INTERMITTENT TWO-POINT DYNAMICS AT THE TRANSITION TO CHAOS FOR RANDOM CIRCLE ENDOMORPHISMS

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Abstract. We establish the existence of intermittent two-point dynamics and infinite stationary measures for a class of random circle endomorphisms with zero Lyapunov exponent, as a dynamical characterisation of the transition from synchronisation (negative Lyapunov exponent) to chaos (positive Lyapunov exponent).

1. Introduction

In dynamical systems theory, the phenomenon of chaos (with hallmark sensitive dependence on initial conditions) has been an important motivation and focal point of research. In particular, the question of when and why chaotic dynamics emerges from more predictable motion, for instance in a parametrized family of dynamical systems, has been a central question in bifurcation theory.

While several routes to chaos have been identified for deterministic dynamical systems (see for instance [26]), the corresponding transition in random dynamical systems (dynamical systems driven by a signal with certain probabilistic characteristics) remains much less understood.

In this paper, we address this problem for a class of random circle endomorphisms, adopting notions of order and chaos in terms of the sign of the Lyapunov exponent, with negative Lyapunov exponent implying synchronisation (almost sure convergence of the distance between different trajectories with different initial conditions) and positive Lyapunov exponent implying chaos (including sensitive dependence on initial conditions). We are led to address the question how this transition happens, and in particular to identify dynamical aspects that accompany the change of sign of the Lyapunov exponent. We focus in this respect on the so-called two-point motion from the topological and (invariant) measure theoretical point of view.

In late 1980s, Baxendale and Stroock [9, 11] studied the dynamics of stochastic differential equations, characterising stationary measures for the two-point motion in different Lyapunov exponent regimes, establishing in particular the existence of an infinite ergodic invariant measure at the zero Lyapunov transition point between synchronisation (stationary probability measure on the diagonal) and chaos (stationary probability measures on and off the diagonal).

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In the early 1990s, Pikovsky [48] and Yu, Ott and Chen [51] studied the transition to chaos in the discrete time setting (random maps), leading to numerical evidence supporting several heuristic conjectures concerning intermittency.

Recently, Homburg and Kalle [32] obtained explicit results about stationary measures for certain random affine iterated function systems on the circle. They also point out the fact that the infinite stationary measure of the two-point motion at the transition corresponds to intermittent dynamics.

This leads to the natural question whether intermittency and associated infinite ergodic stationary measure of the two-point motion are generic features of the transition to chaos in random dynamical systems.

In this paper, we answer this question in the affirmative, in the specific setting of circle endomorphisms of degree two with additive noise. Importantly, we develop techniques to analyse the two-point dynamics near the diagonal, expanding on results developed by Baxendale and Stroock [9, 11] for stochastic differential equations. In particular, we analyse the spectral properties and actions of annealed Koopman operators for one- and two-point motions, allowing us to derive quantitative estimates on various escape times of the quenched process, which are key to our results. In Section 2.2, we present a summary of our techniques.

We anticipate that the techniques introduced in this paper will turn out to be fundamental to settle our question in general.

1.1. **Main results.** Consider a smooth monotone circle endomorphism $T : \mathbb{T} \to \mathbb{T}$ of degree $k > 1$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the circle endowed with the topology induced by the arc-length metric d, and the parameter family of maps defined as

$$T_a(x) := T(x + a \pmod 1).$$

We consider random iteration of maps from this family

$$x_n = T^n_\omega(x_0) := T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} \circ T_{\omega_0}(x_0)$$

where $\omega := (\omega_i)_{i\in\mathbb{N}}$ and ω_i is drawn randomly (i.i.d.) from $[-\vartheta, \vartheta]$ with uniform measure $\mathrm{Leb}/(2\vartheta)$. We denote the corresponding sequence space

$$\Sigma_\vartheta := [-\vartheta, \vartheta]^{\mathbb{N}}.$$

with corresponding product measure $\mathbb{P}$.

Our results require certain mild hypotheses, Hypothesis (H1)-(H5), which we proceed to sketch with reference to Section 2 for details.

We assume that T and ϑ are such that the random dynamical system has a unique stationary measure μ with smooth and everywhere positive density. As a result, the random dynamical system has a single Lyapunov exponent

$$\lambda = \frac{1}{2\vartheta} \int_{\mathbb{T}} \int_{[-\vartheta,\vartheta]} \ln\left(DT_\omega(x)\right)\, d\mathrm{Leb}(\omega) d\mu(x),$$

which can be negative, zero, or positive, depending on T and ϑ. When we consider the case where the random dynamical system depends smoothly on an additional parameter, the Lyapunov exponent also varies smoothly with this parameter (due to our hypotheses).

Then, a transition to chaos corresponds to the Lyapunov exponent traversing zero from below.

We consider the two-point motion to gain insights into the dynamics:

$$(x_n, y_n) = (T_\omega^{(2)})^n (x_0, y_0) \coloneqq (T_\omega^n(x_0), T_\omega^n(y_0)) .$$

Stationary measures for the two-point random dynamical system provide information well beyond stationary measure for (the one-point dynamics of) T_ω. Note that the stationary measure for T_ω is also a stationary measure of the two-point motion, supported on the diagonal

$$\Delta \coloneqq \left\{(x, x) \in \mathbb{T}^2 \, ; \, x \in \mathbb{T}\right\} .$$

We are primarily interested in stationary measures with support outside of Δ, providing detail on the comparison between orbits with different initial conditions. The ε-neighbourhood of the diagonal is denoted as

$$\Delta_\varepsilon \coloneqq \left\{(x, y) \in \mathbb{T}^2 \, ; \, d(x, y) < \varepsilon\right\} .$$

Our main result concerns the following trichotomy in phenomenology from a topological and measure theoretic point of view:

Theorem 1.1 (Topological random dynamics)**.** *Let T_ω be a random dynamical system satisfying hypotheses (H1) to (H5), λ be its Lyapunov exponent, and let $T_\omega^{(2)}$ be the corresponding two-point random dynamical system. Then,*

(1) ***Synchronisation:*** *if $\lambda < 0$, for all $x, y \in \mathbb{T}$,*

$$\lim_{n\to\infty} d\left(T_\omega^n(x), T_\omega^n(y)\right) = 0, \ \mathbb{P} - \text{a.s.}$$

(2) ***Intermittency:*** *if $\lambda = 0$, for all $(x, y) \in \mathbb{T}^2 \setminus \Delta$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(T_\omega^i(x), T_\omega^i(y)\right) = 0 \ \textit{and} \ \limsup_{n\to\infty} d\left(T_\omega^n(x), T_\omega^n(y)\right) > 0, \ \mathbb{P} - \text{a.s.}$$

(3) ***Chaos:*** *if $\lambda > 0$, for all $(x, y) \in \mathbb{T}^2 \setminus \Delta$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(T_\omega^i(x), T_\omega^i(y)\right) > 0 \ \textit{and} \ \liminf_{n\to\infty} d\left(T_\omega^n(x), T_\omega^n(y)\right) = 0, \ \mathbb{P} - \text{a.s.}$$

This result mirrors those in [32], obtained for special random affine circle maps. The most interesting aspect concerns the existence of intermittency in part (2). The existence of synchronisation in the presence of negative Lyapunov exponent has already been well-studied, see e.g. [46], and the properties highlighted under part (3) also align with existing insights. The characterization of intermittency in part (2) is reminiscent of synchronisation on average [29] and that of chaos in part (3) of Li-Yorke chaos [16, 39]. These characterisations are not exhaustive. For instance, recently the positive Lyapunov exponent regime has been associated with the existence of so-called random horse-shoes [35, 36].

The proof of Theorem 1.1, which uses techniques introduced in Sections 2 and 3, is given in Section 4. The results for $\lambda < 0, \lambda = 0$ and $\lambda > 0$ are presented separately in Propositions 4.8, 4.3 and 4.6, respectively.

Theorem 1.2 (Stationary measures)**.** *Let T_ω be as in Theorem 1.1, and let μ denote its unique stationary (probability) measure.*

Then,

(1) *If $\lambda < 0$, μ on Δ is the unique stationary measure for $T_\omega^{(2)}$.*

(2) *If $\lambda = 0$, μ on Δ is the unique stationary probability measure for $T_\omega^{(2)}$. In addition, $T_\omega^{(2)}$ admits an infinite stationary (Radon) measure $\mu^{(2)}$ on $\mathbb{T}^2 \setminus \Delta$, which has full support. Moreover, for each such measure there exist $\alpha, \beta \in (0,\infty)$ such that*

$$\alpha \leq \liminf_{\varepsilon\to 0} \frac{\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon)}{-\ln(\varepsilon)} \leq \limsup_{\varepsilon\to 0} \frac{\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon)}{-\ln(\varepsilon)} \leq \beta.$$

(3) *If $\lambda > 0$, in addition to the unique stationary measure μ on Δ, $T_\omega^{(2)}$ admits a stationary probability measure $\mu^{(2)}$ on $\mathbb{T}^2 \setminus \Delta$, which has full support.*

Moreover, if the non-zero root $\gamma < 0$ of the moment Lyapunov function[1] *associated with T_ω is also larger than $-\frac{1}{2}$, then for each such measure there exists $\alpha, \beta \in (0,\infty)$, such that*

$$\alpha \leq \liminf_{\varepsilon\to 0} \frac{\mu^{(2)}(\Delta_\varepsilon)}{\varepsilon^{-\gamma}} \leq \limsup_{\varepsilon\to 0} \frac{\mu^{(2)}(\Delta_\varepsilon)}{\varepsilon^{-\gamma}} \leq \beta.$$

As in Theorem 1.1, the most interesting aspect of this result is the intermittent case in part (2), where we find that ergodic invariant measures off the diagonal are infinite, together with the estimate on how such measures grow near the diagonal.

The asymptotics of the stationary invariant measure near the diagonal in the positive Lyapunov exponent scenario (in part (3)) relies on the condition that $\gamma \in (-\frac{1}{2}, 0)$. If $\gamma < -\frac{1}{2}$, then the asymptotics may be different due to points being mapped more frequently close to the diagonal, see Proposition 5.5.

The construction of stationary measures off the diagonal for the two-point motion in parts (2) and (3) makes use of an inducing scheme with randomized return times. It should be noted stationary measures constructed in this way do not need to be unique.

The heart of the proof of the above theorem is presented in Section 5, with arguments relying on estimates from Sections 3, 4. The existence of the stationary measure on Δ follows from Proposition 2.1. For $\lambda < 0$, Theorem 1.1 (1) directly implies the unique stationary measure in Theorem 1.2 (1). The results for $\lambda = 0$ and $\lambda > 0$ follow from Proposition 5.1, Proposition 5.4 and Proposition 5.5.

We proceed to illustrate our results in an example.

Example 1.3. We consider a one-parameter family of random circle endomorphism

$$x_n = T^n_{\nu,\omega}(x_0) := T_{\nu,\omega_{n-1}} \circ \cdots \circ T_{\nu,\omega_1} \circ T_{\nu,\omega_0}(x_0)$$

with ω_i drawn i.i.d. from the uniform distribution on a subinterval of the circle $[-\vartheta, \vartheta]$, and

$$T_{\nu,a}(x) := T_\nu(x + a \pmod 1), \tag{1.1}$$

[1] See Eq. (3.1) and Lemma 3.4.

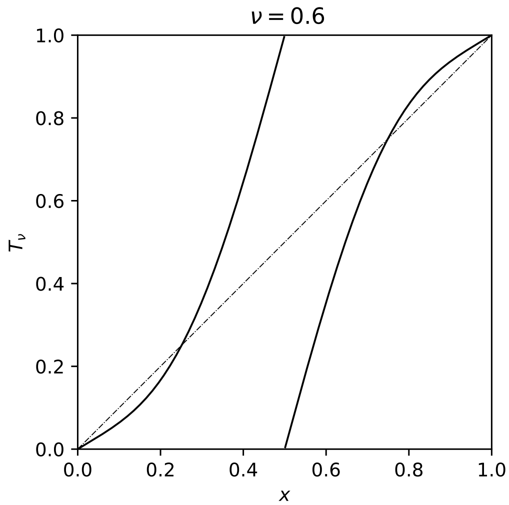


FIGURE 1. The graph of T_ν (1.2) at $\nu = 0.6$.

with

$$T_\nu(x) := \int_0^x \nu + 140(2-\nu)t^3(1-t)^3 dt \pmod 1. \tag{1.2}$$

Note that $T_\nu(0) = 0$ and $DT_\nu(0) = \nu$. In Figure 1 we sketch the graph of T_ν for $\nu = 0.6$.

For parameter ranges discussed in this example $T_{\nu,\omega}$ satisfies the Hypotheses (H1) to (H5), see Example 2.6.

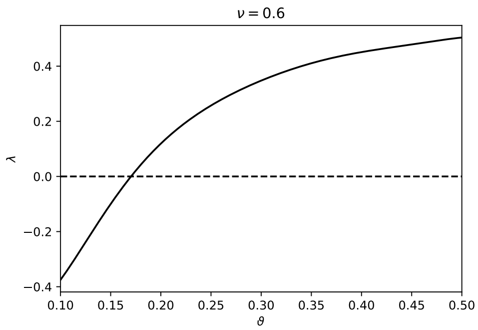
We first consider the behaviour of the Lyapunov exponent at a fixed value of $\nu = 0.6$ with varying ϑ. In Figure 2(a), we observe noise-induced chaos, as the Lyapunov exponent increases monotonically from -0.4 to 0.45 with increasing $\theta \in [0.1, 0.5]$. In this interval, the noise is large enough to ensure the existence of a unique stationary measure for $T_{0.6,\omega}$ on $\mathbb{T}$.

The dynamics of the two-point motions is illustrated by examples of time series for two different values of ϑ in Figure 2(c) ($\theta = 0.17$, close to intermittency) and Figure 2(d) ($\theta = 0.2$, chaos), displaying qualitative differences consistent with Theorem 1.1.

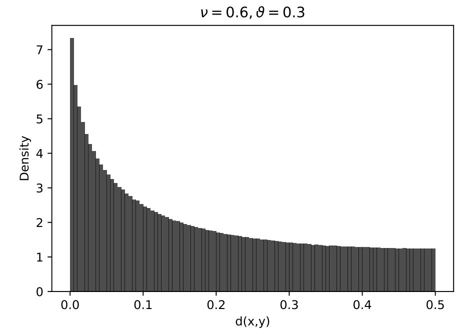
Figure 2(b) shows the distribution of the two-point distance for an example time-series, displaying inverse-power-law behavior near zero, as asserted in Theorem 1.2.

Rather than varying the noise level for fixed parameter ν, one may also consider varying ν at fixed noise level. Fixing the noise level at $\theta = 0.5$ (full noise), yields one-point motion orbits to be random i.i.d. sample from the stationary measure $\mu = (T_\nu)_*(\text{Leb})$, see Figure 3(b).

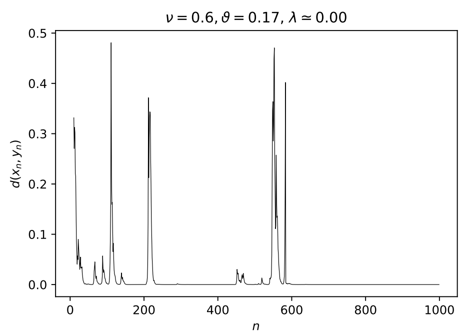
We include this example as it highlights the difference between the one- and two-point motion. In particular, at full noise, the one-point dynamics is essentially a full shift while the dynamics may be synchronising or chaotic. Indeed, varying ν between 0.025 and 0.2 we observe (in Figure 3(a)) a monotonically increasing Lyapunov exponent going from negative (synchronisation) to positive (chaos).



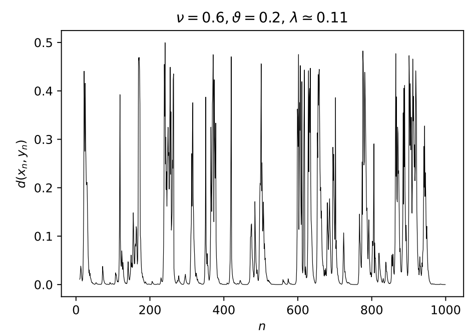
(a) Numerical approximation of the Lyapunov exponent for fixed $\nu = 0.6$ and varying ϑ. The Lyapunov exponent increases monotonically with the noise level, from negative to positive (noise-induced chaos).



(b) Normalized distribution of distance between two orbits (10^6 iterations) with different initial conditions and the same noise realisation, consistent with Theorem 1.2 (3).



(c) Example of a time series of the distance between two orbits for $\vartheta = 0.17$, when $\lambda \approx 0$, close to intermittency.



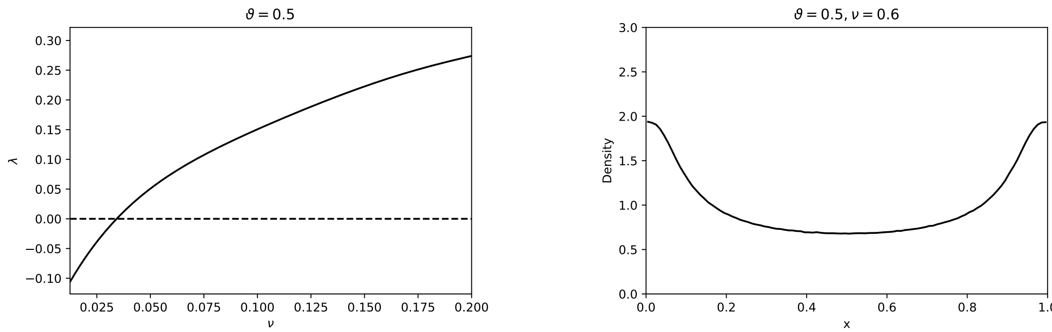
(d) Example of a time series of the distance between two orbits for $\vartheta = 0.2$, when $\lambda \approx 0.11$, in the chaotic regime.

FIGURE 2. Illustration of aspects of the dynamics in Example 1.3 regarding the family of random maps $T_{\nu,\omega}$ with $\nu = 0.6$.

We revisit the full noise case in the context of this example, in Example 3.9 and Example 3.12, where this assumption leads to simplified statements.

1.2. **Context.** In this paper we have established a link between intermittency (of the two-point motion) and the transition to chaos in random dynamical systems. In the deterministic setting, intermittency has been recognized as a hallmark of only one of several routes to chaos [49]. We thus conjecture that the presence of noise "destroys" the others.

Intermittency and infinite ergodic measures are already known to be linked to nonhyperbolic fixed points and zero Lyapunov exponents. For instance, for random dynamical systems on unbounded state spaces, a zero Lyapunov exponent may yield infinite stationary



(a) Numerical approximation of the Lyapunov exponent for fixed $\vartheta = 0.5$ and varying ν. The Lyapunov exponent increases monotonically with ν, from negative to positive.

(b) Numerical approximation of the stationary density, with $\nu = 0.6$ and $\vartheta = 0.5$.

FIGURE 3. Illustration of aspects of the dynamics of Example 1.3 regarding the family of random maps $T_{\nu,\omega}$, with $\vartheta = 0.5$ and varying ν.

measures [5, 18–20, 24]. In random maps on bounded state spaces with common neutral fixed points, research has focused on two cases: invariant neutral fixed points (e.g., in random intermittent maps [6]) and on-average neutral fixed points (e.g., the Pelikan map [47]). Both scenarios exhibit infinite invariant Radon measures on the complement of the fixed point. A number of case studies focus on random logistic maps with zero Lyapunov exponents [1, 4, 28]. For specific results on synchronisation on average (equivalent to our notion of intermittency) in random maps, see also [29, 42].

While the majority of research in ergodic theory of dynamical systems focusses on one-point motions, two-point motions are central to this paper. The importance of two-point motions, or more general n-point motions, has been recognized in the study of stochastic differential equations, see for instance [34], and in particular by Ledrappier & Young [37,38] and Baxendale & Stroock [9, 11] focussing on different Lyapunov exponent regimes. More recently, their importance has been recognized in stochastic fluid dynamics [12–15, 17].

Synchronisation in random dynamical systems constitutes the most elementary type of dynamical behaviour, and has been studied extensively [27,31,43]. In particular, it has been established in the negative Lyapunov exponent regime that synchronisation is equivalent to a mild contractability condition [46]. Specifically, random circle homeomorphism have been discussed in [2, 33, 41, 52].

1.3. **Organization of the paper.** This paper is divided into six sections. In Section 2, we discuss the underlying hypotheses of this paper and provide a summary of the strategy of the proof. In Section 3, the technical machinery involving Koopman operators is developed, which is required in the subsequent sections. Section 4 is dedicated to the proof of Theorem 1.1, where the three different scenarios with zero, positive, and negative Lyapunov exponent are treated, respectively in Subsections 4.1, 4.2, and 4.3. The construction of stationary measures and the derivation of their properties, leading to Theorem 1.2, are contained in Section 5.

2. Setting and strategy of the proof

This section introduces the hypotheses on the system and includes a summary of the strategy of the proof. We start with a circle endomorphism $T : \mathbb{T} \to \mathbb{T}$ with the following property.

