_y(x), 1/f'_y(x)\} < \min_y g'(y) = M$, which is implicit in Theorem 4.1. Consider \mathbf{z} close to \mathbf{y} ; i.e. z_i close to y_i for all $i \in \mathbb{N}$. The branch of g defined near y_{-i-1} for which $g(y_{-i-1}) = g_{-i}$ has an inverse; in the following we write g^{-1} for it with the understanding that we consider orbits near \mathbf{y} . Consider $f_{g^{-n}(\mathbf{y})}^n(x) = f_{g^{-1}(\mathbf{y})} \circ \dots \circ f_{g^{-n}(\mathbf{y})}(x)$ and compute

$$\frac{\partial}{\partial y} f_{g^{-n}(\mathbf{y})}^n(x) = \sum_{i=1}^n \left(f_{g^{-i+1}(\mathbf{y})}^{i-1} \right)' (f_{g^{-n}(\mathbf{y})}^{n-i+1}(x)) \frac{\partial}{\partial y} f_{g^{-i}(\mathbf{y})} (f_{g^{-n}(\mathbf{y})}^{n-i}(x)) (g^{-i})'(\mathbf{y}),$$

which is uniformly bounded by (8). Likewise

$$\frac{\partial}{\partial y} f_{g^{-n}(\mathbf{y})}^{n-l}(x) M^l = \mathcal{O}(1).$$

Now, for a subsequence $n_i \rightarrow \infty$, $f_{y_{-n_i}}^{n_i} \lambda$ converges in the weak star topology to a delta measure $\delta_{\omega^+(\mathbf{y})}$. Take l so that $g^{-l}(\mathbb{T})$ is an interval of length $\mathcal{O}(\varepsilon^s)$ for a positive s . Recall that $f_{g^{-n}(\mathbf{y})}^n$ maps an interval V

of length $1 - \varepsilon$ to an interval I of length ε . Then $f_{\mathbf{y}}^{-l}(I)$ is an interval of length ε^t for some $t > 0$. These estimates imply that for \mathbf{z} near \mathbf{y} , $f_{g^{-n_i}(\mathbf{z})}^{n_i} \lambda$ converges to a delta measure $\delta_{\omega^+(\mathbf{z})}$ depending continuously on \mathbf{z} . The graph transform construction of the strong unstable lamination in fact shows that $(\mathbf{z}, \omega^+(\mathbf{z}))$ is in $W^{uu}(\mathbf{y}, \omega^+(\mathbf{y}))$. See also [24] and [7, Chapter 11].

Consider local center stable manifolds $W^{ss,c}(y) = \{(\dots, y_{-1}, y_0), x\} \in \mathcal{S} \times \mathbb{T}$, $y_0 = y$ in $\mathcal{S} \times \mathbb{T}$. The invariant measure μ has disintegrations μ_y along $W^{ss,c}(y)$, $y \in \mathbb{T}$. If π^{ss} denotes the projection onto the fiber \mathbb{T} ,

$$\pi^{ss}((\dots, y_{-1}, y_0), x) = (y_0, x),$$

then

$$\mu_{y_0}^+ = \pi^{ss} \mu_{y_0}. \quad (13)$$

We claim that the disintegrations μ_{y_0} are u -invariant, meaning that the disintegrations μ_{y_0} are invariant under the holonomy along strong unstable leaves.

Consider F acting on $\Sigma \times \mathbb{T}^2$ (compare Section 2) and take coordinates in which the strong unstable lamination is affine. Take a product measure $m = \nu_2 \times \nu$. A Césaro accumulation point of push-forwards $F^n m$ is a Gibbs u -measure [24] or [7, Chapter 11]. The Césaro accumulation point is a product measure and hence u -invariant (see also [3, Remark 4.1]).

Equation (12) follows by (5) since strong unstable manifolds are dense in $\mathcal{S} \times \mathbb{T}$. If an open set has positive measure, also the image under F has positive measure. Since the measure μ is invariant and the strong unstable lamination is minimal, with (13) this yields (12). Continuous dependence of $\mu_{y_0}^+$ on y_0 is implied by (13) and u -invariance of μ_{y_0} .

