Universal scalings in homoclinic doubling cascades

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Abstract

We study a renormalization operator for families of one dimensional maps close to $x \mapsto p + r(1 - x^{\beta})^2$, $\beta \in (\frac{1}{2}, 1)$. Such functions occur in the study of cascades of homoclinic doubling bifurcations in three dimensional differential equations. For values of β close to $\frac{1}{2}$, we prove the existence of a fixed point of the renormalization operator, whose linearization at the fixed point has two unstable eigenvalues. This is in marked contrast to renormalization theory for period doubling cascades, in which one unstable eigenvalue appears. We derive from the renormalization theory consequences for universal scalings in the bifurcation diagrams.

1 Introduction

Cascades of period doubling bifurcations form one of the most intriguing ways to go from simple to chaotic dynamics on paths of differential equations. Its main characteristics are independent of the details of the differential equations; renormalization theory predicts universal scalings in the bifurcation diagram. A similar scenario, albeit with only saddle periodic orbits, exists with cascades of homoclinic bifurcations [Hom96]. In two parameter families of differential equations, these two scenarios can be joint through cascades of homoclinic doubling bifurcations (a homoclinic doubling bifurcation is a codimension two homoclinic bifurcation, at which in the parameter plane a curve of doubled homoclinic orbits branches from a curve of primary homoclinic orbits). Homoclinic doubling cascades were described and shown to occur persistently in a class of two parameter families of three dimensional vector fields in [HomKokNau97], following numerical investigations in [KokKomOka96]. The analogy with period doubling cascades was pointed out, in

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particular by establishing a relation to one dimensional dynamics. All of this naturally leads to the question of universal scalings as observed in period doubling cascades. Using renormalization theory, we will discuss universal scalings in families of one dimensional model maps appearing in the study of cascades of homoclinic doubling bifurcations in vector fields.

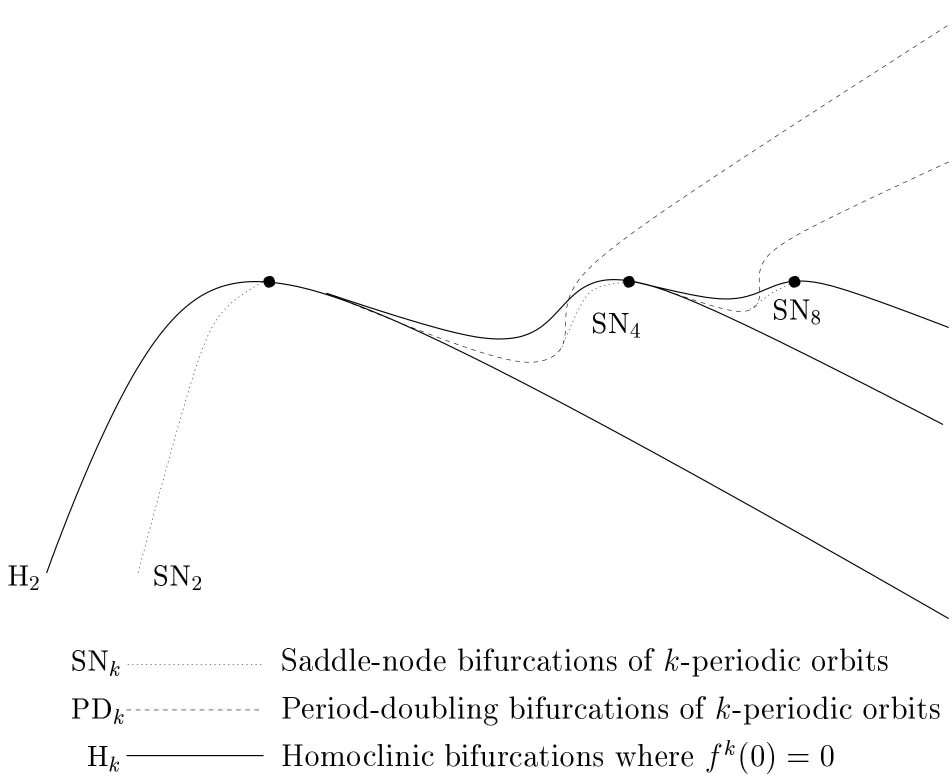

Figure 1: Schematic impression of the expected bifurcation set of $x \mapsto p + r(1 - x^\beta)^2$ in the parameter plane $\{(r,p) \in \mathbb{R}^2\}$. The thick dots indicate three subsequent 'homoclinic doubling' bifurcations.

In [HomKokNau97] and [HomKra98], bifurcation theorems are formulated establishing the existence of homoclinic doubling cascades in unfoldings of certain codimension three homoclinic bifurcations. We refer to [HomKokNau97] and [HomKra98] for a detailed description. Here we restrict ourselves to the following observations. Homoclinic doubling cascades are examined, in models of three dimensional differential equations, through the introduction of a Poincaré return map Π on a two dimensional cross section Σ. By restricting Π to a small domain Σ' and rescaling that to unit size, Π can be written as a small perturbation of the singular map

$$(x, y) \quad \mapsto \quad (p + r(1 - x^\beta)^2, 0),$$

where β is a parameter in $(\frac{1}{2}, 1)$ and (x, y) is from a box of the form $[0, L] \times [-M, M]$ for some $L, M > 0$. This makes Π essentially one dimensional. Compared to a strongly dissipative Hénon map, which is a singular perturbation of the one dimensional quadratic map $(x, y) \mapsto (1 - ax^2, 0)$, we obtain one dimensional maps which are also unimodal but not differentiable in $x = 0$ (and defined only if $x \geq 0$). We remark that the interplay in the dynamics of a point with infinite slope at $x = 0$ and a critical point at $x = 1$, has a profound effect on the dynamics and bifurcations.