> T is a circle endomorphism in $\mathcal{C}^2(\mathbb{T})$, of degree two with derivative positively bounded from above and below; there exist positive real numbers $a_1 < a_2$, such that
> $$a_1 < DT(x) < a_2 \tag{H1}$$
> for all $x \in \mathbb{T}$.

Note that every point has exactly two inverse images under T. Degree two is not essential in this paper, higher degree works the same with minor modifications. Define

$$R_{\min} \coloneqq \max \left\{ r \; ; \; a_1 d(x,y) \le d\left(T(x), T(y)\right) \le a_2 d(x,y) \text{ for all } x, y \in \mathbb{T} \text{ with } d(x,y) \le r \right\}. \tag{2.1}$$

In particular $T(x) \neq T(y)$ if $d(x,y) \le R_{\min}$.

For $\omega \in \mathbb{T}$ write

$$T_\omega(x) = T(x + \omega \pmod 1)$$

for the circle endomorphism obtained by composing T with a translation. We take a random process ω_i, $i \in \mathbb{N}$, where the ω_i are independently drawn from a uniform distribution on

$$\Omega_\vartheta \coloneqq [-\vartheta, \vartheta].$$

This yields a random dynamical system

$$x_{n+1} = T_{\omega_n}(x_n) \tag{2.2}$$

for an initial point $x_0 \in \mathbb{T}$.

Let the sequence space

$$\Sigma_\vartheta \coloneqq \Omega_\vartheta^{\mathbb{N}}$$

be endowed with the product topology. We write $\omega = (\omega_i)_{i \in \mathbb{N}}$ for points in Σ_ϑ. Cylinders are sets $[A_0, A_1 \ldots, A_k] = \{\omega \in \Sigma_\vartheta \; ; \; \omega_i \in A_i, 0 \le i \le k\}$ for Borel sets $A_i \subset \mathbb{T}$. Cylinders are the basis for the product topology. Given the normalized Lebesgue measure $\mathrm{Leb}/(2\vartheta)$ on Ω_ϑ, write $\mathbb{P}$ for the corresponding product measure on Σ_ϑ. For a function X on Σ_ϑ we use common notation such as

$$\mathbb{E}\left[X\right] \coloneqq \int_{\Sigma_\vartheta} X(\omega)\, d\mathbb{P}(\omega).$$

By identifying Ω_ϑ with the cylinder $[\Omega_\vartheta]$ we can also use $\mathbb{P}$ for normalized Lebesgue measure $\mathrm{Leb}/(2\vartheta)$ on Ω_ϑ. So, with a slight abuse of notation, if a function $\omega \mapsto X(\omega)$ depends on a single symbol $\omega \in \Omega_\vartheta$, we write $\mathbb{E}[X] = \int_{\Omega_\vartheta} X(\omega)\, d\mathbb{P}(\omega)$. Recall that a measure μ on $\mathbb{T}$ is called a stationary measure if

$$\int_{\Sigma_\vartheta} \mu\left((T_\omega)^{-1}(A)\right) d\mathbb{P}(\omega) = \mu(A),$$

for any (Borel) measurable set $A \subset \mathbb{T}$.

The skew product map $\Theta : \Sigma_\vartheta \times \mathbb{T} \to \Sigma_\vartheta \times \mathbb{T}$ is defined by

$$\Theta(\omega, x) \coloneqq (\sigma\omega, T_{\omega_0}(x)).$$

Here σ is the left shift operator $\sigma\omega \coloneqq (\omega_{i+1})_{i\in\mathbb{N}}$. With a slight abuse of notation we write

$$T_\omega^n(x) \coloneqq T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0}(x)$$

for iterates.

We compares two different trajectories by studying the random dynamical system. For $\omega \in \Sigma_\vartheta$, the two-point map $(x, y) \mapsto T_\omega^{(2)}(x, y)$ on $\mathbb{T}^2$ is the product

$$(x, y) \mapsto (T_\omega(x), T_\omega(y)).$$

This yields the random dynamical system

$$(x_{n+1}, y_{n+1}) = T_{\omega_n}^{(2)}(x_n, y_n). \tag{2.3}$$

The two-point skew product map $\Theta^{(2)} : \Sigma_\vartheta \times \mathbb{T}^2 \to \Sigma_\vartheta \times \mathbb{T}^2$ is denoted by

$$\Theta^{(2)}(\omega, x, y) = (\sigma\omega, T_\omega^{(2)}(x, y)).$$

A measure $\mu^{(2)}$ on $\mathbb{T}^2$ is a stationary measure of the random dynamical system $T_\omega^{(2)}$ on $\mathbb{T}^2$ if

$$\int_{\Sigma_\vartheta} \mu^{(2)}\left(\left(T_\omega^{(2)}\right)^{-1}(A)\right) d\mathbb{P}(\omega) = \mu^{(2)}(A),$$

for any (Borel) measurable set $A \subset \mathbb{T}^2$.

2.1. **Hypotheses.** We focus on random circle endomorphisms whose trajectories are not confined to subintervals of the circle but spread over the entire circle.

There is $k > 0$ so that for any $x, y \in \mathbb{T}$, there is $\omega \in \Sigma_\vartheta$ so that $T_\omega^k(x) = y$. **(H2)**

This hypothesis guarantees the existance of a unique absolutely continuous stationary measure of full support, but also has further applications that are used throughout the paper.

Proposition 2.1. *Suppose the random dynamical system described by (2.2) with ω_n i.i.d. picked from a uniform distribution for $[-\vartheta, \vartheta]$, adheres to Hypotheses (H1), (H2).*

Then the random dynamical system admits an absolutely continuous stationary measure μ with full support and smooth density.

The proof is omitted and be obtained from standard arguments using Perron-Frobenius operators. A similar setup is in [7], to which we refer for details.

The following assumption provides a local source for contraction. This is needed to achieve a negative or zero Lyapunov exponent.

The map T has a hyperbolic attracting periodic orbit. **(H3)**

We must avoid that points in $\mathbb{T}^2 \setminus \Delta$ are mapped into Δ by $T_a^{(2)}$ with positive probability, and more generally we must control and bound the probability that points are mapped very close to the diagonal by the two-point maps $T_a^{(2)}$. The following hypothesis provides us with an assumption to prevent this.

For every $(x, y) \in \mathbb{T}^2 \setminus \Delta$, the curve $a \mapsto (T_a(x), T_a(y))$ in $\mathbb{T}^2$ intersects the diagonal Δ transversely or with at most a single quadratic tangency. **(H4)**

Hypothesis (H4) is a generic condition.

Lemma 2.2. *There is an open and dense set of degree two circle endomorphisms $T : \mathbb{T} \to \mathbb{T}$ in the $\mathcal{C}^2$ topology so that the corresponding random system satisfies Hypothesis (H4).*

Proof. In the argument to prove denseness we take the random parameter a in T_a from the whole circle $\mathbb{T}$; taking a from a larger interval only shrinks the set of circle endomorphisms that satisfy Hypothesis (H4).

Let $w(x, y)$ denote the signed distance of $x, y \in \mathbb{T}$, The hypothesis can be rephrased as stating that the graph of the function $a \mapsto w(T_a^{(2)}(x, y))$ has only transverse intersections and at most one quadratic tangency to the graph of the zero function.

Present T as an interval map on $[0, 1]$ with two monotone branches $\psi_1 : I_1 \to [0, 1]$ and $\psi_2 : I_2 \to [0, 1]$ on intervals $I_1 = [0, d]$ and $I_2 = [d, 1]$, where both maps are surjective. For $x \in I_1$, there is a unique point $y \in I_2$ with $T(x) = T(y)$. For $R < R_{\min}$, $d(T(x), T(y)) \neq 0$ for all (x, y) with $0 < d(x, y) < R$.

By a small perturbation of T near the boundary points of I_1, we get that $DT(0) \neq DT(d)$. Write $\chi_1 = \psi_1^{-1}$ and $\chi_2 = \psi_2^{-1}$, both defined on $[0, 1]$. Note that $z = \psi_1(x) = \psi_2(y)$ means $\chi_2(z) - \chi_1(z) = y - x$. Using $x \neq y$, note that $T_a(x) = T_a(y)$ or equivalently $T(x + a) = T(y + a)$ can only occur if

$$\chi_2(z) - \chi_1(z) = y - x$$

for $z = T_a(x) = T_a(y)$. Also, a transverse intersection of $a \mapsto w(T_a^{(2)}(x, y))$ with zero occurs if $D\chi_2(z) \neq D\chi_1(z)$, a tangency if $D\chi_2(z) = D\chi_1(z)$. Such a tangency is quadratic if $D^2\chi_2(z) \neq D^2\chi_1(z)$.

A small perturbation of χ_2 (and thus of T) will ensure that the critical points of $z \mapsto \chi_2(z) - \chi_1(z)$ are quadratic (have a nonzero second order derivative) and the critical values are different. So $z \mapsto \chi_2(z) - \chi_1(z) - (y - x)$ is either transverse or at most quadratically tangent to 0, and there is at most one tangency for fixed (x, y). Thus the graph of the function $a \mapsto w(T_a^{(2)}(x,y))$ has only transverse intersections and quadratic tangencies to the graph of the zero function, with at most one such tangency. As (x,y) can be taken from the compact region $\mathbb{T}^2 \setminus \Delta_R$, the numbers of transverse intersections and of tangencies are bounded uniformly in (x,y) from $\mathbb{T}^2 \setminus \Delta_\varepsilon$. For the same reason, there is a uniform lower bound on the second order derivatives at tangencies. Openness is now clear. □

The integrability statement in the following lemma is a consequence of Hypothesis (H4), and results in an estimate that is required in later arguments (Proposition 4.6, Proposition 5.4 and Proposition 5.5).

Lemma 2.3. *For all $R > 0$ there exists $C_1 > 0$ with*

$$\mathbb{E}\left[-\ln(d(T_\omega^{(2)}(x,y)))\right] < C_1, \tag{2.4}$$

for all $(x,y) \in \mathbb{T}^2 \setminus \Delta_R$.

Proof. As earlier we let $w(x,y)$ denote the signed distance of $x, y \in \mathbb{T}$. Consider the real valued function

$$a \mapsto f(a) \coloneqq w\left(T_a^{(2)}(x,y)\right)$$

on $\mathbb{T}$. Hypothesis (H4) implies that (see also the proof of Lemma 2.2) for $(x,y) \in \mathbb{T}^2 \setminus \Delta_\delta$, there exist $\delta > 0, C > 0$ so that

$$\left|D^2 f(a)\right| \geq C$$

whenever $|f(a)| < \delta$ and $|Df(a)| < \delta$.

If $I \subset \mathbb{T}$ is an interval so that for $a \in I$, $|f(a)| < \delta$ and $|Df(a)| \geq \delta$, then $\int_I -\ln(|f(a)|)\, da$ is bounded by some $C_\delta > 0$.

To prove the lemma it remains to consider f on an interval $J \subset \mathbb{T}$ so that $|f(a)| < 2\delta$ for $a \in J$ and J contains a point $d \in J$ with $Df(d) = 0$. If $f(d) = 0$ and $Df(d) = 0$, we have a bound $0 < c(a-d)^2 < |f(a)|$ for some $c > 0$. The same bound applies if $Df(d) = 0$ and f is never zero on J. The possibility that is left is where f has two zeros $f(d_1) = f(d_2) = 0$ for nearby points d_1, d_2 on different sides of d. Then a bound $0 < c|(a-d_1)(a-d_2)| < |f(a)|$ for some $c > 0$ holds. In all cases c is uniformly bounded away from 0. As the logarithms of $c(a-d)^2$ and $c|(a-d_1)(a-d_2)|$ are integrable, we find that $\int_J -\ln(|f(a)|)\, da$ is bounded by some $C_\delta > 0$. Noting that $-\ln(|f(a)|)$ is bounded if $|f(a)| > \delta$, proves the lemma. □

We make a final assumption that will be used to prove full support of stationary measures for the two-point motion. The random two-point system maps a point (x,y) to a curve $a \mapsto (T_a(x), T_a(y))$ in $\mathbb{T}^2$. For the composition of two iterates, we have

$$\begin{aligned} H_{a_0,a_1}(x,y) &\coloneqq \det\left(\frac{\partial}{\partial(a_0,a_1)} T_{a_1}^{(2)} \circ T_{a_0}^{(2)}(x,y)\right) \\ &= DT_{a_1}(T_{a_0}(x)) DT_{a_1}(T_{a_0}(y)) \left(DT_{a_0}(x) - DT_{a_0}(y)\right). \end{aligned} \tag{2.5}$$

Consider now the following hypothesis.

For each $(x,y) \in \mathbb{T}^2 \setminus \Delta$, there are a_0, a_1 so that $H_{a_0,a_1}(x,y) \neq 0$. **(H5)**

This condition implies that for each $(x, y) \in \mathbb{T}^2 \setminus \Delta$, the image of

$$(a_0, a_1) \mapsto T^{(2)}_{a_1} \circ T^{(2)}_{a_0}(x, y),$$

with $a_0, a_1 \in [-\vartheta, \vartheta]$, contains an open set.

Recall that a map $F : X \to X$ on a topological space X is called topologically exact if for any open $U \subset X$, there is $n \in \mathbb{N}$ so that $F^n(U) = X$.

Lemma 2.4. *The skew product systems $\Theta : \Sigma_\vartheta \times \mathbb{T} \to \Sigma_\vartheta \times \mathbb{T}$ and $\Theta^{(2)} : \Sigma_\vartheta \times \mathbb{T}^2 \to \Sigma_\vartheta \times \mathbb{T}^2$ are topologically exact.*

Proof. We first consider Θ and we start with the following statement: for any open interval $I \subset \mathbb{T}$ there is $\omega \in \Sigma_\vartheta$ and $k \in \mathbb{N}$ so that $T^k_\omega(I) = \mathbb{T}$. The reasoning will also show that there is a cylinder $D \subset \Sigma_\vartheta$ that contains ω, so that $T^k_\zeta(I) = \mathbb{T}$ for any $\zeta \in D$.

Given the interval I, take $x \in I$. Let $y \in \mathbb{T}$ be a periodic point in an uncountable transitive hyperbolic repelling set Λ of a map T_a [44]. Write k for the period of y. Given an open neighborhood V of y, there is an iterate T^{kn}_a that maps V onto all of $\mathbb{T}$ (if $\cup_n T^{kn}_a(V)$ is an interval, then T^k_a is a diffeomorphism on it, but it also admits an uncountable hyperbolic repelling set, which is not possible). By Hypothesis (H2) there is $\varsigma \in \Sigma_\vartheta$ and $m \in \mathbb{N}$ so that $T^m_\varsigma(x) = y$. Hence there is $n \in \mathbb{N}$ so that $T^{kn}_a \circ T^m_\varsigma$ maps I onto $\mathbb{T}$. By continuity the same applies to T^{m+kn}_ν for ν in a suitable cylinder D that contains $\varsigma_0 \cdots \varsigma_{m-1} a \cdots a$ (with nk times a). This proves the statement with which we started.

To conclude the proof of topological exactness of Θ, let $U \subset \Sigma_\vartheta \times \mathbb{T}$ be an open set. By shrinking U we may assume that U is a product set $U = C \times J$ of a cylinder C and an open interval J. There is an iterate $\Theta^j(U)$ that contains an open set $\Sigma_\vartheta \times I$ for an open interval $I \subset \mathbb{T}$. Now use the above statement to establish that a further iterate covers $\Sigma_\vartheta \times \mathbb{T}$.

Next we consider $\Theta^{(2)}$. Take an open set $U \subset \Sigma_\vartheta \times \mathbb{T}^2$. By shrinking U we may assume that U is a product set $U = C \times I \times J$ of a cylinder C and open intervals I, J.

By the above reasoning, there is $l \in \mathbb{N}$ and $\omega \in C$ so that $\left(T^{(2)}_\omega\right)^l (I \times J)$ contains $\mathbb{T} \times K$ for an open interval $K \subset \mathbb{T}$. Applying the above reasoning again, there is $l \in \mathbb{N}$ so that $\left(T^{(2)}_\omega\right)^l (I \times J)$ equals $\mathbb{T}^2$. The proof of topological exactness of $\Theta^{(2)}$ is concluded as above. □

Hypotheses (H3), (H5) and Lemma 2.4 show that $\mathbb{P}$-almost surely all orbits of the two point motion get arbitrarily close to the diagonal Δ. The following lemma formalizes this.

Lemma 2.5. *For all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$, $C > 0$ such that for all $(x, y) \in \mathbb{T}^2$, we have*

$$\mathbb{P}\left(\left\{\omega \in \Sigma_\vartheta \; ; \; d(T^i_\omega(x), T^i_\omega(y)) < \varepsilon \text{ for some } 0 \le i \le N\right\}\right) > C.$$

Proof. Take $(x, y) \in \mathbb{T}^2 \setminus \Delta_\varepsilon$. By Hypothesis (H5) we find that $\omega \mapsto \left(T^{(2)}_\omega\right)^2 (x, y)$ contains an open product set $I_x \times I_y \subset \mathbb{T}^2$. By compactness we find $\delta > 0$ so that I_x and I_y have diameter at least δ, uniformly in $x, y \in \mathbb{T}^2 \setminus \Delta_\varepsilon$. The argument of Lemma 2.4 shows that for any $(x, y) \in \mathbb{T}^2 \setminus \Delta_\varepsilon$ there is an open neighborhood U of (x, y) and a positive integer

N so that $\left(T_\omega^{(2)}\right)^N(x',y')$ covers $\mathbb{T}^2$ for any $(x',y') \in U$. By compactness N is bounded uniformly in $(x,y) \in \mathbb{T}^2 \setminus \Delta_\varepsilon$. This implies the lemma. □

Example 2.6. We continue from Example 1.3, giving concise justifications that the required hypotheses hold. Hypotheses (H1) and (H3) are trivially satisfied. Hypothesis (H2) can be checked via a simulation or some naive bounds.

Regarding Hypothesis (H4), observe that there is exactly one point $y \in \mathbb{T}$ that has two preimages $x_1, x_2 \in \mathbb{T}$ with

$$DT(x_1) = DT(x_2).$$

At this point y, the second derivatives $DT(x_1)$ and $DT(x_2)$ have opposite signs. The curvature of the curve $a \mapsto (T_a(x_1), T_a(x_2))$ at the point of tangency with Δ is therefore nonzero, and the tangency is quadratic.

Finally, by Equation (2.5), and the fact that for all $(x,y) \in \mathbb{T}^2 \setminus \Delta$, there exists an $a_0 \in [-\vartheta, \vartheta]$ such that $DT_{a_0}(x) - DT_{a_0}(y) \neq 0$, Hypothesis (H5) follows. ■

2.2. **Strategy of the proof.** This section gives a helicopter view on the reasoning in Sections 3 and 4 that leads to Theorem 1.1 on dynamics of the random two-point maps. For the study of orbits of the random two-point maps it is crucial to understand the duration of trajectories of two nearby points staying close to each other. In terms of the two-point motion this is the duration of trajectories of the two-point motion staying close to the diagonal. Suppose $x_0, y_0 \in \mathbb{T}$ are close, so that $d_0 = d(x_0, y_0)$ is close to zero. With $x_n = T_\omega^n(x_0)$, $y_n = T_\omega^n(y_0)$ and $d_n = d(x_n, y_n)$, we find that as long as d_n is small,

$$d_{n+1} \approx DT_{\omega_n}(x_n) d_n. \tag{2.6}$$

Taking logarithms $u_n = \ln(d_n)$ this reads

$$u_{n+1} \approx u_n + \ln\left(DT_{\omega_n}(x_n)\right), \tag{2.7}$$

which is a random walk driven by the one-point motion $x_{n+1} = T_{\omega_n}(x_n)$.

In the special case where x_n is identically and independently distributed (see Example 1.3, with $\vartheta = 0.5$), the approximation is a random walk with i.i.d. steps. But this is not true in general. We derive estimates for the duration of passages near the diagonal by adapting reasoning in [9, 11] for continuous time settings to our discrete time setting. We base our analysis on properties of the Koopman operator for the two-point system.

For the one-point motion, the annealed Koopman operator P acting on a real valued function ϕ on $\mathbb{T}$ is defined as

$$P\phi(x) = \mathbb{E}\left[\phi(T_\omega(x))\right].$$

Here $\mathbb{E}$ stands for an expectation over $\omega \in \Sigma_\vartheta$. Analogously the annealed two-point Koopman operator $P_{(2)}$ acting on a real valued function ϕ on $\mathbb{T}^2 \setminus \Delta$ is defined as

$$P_{(2)}\phi(x,y) = \mathbb{E}\left[\phi(T_\omega^{(2)}(x,y))\right].$$

To estimate stopping times for trajectories of the two-point motion from strips $\Delta_\delta \setminus \Delta_\varepsilon$ near the diagonal (so with $0 < \varepsilon < \delta$ small) we construct sub- and supermartingales for the stopped dynamics. The key statements are Proposition 3.14 and Proposition 3.16 below. The constructions rely on an analysis of $P_{(2)}$, which in turn is facilitated by the approximations (2.6) and (2.7) and a study of the corresponding linearized Koopman operator, defined for continuous real valued functions ϕ on $\mathbb{T} \times \mathbb{R}^+$ by

$$TP\phi(x,u) = \mathbb{E}\left[\phi(T_\omega(x), DT_\omega(x)u)\right].$$

Formulas for the mentioned sub- and supermartingales are obtained from a study of the twisted Koopman operator P_q whose action on a real valued function ψ on $\mathbb{T}$ is defined as

$$P_q\psi(x) = \mathbb{E}\left[\frac{\psi\left(T_\omega(x)\right)}{(DT_\omega(x))^q}\right].$$

This in turn connects to the moment Lyapunov exponent

$$\Lambda(q) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{E}\left[DT_\omega^n(x)^q\right]\right).$$

The moment Lyapunov exponent plays a central role in our analysis, similar to [9, 11]. Originally introduced by Molchanov [45] and subsequently developed in particular by Arnold [3], it can be viewed as a generalization of the Lyapunov exponent [21]. The relationship between the moment Lyapunov exponent and the Lyapunov exponent was first described in [22].