It remains to prove that $\mu_{y_0}^+$ is diffuse. In coordinates in which $y \mapsto g(y)$ is a linear expanding circle map $y \mapsto ky \pmod{1}$,

$$\mu_{y_0}^+ = \frac{1}{k} \sum_{\{z : g(z) = y_0\}} \mu_z^+, \quad (14)$$

by invariance of μ^+ . Take an atom (y_0, a) in the fiber over y_0 with maximal mass for $\mu_{y_0}^+$. Then by (14), the points in $F_+^{-1}(y_0, a)$ each possess the same mass. The inverse images $F_+^{-i}(y_0, a)$ lie dense in \mathbb{T}^2 by minimality of the strong stable foliation. Atoms with the same positive mass in their fibers thus accumulate onto the fiber over y_0 . By continuity of the measures $\mu_{y_0}^+$ in y_0 , this contradicts $\mu_{y_0}^+(\mathbb{T}) = 1$. Hence $\mu_{y_0}^+$ is diffuse as claimed. \square

Lemma 5.3. *For ν -almost all $\mathbf{y} \in \mathcal{S}$,*

$$\lim_{n \rightarrow \infty} f_{g^{-n}(\mathbf{y})}^n \mu_{g^{-n}(\mathbf{y})}^+ = \lim_{n \rightarrow \infty} f_{g^{-n}(\mathbf{y})}^n \lambda = \delta_{\omega^+(\mathbf{y})}$$

for a delta measure $\delta_{\omega^+(\mathbf{y})}$.

Proof. Recall that $f_{g^{-n}(\mathbf{y})}^n \lambda$ has a delta measure $\delta_{\omega^+(\mathbf{y})}$ as accumulation point. Further, $f_{g^{-n}(\mathbf{y})}^n \mu_{g^{-n}(\mathbf{y})}^+$ converges by Lemma 2.1. By Lemma 5.2, $f_{g^{-n}(\mathbf{y})}^n \lambda$ and $f_{g^{-n}(\mathbf{y})}^n \mu_{g^{-n}(\mathbf{y})}^+$ converge to $\delta_{\omega^+(\mathbf{y})}$. \square

We have constructed an invariant graph $\{(\mathbf{y}, \omega^+(\mathbf{y}))\}$ for $F : \mathcal{S} \times \mathbb{T} \rightarrow \mathcal{S} \times \mathbb{T}$. To prove its attraction property, we need to consider iterates from time zero to time $n > 0$. Invertibility of the maps in the fibers implies that if $f_{\mathbf{y}}^n \lambda$ is close to a delta measure, then also $f_{g^n(\mathbf{y})}^{-n} \lambda$ is close to a delta measure. So we can also consider iterates from time $n > 0$ to time 0, for which we consider the inverse skew product map F^{-1} . One can largely follow the previous reasoning to construct an invariant graph $\{(\mathbf{y}, \omega^-(\mathbf{y}))\}$ for $F^{-1} : \mathcal{S} \times \mathbb{T} \rightarrow \mathcal{S} \times \mathbb{T}$.

We give the lemma's that correspond to Lemma's 5.2 and 5.3. Recall the last part of Section 2 on ergodic properties of F^{-1} .

Lemma 5.4. For each $\omega \in \Sigma$,

$$\text{supp } \zeta_{\omega}^{-} = \mathbb{T}. \quad (15)$$

Moreover, ζ_{ω}^{-} is diffuse and depends continuously on ω in the weak star topology.

Lemma 5.5. For ν -almost all $\mathbf{y} \in \mathcal{S}$,

$$\lim_{n \rightarrow \infty} k_{g^n(\mathbf{y})}^{-n} \zeta_{g^n(\mathbf{y})}^{-} = \lim_{n \rightarrow \infty} f_{g^n(\mathbf{y})}^{-n} \lambda = \delta_{\omega^{-}(\mathbf{y})}$$

for a delta measure $\delta_{\omega^{-}(\mathbf{y})}$.

Lemma 5.5 implies that the graph of ω^{+} , whose existence is given by Lemma 5.3, is attracting. It attracts all points lying outside the graph of ω^{-} . This is true even if $\omega^{+} = \omega^{-}$, but in fact $\omega^{+}(\mathbf{y}) \neq \omega^{-}(\mathbf{y})$ for ν -almost all \mathbf{y} . This can be seen by writing $\mathcal{S} = \Sigma \times I / \sim$ as in (6), so that in $\mathbf{y} = \omega \times y$ the “past” ω and the “future” y are independent. The resulting positions $\omega^{+}(\mathbf{y})$ and $\omega^{-}(\mathbf{y})$ depend on past and future only, respectively, and vary according to Lemma 5.3 and Lemma 5.5.