The above comments on methodology refer to a study of the two-point motion near the diagonal. Our setting of endomorphisms, instead of diffeomorphisms, brings the effect that points away from the diagonal may be mapped directly onto the diagonal by an application of the two-point map. Hypothesis (H4) controls the probability with which points are mapped onto or near the diagonal by the two-point maps.

3. Koopman operators

This section develops theory of Koopman operators needed for our analysis on random dynamics. The contents of this section are crucial for the arguments in the following sections, although the statements in the lemmas and propositions in the following sections can be read without reference to this section.

Below we define in particular the annealed Koopman operator and the linearized Koopman operator for the random one-point maps, and the annealed Koopman operator for the random two-point maps. We consider the twisted Koopman operator and the moment Lyapunov exponent and use it to obtain eigenfunctions for the linearized Koopman operator. The approximation of the two-point random maps applied to nearby points by the linearized random map allows us to obtain, from these eigenfunctions, functions on which the two-point Koopman operator acts in a desired way: these functions appear in the construction of sub- and supermartingales for the random two-point maps considered near the diagonal.

3.1. **Twisted Koopman operator and moment Lyapunov function.** Write $\mathcal{C}(\mathbb{T},\mathbb{R})$ for the space of real valued continuous functions on $\mathbb{T}$. Denote the $\mathcal{C}^0$-norm as $\|\cdot\|$. The *(annealed) Koopman operator* $P : \mathcal{C}(\mathbb{T},\mathbb{R}) \to \mathcal{C}(\mathbb{T},\mathbb{R})$ is defined as

$$P\phi(x) \coloneqq \mathbb{E}\left[\phi(T_\omega(x))\right].$$

The *twisted Koopman operator*, $P_q : \mathcal{C}(\mathbb{T},\mathbb{R}) \to \mathcal{C}(\mathbb{T},\mathbb{R})$ with $q \in \mathbb{R}$, is defined for continuous functions $\psi : \mathbb{T} \to \mathbb{R}$ by

$$P_q\psi(x) := \mathbb{E}\left[\frac{\psi\left(T_\omega(x)\right)}{(DT_\omega(x))^q}\right].$$

Note that $P = P_0$.

The following lemma establishes basic properties of P_q. Similar statements in other settings are in [3, 30].

Lemma 3.1. *For $q \in \mathbb{R}$, the operator P_q on $\mathcal{C}(\mathbb{T},\mathbb{R})$ is positive, compact and irreducible.*

Proof. It is clear that $P_q\psi \geq 0$ if $\psi \geq 0$, that is, that P_q is positive. Hypothesis (H2) yields the following property. There exists $n \in \mathbb{N}$, so that for any $\psi \in \mathcal{C}(\mathbb{T},\mathbb{R})$ with $\psi \geq 0$ and $\psi \neq 0$, $P_q^n\psi > 0$ everywhere. This means that P_q is irreducible.

Let $\mathcal{B}$ be the unit ball in $\mathcal{C}(\mathbb{T},\mathbb{R})$. For $\psi \in \mathcal{B}$, note that (for readability skipping the (mod 1) from the arguments)

$$P_q\psi(x) = \frac{1}{2\vartheta}\int_{-\vartheta}^{\vartheta} DT(x+\omega)^{-q}\psi(T(x+\omega))\,d\omega = \frac{1}{2\vartheta}\int_{x-\vartheta}^{x+\vartheta} DT(y)^{-q}\psi(T(y))\,dy$$

is a continuously differentiable function of $x \in \mathbb{T}$. As $\|P_q\psi\| \leq C$ for a constant C (for $q > 0$ we can take $C = a_2^q/\vartheta$, for $q < 0$ we can take $C = a_1^q/\vartheta$ with a_1, a_2 from Hypothesis (H1)), P_q is a bounded operator. Now

$$\begin{aligned}|DP_q\psi(x)| &= \frac{1}{2\vartheta}\left|DT(x+\vartheta)^{-q}\psi(T(x+\vartheta)) - DT(x-\vartheta)^{-q}\psi(T(x-\vartheta))\right| \\ &\leq C\|\psi\|\end{aligned}$$

for the same constant C. It follows that $P_q\psi$ for $\psi \in \mathcal{B}$ is an equicontinuous family of functions. By the Arzela–Ascoli theorem we get that P_q is a compact operator. □

Using this lemma we get the following results on the dominant eigenvalue and corresponding eigenvector of P_q.

We make use of the qth moment Lyapunov exponent $\Lambda(q)$ (see [3, 10, 30]) defined as

$$\Lambda(q) := \lim_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{E}\left[DT_\omega^n(x)^q\right]\right). \tag{3.1}$$

We also write moment Lyapunov function, especially when discussing its dependence on q. We will find that the limit does not depend on x, and that it exists as an analytic function of q. Denote $\mathbb{1}_\mathbb{T} : \mathbb{T} \to \mathbb{R}$ as the constant function equal to 1 on the circle.

Proposition 3.2. *For $q \in \mathbb{R}$, P_q has a dominant simple eigenvalue $e^{\Lambda(-q)}$. The rest of the spectrum of P_q is contained in a disk of radius less then $e^{\Lambda(-q)}$. Write $\phi_q \in \mathcal{C}(\mathbb{T},\mathbb{R})$ for the dominant eigenfunction of P_q, so*

$$P_q\phi_q = e^{\Lambda(-q)}\phi_q. \tag{3.2}$$

Then

(1) $\phi_0 = \mathbb{1}_\mathbb{T}$ *and* $e^{\Lambda(0)} = 1$,
(2) ϕ_q *is a positive function,*

(3) $\Lambda(q)$ *and* ϕ_q *depend analytically on* q*,*
(4) Λ *is convex,*
(5) $\Lambda(q) \geq \lambda q$*.*

Proof. The proof follows ideas as in [3, 30]. The spectral properties of P_q follow from Lemma 3.1 and the Krein-Rutman theorem (see for instance [23, Section 19.5]). Write $r(q) = \sigma(P_q)$ for the spectral radius of P_q. By the Krein-Rutman theorem, this equals the dominant eigenvalue of P_q. We have

$$P_q^n \mathbb{1}_{\mathbb{T}}(x) = \langle k_q, \mathbb{1}_{\mathbb{T}} \rangle r(q)^n \phi_q(x) + o(r(q)^n),$$

as $n \to \infty$, uniformly in x, where $k_q \in \mathcal{C}(\mathbb{T}, \mathbb{R})^*$ is a probability measure, see [3]. Now we rescale ϕ_q, such that $\langle k_q, \mathbb{1}_{\mathbb{T}} \rangle = 1$ and using this, calculate

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{E}\left[(DT_\omega^n(x))^{-q}\right]\right) &= \lim_{n\to\infty} \frac{1}{n} \ln\left(P_q^n \mathbb{1}_{\mathbb{T}}(x)\right) \\
&= \lim_{n\to\infty} \ln\left(\left(P_q^n \mathbb{1}_{\mathbb{T}}(x)\right)^{1/n}\right) \\
&= \lim_{n\to\infty} \ln\left(\left(r(q)^n \phi_q(x) + o(r(q)^n)\right)^{1/n}\right) \\
&= \ln(r(q)).
\end{aligned}$$

We find that $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{E}\left[(DT_\omega^n(x))^{-q}\right]\right)$ does not depend on x. The remaining properties of $\Lambda(-q)$ and ϕ_q follow from [3, 30]. $\square$

The following lemma connects the moment Lyapunov exponent and the Lyapunov exponent.

Lemma 3.3. *The first derivative of the moment Lyapunov function at* $q = 0$ *is equal to the Lyapunov exponent*

$$\Lambda'(0) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\ln\left(DT_\omega^n(x)\right)\right] = \lambda. \tag{3.3}$$

Proof. This follows from the fact that $\Lambda(0) = 0$, $\Lambda(q) \geq \lambda q$ and the analyticity of Λ. For the complete argument, see [3, 30]. $\square$

We have established that Λ is a convex function that vanishes at 0. The following lemma shows the existence of a second zero if $\lambda \neq 0$. This second zero plays a prominent role in our analysis and also appears in the statements of the main theorem on stationary measures.

Lemma 3.4. *If* $\lambda \neq 0$ *then there is a unique* $\gamma \neq 0$*, with opposite sign, such that* $\Lambda(\gamma) = 0$*. (We set* $\gamma = 0$ *if* $\lambda = 0$*.)*

Proof. Assume that $\lambda > 0$, which implies by (3.3) that $\Lambda'(0) > 0$. By Hypothesis (H3) there is a contracting periodic point x_c of T. Let k_c denote its period. By continuity of map, there exist $\varepsilon > 0$ and $\delta > 0$ such that the set

$$\mathcal{H}^c = \left\{\omega \in \Sigma_\vartheta \; ; \; x \in B_\varepsilon(x_c) \text{ implies } T_\omega^i(x) \in B_\varepsilon(T^i(x_c)), 1 \leq i \leq k_c, \; \ln(DT_\omega^{k_c}(x)) < -\delta\right\}$$

has positive measure $\mathbb{P}(\mathcal{H}^c) > 0$.

By Hypothesis (H2), we know that for any $x \in \mathbb{T}$, there is a positive probability of reaching $B_\varepsilon(x_c)$ in k steps. For any $m = k + nk_c$ sufficiently large, we have

$$\mathbb{E}\left[(DT_\omega^m(x))^{-q}\right] \geq \mathbb{P}(x_k \in B_\varepsilon(x_c))\, e^{n\delta q}\, \mathbb{P}(\mathcal{H}^c)^n.$$

Taking the logarithm and applying the limit $\lim_{n\to\infty} \frac{\ln(\cdot)}{n}$, it follows that $\Lambda(q) \to \infty$ as $q \to -\infty$.

By the continuity of $\Lambda(q)$ and the fact that $\Lambda'(0) > 0$, there must exist a unique $\gamma < 0$ such that $\Lambda(\gamma) = 0$. The argument in the case where $\lambda < 0$ is analogous, using an expanding periodic orbit instead of the contracting periodic orbit. The existence of an expanding periodic orbit follows from [44]. □

We write $\partial_q \phi_q$ for the derivative of ϕ_q with respect to q, and likewise $\partial_q^2 \phi_q$ for the second order derivative with respect to q. Recall from (3.3) that $\lambda = \Lambda'(0)$ and let

$$V \coloneqq \Lambda''(0).$$

Lemma 3.5. *We have the following two equalities for $x \in \mathbb{T}$:*

$$\mathbb{E}\left[\ln(DT_\omega(x))\right] - \lambda = (P_0 - I)\, \partial_q \phi_0(x) \tag{3.4}$$

and, if $\lambda = 0$,

$$V = (P_0 - I)\, \partial_q^2 \phi_0(x) - 2\mathbb{E}\left[\partial_q \phi_0(T_\omega(x)) \ln(DT_\omega(x))\right] + \mathbb{E}\left[\ln^2(DT_\omega(x))\right]. \tag{3.5}$$

Proof. Differentiating (3.2) once with respect to q, yields for $x \in \mathbb{T}$,

$$\begin{aligned} e^{\Lambda(-q)} \partial_q \phi_q(x) - e^{\Lambda(-q)} \Lambda'(-q) \phi_q(x) &= \mathbb{E}\left[\partial_q \left(\frac{\phi_q\left(T_\omega(x)\right)}{DT_\omega(x)^q}\right)\right] \\ &= (P_q \partial_q \phi_q)(x) - \mathbb{E}\left[\frac{\phi_q\left(T_\omega(x)\right) \ln(DT_\omega(x))}{DT_\omega(x)^q}\right]. \end{aligned} \tag{3.6}$$

Equation (3.6) evaluated at $q = 0$ yields

$$\partial_q \phi_0(x) - \Lambda'(0) = (P_0 \partial_q \phi_0)(x) - \mathbb{E}\left[\ln(DT_\omega(x))\right].$$

Rewriting this proves (3.4).

Next, differentiating (3.6) with respect to q yields

$$\begin{aligned} &e^{\Lambda(-q)} \left(\partial_q^2 \phi_q - 2\Lambda'(-q) \partial_q \phi_q + \left(\Lambda'(-q)^2 + \Lambda''(-q)\right) \phi_q\right)(x) \\ &\quad = P_q \partial_q^2 \phi_q(x) - 2\mathbb{E}\left[\frac{\partial_q \phi_q\left(T_\omega(x)\right) \ln(DT_\omega(x))}{DT_\omega(x)^q}\right] + \mathbb{E}\left[\frac{\phi_q\left(T_\omega(x)\right) \ln^2(DT_\omega(x))}{DT_\omega(x)^q}\right]. \end{aligned} \tag{3.7}$$

Evaluating (3.7) for $q = 0$ yields

$$\begin{aligned} &- 2\Lambda'(0) \partial_q \phi_0(x) + \Lambda'(0)^2 + \Lambda''(0) \\ &\quad = (P_0 - I)\, \partial_q^2 \phi_0(x) - 2\mathbb{E}\left[\partial_q \phi_0\left(T_\omega(x)\right) \ln(DT_\omega(x))\right] + \mathbb{E}\left[\ln^2(DT_\omega(x))\right]. \end{aligned} \tag{3.8}$$

Rewriting, and plugging in $\lambda = 0$, we obtain (3.5). □

It is immediate from the convexity of Λ that $V \geq 0$. Our set-up implies that in fact $V > 0$, which we require in our analysis of the case with a zero Lyapunov exponent λ (see Lemma 4.1 below).

Lemma 3.6. *We have $V > 0$.*

Proof. Following [10] we establish that $V = 0$ implies $\Lambda(q) = \lambda q$, which will lead to a contradiction.

Rewriting (3.8), we obtain

$$\begin{aligned} V + (P_0 - I)\left((\partial_q\phi_0)^2 - \partial_q^2\phi_0\right)(x) \\ &= \mathbb{E}\left[(\partial_q\phi_0(T_\omega(x)) - \ln(DT_\omega(x)))^2\right] - (\partial_q\phi_0(x) - \lambda)^2 \\ &= \mathbb{E}\left[(\partial_q\phi_0(T_\omega(x)) - \ln(DT_\omega(x)))^2\right] - \left(\mathbb{E}\left[\partial_q\phi_0(T_\omega(x)) - \ln(DT_\omega(x))\right]\right)^2. \end{aligned} \quad (3.9)$$

The last step uses (3.4). We conclude that

$$V + (P_0 - I)\left((\partial_q\phi_0)^2 - \partial_q^2\phi_0\right)(x) \geq 0.$$

Now suppose $V = 0$, for the sake of contradiction. From $(P_0 - I)\left((\partial_q\phi_0)^2 - \partial_q^2\phi_0\right)(x) \geq 0$ we conclude that $(\partial_q\phi_0)^2 - \partial_q^2\phi_0$ is constant (to see this, note that if a continuous function $\chi : \mathbb{T} \to \mathbb{R}$ takes a maximum at x, then the property $\mathbb{E}\left[\chi(T_\omega(x))\right] \geq \chi(x)$ with Hypothesis (H2) implies that χ is maximal at every point of $\mathbb{T}$). So in fact

$$(P_0 - I)\left((\partial_q\phi_0)^2 - \partial_q^2\phi_0\right)(x) = 0$$

and (3.9) implies that

$$\partial_q\phi_0(T_\omega(x)) - \ln(DT_\omega(x)) = \partial_q\phi_0(x) - \lambda$$

for all x and ω. Then also

$$\partial_q\phi_0(T_\omega^n(x)) - \ln(DT_\omega^n(x)) = \partial_q\phi_0(x) - n\lambda.$$

As $\Lambda(q) = \lim_{n\to\infty} \frac{1}{n}\ln\left(\mathbb{E}\left[(DT_\omega^n(x))^q\right]\right)$ we find $\Lambda(q) = q\lambda$. Lemma 3.4 shows that this is not the case in our set-up. ∎

3.2. **Linearized Koopman operator.** We introduce the *linearized Koopman operator*, defined for continuous real valued functions ϕ on $\mathbb{T} \times \mathbb{R}^+$ by

$$TP\phi(x, u) \coloneqq \mathbb{E}\left[\phi(T_\omega(x), DT_\omega(x)u)\right].$$

Recall from Proposition 3.2 that ϕ_q is the dominant eigenfunction of P_q. Define $\tilde{W}_q : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\tilde{W}_q(x, u) = u^{-q}\phi_q(x). \quad (3.10)$$

We reserve the *tilde* notation for functions on $\mathbb{T} \times \mathbb{R}^+$.

Lemma 3.7. *The function $\tilde{W}_q$ is an eigenfunction of TP, with eigenvalue $e^{\Lambda(-q)}$:*

$$TP\tilde{W}_q(x, u) = e^{\Lambda(-q)}\tilde{W}_q(x, u). \quad (3.11)$$

Proof. This follows from a straightforward computation. For $(x,u) \in \mathbb{T} \times \mathbb{R}^+$,

$$\begin{aligned} TP\tilde{W}_q(x,u) &= \mathbb{E}\left[\tilde{W}_q(T_\omega(x), DT_\omega(x)u)\right] \\ &= \mathbb{E}\left[(DT_\omega(x)u)^{-q}\phi_q\left(T_\omega(x)\right)\right] \\ &= u^{-q}\mathbb{E}\left[DT_\omega(x)^{-q}\phi_q\left(T_\omega(x)\right)\right] \\ &= u^{-q}P_q\phi_q(x) \\ &= e^{\Lambda(-q)}u^{-q}\phi_q(x) \\ &= e^{\Lambda(-q)}\tilde{W}_q(x,u). \end{aligned}$$

□

Remark 3.8. Consider a random sequence

$$(x_{n+1}, u_{n+1}) = \left(T_{\sigma^n\omega}(x_n), DT_{\sigma^n\omega}(x_n)u_n\right),$$

with $x_0, u_0 \in \mathbb{T} \times \mathbb{R}^+$. Take γ as in Lemma 3.4 such that $e^{\Lambda(\gamma)} = 1$. Then

$$(TP - I)\tilde{W}_\gamma(x,u) = 0,$$

which shows that $\tilde{W}_\gamma(x_n, u_n)$ is a martingale. ■

Example 3.9. In the case of Example 1.3, with $\vartheta = 0.5$, we get

$$\tilde{W}_q(x,u) = \|\phi_q\| u^{-q},$$

which does not depend on x, as the one point dynamics is essentially a full shift. ■

Lemma 3.10. *There exist $K > 0$ and two continuous functions*

$$\tilde{\phi}, \tilde{\eta} \in \mathcal{C}^0(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}),$$

such that the following holds for $(x,u) \in \mathbb{T} \times \mathbb{R}^+$. For $\tilde{\phi}$ we have

$$|\tilde{\phi}(x,u) - \ln(u)| \leq K, \tag{3.12}$$

$$(TP - I)\,\tilde{\phi}(x,u) = \lambda. \tag{3.13}$$

Assume $\lambda = 0$. Then for $\tilde{\eta}$ we have

$$|\tilde{\eta}(x,u) - \ln^2(u)| \leq K|\ln(u)|, \tag{3.14}$$

$$(TP - I)\,\tilde{\eta}(x,u) = V. \tag{3.15}$$

Concerning notation in the following proof and further below, π_1 is the function $\pi_1(x,u) = x$ on $\mathbb{T} \times \mathbb{R}^+$, π_2 stands for the function $\pi_2(x,u) = u$ on $\mathbb{T} \times \mathbb{R}^+$.

Proof of Lemma 3.10. We give a constructive proof for Lemma 3.10, where we use Proposition 3.2 and Lemma 3.5. We first find the function $\tilde{\phi}$ for which (3.12) and (3.13) holds. Subsequently we find $\tilde{\eta}$ for which (3.14) and (3.15) holds.

For $(x,u) \in \mathbb{T} \times \mathbb{R}^+$, let

$$\tilde{\phi}(x,u) = \ln(u) - \partial_q\phi_0(x). \tag{3.16}$$

Note that

$$\tilde{\phi} \in \mathcal{C}^0(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}).$$

Furthermore, for all $(x, u) \in \mathbb{T} \times \mathbb{R}^+$, we have

$$\begin{aligned}(TP - I)\,\tilde{\phi}(x,u) &= (TP - I)\ln(\pi_2)(x,u) - (P_0 - I)\,(\partial_q \phi_0)\,(x) \\ &= \mathbb{E}\left[\ln(DT_\omega(x)u)\right] - \ln(u) - \mathbb{E}\left[\ln(DT_\omega(x))\right] + \lambda \\ &= \lambda.\end{aligned}$$

In the first line we use that $\partial_q \phi_0(x)$ only depends on x. For the second equality we apply (3.4) in Lemma 3.5.

For the second part of the proof, we take $\lambda = 0$ and let

$$\tilde{\eta}(x,u) = \ln^2(u) - 2\ln(u)\partial_q \phi_0(x) + \partial_q^2 \phi_0(x).$$

And again note that

$$\tilde{\eta} \in \mathcal{C}^0(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}).$$

From a straightforward computation, we obtain, for $(x, u) \in \mathbb{T} \times \mathbb{R}^+$

$$\begin{aligned}&(TP - I)\,\eta(x,u) = \\ &\quad (TP - I)\ln^2(\pi_2)(x,u) - 2\,(TP - I)\ln(\pi_2)\partial_q\phi_0(\pi_1)(x,u) + (TP - I)\,\partial_q^2\phi_0(\pi_1)(x,u).\end{aligned}$$

We analyse the terms on the right hand side separately. For the first term, we have

$$\begin{aligned}(TP - I)\left(\ln^2(\pi_2)\right)(x,u) &= \mathbb{E}\left[\ln^2(DT_\omega(x)u)\right] - \ln^2(u) \\ &= \mathbb{E}\left[\left(\ln(DT_\omega(x)) + \ln(u)\right)^2\right] - \ln^2(u) \\ &= \mathbb{E}\left[\ln^2(DT_\omega(x))\right] + 2\ln(u)\mathbb{E}\left[\ln(DT_\omega(x))\right]. \end{aligned} \tag{3.17}$$

For the second term we have

$$\begin{aligned}2\,(TP - I)\ln(\pi_2)\partial_q\phi_0(\pi_1)(x,u) &= 2\mathbb{E}\left[\ln(DT_\omega(x)u)\partial_q\phi_0(T_\omega(x))\right] - 2\ln(u)\partial_q\phi_0(x) \\ &= 2\mathbb{E}\left[\ln(DT_\omega(x))\partial_q\phi_0(T_\omega(x))\right] + 2\ln(u)\left(\mathbb{E}\left[\partial_q\phi_0(T_\omega(x))\right] - \partial_q\phi_0(x)\right) \\ &= 2\mathbb{E}\left[\ln(DT_\omega(x))\partial_q\phi_0(T_\omega(x))\right] + 2\ln(u)(P - I)\partial_q\phi_0(x) \\ &= 2\mathbb{E}\left[\ln(DT_\omega(x))\partial_q\phi_0(T_\omega(x))\right] + 2\ln(u)\mathbb{E}\left[\ln(DT_\omega(x))\right], \end{aligned} \tag{3.18}$$

where in the third to fourth line we make use of the properties of $\partial_q\phi_0$, as described by (3.4) in Lemma 3.5. For the third term, we have

$$(TP - I)\,\partial_q^2\phi_0(\pi_1)(x,u) = V + 2\mathbb{E}\left[\ln(DT_\omega(x))\partial_q\phi_0\left(T_\omega(x)\right)\right] - \mathbb{E}\left[\ln^2(DT_\omega(x))\right]. \tag{3.19}$$

Here we use (3.5) in Lemma 3.5. Combining (3.17), (3.18) and (3.19), we conclude, for $(x, u) \in \mathbb{T} \times \mathbb{R}^+$,

$$(TP - I)\,\tilde{\eta}(x,u) = V. \tag{3.20}$$

Finally, (3.12) and (3.14) follow from the analyticity in q of $\phi_q(x)$(see Proposition 3.2). $\square$

Remark 3.11. Consider a random sequence

$$(x_{n+1}, u_{n+1}) = (T_{\sigma^n\omega}(x_n), DT_{\sigma^n\omega}(x_n)u_n)\,,$$

with $(x_0, u_0) \in \mathbb{T} \times \mathbb{R}^+$. Then (3.13) expresses that the function $\tilde{\phi}(x_n, u_n) - n\lambda$ is a martingale. Assuming $\lambda = 0$, (3.15) expresses that the function $\tilde{\eta}(x_n, u_n) - nV$ is a martingale. ■

Example 3.12. In the case of Example 1.3, with $\vartheta = 0.5$, we get $\tilde{\phi}(x, u) = \ln(u) + \|\partial_q\phi_0\|$ and $\tilde{\eta}(x, u) = \ln^2(u) - 2\|\partial_q\phi_0\|\ln(u) + \|\partial_q^2\phi_0\|$. Both functions are independent of x, as the one point dynamics is essentially a full shift. ■

3.3. **Two-point Koopman operator.** The *(annealed) two-point Koopman operator* $P_{(2)}$ is defined for a real valued function ϕ on $\mathbb{T}^2 \setminus \Delta$ by

$$P_{(2)}\phi(x, y) := \mathbb{E}\left[\phi(T_\omega^{(2)}(x, y))\right].$$

The following lemma addresses the approximation of $P_{(2)}$ by TP.

Define $\phi : \mathbb{T}^2 \setminus \Delta \to \mathbb{R}$ by

$$\phi(x, y) := \ln(d(x, y)) - \partial_q\phi_0(x).$$

With $\tilde{\phi}$ from Lemma 3.10 (see (3.16)) we have, for $(x, y) \in \mathbb{T}^2 \setminus \Delta$,

$$\phi(x, y) = \tilde{\phi}(x, d(x, y)). \tag{3.21}$$

Let $\eta : \mathbb{T}^2 \setminus \Delta \to \mathbb{R}$ be defined as

$$\eta(x, y) := \ln^2(d(x, y)) - 2\partial_q\phi_0(x)\ln(d(x, y)) + \partial_q\phi_0(x).$$

Note that for $(x, y) \in \mathbb{T}^2 \setminus \Delta$,

$$\eta(x, y) = \tilde{\eta}(x, d(x, y)). \tag{3.22}$$

Define $W_q : \mathbb{T}^2 \setminus \Delta \to \mathbb{R}$ by

$$W_q(x, y) := d(x, y)^{-q}\phi_q(x). \tag{3.23}$$

With $\tilde{W}_q$ from (3.10) we have for $(x, y) \in \mathbb{T}^2 \setminus \Delta$,

$$W_q(x, y) = \tilde{W}_q(x, d(x, y)). \tag{3.24}$$

The next lemma compares the action of TP and $P_{(2)}$.

Lemma 3.13. *There exist $R, B > 0$, such that we have the following bounds, for $(x, y) \in \Delta_R \setminus \Delta$,*

$$\left|TP\tilde{\phi}(x, d(x, y)) - P_{(2)}\phi(x, y)\right| \le Bd(x, y), \tag{3.25}$$

$$\left|TP\tilde{\eta}(x, d(x, y)) - P_{(2)}\eta(x, y)\right| \le Bd(x, y)|\ln(d(x, y))|, \tag{3.26}$$

and for $q \in [-|\gamma| - 1, |\gamma| + 1]$,

$$\left|TP\tilde{W}_q(x, d(x, y)) - P_{(2)}W_q(x, y)\right| \le Bd(x, y)^{-q+1}. \tag{3.27}$$

Proof. We will work out the estimates for (3.25), then the computation for (3.26) is analogous, and will be left to the reader. We will sketch the proof for (3.27).

Recall that $w(x,y)$ denotes the signed distance between x and y, for nearby points $x, y \in \mathbb{T}$. Take $R > 0$ small enough so that T_ω is injective on all intervals of length R. For $(x,y) \in \Delta_R \setminus \Delta$,

$$
\begin{aligned}
TP\tilde{\phi}(x, |w(x,y)|) &= \mathbb{E}\left[\ln(|DT_\omega(x)w(x,y)|)\right] - (P_0\partial_q\phi_0)(x) \\
&= \ln(|w(x,y)|) + \mathbb{E}\left[\ln(DT_\omega(x))\right] - \mathbb{E}\left[\partial_q\phi_0\left(T_\omega(x)\right)\right].
\end{aligned} \tag{3.28}
$$

To determine $P_{(2)}\phi$, we use a Taylor expansion for T_ω. We obtain,

$$
w(T_\omega^{(2)}(x,y)) = DT_\omega(x)w(x,y) + \frac{1}{2}D^2T_\omega(\xi)w(x,y)^2,
$$

for a ξ in the interval between x and $x + w(x,y)$. Therefore, for $(x,y) \in \Delta_R \setminus \Delta$ with $R > 0$ small enough,

$$
\begin{aligned}
P_{(2)}\phi(x,y) &= \mathbb{E}\left[\ln\left(\left|w(T_\omega^{(2)}(x,y))\right|\right)\right] - P_0\partial_q\phi_0(x) \\
&= \mathbb{E}\left[\ln\left(\left|w(x,y)DT_\omega(x) + \frac{D^2T_\omega(\xi)w(x,y)^2}{2}\right|\right)\right] - P_0\partial_q\phi_0(x) \\
&= \ln(|w(x,y)|) + \mathbb{E}\left[\ln\left(DT_\omega(x)\right)\right] + \mathbb{E}\left[\ln\left(\left|1 + \frac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}\right|\right)\right] - \mathbb{E}[\partial_q\phi_0(T_\omega(x))].
\end{aligned} \tag{3.29}
$$

Combining (3.28) and (3.29) yields, for $(x,y) \in \Delta_R \setminus \Delta$,

$$
\left|P_{(2)}\phi(x,y) - TP\phi(x, |w(x,y)|)\right| \leq \mathbb{E}\left[\ln\left(\left|1 + \frac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}\right|\right)\right]. \tag{3.30}
$$

The right hand side of (3.30) can be bounded using the standard inequalities $x/(1+x) \leq \ln(1+x) \leq x$ for $x > -1$. For R small we have $\left|\frac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}\right| < 1$ for $(x,y) \in \Delta_R \setminus \Delta$. Then for the upper bound we obtain

$$
\mathbb{E}\left[\ln\left(1 + \frac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}\right)\right] \leq \frac{|w(x,y)|\|D^2T_\omega\|}{2a_1}.
$$

Similarly for the lower bound, when we take R small enough so that $|w(x,y)|\|D^2T_\omega\| < a_1$ for $(x,y) \in \Delta_R \setminus \Delta$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\ln\left(1+\frac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}\right)\right] &\geq \mathbb{E}\left[\frac{\dfrac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}}{1+\dfrac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)}}\right] \\
&\geq \mathbb{E}\left[\frac{D^2T_\omega(\xi)w(x,y)}{2DT_\omega(x)+D^2T_\omega(\xi)w(x,y)}\right] \\
&\geq -|w(x,y)|\|D^2T_\omega\|\mathbb{E}\left[\frac{1}{2DT_\omega(x)+D^2T_\omega(\xi)w(x,y)}\right] \\
&\geq \frac{-|w(x,y)|\|D^2T_\omega\|}{a_1}.
\end{aligned}$$

Setting $B = \frac{\|D^2T_\omega\|}{a_1}$ finishes the first part.

In a similar manner, we can choose B such that (3.26) holds for $(x,y) \in \Delta_R \setminus \Delta$ with R small enough.

For (3.27), take $(x,y) \in \Delta_R \setminus \Delta$ with R small and assume, without loss of generality, $x < y$. Then

$$\begin{aligned}
&\left|TP\tilde{W}_q(x,d(x,y)) - P_{(2)}W_q(x,y)\right| \\
&\qquad \leq \|\phi_q\|\,\mathbb{E}\left[\left||DT_\omega(x)w(x,y)|^{-q} - \left|DT_\omega(x)w(x,y)+\frac{1}{2}D^2T_\omega(\xi)w(x,y)^2\right|^{-q}\right|\right].
\end{aligned}$$

With (3.3) and the mean value theorem for $a \mapsto a^{-q}$, we get

$$\begin{aligned}
\left|TP\tilde{W}_q(x,w(x,y)) - P_{(2)}W_q(x,y)\right| &\leq C_q|q|\mathbb{E}\left[|w(x,y)|^{-q-1}|D^2T_\omega(\xi)w(x,y)|^2\right] \\
&\leq C_q|q|\|D^2T\|^2|w(x,y)|^{-q+1} \\
&\leq Bd(x,y)^{-q+1},
\end{aligned}$$

for some positive constant C_q. As $C_q|q|$ depends continuous on q, restricting q to $[-|\gamma|-1, |\gamma|+1]$ allows us to uniformly bound it with a constant B. This completes the proof. ☐

The following lemma is a key lemma that adapts the equalities and estimates in Lemma 3.13 for TP to the setting for $P_{(2)}$, but only near the diagonal Δ, using that near Δ, $P_{(2)}$ can be approximated by TP (Lemma 3.13).

Proposition 3.14. *There exists a $R, K > 0$, and continuous integrable functions*

$$\phi^{\pm}, \eta^{\pm} \in \mathcal{C}^0(\mathbb{T}^2 \setminus \Delta, \mathbb{R}),$$

such that the following holds for $(x,y) \in \Delta_R \setminus \Delta$.

For $\phi^\pm$ we have

$$|\phi^\pm(x,y) - \ln(d(x,y))| \le K, \tag{3.31}$$

$$\left(P_{(2)} - I\right)\phi^-(x,y) \le \lambda \le \left(P_{(2)} - I\right)\phi^+(x,y). \tag{3.32}$$

Assume $\lambda = 0$. Then for $\eta^\pm$ we have

$$|\eta^\pm(x,y) - \ln^2(d(x,y))| \le K|\ln(d(x,y))|, \tag{3.33}$$

$$\left(P_{(2)} - I\right)\eta^-(x,y) \le V \le \left(P_{(2)} - I\right)\eta^+(x,y). \tag{3.34}$$

Proof. We can bound $\left(P_{(2)} - I\right)\phi(x,y)$, by applying the triangle inequality and using (3.13), (3.21) and (3.25):

$$\left|\left(P_{(2)} - I\right)\phi(x,y)\right| \le \overbrace{\left|TP\tilde\phi(x,d(x,y)) - P_{(2)}\phi(x,y)\right|}^{\le Bd(x,y) \text{ by } (3.25)} + \overbrace{\left|(TP-I)\,\tilde\phi(x,d(x,y))\right|}^{=\lambda \text{ by } (3.13)} + \overbrace{\left|\phi(x,y) - \tilde\phi(x,d(x,y))\right|}^{=0 \text{ by } (3.21)},$$

for $(x,y) \in \Delta_R \setminus \Delta$. So

$$\left|\left(P_{(2)} - I\right)\phi(x,y) - \lambda\right| \le Bd(x,y), \tag{3.35}$$

for $(x,y) \in \Delta_R \setminus \Delta$.

This basically means that $\left(P_{(2)} - I\right)\phi$ near the diagonal is close to λ. To obtain functions $\phi^\pm$ so that $P_{(2)} - I$ applied to them, after subtracting λ, has a definite sign, we add suitable functions to ϕ: we take $\phi^\pm : \mathbb{T}^2 \setminus \Delta \to \mathbb{R}$ of the form

$$\phi^\pm(x,y) = \phi(x,y) \pm c_1 W_{-q_0}(x,y), \tag{3.36}$$

for a small positive value $q_0 \in (0,1/2)$, such that $\Lambda(q_0) \ne 0$, and $|c_1|$ large enough (will be defined below), with sign such that $c_1(e^{\Lambda(q_0)} - 1) > 0$. Note that $\phi^\pm \in \mathcal{C}^0(\mathbb{T}^2 \setminus \Delta, \mathbb{R})$.

By Proposition 3.2, we obtain

$$(TP - I)\,\tilde W_{-q_0}(x,u) = \left(e^{\Lambda(q_0)} - 1\right)u^{q_0}\phi_{-q_0}(x), \tag{3.37}$$

for $(x,u) \in \mathbb{T} \times \mathbb{R}^+$.

Combining (3.35), (3.37), (3.27) and (3.24) allows us to prove the first inequality of (3.32), by choosing $|c_1|$ in (3.36) large enough. For $(x,y) \in \Delta_R \setminus \Delta$,

$$\begin{aligned}
\left(P_{(2)} - I\right)\phi^+(x,y) - \lambda &= \left(P_{(2)} - I\right)\phi(x,y) - \lambda + c_1\left(P_{(2)} - I\right)W_{-q_0}(x,y) \\
&= \overbrace{\left(P_{(2)} - I\right)\phi(x,y) - \lambda}^{\ge -Bd(x,y) \text{ by } (3.35)} \quad + \overbrace{c_1(TP-I)\tilde W_{-q_0}(x,d(x,y))}^{\ge c_1\left(e^{\Lambda(q_0)}-1\right)d(x,y)^{q_0}/C \text{ by } (3.37)} \\
&\quad + \overbrace{c_1\left(P_{(2)}W_{-q_0}(x,y) - TP\tilde W_{-q_0}(x,d(x,y))\right)}^{\ge -|c_1|Bd(x,y)^{q_0+1} \text{ by } (3.27)} \quad + \quad \overbrace{c_1\left(W_{-q_0}(x,y) - \tilde W_{-q_0}(x,d(x,y))\right)}^{=0 \text{ by } (3.24)} \\
&\ge -Bd(x,y) + c_1\left(e^{\Lambda(q_0)} - 1\right)d(x,y)^{q_0}/C - |c_1|Bd(x,y)^{q_0+1},
\end{aligned} \tag{3.38}$$

so that

$$\left(P_{(2)} - I\right) \phi^+(x,y) \geq \lambda,$$

if R is small and $|c_1|$ is chosen large enough. The second inequality regarding ϕ^- is obtained using similar bounds. The bound (3.31) for suitable $K > 0$ is immediate from the expressions for $\phi^\pm$.

We take $\lambda = 0$ and proceed with the construction of $\eta^\pm$. We can bound $\left(P_{(2)} - I\right) \eta(x,y)$ by applying the triangle inequality and using (3.20), (3.26) and (3.22). This yields

$$\begin{aligned}\left|\left(P_{(2)} - I\right) \eta(x,y) - V\right| \leq & \overbrace{\left|\left(TP\tilde{\eta}(x, d(x,y)) - P_{(2)}\phi(x,y)\right)\right|}^{\leq Bd(x,y)|\ln(d(x,y))| \text{ by (3.20)}} + \\ & \overbrace{\left|\left(TP - I\right) \phi(x, d(x,y)) - V\right|}^{=0 \text{ by (3.26)}} + \overbrace{\left|\phi(x,y) - \tilde{\phi}(x, d(x,y))\right|}^{=0 \text{ by (3.22)}}\end{aligned}$$

so that

$$\left|\left(P_{(2)} - I\right) \eta(x,y) - V\right| \leq Bd(x,y)|\ln(d(x,y))|, \tag{3.39}$$

for all $(x,y) \in \Delta_R \setminus \Delta$. We may not have that $\left(P_{(2)} - I\right) \eta(x,y) - V$ has a definite sign. We therefore add functions to η (as we did to obtain $\phi^\pm$ from ϕ) and let

$$\eta^\pm(x,y) = \eta(x,y) \pm c_2 W_{-q_0}(x, d(x,y)), \tag{3.40}$$

with $|c_2|$ large enough (will be defined below), with sign such that $c_2(e^{\Lambda(q_0)} - 1) > 0$. We clearly have that $\eta^\pm \in \mathcal{C}^0(\mathbb{T}^2 \setminus \Delta, \mathbb{R})$.

Combining (3.20),(3.27) and (3.37) allows us to prove the first inequality of (3.34), by choosing c_2 in (3.40) large enough. For $(x,y) \in \Delta_R \setminus \Delta$, we get through a straightforward combination of previous inequalities,

$$\begin{aligned}&\left(P_{(2)} - I\right) \eta^+(x,y) \\ &= \overbrace{\left(P_{(2)} - I\right) \eta(x,y)}^{\geq V - Bd(x,y)|\ln(d(x,y))| - CBd(x,y) \text{ by (3.39)}} + \overbrace{c_2 \left(P_{(2)} - I\right) W_{-q_0}(x,y)}^{\geq c_2\left(e^{\Lambda(q_0)}-1\right)d(x,y)^{q_0}/C - |c_2|Bd(x,y)^{q_0+1} \text{ by (3.38)}},\end{aligned}$$

so that

$$\left(P_{(2)} - I\right) \eta^+(x,y) \geq V,$$

for R small enough, by choosing c_2 large enough. Here we use the fact that for $0 < R < 1$, and $q_0 \in (0,1)$, there exists a $C > 0$, such that for all $x \in (0,R)$, $Cx_0^q > -x\ln(x)$. The second inequality in (3.34) for η^- is obtained using similar bounds.