This finishes the proof of Theorem 5.1. □

References

- [1] V. A. Antonov. Modeling of processes of cyclic evolution type. synchronization by a random signal. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 2:67–76, 1984.
- [2] L. Arnold. *Random dynamical systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
- [3] A. Avila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. *Invent. Math.* 181:115–174 (2010).
- [4] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou. The synchronization of chaotic systems. *Phys. Rep.*, 366(1-2):1–101, 2002.
- [5] C. Bonatti and L. J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math. (2)*, 143(2):357–396, 1996.
- [6] C. Bonatti, L. J. Díaz, and A. S. Gorodetskiĭ. Non-hyperbolic ergodic measures with large support. *Nonlinearity*, 23(3):687–706, 2010.
- [7] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005.
- [8] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.
- [9] K. Burns, D. Dolgopyat, Ya. Pesin, and M. Pollicott. Stable ergodicity for partially hyperbolic attractors with negative central exponents. *J. Mod. Dyn.*, 2(1):63–81, 2008.
- [10] H. Crauel. Extremal exponents of random dynamical systems do not vanish. *J. Dynam. Differential Equations*, 2:245–291, 1990.
- [11] B. Deroin, V. A. Kleptsyn, and A. Navas. Sur la dynamique unidimensionnelle en régularité intermédiaire. *Acta Math.*, 199(2):199–262, 2007.
- [12] L. J. Díaz and A. S. Gorodetskiĭ. Non-hyperbolic ergodic measures for non-hyperbolic homoclinic classes. *Ergodic Theory Dynam. Systems* 29(5):1479–1513, 2009.

- [13] G. H. Ghane, A. J. Homburg, and A. Sarizadeh. C^1 robustly minimal iterated function systems. *Stoch. Dyn.*, 10(1):155–160, 2010.
- [14] A. S. Gorodetskiĭ and Yu. S. Il'yashenko. Certain properties of skew products over a horseshoe and a solenoid. Grigorchuk, R. I. (ed.), *Dynamical systems, automata, and infinite groups*. Proc. Steklov Inst. Math. 231, 90-112 (2000).
- [15] A. S. Gorodetskiĭ, Yu. S. Il'yashenko, V. A. Kleptsyn, and M. B. Nal'skiĭ. Nonremovability of zero Lyapunov exponents. *Funct. Anal. Appl.*, 39(1):21–30, 2005.
- [16] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin, 1977.
- [17] T. H. Jäger. Strange non-chaotic attractors in quasiperiodically forced circle maps. *Comm. Math. Phys.*, 289(1):253–289, 2009.
- [18] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems. With a supplement by Anatole Katok and Leonardo Mendoza*. Encyclopedia of Mathematics and Its Applications. 54. Cambridge: Cambridge University Press. xviii, 802 p., 1997.
- [19] V.A. Kleptsyn and M.B. Nalskii. Contraction of orbits in random dynamical systems on the circle. *Funct. Anal. Appl.*, 38(4):267–282, 2004.
- [20] Y. Le Jan. Equilibre statistique pour les produits de difféomorphismes aléatoires indépendants. *Ann. Inst. H. Poincaré Probab. Statist.*, 23:111–120, 1987.
- [21] R. Mañé. *Ergodic theory and differentiable dynamics*, volume 8 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.
- [22] A. Navas. *Groups of circle diffeomorphisms*. The university of Chicago press, 2011.
- [23] J. Palis and F. Takens. *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations* Cambridge University Press, 1993.
- [24] Ya. B. Pesin and Ya. G. Sinaĭ. Gibbs measures for partially hyperbolic attractors. *Ergodic Theory Dynam. Systems*, 2(3-4):417–438 (1983), 1982.
- [25] Ya. B. Pesin. *Lectures on partial hyperbolicity and stable ergodicity*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.
- [26] C. Pugh, M. Shub and A. Wilkinson. Hölder foliations. *Duke Math. J.* 86(3):517–546 (1997). Correction to: “Hölder foliations” [Duke Math. J. 86 (1997), no. 3, 517–546] *Duke Math. J.*, 105(1):105–106 (2000).
- [27] D. Ruelle. Perturbation theory for Lyapunov exponents of a toral map: extension of a result of Shub and Wilkinson. *Israel J. Math.*, 134:345–361, 2003.
- [28] D. Ruelle and A. Wilkinson. Absolutely singular dynamical foliations. *Comm. Math. Phys.*, 219(3):481–487, 2001.
- [29] M. Shub. Endomorphisms of compact differentiable manifolds. *Amer. J. Math.*, 91(1):175–199, 1969.
- [30] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.*, 139(3):495–508, 2000.
- [31] M. Tsujii. Fat solenoidal attractors. *Nonlinearity*, 14(5):1011–1027, 2001.
- [32] M. Tsujii. Physical measures for partially hyperbolic surface endomorphisms. *Acta Math.*, 194(1):37–132, 2005.
- [33] R. F. Williams. Expanding attractors. *Inst. Hautes Études Sci. Publ. Math.*, (43):169–203, 1974.

- [34] H. Zmarrou and A. J. Homburg. Dynamics and bifurcations of random circle diffeomorphisms. *Discrete Contin. Dyn. Syst. Ser. B*, 10(2-3):719–731, 2008.