Finally, (3.33) for K large enough is clear from the expression for $\eta^\pm$, using Proposition 3.2. □

Remark 3.15. We can use (3.32) to get

$$\pm\mathbb{E}\left[\phi^+(T_\omega(x), T_\omega(y))\right] - \lambda \geq \pm\phi^\pm(x,y),$$

for $d(x,y)$ small enough. Similarly if we assume $\lambda = 0$, we get

$$\pm\mathbb{E}\left[\eta^+(T_\omega(x), T_\omega(y))\right] - V \geq \pm\eta^\pm(x,y),$$

for $d(x,y)$ small enough. Applying Doob's stopping time theorem (we refer to standard references such as [50] or [25]) with suitable stopping times we build sub- and supermartingales for such random sequences from the functions ϕ^+, ϕ^-, η^+ and η^-. This is done in the proof of Lemma 4.1. ■

Proposition 3.16. *There exists a $R, K > 0$, and a family of continuous functions,*

$$W_q^\pm \in \mathcal{C}^0(\mathbb{T}^2 \setminus \Delta, \mathbb{R}),$$

for $q \in [-|\gamma| - 1/2, |\gamma| + 1/2]$, such that the following holds for all $(x,y) \in \Delta_R \setminus \Delta$. For $W_q^\pm$ we have

$$\frac{1}{K} d(x,y)^{-q} \leq W_q^\pm(x,y) \leq K d(x,y)^{-q}, \tag{3.41}$$

$$\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q^-(x,y) \leq 0 \leq \left(P_{(2)} - e^{\Lambda(-q)}\right) W_q^+(x,y). \tag{3.42}$$

Proof. We can bound $\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q(x,y)$, by applying the triangle inequality and using Lemma 3.13:

$$\begin{aligned}\left|\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q(x,y)\right| \leq \\ \overbrace{\left|P_{(2)} W_q(x,y) - TP\tilde{W}_q(x, d(x,y))\right|}^{\leq B d(x,y)^{-q+1} \text{ by } (3.27)} + \overbrace{\left|\left(TP - e^{\Lambda(-q)}\right) \tilde{W}_q(x, d(x,y))\right|}^{=0 \text{ by } (3.11)} \\ + e^{\Lambda(-q)} \overbrace{\left|W_q(x,y) - \tilde{W}_q(x, d(x,y))\right|}^{=0 \text{ by } (3.24)},\end{aligned}$$

for $(x,y) \in \Delta_R \setminus \Delta$. So

$$\left|\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q(x,y)\right| \leq B d(x,y)^{-q+1}, \tag{3.43}$$

for $(x,y) \in \Delta_R \setminus \Delta$.

This basically means that $\left(P_{(2)} - e^{\Lambda(-p)}\right) W_q$ near the diagonal is close to 0. To obtain functions $W_q^\pm$ so that $P_{(2)} - e^{\Lambda(-p)}$ applied to them, has a definite sign, we add suitable functions to W_q: we take $W_q^\pm : T^2 \setminus \Delta \to \mathbb{R}$ of the form

$$W_q^\pm(x,y) = W_q(x,y) \pm c_3 W_{q_1}(x,y),$$

for $q_1 \in (q - 1/2, q)$, such that $\Lambda(-q) - \Lambda(-q_1) \neq 0$, and $|c_3|$ large enough (will be defined below), with sign such that $c_3(e^{\Lambda(-q)} - e^{\Lambda(-q_1)}) > 0$.

By applying (3.43) we get for $(x,y) \in \Delta_R \setminus \Delta$,

$$\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q^+(x,y) = \overbrace{\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q(x,y)}^{\geq -Bd(x,y)^{-q+1} \text{ by } (3.43)} + \overbrace{c_3 \left(P_{(2)} - e^{\Lambda(-q_1)}\right) W_{q_1}(x,y)}^{\geq -c_3\left(Bd(x,y)^{-q_1+1}\right) \text{ by } (3.43)} + \overbrace{c_3(e^{\Lambda(-q_1)} - e^{\Lambda(-q)})W_{q_1}}^{\geq c_3(e^{\Lambda(-q_1)}-e^{\Lambda(-q)})d(x,y)^{-q_1}/K \text{ by } (3.23)}.$$

Picking c_3 large enough and R small enough yields,

$$\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q^+(x,y) \geq 0, \quad \text{for } (x,y) \in \Delta_R \setminus \Delta.$$

Similarly we get

$$\left(P_{(2)} - e^{\Lambda(-q)}\right) W_q^-(x,y) \leq 0, \quad \text{for } (x,y) \in \Delta_R \setminus \Delta.$$

Now set K again large enough such that (3.41) holds. □

Remark 3.17. Suppose γ is such that $e^{\Lambda(\gamma)} = 1$. We can use (3.42) to get

$$\pm \mathbb{E}\left[W_{-\gamma}^{\pm}(T_\omega(x), T_\omega(y))\right] \geq \pm W_{-\gamma}^{\pm}(x,y),$$

for $d(x,y)$ small enough. Applying Doob's stopping time theorem with suitable stopping times we build sub- and supermartingales for such random sequences from the functions W_γ^+ and W_γ^-. This is done in the proof of Lemma 4.5.

4. Topological random dynamics

In this section we prove Theorem 1.1(3) and construct tools to prove Theorem 1.2 in Section 5, for $\lambda \geq 0$. We look separately at cases with zero Lyapunov exponent, positive Lyapunov exponent and negative Lyapunov exponent.

4.1. Zero Lyapunov exponent.

The next part of the analysis is to calculate escape probabilities and expected escape times for escape from neighborhoods of the diagonal and strips near the diagonal. The proofs in this section rely on an analysis of Koopman operators, which is developed in Section 3. The statements can be read without reference to Section 3, but for the proofs the reader has to familiarize with the results in Section 3.

For suitable small R, numbers $0 < \varepsilon < \delta < R$ and points $(x,y) \in \Delta_R$ with $\varepsilon < d(x,y) < \delta$, we define stopping times

$$\tau_{\delta,+}(x,y) = \min\{n \in \mathbb{N}\,;\ d(T_\omega^n(x), T_\omega^n(y)) > \delta\}, \tag{4.1}$$

$$\tau_{\varepsilon,-}(x,y) = \min\{n \in \mathbb{N}\,;\ d(T_\omega^n(x), T_\omega^n(y)) < \varepsilon\}. \tag{4.2}$$

The following lemma addresses statistics of these stopping times.

Lemma 4.1. *For $\lambda = 0$, there exists an sufficiently small $R > 0$, and sufficiently large $K > 0$, such that if $0 < \varepsilon < d(x,y) < \delta < R$, then*

$$\mathbb{P}(\min\{\tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\} < \infty) = 1, \tag{4.3}$$

as well as

$$\frac{\ln\left(\frac{\delta}{d(x,y)}\right) - 2K}{\ln\left(\frac{\delta}{\varepsilon}\right)} \leq \mathbb{P}\left(\{\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y))\}\leq \frac{\ln\left(\frac{\delta}{d(x,y)}\right) + 2K}{\ln\left(\frac{\delta}{\varepsilon}\right)} \tag{4.4}$$

and

$$\begin{aligned}\frac{1}{V}&\left(\ln\left(\frac{\delta}{d(x,y)}\right)\ln\left(\frac{d(x,y)}{\varepsilon}\right) - 6K|\ln(\varepsilon)| - 2K^2\right)\\ &\leq \mathbb{E}\left[\min\{\tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\}\right]\\ &\leq \frac{1}{V}\left(\ln\left(\frac{\delta}{d(x,y)}\right)\ln\left(\frac{d(x,y)}{\varepsilon}\right) + 6K|\ln(\varepsilon)| + 2K^2\right).\end{aligned} \tag{4.5}$$

Proof. We follow reasoning of [9, 11]. As indicated above, we will use statements from Section 3. The functions $\phi^{\pm}$ and $\eta^{\pm}$ come directly from Proposition 3.14. Denote $(x_n, y_n) = T^n_\omega(x_0, y_0)$, for $(x_0, y_0) \in \mathbb{T}^2 \setminus \Delta$, such that $0 < \varepsilon < d(x,y) < \delta < R$. Now $\eta^+(x_n, y_n)$ stopped at $\min\{\tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\}$ is a submartingale, by Proposition 3.14, Remark 3.15 and applying Doob's stopping time theorem. So, with

$$\tilde{n} = \min\{n, \tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\},$$

we have

$$\eta^+(x,y) \leq \mathbb{E}\left[\eta^+(x_{\tilde{n}}, y_{\tilde{n}}) - \tilde{n}V\right].$$

Rewriting, we obtain

$$\begin{aligned}\mathbb{E}[\tilde{n}]V &\leq \mathbb{E}\left[\eta^+(x_{\tilde{n}}, y_{\tilde{n}})\right] - \eta^+(x,y)\\ &\leq \ln^2(d(x,y)) + \ln^2(\varepsilon a_1) + K|\ln(\varepsilon a_1)| < \infty.\end{aligned}$$

This implies that $\mathbb{P}(\tilde{n} = \infty) = 0$, which then implies (4.3).

By Proposition 3.14 and Remark 3.15 and applying Doob's stopping time theorem, $\phi^+(x_n, y_n)$ stopped at $\min\{\tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\}$ is a submartingale. Therefore we have

$$\phi^+(x,y) \leq \mathbb{E}\left[\phi^+(x_{\tilde{n}}, y_{\tilde{n}})\right].$$

Letting n go to infinity, and conditioning separately on $\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)$ or $\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)$, we get

$$\begin{aligned}\phi^+(x,y) &\leq \lim_{\tilde{n}\to\infty} \mathbb{E}\left[\phi^+(x_{\tilde{n}}, y_{\tilde{n}})\right]\\ &= \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[\phi^+(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right]\\ &+ \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[\phi^+(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right].\end{aligned} \tag{4.6}$$

By (4.3) we have

$$P\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right) = 1 - \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right). \tag{4.7}$$

From (3.31) we get

$$\ln(d(x,y)) - K \leq \phi^+(x,y). \tag{4.8}$$

Note that once the distance $d(x_{\tilde{n}}, y_{\tilde{n}})$ is outside of (ε, δ), the distance is either in $(\varepsilon a_1, \varepsilon]$ or in $[\delta, \delta a_2)$ by Hypothesis (H1). Therefore we can bound the conditional expectation in (4.6) in the following manner,

$$\mathbb{E}\left[\phi^+(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \leq \sup_{r\in[a_1\varepsilon,\varepsilon)} \{\ln(r) + K\} \leq \ln(\varepsilon) + K \tag{4.9}$$

and

$$\mathbb{E}\left[\phi^+(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right] \leq \sup_{r\in[\delta,a_2\delta]} \{\ln(r) + K\} \leq \ln(\delta) + 2K, \tag{4.10}$$

for K chosen large enough. Using the bounds (4.8), (4.9), (4.10) and the equality (4.7) in (4.6), we get

$$\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right) \leq \frac{\ln\left(\dfrac{d(x,y)}{\delta}\right) - 4K}{\ln\left(\dfrac{\varepsilon}{\delta}\right)}.$$

In a similar fashion we get the counterpart of (4.6) for ϕ^-,

$$\begin{aligned}
\phi^-(x,y) &\geq \lim_{n\to\infty} \mathbb{E}\left[\phi^-(x_{\tilde{n}}, y_{\tilde{n}})\right] \\
&= \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right) \mathbb{E}\left[\phi^-(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\
&\quad + \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right) \mathbb{E}\left[\phi^-(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right],
\end{aligned}$$

from which we obtain a bound

$$\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right) \geq \frac{\ln\left(\dfrac{d(x,y)}{\delta}\right) + 4K}{\ln\left(\dfrac{\varepsilon}{\delta}\right)}, \tag{4.11}$$

again assuming that K is chosen large enough. This finishes the proof of (4.4).

For (4.5) we use the submartingale property of η^+ stopped at $\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)$ or $\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)$ (see Remark 3.15 and apply Doob's stopping time theorem.) and we let time go to infinity. Denoting

$$\tau = \min\{\tau_{\delta,+}(x,y), \tau_{\varepsilon,-}(x,y)\},$$

this yields

$$\eta^+(x,y) \leq \mathbb{E}\left[\eta^+(x_\tau, y_\tau) - V\tau\right].$$

Rewriting and conditioning on $\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)$ and $\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)$ separately, we obtain

$$\begin{aligned} V\mathbb{E}\left[\tau\right] \leq \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[\eta^{+}(x_\tau,y_\tau)|\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\ + \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[\eta^{+}(x_\tau,y_\tau)|\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right] - \eta^{+}(x,y). \end{aligned} \quad (4.12)$$

As above we have bounds

$$\begin{aligned} &\mathbb{E}\left[\eta^{+}(x_\tau,y_\tau)|\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \leq \ln^2(a_1\varepsilon) + K\left|\ln(a_1\varepsilon)\right| \leq \ln^2(\varepsilon) + 3K\left|\ln(\varepsilon)\right| + 2K^2, \\ &\mathbb{E}\left[\eta^{+}(x_\tau,y_\tau)|\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right] \leq \ln^2(\delta) + K\left|\ln(\delta)\right|, \end{aligned}$$

Now for a different, larger K, we have $\mathbb{E}\left[\eta^{+}(x_\tau,y_\tau)|\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \leq \ln^2(\varepsilon) + K|\log(\varepsilon)|$ and we have

$$-\eta^{+}(x,y) \leq -\ln^2(d(x,y)) + K\left|\ln(d(x,y))\right|.$$

Plugging this and (4.7), (4.11), (3.33) into the estimate (4.12), we obtain

$$\begin{aligned} V\mathbb{E}\left[\tau\right] \leq \frac{\ln\left(\dfrac{d(x,y)}{\delta}\right) + 4K}{\ln\left(\dfrac{\varepsilon}{\delta}\right)}\left(\ln^2\left(\varepsilon\right) - \ln^2\left(\delta\right) + K\left|\ln\left(\frac{\varepsilon}{\delta}\right)\right|\right) \\ + \ln^2(\delta) + 2K|\ln(\delta)| - \ln^2(d(x,y)) + 2K|\ln(d(x,y))|. \end{aligned}$$

Note that $\ln^2\left(\varepsilon\right) - \ln^2\left(\delta\right) = \ln\left(\frac{\varepsilon}{\delta}\right)\ln(\varepsilon\delta)$ and $\ln^2(\delta) - \ln^2(d(x,y)) = \ln\left(\frac{\delta}{d(x,y)}\right)\ln(d(x,y)\delta)$. As $|\ln(x)|$ is a decreasing function for $0 < x < 1$, we have $|\ln(\varepsilon)| < |\ln(d(x,y))| < |\ln(\delta)|$. Using these identities and estimates we get

$$\begin{aligned} V\mathbb{E}\left[\tau\right] &\leq \ln\left(\frac{d(x,y)}{\delta}\right)\ln(\varepsilon\delta) - \ln\left(\frac{d(x,y)}{\delta}\right)\ln\left(d(x,y)\delta\right) + 12K|\ln(\varepsilon)| + 8K^2 \\ &\leq \ln\left(\frac{d(x,y)}{\delta}\right)\ln\left(\frac{\varepsilon}{d(x,y)}\right) + 12K|\ln(\varepsilon)| + 8K^2. \end{aligned}$$

In a similar fashion we get

$$\begin{aligned} V\mathbb{E}\left[\tau\right] \geq \frac{\ln\left(\dfrac{d(x,y)}{\delta}\right) - 4K}{\ln\left(\dfrac{\varepsilon}{\delta}\right)}\left(\ln^2\left(\varepsilon\right) - \ln^2\left(\delta\right) + 2K\left|\ln\left(\frac{\varepsilon}{\delta}\right)\right|\right) \\ + \ln^2(\delta) - 2K|\ln(\delta)| - \ln^2(d(x,y)) + 2K|\ln(d(x,y))|, \end{aligned}$$

and from this,

$$V\mathbb{E}\left[\tau\right] \geq \ln\left(\frac{d(x,y)}{\delta}\right)\ln\left(\frac{\varepsilon}{d(x,y)}\right) - 12K|\ln(\varepsilon)| - 8K^2.$$

Replace $2K$ by K to get the statement of the lemma. This finishes the proof. □

From here on, R, K are fixed as in Lemma 4.1. We now formulate a result stating that trajectories of points $(x, y) \in \Delta_\delta$ will almost surely escape from Δ_δ. However, the expected escape time will be infinite for $d(x, y)$ sufficiently small.

Lemma 4.2. *Let R, K be as in Lemma 4.1. Then for all $(x, y) \in \mathbb{T}^2$, with $0 < d(x, y) < \delta < R$,*

$$\mathbb{P}\left(\tau_{\delta,+}(x, y) < \infty\right) = 1 \tag{4.13}$$

and

$$\mathbb{E}\left[\tau_{\delta,+}(x, y)\right] = \infty \text{ whenever } d(x, y) < e^{-6K}\delta.$$

Proof. We apply Lemma 4.1. The first statement follows from (4.3). For the second statement, by Lemma 2.5 we find that for (x, y) with $d(x, y) < R$, there is a positive probability to enter $\Delta_{e^{-6K}\delta}$. Now let $\varepsilon \to 0$ in (4.5). □

From this lemma we know that for $(x_0, y_0) \in \Delta_\delta$, the expected number of orbit points

$$(x_n, y_n) = \left(T_\omega^{(2)}\right)^n (x_0, y_0)$$

before escaping Δ_δ, with $(x_0, y_0) \in \Delta_{e^{-6K}\delta}$, is infinite.

The above results allow to conclude Theorem 1.1(2).

Proposition 4.3. *If $\lambda = 0$, for all $(x, y) \in \mathbb{T}^2 \setminus \Delta$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(T_\omega^i(x), T_\omega^i(y)\right) = 0 \text{ and } \limsup_{n\to\infty} d\left(T_\omega^n(x), T_\omega^n(y)\right) > 0, \ \mathbb{P} - \text{a.s.}$$

Proof. Let $\varepsilon > 0$. For $(x, y) \in \mathbb{T}^2 \setminus \Delta$ and $\omega \in \Sigma_\vartheta$, consider the empirical count of iterates in Δ_ε,

$$N_\varepsilon(x, y, \omega) = \lim_{n\to\infty} \frac{\#\{i \in \mathbb{N}, 0 \le i \le n-1 : d(T_\omega^i(x), T_\omega^i(y)) < \varepsilon\}}{n}.$$

By Lemma 4.2, Lemma 2.5 and the strong law of large numbers, we get that $\mathbb{P}$-almost surely $N_\varepsilon(x, y, \omega) = 1$, for all $\varepsilon > 0$ implying the first part of the statement. See for example [4] for a worked out argument. The second part follows from Lemma 2.5. □

4.2. **Positive Lyapunov exponent.** In this section we consider positive Lyapunov exponent $\lambda > 0$. This will be assumed to hold throughout the section. Furthermore, we will denote the second zero of the moment Lyapunov function Λ as γ.

Consider $0 < \delta < R$ and $(x, y) \in \mathbb{T}^2 \setminus \Delta$ with $d(x, y) < \delta$. Let $\tau(x, y, \omega)$ be the minimal time with

$$d\left(\left(T_\omega^{(2)}\right)^{\tau(x,y,\omega)} (x, y)\right) > \delta,$$

with $\tau(x, y, \omega) = \infty$ if this does not exist.

Lemma 4.4. *Let R, K be as in Lemma 4.1. Suppose $\lambda > 0$. For $0 < \delta < R$ and $(x, y) \in \mathbb{T}^2 \setminus \Delta$ with $d(x, y) < \delta$ we have*

$$\mathbb{P}\left(d(T_\omega^n(x), T_\omega^n(y)) > \delta \text{ for some } n \in \mathbb{N}\right) = 1.$$

Moreover, for some $K > 0$,

$$\frac{1}{\lambda}\left(\ln\left(\frac{\delta}{d(x,y)}\right) - 2K\right) \le \mathbb{E}\left[\tau(x,y,\omega)\right] \le \frac{1}{\lambda}\left(\ln\left(\frac{\delta}{d(x,y)}\right) + 2K\right). \tag{4.14}$$

Proof. Write $x_n = T^n_\omega(x)$ and $y_n = T^n_\omega(x)$. We also use shorthand notation τ for $\tau(x,y,\omega)$. By Proposition 3.14 and Remark 3.15 we find, applying Doob's stopping time theorem,

$$\lambda\mathbb{E}[\tau] \le \mathbb{E}\left[\phi^+(x_\tau,y_\tau) - \phi^+(x,y)\right] \le \ln(\delta) - \ln(d(x,y)) + 2K$$

and

$$\lambda\mathbb{E}[\tau] \ge \mathbb{E}\left[\phi^-(x_\tau,y_\tau) - \phi^-(x,y)\right] \ge \ln(\delta) - \ln(d(x,y)) - 2K.$$

The lemma follows. □

More detailed estimates can be obtained by analysing escape times from strips $\Delta_\delta \setminus \Delta_\varepsilon$. The next lemma is the equivalent of Lemma 4.1 for positive Lyapunov exponent. Stopping times $\tau_{\delta,+}(x,y)$ and $\tau_{\varepsilon,-}(x,y)$ are defined as before in (4.1), (4.2).

Lemma 4.5. *Let R, K be as in Lemma 4.1. Suppose $\lambda > 0$. Let $\gamma \neq 0$ be the negative value so that $\Lambda(\gamma) = 0$. If $0 < \varepsilon < d(x,y) < \delta < R$, then*

$$\mathbb{P}(\min\{\tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\} < \infty) = 1. \tag{4.15}$$

Furthermore, there exists a $\kappa \in (0,1)$, such that if $0 < \varepsilon < d(x,y) < \kappa\delta < \kappa R$, then

$$\frac{1}{K}\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma \le \mathbb{P}\left(\{\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\}\right) \le K\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma. \tag{4.16}$$

Proof. The proof is similar to the proof of Lemma 4.1. Let $(x,y) \in \Delta_\delta \setminus \Delta_\varepsilon$, so with $\varepsilon < d(x,y) < \delta$. Let $\tau_{\varepsilon,-}$ and $\tau_{\delta,+}$ as in (4.1) and (4.2). As before we get (4.15), that is, $\min\{\tau_{\varepsilon,-}, \tau_{\delta,+}\} < \infty$ for almost all ω.

We start with the upper bound in (4.16). Using Remark 3.17 and applying Doob's stopping time theorem, we get

$$\begin{aligned} W^-_{-\gamma}(x,y) \ge{}& \\ &\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\ &+ \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right]. \end{aligned} \tag{4.17}$$

As before, since $\min\{\tau_{\varepsilon,-}, \delta\}$ is almost surely finite (see Lemma 4.1), we have

$$\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right) = 1 - \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right).$$

Furthermore, for some $K > 1$ as in Proposition 3.16, we get

$$\begin{aligned} &\mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \ge \frac{\varepsilon^\gamma}{K}, \\ &\mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right] \ge \frac{\delta^\gamma}{K}, \end{aligned}$$

and also $Kd(x,y)^\gamma \geq W^-_{-\gamma}(x,y)$. Then (4.17) yields

$$\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right) \leq \left(K^2 d(x,y)^\gamma - \delta^\gamma\right) / \left(\varepsilon^\gamma - \delta^\gamma\right).$$

From this and a condition $d(x,y) \leq \kappa\delta$ for small enough κ, we get the stated bound. The lower bound in (4.16) is derived similarly. □

Using the above lemma we get the following statement in which we discuss, for orbit pieces starting in $\Delta_{\kappa\delta} \setminus \Delta_\varepsilon$ until escape from Δ_δ, the expected number of points in Δ_ε. It is similar to Lemma 5.3, but for positive Lyapunov exponent.

The results in this section easily allow to conclude Theorem 1.1(3).

Proposition 4.6. *If $\lambda > 0$, for all $(x,y) \in \mathbb{T}^2 \setminus \Delta$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(T^i_\omega(x), T^i_\omega(y)\right) > 0 \text{ and } \liminf_{n\to\infty} d\left(T^n_\omega(x), T^n_\omega(y)\right) = 0, \ \mathbb{P}-\text{a.s.}$$

Proof. The reasoning follows Proposition 4.3. Let $\varepsilon > 0$. For $(x,y) \in \mathbb{T}^2 \setminus \Delta$ and $\omega \in \Sigma_\vartheta$, consider the empirical count of iterates in Δ_ε,

$$N_\varepsilon(x,y,\omega) = \lim_{n\to\infty} \frac{\#\{i \in \mathbb{N}, 0 \leq i \leq n-1 : d(T^i_\omega(x), T^i_\omega(y)) < \varepsilon\}}{n}.$$

By Lemma 4.4 and Lemma 2.3 and the strong law of large numbers, we get that $\mathbb{P}$-almost surely $N_\varepsilon(x,y,\omega) < 1$, for a $\varepsilon > 0$ implying the first part of the statement. The second part follows from Lemma 2.5. □

4.3. **Negative Lyapunov exponent.** In this section we consider negative Lyapunov exponent $\lambda < 0$. This will be assumed to hold throughout the section. The following lemma can be obtained by the construction of local stable manifolds for T^n_ω of points in Δ, which exists for almost all $\omega \in \Sigma_\vartheta$ [40]. We provide a proof along the lines of our reasoning in previous sections, compare also [8].

Lemma 4.7. *Let R, K be as in Lemma 4.1. Suppose $\lambda < 0$. There is $0 < \chi < 1$ so that the following holds. For each $0 < \delta < R$ there is $0 < \delta' < \delta$, so that for (x,y) with $d(x,y) < \delta'$,*

$$\mathbb{P}\left(d(T^n_\omega(x), T^n_\omega(y)) < \delta \text{ for all } n \text{ and } \lim_{n\to\infty} d(T^n_\omega(x), T^n_\omega(y)) = 0\right) > \chi.$$

Proof. Let $\gamma > 0$ given by Lemma 3.4 be so that $\Lambda(\gamma) = 0$. The first part of the proof is similar to the proof of Lemma 4.1 and Lemma 4.5. Take $(x,y) \in \Delta_\delta \setminus \Delta_\varepsilon$, so with $\varepsilon \leq d(x,y) < \delta$. Let $\tau_{\varepsilon,-}(x,y)$ and $\tau_{\delta,+}(x,y)$ as in (4.1) and (4.2). As before we get (4.15), that is, $\min\{\tau_{\varepsilon,-}(x,y), \tau_{\delta,+}(x,y)\} < \infty$ for almost all ω. Thus, as in Lemma 4.1,

$$\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right) = 1 - \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right). \tag{4.18}$$

Using Remark 3.17 and applying Doob's stopping time theorem, we get

$$\begin{aligned} W^-_{-\gamma}(x,y) \geq{}& \\ &\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right) \mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\ &+ \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right) \mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right]. \end{aligned} \tag{4.19}$$

For some $K > 1$ as in Proposition 3.16, we get

$$\mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\varepsilon,-}(x,y)}, y_{\tau_{\varepsilon,-}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \geq \frac{\varepsilon^\gamma}{K},$$
$$\mathbb{E}\left[W^-_{-\gamma}(x_{\tau_{\delta,+}(x,y)}, y_{\tau_{\delta,+}(x,y)}) \mid \tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right] \geq \frac{\delta^\gamma}{K},$$

and also $Kd(x,y)^\gamma \geq W^-_{-\gamma}(x,y)$. Using (4.18), (4.19) yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right) &\geq \lim_{\varepsilon \to 0} \left(K^2 d(x,y)^\gamma - \delta^\gamma\right) / \left(\varepsilon^\gamma - \delta^\gamma\right) \\ &= 1 - K^2 \left(\frac{d(x,y)}{\delta}\right)^\gamma. \end{aligned} \tag{4.20}$$

The limit exists because the probability is monotone decreasing as $\varepsilon \to 0$. This lower bound for $\mathbb{P}$ is strictly positive if $d(x,y)/\delta$ is small enough. The computation means that there is a strictly positive probability

$$\mathbb{P}\left(d(x_n,y_n) < \delta \text{ for all } n \in \mathbb{N} \text{ and } \liminf_{n\to\infty} d(x_n,y_n) = 0\right) \geq 1 - K^2\left(\frac{d(x,y)}{\delta}\right)^\gamma,$$

at least for $d(x,y)/\delta$ small enough. Note that $\mathbb{P}$ goes to one if $d(x,y)/\delta$ goes to zero.

We obtain the lemma from the observation that the above argument applies to any δ. To finish the argument, take $0 < d_2 < \delta_2 < d(x,y) < \delta$ and, using (4.20), consider

$$\begin{aligned}&\mathbb{P}\left(\tau_{d_2,-}(x,y) < \tau_{\delta,+}(x,y) \text{ and } d(x_i,y_i) < \delta_2 \text{ for all } i \geq \tau_{d_2,-}(x,y)\right) \\ &\qquad \geq \left(1 - K^2\left(\frac{d(x,y)}{\delta}\right)^\gamma\right)\left(1 - K^2\left(\frac{d_2}{\delta_2}\right)^\gamma\right).\end{aligned}$$

The lemma follows by taking d_2 and δ_2 to zero, with d_2/δ_2 small enough, and noting that the bound on the right hand side stays strictly positive. □

Synchronisation of typical orbits expressed by Theorem 1.11 is a consequence of the above lemma and our hypotheses on the random dynamical system.

Proposition 4.8. *Suppose $\lambda < 0$. For all $x, y \in \mathbb{T}$, for $\mathbb{P}$-almost all $\omega \in \Sigma_\vartheta$,*

$$\lim_{n\to\infty} d(T^n_\omega(x), T^n_\omega(y)) = 0.$$

Proof. Take $0 < \delta' < \delta$ as in Lemma 4.7. By Lemma 2.5 there is a strictly positive probability for an orbit $T^n_\omega(x,y)$ to enter $\Delta_{\delta'}$ in finitely many steps. That is, there exists $C > 0$ and $N > 0$ so that for $(x,y) \in \mathbb{T}^2 \setminus \Delta_{\delta'}$,

$$\mathbb{P}\left((T^n_\omega(x), T^n_\omega(y)) \in \Delta_{\delta'} \text{ for some } 0 < n < N\right) > C.$$

We find that for $(x,y) \in \mathbb{T}^2 \setminus \Delta_{\delta'}$,

$$\mathbb{P}\left((T^n_\omega(x), T^n_\omega(y)) \notin \Delta_{\delta'} \text{ for all } 0 \leq n < kN\right) \geq (1-C)^k,$$

so that $\mathbb{P}\left((T^n_\omega(x), T^n_\omega(y)) \notin \Delta_{\delta'} \text{ for all } n\right) = 0$. There are therefore almost surely infinitely many orbit points from $(T^n_\omega(x), T^n_\omega(y))$ in $\Delta_{\delta'}$.

As before, for $0 < \varepsilon < \delta$ and $(x,y) \in \mathbb{T}^2$ with $\varepsilon < d(x,y) < \delta$, $(T^n_\omega(x), T^n_\omega(y))$ will be outside $\Delta_\delta \setminus \Delta_\varepsilon$ for some $n > 0$, almost surely. Combined with the above we see that almost surely, orbit points $(T^n_\omega(x), T^n_\omega(y))$ are in Δ_ε infinitely often. This holds for any $\varepsilon > 0$, so that

$$\liminf_{n\to\infty} d(T^n_\omega(x), T^n_\omega(y)) = 0$$

almost surely. That is, given a sequence ε_k that is decreasing to zero, there is a sequence of first iterates $(T^{n_k}_\omega(x), T^{n_k}_\omega(y))$ inside Δ_{ε_k}. Lemma 4.7 now implies synchronisation. □

5. Stationary measures for the two-point motion

This section contains both the construction of stationary measures on $\mathbb{T}^2 \setminus \Delta$ for the random two-point maps, in the case of zero and positive Lyapunov exponent, and the derivation of their asymptotics at the diagonal. The results apply to zero and positive Lyapunov exponent.

5.1. **Construction by inducing.** We construct a stationary Radon measure on $\mathbb{T}^2 \setminus \Delta$ for the two-point random dynamical system with Lyapunov exponent greater than or equal to zero. We do this by inducing: we construct a stationary measure for a first return map on a domain away from the diagonal Δ by the Krylov-Bogolyubov method. Following work by Deroin, Kleptsyn, Navas and Parwani [24], who study random walks on the real line, we introduce random stopping times in order to be able to apply a Krylov-Bogolyubov method to find stationary measures.

A stationary measure is obtained as usual by pushing forward the stationary measure for the first return map. For a zero Lyapunov exponent, Lemma 4.2 implies that it takes in expectation infinite time to get away from the diagonal and therefore the stationary measure is not finite. For a positive Lyapunov exponent, a stationary measure for the two-point motion can be constructed just as in the case of zero Lyapunov exponent. The finite expectation of the escape time from Δ_δ as expressed by Lemma 4.4, shows that the total measure is finite. Normalizing the measure, we derive the existence of a stationary probability measure.

Proposition 5.1. *The random dynamical system* (2.3) *has the following properties:*

- *if* $\lambda = 0$*, then the two-point motion admits an infinite stationary Radon measure* $\mu^{(2)}$ *on* $\mathbb{T}^2 \setminus \Delta$*, and*
- *if* $\lambda > 0$*, then the two-point motion admits a stationary probability measure* $\mu^{(2)}$ *on* $\mathbb{T}^2 \setminus \Delta$*.*

Proof. For a fixed small $\varepsilon > 0$ take the compact set

$$\mathcal{K} = \mathbb{T}^2 \setminus \Delta_\varepsilon.$$

Because T is of degree two, each element in $\mathbb{T}$ has two distinct pre-images. The minimal distance between pre-images is smaller than $R_{\min}$ (recall (2.1)). Take $\varepsilon < R_{\min}$, so that points in $\mathbb{T}^2 \setminus (\mathcal{K} \cup \Delta)$ can not be mapped into Δ by $T^{(2)}_a$. Note that there will be points in $\mathcal{K}$ that are mapped into Δ by some $T^{(2)}_a$. Namely, any $(x, y) \in \mathbb{T}^2$ with $x \neq y$ and $T_a(x) = T_a(y)$ lies in $\mathcal{K}$ and will be mapped into Δ by $T^{(2)}_a$. By Lemma 4.2, for any

$(x_0, y_0) \in \mathcal{K}$, of the random orbit $(x_n, y_n) = T^{(2)}_\omega(x_0, y_0)$ has almost surely infinitely many points contained in $\mathcal{K}$.

Fix a smooth function $\xi : \mathbb{T}^2 \to [0, 1]$ with support disjoint from Δ and with $\xi \equiv 1$ on $\mathcal{K}$. For an initial point (x_0, y_0) consider the random stopping time $V(\omega) \geq 1$ (suppressing dependence on (x_0, y_0)) so that the probability $\mathbb{P}(V = n + 1 \mid V \geq n)$ equals $\xi(x_{n+1}, y_{n+1})$. So when the random orbit arrives at (x_{n+1}, y_{n+1}) we stop with probability $\xi(x_{n+1}, y_{n+1})$ and continue with probability $1 - \xi(x_{n+1}, y_{n+1})$.

We use $\mathbb{E}$ to denote the expectation both over Σ_ϑ and over the random process defining the random stopping time. So $\mathbb{E}\left[\delta_{(x_V, y_V)}\right]$ is the distribution of the stopped point (x_V, y_V). It is an element of the space $\mathcal{P}(\operatorname{supp}\xi)$ of probability measures on $\operatorname{supp}\xi$, which we endow with the weak star topology.

We claim that this distribution depends continuously on $(x_0, y_0) \in \mathcal{K}$ in the weak star topology. Namely, consider a sequence of distributions of stopped points for a converging sequence of initial points (x_0^n, y_0^n). From (4.13) we get that for $\zeta > 0$ small there exists $N > 0$ so that with probability at least $1 - \zeta$, the stopping time V is smaller than N. As the maps T_a depend continuously on a, the points (x_V, y_V), $V < N$, for (x_0^n, y_0^n) are close to those for (x_0, y_0). Consequently, the distribution $\mathbb{E}\left[\delta_{(x_V, y_V)}\right]$ of the stopped point depends continuously on (x_0, y_0), for $(x_0, y_0) \in \mathcal{K}$, in the weak star topology.

Define

$$P_\xi \mu = \int_{\mathbb{T}^2} \mathbb{E}\left[\delta_{(x_V, y_V)}\right] \, d\mu(x_0, y_0) \tag{5.1}$$

acting on $\mathcal{P}(\operatorname{supp}\xi)$. Then (5.1) is a continuous map from $\mathcal{P}(\operatorname{supp}\xi)$ to itself. We can therefore apply the Krylov-Bogolyubov procedure of taking a converging subsequence of Césaro averages, to find a P_ξ invariant probability measure ς_0,

$$P_\xi \varsigma_0 = \varsigma_0,$$

with $\sup \varsigma_0 \subset \operatorname{supp}\xi$.

We will use this to construct a stationary Radon measure on $\mathbb{T}^2 \setminus \Delta$. Consider the average measures

$$\overline{m}_{(x_0, y_0)} := \mathbb{E}\left[\sum_{j=0}^{V(\omega)-1} \delta_{(x_j, y_j)}\right]. \tag{5.2}$$

Because $1 - \xi$ vanishes on $\mathcal{K}$, we can write , with $(x_j, y_j) = T^{(2)}_{\omega_{j-1}} \circ \cdots \circ T^{(2)}_{\omega_0}(x_0, y_0)$,

$$\overline{m}_{(x_0, y_0)} = \sum_{n=0}^{\infty} \iint_{\substack{\omega_0, \ldots, \omega_{n-1} \\ \in \mathbb{T}}} \prod_{j=1}^{n} (1 - \xi(x_j, y_j)) \, \delta_{(x_n, y_n)} \, d\mathbb{P}(\omega_0) \cdots d\mathbb{P}(\omega_{n-1}). \tag{5.3}$$

Integrate over (x_0, y_0) taken from the measure ς_0 to obtain

$$\mu^{(2)} := \int_{\mathbb{T}^2} \overline{m}_{(x_0, y_0)} \, d\varsigma_0(x_0, y_0). \tag{5.4}$$

Written out this reads

$$\mu^{(2)}(A) = \int_{\mathbb{T}^2} \mathbb{E}\left[\sum_{j=0}^{V(\omega)-1} \mathbb{1}_A(T^j_\omega(x), T^j_\omega(y))\right] d\varsigma_0(x_0, y_0)$$

for Borel sets $A \subset \mathbb{T}^2 \setminus \Delta$.

We claim that for any continuous function $\psi : \mathbb{T}^2 \to \mathbb{R}$ with support disjoint from Δ,

$$\int_{\mathbb{T}^2} \psi \, dm = \int_{\mathbb{T}^2} \mathbb{E}\left[\sum_{j=0}^{V(\omega)-1} \psi(x_j)\right] d\varsigma_0(x_0, y_0) \tag{5.5}$$

is well defined and finite. We conclude from this that $\mu^{(2)}$ is a Radon measure that assigns finite measure to compact sets disjoint from Δ. To establish the claim note that there are $N > 0$ and $q > 0$ so that with probability at least q an orbit starting in $\operatorname{supp}\psi$ hits $\mathcal{K}$ in at most N steps. Note that iterates outside $\operatorname{supp}\psi$ do not contribute to the right hand side of (5.5). We find an estimate

$$\mathbb{E}\left[\sum_{j=0}^{V(\omega)-1} \psi(x_j, y_j)\right] < \sum_{i=1}^{\infty} \max_{(x,y)\in\mathbb{T}^2} |\psi(x,y)| N(1-q)^{i-1} < \infty.$$

That is, $\mathbb{E}\left[\sum_{j=0}^{V(\omega)-1} |\psi(x_j, y_j)|\right]$ is finite and bounded uniformly on $\operatorname{supp}\psi$. This implies that the right hand side of (5.5) is finite.

We will establish that $\mu^{(2)}$ is stationary for the two-point maps. We must thus show $P\mu^{(2)} = \mu^{(2)}$ with

$$P\mu^{(2)} := \int_{\mathbb{T}} \left(T^{(2)}_a\right)_* \mu^{(2)} \, d\mathbb{P}(a).$$

Applying P yields, with $(x_j, y_j) = T^{(2)}_{\omega_{j-1}} \circ \cdots \circ T^{(2)}_{\omega_0}(x_0, y_0)$,

$$\begin{aligned}
P\overline{m}_{x_0,y_0} &= \int_{\mathbb{T}} \left(T^{(2)}_a\right)_* \overline{m}_{x_0,y_0} \, d\mathbb{P}(a) \\
&= \sum_{n=0}^{\infty} \iint\limits_{\substack{a,\omega_0,\ldots,\omega_{n-1}\\ \in\mathbb{T}}} \prod_{j=1}^{n} (1 - \xi(x_j, y_j)) \left(T^{(2)}_a\right)_* \delta_{(x_n,y_n)} \, d\mathbb{P}(a) d\mathbb{P}(\omega_0) \cdots d\mathbb{P}(\omega_{n-1}) \\
&= \sum_{n=0}^{\infty} \iint\limits_{\substack{\omega_0,\ldots,\omega_n\\ \in\mathbb{T}}} \prod_{j=1}^{n} (1 - \xi(x_j, y_j)) \, \delta_{(x_{n+1},y_{n+1})} \, d\mathbb{P}(\omega_0) d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_n) \\
&= \mathbb{E}\left[\sum_{j=1}^{V(\omega)} \delta_{(x_j,y_j)}\right]
\end{aligned}$$

(again using that $1 - \xi$ vanishes on $\mathcal{K}$).

Compared to (5.2), the index j is counting from 1 to $V(\omega)$ instead of from 0 to $V(\omega)-1$. We thus find

$$P\overline{m}_{(x_0,y_0)} = \overline{m}_{(x_0,y_0)} - \delta_{(x_0,y_0)} + \mathbb{E}\left[\delta_{(x_{V(\omega)},y_{V(\omega)})}\right].$$

Integration over ς_0 yields

$$\begin{aligned}
P\mu^{(2)} &= P\int_{\mathbb{T}^2} \overline{m}_{(x_0,y_0)}\, d\varsigma_0(x_0,y_0)\\
&= \int_{\mathbb{T}^2} P\overline{m}_{(x_0,y_0)}\, d\varsigma_0(x_0,y_0)\\
&= \int_{\mathbb{T}^2} \overline{m}_{(x_0,y_0)}\, d\varsigma_0(x_0,y_0) - \int_{\mathbb{T}^2} \delta_{(x_0,y_0)}\, d\varsigma_0(x_0,y_0) + \int_{\mathbb{T}^2} \mathbb{E}\left[\delta_{(x_{V(\omega)},y_{V(\omega)})}\right]\, d\varsigma_0(x_0,y_0)\\
&= \mu^{(2)} - \varsigma_0 + P_\xi\varsigma_0\\
&= \mu^{(2)},
\end{aligned}$$

the last step by P_ξ invariance of ς_0.

Note that the argument includes the observation that $\mu^{(2)}$ assigns finite measure to compact sets disjoint from Δ.

For $\lambda = 0$, Lemma 2.5 shows that iterates of points $(x,y)\in\mathbb{T}^2$ enter any small neighborhood of Δ with positive probability. Combining this with Lemma 4.2 shows

$$\begin{aligned}
\mu^{(2)}(\mathbb{T}^2) &= \int_{\mathbb{T}^2} \mathbb{E}\left[\sum_{j=0}^{V(\omega)-1} \mathbb{1}_{\mathbb{T}^2}(T^j_\omega(x), T^j_\omega(y))\right] d\varsigma_0(x_0,y_0)\\
&= \infty.
\end{aligned}$$

For $\lambda > 0$, by Lemma 4.4, we see that

$$X = \int_{\mathbb{T}^2} \mathbb{E}\left[\sum_{j=0}^{V(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2}(T^j_\omega(x), T^j_\omega(y))\right] d\varsigma_0(x,y) = \int_{\mathbb{T}^2} \mathbb{E}\left[V(x,y,\omega)\right]\, d\varsigma_0(x,y) < \infty.$$

Now $\mu^{(2)}$ given by

$$\mu^{(2)}(A) = \frac{1}{X}\int_{\mathbb{T}^2} \mathbb{E}\left[\sum_{j=0}^{V(x,y,\omega)-1} \mathbb{1}_{A}(T^j_\omega(x), T^j_\omega(y))\right] d\varsigma_0(x,y)$$

is a stationary measure. Since $\mu^{(2)}(\mathbb{T}^2) = 1$, we find from this expression that $\mu^{(2)}$ is a probability measure. □

Remark 5.2. The stationary measure $\mu^{(2)}$ is obtained by pushing forward the measure ς_0 on the support of a test function ξ. The test function is constant one on a suitable set $\mathcal{K}$. The construction shows that $\mu^{(2)}$ restricted to $\mathcal{K}$ equals ς_0 restricted to $\mathcal{K}$ (see (5.3) and

(5.4)). We can therefore also obtain $\mu^{(2)}$ from ς_0 restricted to $\mathcal{K}$ by pushing forward. That is, with

$$\widehat{m}_{(x_0,y_0)} = \sum_{n=0}^{\infty} \underset{\substack{\omega_0,\dots,\omega_{n-1}\\ \in\mathbb{T}}}{\iint} \prod_{j=1}^{n} (1 - \mathbb{1}_{\mathcal{K}}(x_j, y_j))\, \delta_{(x_n,y_n)}\, d\lambda(\omega_0)\cdots d\lambda(\omega_{n-1})$$

for $(x_0, y_0) \in \mathcal{K}$, we find

$$\mu^{(2)} = \int_{\mathcal{K}} \widehat{m}_{(x_0,y_0)}\, d\varsigma_0(x_0, y_0).$$

■

5.2. **The support of the stationary measure.** Hypothesis (H5) and Lemma 2.4 show that for all $\varepsilon > 0$, there exists a $k \in \mathbb{N}$, such that images under $\Theta^{(2)}$ of a set $\Sigma_\vartheta \times \{(x, y)\}$ for $(x, y) \in \mathbb{T}^2 \setminus \Delta_\varepsilon$, cover $\Sigma_\vartheta \times \mathbb{T}^2$. This implies that stationary measures for the two-point motion have full support, if they are obtained from an inducing scheme as in Section 5.1.

5.3. **The growth rate of the stationary measure at the diagonal for $\lambda = 0$.** The following lemma discusses the expected number of such orbit points that lie inside strips $\Delta_\delta \setminus \Delta_\varepsilon$, in the limit of ε going to zero. The obtained bounds will be used below to derive the growth-rate near the diagonal of stationary measures for the two-point motion.

Lemma 5.3. *Let R, K be as in Lemma 4.1. Suppose $\lambda = 0$. Assume $0 < \varepsilon < \delta < R$. For $(x, y) \in \mathbb{T}^2$ with $0 < d(x, y) < \delta$, define*

$$g_{\varepsilon,\delta}(x, y) = \mathbb{E}\left[\sum_{i=0}^{\tau_{\delta,+}(x,y)} \mathbb{1}_{[\varepsilon,\infty)}(d(T^i_\omega(x), T^i_\omega(y)))\right].$$

Then

$$\frac{1}{V}\left(\ln\left(\frac{\delta a_1}{d(x,y)}\right) - 6K\right) \le \liminf_{\varepsilon\to 0} \frac{g_{\varepsilon,\delta}(x,y)}{-\ln(\varepsilon)} \le \limsup_{\varepsilon\to 0} \frac{g_{\varepsilon,\delta}(x,y)}{-\ln(\varepsilon)} \le \frac{1}{V}\left(\ln\left(\frac{\delta a_2}{d(x,y)}\right) + 6K\right).$$

Proof. We follow [9, Proposition 5.6], with modifications needed for the discrete time setting. Recall that a_1, a_2 are given in Hypothesis (H1). Define, for $\varepsilon < r < \delta$,

$$g^-_{\varepsilon,\delta}(r) = \inf\{g_{\varepsilon,\delta}(x, y)\ ;\ ra_1 \le d(x, y) \le r\},$$
$$g^+_{\varepsilon,\delta}(r) = \sup\{g_{\varepsilon,\delta}(x, y)\ ;\ r \le d(x, y) \le ra_2\}.$$

Observe that $r \mapsto g^\pm_{\varepsilon,\delta}(r)$ is a monotone non-increasing function on $[\varepsilon, \delta]$, and constant on $(0, \varepsilon]$.

We first focus on $g^-_{\varepsilon,\delta}$. Conditioning on the smallest stopping time $\tau_{\delta,+}$ or $\tau_{\varepsilon,-}$, which have been defined in (4.1) and (4.2), we obtain

$$\begin{aligned} g_{\varepsilon,\delta}(x,y) &= \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) > \tau_{\delta,+}(x,y)\right)\mathbb{E}[\tau_{\delta,+}(x,y)] \\ &\quad + \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,-}(x,y)\right)\mathbb{E}[\tau_{\varepsilon,-}(x,y)] \\ &\quad + \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,-}(x,y)\right)\mathbb{E}\left[g_{\varepsilon,\delta}(x_{\tau_{\varepsilon,-}}, y_{\tau_{\varepsilon,-}}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\ &= \mathbb{E}\left[\min\{\tau_{\delta,+}(x,y), \tau_{\varepsilon,-}(x,y)\}\right] \\ &\quad + \mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right)\mathbb{E}\left[g_{\varepsilon,\delta}(x_{\tau_{\varepsilon,-}}, y_{\tau_{\varepsilon,-}}) \mid \tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\ &\geq \mathbb{E}\left[\min\{\tau_{\delta,+}(x,y), \tau_{\varepsilon,-}(x,y)\}\right] + g^-_{\varepsilon,\delta}(\varepsilon)\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right). \end{aligned}$$

Due to monotonicity of $g^-_{\varepsilon,\delta}(r)$ in r, we have for $\varepsilon < r < \delta$,

$$\begin{aligned} g^-_{\varepsilon,\delta}(\varepsilon) &\geq g^-_{\varepsilon,\delta}(r) \\ &\geq \inf\left\{g_{\varepsilon,\delta}(x,y)\ ;\ a_1 r \leq d(x,y) \leq r\right\} \\ &\geq \inf_{ra_1 \leq d(x,y) \leq r}\left\{\mathbb{E}\left[\min\{\tau_{\delta,+}(x,y), \tau_{\varepsilon,-}(x,y)\}\right] + g^-_{\varepsilon,\delta}(\varepsilon)\mathbb{P}\left(\tau_{\varepsilon,-}(x,y) < \tau_{\delta,+}(x,y)\right)\right\}. \end{aligned}$$

We get the lower bound for $g_{\varepsilon,\delta}(x,y)$ by setting $r = d(x,y)$, using (4.4) and (4.5) from Lemma 4.1, and $g^-_{\varepsilon,\delta}(\varepsilon) \geq g^-_{\varepsilon,\delta}(r)$,

$$\begin{aligned} g_{\varepsilon,\delta}(x,y) &\geq g^-_{\varepsilon,\delta}(r) \\ &\geq \inf_{ra_1 \leq d(x,y) \leq r}\left\{\frac{1}{V}\left(\ln\left(\frac{\delta}{d(x,y)}\right)\ln\left(\frac{d(x,y)}{\varepsilon}\right) - 6K|\ln\varepsilon| + 2K^2\right)\right. \\ &\qquad \left. + g^-_{\varepsilon,\delta}(\varepsilon)\frac{\ln\left(\dfrac{\delta}{d(x,y)}\right) - 2K}{\ln\left(\dfrac{\delta}{\varepsilon}\right)}\right\} \\ &\geq \frac{1}{V}\inf_{r/a \leq d(x,y) \leq r}\left\{\ln\left(\frac{\delta}{d(x,y)}\right)\ln\left(\frac{d(x,y)}{\varepsilon}\right)\right\} \\ &\qquad - 6K|\ln(\varepsilon)| + 2K^2 + g^-_{\varepsilon,\delta}(r)\frac{\ln\left(\dfrac{\delta}{ra_1}\right) - 2K}{\ln\left(\dfrac{\delta}{\varepsilon}\right)}. \end{aligned}$$

Divide by $|\ln(\varepsilon)|$ and take $\liminf_{\varepsilon\to 0}$. This yields

$$\liminf_{\varepsilon\to 0}\frac{g^-_{\varepsilon,\delta}(\delta)}{|\ln(\varepsilon)|} \geq \inf_{a_1 r < d(x,y) < r}\frac{\ln\left(\frac{\delta}{d(x,y)}\right) - 6K}{V} \geq \frac{\ln\left(\frac{\delta}{r/a}\right) - 6K}{V}.$$

The bound for $g^+_{\varepsilon,\delta}(r)$ is obtained by following a similar scheme. □

The following proposition allows us to estimate on the growth-rate of the stationary measure $\mu^{(2)}$ near the diagonal, in the case $\lambda = 0$.

Proposition 5.4. *Suppose $\lambda = 0$. Given the stationary measure $\mu^{(2)}$, there exist $\alpha, \beta \in (0, \infty)$, such that*

$$\alpha \leq \liminf_{\varepsilon \to 0} \frac{\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon)}{-\ln(\varepsilon)} \leq \limsup_{\varepsilon \to 0} \frac{\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon)}{-\ln(\varepsilon)} \leq \beta. \tag{5.6}$$

Proof. Temporary fix $\varepsilon > 0$ small. For $0 < \varepsilon < \delta$ for suitable small δ, set $\mathcal{K} = \mathbb{T}^2 \setminus \Delta_\delta$. For $(x, y) \in \mathcal{K}$, write

$$N(x, y, \omega) = \min\{n > 0 \; ; \; (T^n_\omega(x), T^n_\omega(y)) \in \mathcal{K}\}$$

for the first return time to $\mathcal{K}$. By Remark 5.2 we have that for all stationary Radon measures $\mu^{(2)}$ on $\mathbb{T}^2 \setminus \Delta$, the restricted measure $\mu_\mathcal{K} = \mu^{(2)}|_\mathcal{K}$ is a stationary measure for the induced process

$$(x_{n+1}, y_{n+1}) = \left(T^{(2)}_\omega\right)^{N(x_n, y_n, \omega)} (x_n, y_n)$$

on $\mathcal{K}$. For convenience we rescale $\mu_\mathcal{K}$ so that it becomes a probability measure,

$$\mu_\mathcal{K}(\mathcal{K}) = 1.$$

Denote the set $\mathcal{G} \subset \mathcal{K} \times \Sigma_\vartheta$ as the union of $(x, y, \omega) \in \mathcal{K} \times \Sigma_\vartheta$, such that there exists $\tau(x, y, \omega) \in \mathbb{N}$ with the following properties,

(1) for $0 < i < \tau$, $(T^i_\omega(x), T^i_\omega(y)) \notin \mathcal{K}$,
(2) $(T^{\tau(x,y,\omega)}_\omega(x), T^{\tau(x,y,\omega)}_\omega(y)) \in \Delta_{e^{-7K} a_1 \delta}$.

Here K is as in Lemma 4.1. By Lemma 2.5, we have $(\mu_\mathcal{K} \times \mathbb{P})(\mathcal{G}) > 0$.

Now we have the ingredients to prove the lower bound for the lim inf. For all measurable sets $A \subset \mathbb{T}^2$, from Remark 5.2 we obtain

$$\mu^{(2)}(A) = \int_{\mathcal{K} \times \Sigma_\vartheta} \sum_{j=0}^{N(x,y,\omega)-1} \mathbb{1}_A(T^j_\omega(x), T^j_\omega(y)) \, d(\mu_\mathcal{K} \times \mathbb{P})(x, y, \omega). \tag{5.7}$$

By (5.7) and Lemma 5.3,

$$\begin{aligned}
\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon) &= \int_{\mathcal{K} \times \Sigma_\vartheta} \sum_{j=0}^{N(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2 \setminus \Delta_\varepsilon}(T^j_\omega(x), T^j_\omega(y)) \, d(\mu_\mathcal{K} \times \mathbb{P})(x, y, \omega) \\
&\geq \int_\mathcal{G} \sum_{j=0}^{N(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2 \setminus \Delta_\varepsilon}(T^j_\omega(x), T^j_\omega(y)) \, d(\mu_\mathcal{K} \times \mathbb{P})(x, y, \omega) \\
&\geq \int_\mathcal{G} \sum_{j=\tau(x,y,\omega)}^{N(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2 \setminus \Delta_\varepsilon}(T^j_\omega(x), T^j_\omega(y)) \, d(\mu_\mathcal{K} \times \mathbb{P})(x, y, \omega) \\
&\geq \int_\mathcal{G} g_{\varepsilon,\delta}\left(\left(T^{(2)}_\omega\right)^{\tau(x,y,\omega)} (x, y) \right) d(\mu_\mathcal{K} \times \mathbb{P})(x, y, \omega).
\end{aligned}$$

To get the first inequality of (5.6) we divide both sides by $-\ln(\varepsilon)$, take the lim inf, and apply Lemma 5.3. This yields

$$\liminf_{\varepsilon\to 0} \frac{\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon)}{-\ln(\varepsilon)} \geq \frac{(K+\ln(a_1))(\mu_K \times \mathbb{P})(\mathcal{G})}{V}.$$

This proves the lower bound for the lim inf.

To get the upper bound for the lim sup we use a similar technique. Here we have to incorporate the possibility that points in $\mathcal{K}$ are mapped onto, or close to, Δ by a single iterate of $T^{(2)}_\omega$. When pushing forward $\mu_\mathcal{K}$, this moves mass directly to small neighborhoods of Δ. Write $\mathcal{K} \times \Sigma_\vartheta = \mathcal{R}_0 \cup \mathcal{R}_+$ as a union of sets on which N is either equal to 1 or is bigger than 1,

$$\mathcal{R}_0 = \{(x,y,\omega) \in \mathcal{K} \times \Sigma_\vartheta \ ;\ N(x,y,\omega) = 1\},$$
$$\mathcal{R}_+ = \{(x,y,\omega) \in \mathcal{K} \times \Sigma_\vartheta \ ;\ N(x,y,\omega) > 1\}.$$

The set $\mathcal{R}_+$ is a disjoint union of a set

$$\mathcal{G}_d = \mathcal{R}_+ \cap (\Delta_{R_{\min}} \times \Sigma_\vartheta)$$

(recall (2.1) for the definition of $R_{\min}$) and its complement $\mathcal{G}_c$, which is contained in $(\Delta_{a_2\delta} \setminus \Delta_\delta) \times \Sigma_\vartheta$. For $(x,y,\omega) \in \mathcal{G}_c$ we find $T^{(2)}_\omega(x,y) \subset \Delta_\delta \setminus \Delta_{a_1\delta}$. We have

$$\begin{aligned}
\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon) &= \int_\mathcal{K} \int_{\Sigma_\vartheta} \sum_{j=0}^{N(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2\setminus\Delta_\varepsilon}(T^j_\omega(x), T^j_\omega(y))\, d\mathbb{P}(\omega) d\mu_\mathcal{K}(x,y) \\
&= \int_\mathcal{K} \int_{\Sigma_\vartheta} \sum_{j=0}^{N(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2\setminus\Delta_\varepsilon}(T^j_\omega(x), T^j_\omega(y))\, d\mathbb{P}(\omega) d\mu_\mathcal{K}(x,y) \\
&\leq \iint_{\mathcal{R}_0} d\mathbb{P}(\omega) d\mu_\mathcal{K}(x,y) + \iint_{\mathcal{R}_+} \sum_{j=1}^{N(x,y,\omega)-1} \mathbb{1}_{\mathbb{T}^2\setminus\Delta_\varepsilon}(T^j_\omega(x), T^j_\omega(y))\, d\mathbb{P}(\omega) d\mu_\mathcal{K}(x,y) \\
&\leq \mu_\mathcal{K}(\mathcal{K}) + \iint_{\mathcal{R}_+} g_{\varepsilon,\delta}(T^{(2)}_\omega(x,y))\, d\mathbb{P}(\omega) d\mu_\mathcal{K}(x,y). \qquad (5.8)
\end{aligned}$$

Recall that by Lemma 2.3, we have the existence of $C_1 > 0$ with

$$\mathbb{E}\left[-\ln(d(T^{(2)}_\omega(x,y)))\right] < C_1, \qquad (5.9)$$

for all $(x,y) \in \mathcal{K}$.

To conclude the proposition we divide (5.8) by $-\ln(\varepsilon)$ and take the lim sup. Doing this, applying Fatou's lemma and using Lemma 5.3 and (2.4), yields

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \frac{\mu^{(2)}(\mathbb{T}^2 \setminus \Delta_\varepsilon)}{-\ln(\varepsilon)} \\
&\qquad \le \limsup_{\varepsilon\to 0} \frac{\mu_{\mathcal{K}}(\mathcal{K})}{-\ln(\varepsilon)} + \limsup_{\varepsilon\to 0} \iint_{\mathcal{G}_c\cup\mathcal{G}_d} \frac{g_{\varepsilon,\delta}(T_\omega^{(2)}(x,y))}{-\ln(\varepsilon)}\, d\mathbb{P}(\omega) d\mu_{\mathcal{K}}(x,y) \\
&\qquad \le \iint_{\mathcal{G}_c\cup\mathcal{G}_d} \limsup_{\varepsilon\to 0} \frac{g_{\varepsilon,\delta}(T_\omega^{(2)}(x,y))}{-\ln(\varepsilon)}\, d\mathbb{P}(\omega) d\mu_{\mathcal{K}}(x,y) \\
&\qquad \le \iint_{\mathcal{G}_c\cup\mathcal{G}_d} \frac{1}{V}\left(\ln\left(\frac{\delta a_2}{d(T_\omega^{(2)}(x,y))}\right) + 6K\right) d\mathbb{P}(\omega) d\mu_{\mathcal{K}}(x,y) \\
&\qquad \le \frac{\ln(\delta a_2) + 6K + C_1}{V},
\end{aligned}$$

which finishes the proof. □

5.4. **The growth rate of the stationary measure at the diagonal for $\lambda > 0$.** The following proposition allows us to estimate on the growth-rate of the stationary measure $\mu^{(2)}$ near the diagonal, in the case of $\lambda > 0$.

Proposition 5.5. *Suppose $\lambda > 0$ and let γ be the negative value so that $\Lambda(\gamma) = 0$. Suppose $\gamma \in (-1/2, 0)$. Given the stationary measure $\mu^{(2)}$, there exist $\alpha, \beta \in (0,\infty)$, such that*

$$\alpha \le \liminf_{\varepsilon\to 0} \frac{\mu^{(2)}(\Delta_\varepsilon)}{\varepsilon^{-\gamma}} \le \limsup_{\varepsilon\to 0} \frac{\mu^{(2)}(\Delta_\varepsilon)}{\varepsilon^{-\gamma}} \le \beta. \tag{5.10}$$

To prove the above proposition, we first consider orbits near the diagonal.

Lemma 5.6. *Let R be as in Lemma 4.1. Suppose $\lambda > 0$. Assume $0 < \varepsilon < \delta < R$. For $(x, y) \in \mathbb{T}^2$ with $0 < d(x, y) < \delta$, define*

$$f_{\varepsilon,\delta}(x,y) = \mathbb{E}\left[\sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T_\omega^i(x), T_\omega^i(y)))\right].$$

There exist $K < \infty$, $\kappa \in (0,1)$, such that for $(x,y) \in \mathbb{T}^2$, if $0 < \varepsilon < d(x,y) < \kappa\delta < \kappa R$, then

$$\frac{1}{K}\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma \le f_{\varepsilon,\delta}(x,y) \le K\left(\frac{d(x,y)}{\varepsilon}\right)^\gamma.$$

Proof. Let κ be as in Lemma 4.5. Let K_0 be K from Lemma 4.5 and set $K_1 = e^{2\lambda K_0}$. By a straightforward computation for the lower bound, comparable to the proof of Lemma 5.3, we obtain

$$\begin{aligned}
f_{\varepsilon,\delta}(x,y) &\ge \mathbb{P}\left(\tau_{\varepsilon/K_1,-}(x,y) < \tau_{\delta,+}(x,y)\right) \mathbb{E}\left[\tau_{\varepsilon,+}(T_\omega^{(2)\tau_{\varepsilon/K_1}}(x,y)) \mid \tau_{\varepsilon/K_1,-}(x,y) < \tau_{\delta,+}(x,y)\right] \\
&\ge \frac{1}{K_0}\left(\frac{K_1 d(x,y) a_1}{\varepsilon}\right)^\gamma \left(\frac{1}{\lambda}\left(\ln\left(\frac{a_1}{K_1}\right) - 2K_0\right)\right).
\end{aligned}$$

Set

$$f^+_{\varepsilon,\delta}(r) = \sup_{ra_1<d(x,y)\le r} \mathbb{E}\left[\sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x),T^i_\omega(y)))\right] \tag{5.11}$$

and observe that $f^+_{\varepsilon,\delta}$ is monotone decreasing in r. Let $K_2 = \left(\frac{1}{2K_0}\right)^{1/\gamma}/a_2$. When $K_2\varepsilon < \kappa R$, we have

$$\begin{aligned} f^+_{\varepsilon,\delta}(\varepsilon) &\le \sup_{\varepsilon a_1<d(x,y)\le\varepsilon} \mathbb{E}\left[\tau_{(K_2\varepsilon),+}(x,y)\right] + \sup_{\varepsilon K_2<d(x,y)\le\varepsilon K_2 a_2} \mathbb{P}\left(\tau_{\varepsilon,-}(x,y)<\tau_{\delta,+}(x,y)\right) f^+_{\varepsilon,\delta}(\varepsilon) \\ &\le \frac{1}{\lambda}\left(\ln\left(\frac{K_2}{a_1}\right)+2K_0\right) + K_0\left(K_2 a_2\right)^\gamma f^+_{\varepsilon,\delta}(\varepsilon). \end{aligned}$$

Therefore $f^+_{\varepsilon,\delta}(\varepsilon) < \infty$. So, using $f_{\varepsilon,\delta}(x,y) \le \mathbb{P}\left(\tau_{\varepsilon,-}(x,y)<\tau_{\delta,+}(x,y)\right) f^+_{\varepsilon,\delta}(\varepsilon)$, we get the desired result. □

Proof of Proposition 5.5. We follow reasoning of Proposition 5.4, using Lemma 4.4 and Lemma 5.6. Let $R, K > 0$ be as in Lemma 5.6. For $0 < \delta < R$ small, take $\mathcal{K} = \mathbb{T}^2 \setminus \Delta_\delta$. As in Proposition 5.1, a finite measure σ_0 is constructed and from the restriction $\mu_\mathcal{K}$ of σ_0 to $\mathcal{K}$, the measure $\mu^{(2)}$ is obtained by pushing forward $\mu_\mathcal{K}$ by the two-point maps. See Remark 5.2.

By rescaling $\mu_\mathcal{K}$ we may assume

$$\int_{\mathbb{T}^2} \mathbb{E}\left[V(x,y,\omega)\right]\, d\mu_\mathcal{K}(x,y) = 1.$$

Fix $\varepsilon > 0$ small enough and let $\delta < R$. For the stationary measure of the two-point motion we have by Proposition 5.1 that

$$\mu^{(2)}(\Delta_\varepsilon) = \int_{\mathbb{T}^2} \mathbb{E}\left[\sum_{j=0}^{V(x,y,\omega)-1} \mathbb{1}_{\Delta_\varepsilon}(T^j_\omega(x),T^j_\omega(y))\right] d\mu_\mathcal{K}(x_0,y_0).$$

We remark that the condition $|\gamma| < 1/2$ is to bound the mass that is transported to neighborhoods of the diagonal from outside $\Delta_{R_{\min}}$ by this construction (recall (2.1) for the definition of $R_{\min}$).

As before, see the proof of Proposition 5.4, denote the set of $\mathcal{G} \subset (\mathbb{T}^2 \setminus \Delta_\delta) \times \Sigma_\vartheta$ as the union of $(x,y,\omega) \in (\mathbb{T}^2 \setminus \Delta_\delta) \times \Sigma_\vartheta$, such that there exists an $\tau(x,y,\omega) \in \mathbb{N}$, with the following properties:

(1) for all $i \in \mathbb{N}, 0 < i < \tau$, $(T^i_\omega(x), T^i_\omega(y)) \notin (\mathbb{T}^2 \setminus \Delta_\delta)$,
(2) $(T^{\tau(x,y,\omega)}_\omega(x), T^{\tau(x,y,\omega)}_\omega(y)) \in \Delta_{\kappa\delta}$.

Here κ as in the proof of Lemma 5.6.

For the lower bound in (5.10) we consider only orbit pieces near Δ. From Lemma 2.5 we know that $(\mu_\delta \times \mathbb{P})(\mathcal{G}) > 0$.

$$
\begin{aligned}
\mu^{(2)}(\Delta_\varepsilon) &\geq \int_{\mathbb{T}^2 \setminus \Delta_\delta \times \Sigma_\vartheta} \sum_{i=0}^{\tau_{\delta,+}(T_\omega(x),T_\omega(y),\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \\
&\geq \int_{\mathcal{G}} \sum_{i=0}^{\tau_{\delta,+}(T_\omega(x),T_\omega(y),\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \\
&\geq \int_{\mathcal{G}} \sum_{i=\tau(x,y,\omega)}^{\tau_{\delta,+}(T_\omega(x),T_\omega(y),\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \\
&\geq \frac{\varepsilon^{-\gamma}}{K(\kappa\delta)^{-\gamma}} \left(\mu_{\mathcal{K}} \times \mathbb{P}\right)(\mathcal{G}).
\end{aligned}
$$

We proceed with an upper bound. Here we must include orbits that start outside $\Delta_{R_{\min}}$ and jump directly into Δ_δ, and also directly into Δ_ε. This can be seen as pushing forward mass from $\mathcal{K}$ into Δ_ε in a single iterate. We divide the set $\mathcal{G}$ into two subsets, $\mathcal{G} = \mathcal{G}_d \cup \mathcal{G}_c$, with a set

$$
\mathcal{G}_d = \mathcal{G} \cap \left(\mathbb{T}^2 \setminus \Delta_{R_{\min}} \times \Sigma_\vartheta\right)
$$

and its complement $\mathcal{G}_c \subset \left(\Delta_{\delta/a_1} \setminus \Delta_\delta\right) \times \Sigma_\vartheta$.

Using this division, we get

$$
\begin{aligned}
\mu^{(2)}(\Delta_\varepsilon) &\leq \int_{\mathbb{T}^2 \setminus \Delta_\delta \times \Sigma_\vartheta} \sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \\
&\leq \int_{\mathcal{G}_c} \sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \\
&\qquad + \int_{\mathcal{G}_d} \sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega).
\end{aligned}
$$

The first integral can be bounded from above by using Lemma 5.6. Recall that $f^+_{\varepsilon,\delta}$ defined in (5.11) is a monotone function. Now

$$
\begin{aligned}
&\int_{\mathcal{G}_c} \sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \\
&\qquad \leq \int_{\mathcal{G}_c} f^+_{\varepsilon,\delta}(d(T^{(2)}_\omega(x,y))) d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \leq f^+_{\varepsilon,\delta}(\kappa\delta) \leq K\left(\frac{\varepsilon}{a_1\kappa\delta}\right)^{-\gamma}.
\end{aligned}
$$

To handle the second integral we consider separately the subset $\mathcal{G}_\varepsilon \subset \mathcal{G}_d$ of points that are mapped directly in Δ_ε, thus

$$\mathcal{G}_\varepsilon = \left\{ (x,y,\omega) \in \mathcal{G}_d \; ; \; T^{(2)}_\omega(x,y) \in \Delta_\varepsilon \right\}.$$

For $(x,y) \in \Delta_\varepsilon$ let, similar to (4.1),

$$\begin{aligned}\tau_{\varepsilon,+}(x,y) &= \min\{n \in \mathbb{N} \; ; \; d(T^n_\omega(x), T^n_\omega(y)) > \varepsilon\},\\ \tau_{\delta,+}(x,y) &= \min\{n \in \mathbb{N} \; ; \; d(T^n_\omega(x), T^n_\omega(y)) > \delta\}.\end{aligned}$$

By Lemma 4.4 we have an estimate

$$\frac{1}{\lambda}\left(\ln\left(\frac{\varepsilon}{d(x,y)}\right) - 2K\right) \le \mathbb{E}\left[\tau_{\varepsilon,+}(x,y)\right] \le \frac{1}{\lambda}\left(\ln\left(\frac{\varepsilon}{d(x,y)}\right) + 2K\right)$$

for some constant $K > 0$. For $(x,y,\omega) \in \mathcal{G}_\varepsilon$ we find $\tau_{\varepsilon,+}(x,y,\omega) < \tau_{\delta,+}(x,y,\omega)$ almost surely. This allows us to write

$$\begin{aligned}
&\int_{\mathcal{G}_d} \sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega)\\
&\quad\le \int_{\mathcal{G}_\varepsilon} \left(\sum_{i=0}^{\tau_{\varepsilon,+}(x,y,\omega)-1} + \sum_{i=\tau_{\varepsilon,+}(x,y,\omega)}^{\tau_{\delta,+}(x,y,\omega)-1} \right) \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega)\\
&\qquad\quad + \int_{\mathcal{G}_d \setminus \mathcal{G}_\varepsilon} \sum_{i=0}^{\tau_{\delta,+}(x,y,\omega)-1} \mathbb{1}_{(0,\varepsilon]}(d(T^i_\omega(x), T^i_\omega(y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega)\\
&\quad\le \int_{\mathcal{G}_\varepsilon} \tau_{\varepsilon,+}(T^{(2)}_\omega(x,y))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) + \int_{\mathcal{G}_\varepsilon} f^+_{\varepsilon,\delta}(\varepsilon)\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega)\\
&\qquad\qquad\qquad + \int_{\mathcal{G}_d \setminus \mathcal{G}_\varepsilon} f^+_{\varepsilon,\delta}(d(T^{(2)}_\omega(x,y)))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega).
\end{aligned}$$

To bound the first integral in the final expression we use Lemma 4.4, Lemma 2.3 and Hypothesis (H4) to get or any $0 < s < 1$ the existence of a constant $\tilde{C} > 0$, such that

$$\int_{\mathcal{G}_d} \tau_{\delta,+}(T^{(2)}_\omega(x,y))\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \le \tilde{C}\varepsilon^s.$$

Hypothesis (H4) shows the existence of a constant $C > 0$ with $(\mu_{\mathcal{K}} \times \mathbb{P})(\mathcal{G}_\varepsilon) < C\sqrt{\varepsilon}$. Lemma 4.4 shows that

$$f^+_{\varepsilon,\delta}(\varepsilon) \le C\frac{1}{\lambda}$$

for some $C > 0$. We get therefore a bound

$$\int_{\mathcal{G}_\varepsilon} f^+_{\varepsilon,\delta}(\varepsilon)\, d(\mu_{\mathcal{K}} \times \mathbb{P})(x,y,\omega) \le C\sqrt{\varepsilon}$$

for some $C > 0$. The third integral is bounded by

$$\int_{\mathcal{G}_d \setminus \mathcal{G}_\varepsilon} f^+_{\varepsilon,\delta}(d(T^{(2)}_\omega(x,y)))\, d(\varsigma_\delta \times \mathbb{P})(x,y,\omega)$$
$$\leq \int_{\mathcal{G}_d \setminus \mathcal{G}_\varepsilon} K \left(\frac{\varepsilon}{d(T^{(2)}_\omega(x,y))} \right)^{-\gamma} d(\varsigma_\delta \times \mathbb{P})$$
$$\leq K\varepsilon^{-\gamma} \int_{\mathcal{G}_d \setminus \mathcal{G}_\varepsilon} d(T^{(2)}_\omega(x,y))^\gamma d(\varsigma_\delta \times \mathbb{P})(x,y,\omega) \leq C\varepsilon^\gamma,$$

for some $C > 0$. Here we use Lemma 5.6 and Hypothesis (H4), and $\gamma < 1$. Combining the above analysis yields

$$\limsup_{\varepsilon \to 0} \frac{\mu^{(2)}(\Delta_\varepsilon)}{\varepsilon^{-\gamma}} \leq C$$

for some $C > 0$, if $\gamma \in (-1/2, 0)$. □

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References

[1] N. Abbasi, M. Gharaei, and A. J. Homburg. Iterated function systems of logistic maps: synchronization and intermittency. *Nonlinearity*, 31(8):3880–3913, 2018.

[2] V. A. Antonov. Modeling of processes of cyclic evolution type. Synchronization by a random signal. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, pages 67–76, 1984.

[3] L. Arnold. A formula connecting sample and moment stability of linear stochastic systems. *SIAM J. Appl. Math.*, 44(4):793–802, 1984.

[4] K. B. Athreya and H.-J. Schuh. Random logistic maps. II. The critical case. *J. Theoret. Probab.*, 16(4):813–830, 2003.

[5] M. Babillot, P. Bougerol, and L. Elie. The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. *Ann. Probab.*, 25(1):478–493, 1997.

[6] W. Bahsoun and C. Bose. Mixing rates and limit theorems for random intermittent maps. *Nonlinearity*, 29(4):1417–1433, 2016.

[7] V. Baladi and L.-S. Young. On the spectra of randomly perturbed expanding maps. *Comm. Math. Phys.*, 156(2):355–385, 1993.

[8] P. H. Baxendale. Asymptotic behaviour of stochastic flows of diffeomorphisms. In *Stochastic processes and their applications (Nagoya, 1985)*, volume 1203 of *Lecture Notes in Math.*, page 1–19. Springer, Berlin, 1986.

[9] P. H. Baxendale. Statistical equilibrium and two-point motion for a stochastic flow of diffeomorphisms. In *Spatial stochastic processes*, volume 19 of *Progr. Probab.*, page 189–218. Birkhäuser Boston, Boston, MA, 1991.
[10] P. H. Baxendale and R. Z. Khasminskii. Stability index for products of random transformations. *Adv. in Appl. Probab.*, 30(4):968–988, 1998.
[11] P. H. Baxendale and D. W. Stroock. Large deviations and stochastic flows of diffeomorphisms. *Probab. Theory Related Fields*, 80(2):169–215, 1988.
[12] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith. Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier-Stokes. *Probab. Theory Related Fields*, 179(3-4):777–834, 2021.
[13] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith. Almost-sure exponential mixing of passive scalars by the stochastic Navier-Stokes equations. *Ann. Probab.*, 50(1):241–303, 2022.
[14] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith. The Batchelor spectrum of passive scalar turbulence in stochastic fluid mechanics at fixed Reynolds number. *Comm. Pure Appl. Math.*, 75(6):1237–1291, 2022.
[15] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith. Lagrangian chaos and scalar advection in stochastic fluid mechanics. *J. Eur. Math. Soc. (JEMS)*, 24(6):1893–1990, 2022.
[16] F. Blanchard, E. Glasner, S. Kolyada, and A. Maass. On Li-Yorke pairs. *J. Reine Angew. Math.*, 547:51–68, 2002.
[17] A. Blumenthal, M. Coti Zelati, and R. S. Gvalani. Exponential mixing for random dynamical systems and an example of Pierrehumbert. *Ann. Probab.*, 51(4):1559 – 1601, 2023.
[18] S. Brofferio and D. Buraczewski. On unbounded invariant measures of stochastic dynamical systems. *Ann. Probab.*, 43(3):1456–1492, 2015.
[19] S. Brofferio, D. Buraczewski, and E. Damek. On the invariant measure of the random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(2):377–395, 2012.
[20] S. Brofferio, D. Buraczewski, and T. Szarek. On uniqueness of invariant measures for random walks on $\mathrm{HOMEO}^+(\mathbb{R})$. *Ergodic Theory Dynam. Systems*, 42(7):2207–2238, 2022.
[21] A. Carverhill. Flows of stochastic dynamical systems: ergodic theory. *Stochastics*, 14(4):273–317, 1985.
[22] A. Carverhill. A formula for the Lyapunov numbers of a stochastic flow. Application to a perturbation theorem. *Stochastics*, 14(3):209–226, 1985.
[23] K. Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.
[24] B. Deroin, V. Kleptsyn, A. Navas, and K. Parwani. Symmetric random walks on $\mathrm{Homeo}^+(\mathbf{R})$. *Ann. Probab.*, 41(3B):2066–2089, 2013.
[25] R. Durrett. *Probability—theory and examples*, volume 49 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2019.
[26] J.-P. Eckmann. Roads to turbulence in dissipative dynamical systems. *Rev. Modern Phys.*, 53(4):643–654, 1981.
[27] F. Flandoli, B. Gess, and M. Scheutzow. Synchronization by noise. *Probab. Theory Related Fields*, 168(3-4):511–556, 2017.
[28] M. Gharaei and A. J. Homburg. Random interval diffeomorphisms. *Discrete Contin. Dyn. Syst. Ser. S*, 10(2):241–272, 2017.
[29] A. Gorodetski and V. Kleptsyn. Synchronization properties of random piecewise isometries. *Comm. Math. Phys.*, 345(3):781–796, 2016.
[30] P. Homblé. Sample and moment Lyapunov exponents of discrete linear dynamical systems. *Stochastic Anal. Appl.*, 12(3):329–354, 1994.
[31] A. J. Homburg. Synchronization in minimal iterated function systems on compact manifolds. *Bull. Braz. Math. Soc. (N.S.)*, 49(3):615–635, 2018.
[32] A. J. Homburg and C. Kalle. Iterated function systems of affine expanding and contracting maps on the unit interval, 2022.
[33] V. A. Kleptsyn and M. B. Nalskiĭ. Convergence of orbits in random dynamical systems on a circle. *Funktsional. Anal. i Prilozhen.*, 38(4):36–54, 95–96, 2004.

[34] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[35] J. S. W. Lamb, G. Tenaglia, and D. Turaev. Horseshoes for a Class of Nonuniformly Expanding Random Circle Maps. *Annales Henri Poincaré*, 2025.

[36] J. S. W. Lamb, G. Tenaglia, and D. V. Turaev. Non uniform expansion and additive noise imply random horseshoe, 2025.

[37] F. Ledrappier and L.-S. Young. Dimension formula for random transformations. *Comm. Math. Phys.*, 117(4):529–548, 1988.

[38] F. Ledrappier and L.-S. Young. Entropy formula for random transformations. *Probab. Theory Related Fields*, 80(2):217–240, 1988.

[39] T. Y. Li and J. A. Yorke. Period three implies chaos. *Amer. Math. Monthly*, 82(10):985–992, 1975.

[40] P.-D. Liu and M. Qian. *Smooth ergodic theory of random dynamical systems*, volume 1606 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[41] D. Malicet. Random walks on Homeo(S^1). *Comm. Math. Phys.*, 356(3):1083–1116, 2017.

[42] E. Matias and Í. Melo. Random products of maps synchronizing on average. *Commun. Contemp. Math.*, 22(8):1950086, 18, 2020.

[43] E. Matias and E. Silva. Random iterations of maps on $\mathbb{R}^k$: asymptotic stability, synchronisation and functional central limit theorem. *Nonlinearity*, 34(3):1577–1597, 2021.

[44] R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Comm. Math. Phys.*, 100(4):495–524, 1985.

[45] S. A. Molchanov. The structure of eigenfunction of the one-dimensional unordered structures. *Math. USSR Izv.*, 12(1):69–101, 1978.

[46] J. Newman. Necessary and sufficient conditions for stable synchronization in random dynamical systems. *Ergodic Theory Dynam. Systems*, 38(5):1857–1875, 2018.

[47] S. Pelikan. Invariant densities for random maps of the interval. *Trans. Amer. Math. Soc.*, 281(2):813–825, 1984.

[48] A. S. Pikovsky. Statistics of trajectory separation in noisy dynamical systems. *Phys. Lett. A*, 165(1):33–36, 1992.

[49] Y. Pomeau and P. Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.*, 74(2):189–197, 1980.

[50] A. N. Shiryaev. *Probability*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[51] L. Yu, E. Ott, and Q. Chen. Transition to chaos for random dynamical systems. *Phys. Rev. Lett.*, 65:2935–2938, 1990.

[52] H. Zmarrou and A. J. Homburg. Dynamics and bifurcations of random circle diffeomorphisms. *Discrete Contin. Dyn. Syst. Ser. B*, 10(2-3):719–731, 2008.

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