Part of the speculated bifurcation diagram of the family of unimodal maps $x \mapsto p + r(1 - x^\beta)^2$ (and of Π as well), is depicted in figure 1. Among the bifurcation curves are curves of parameter values at which 0 is a periodic point. Parameter values on these curves are called homoclinic bifurcation values because of their interpretation as such for the original differential equation. Choosing parameters on a line in the right hand side of the picture, one encounters a period doubling cascade as well as a cascade of homoclinic bifurcations where $f^{2^n}(0) = 0$, a phenomenon described in [Hom96]. These two scenarios are joint through 'homoclinic doubling' bifurcations; a homoclinic doubling bifurcation is a codimension two bifurcation at which 0 is a periodic point whose orbit goes through the critical point.

In this paper we consider universal scalings in the bifurcation diagram of families near the one dimensional model family $x \mapsto p + r(1 - x^\beta)^2$ for values of β slightly larger then $\frac{1}{2}$. These models occur naturally in the bifurcation study of a codimension three homoclinic bifurcation, an 'inclination flip at resonance' [HomKra98].

The renormalization analysis we perform provides evidence that the bifurcation diagram of such families schematically is like depicted in figure 1, and establishes scaling structures in the bifurcation diagram. In the next section we will define renormalization, state and prove the renormalization Theorem 2.1. In section 3 we discuss the consequences of the renormalization theorem for universal scalings in the bifurcation diagram (see Theorem 3.1). We illustrate the results by numerically computed bifurcation diagrams.

2 The renormalization operator

For $0 < \sigma < 1$, let $\mathfrak{C}^{2+\sigma}$ denote the set of functions $x \mapsto f(x^\beta)$, defined on a compact interval $I = [0, L]$, $L > 1$, with $u \mapsto f(u)$ a unimodal $C^{2+\sigma}$ function (f'' is σ-Hölder with bounded Hölder constant) with a unique quadratic minimum at $x = 1$. We can write a function $f \in \mathfrak{C}^{2+\sigma}$ as

$$f(x) \quad = \quad p + r(x^\beta)\left(1 - x^\beta\right)^2,$$

where p is a real number and $u \mapsto r(u)$ is a positive $C^{2+\sigma}$ function. We will be interested in such functions for which β is close to $\frac{1}{2}$, and for which p is small and r is close to 1 on a compact interval. We will write

$$r(u) \quad = \quad 1 + \epsilon(u).$$

If $f(x) = g(x^\beta)$, let

$$\|f\|_{2+\sigma} \quad = \quad \max \left\{ \sup_{\substack{u = x^\beta \\ x \in I}} |g(u)|, |g'(u)|, |g''(u)|, \sup_{\substack{x^\beta = u \neq v = y^\beta \\ x, y \in I}} \frac{|g''(u) - g''(v)|}{|u - v|^\sigma} \right\}.$$

Equipped with the norm $\|.\|_{2+\sigma}$, $\mathfrak{C}^{2+\sigma}$ is a cone in a Banach space.

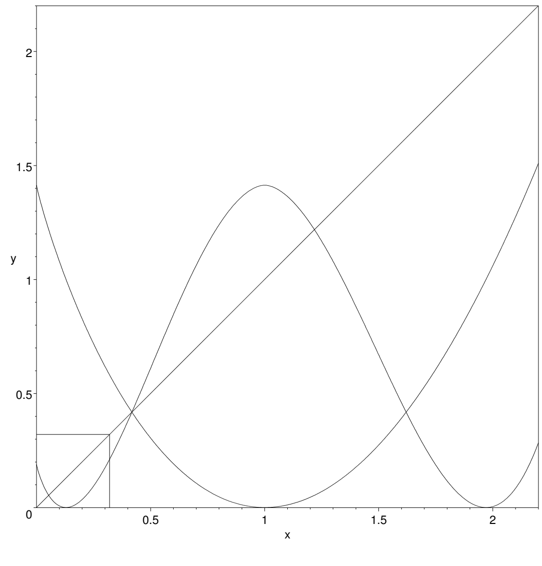

Figure 2: The graphs of f and f^2 for $f(x) = r(1 - x^\beta)^2$ with $\beta = 0.9$ and $r = 1.413178$.

The renormalization $\mathcal{R}(f)$ of f is a rescaling of the second iterate f^2. It will be defined on an open subset of $\mathfrak{C}^{2+\sigma}$, given by the following conditions.

- $f(0) > 1$,
- $f(0) \leq L$,
- $f(1) < 1$,
- $f(aL) < 1$, where $a \in (0, 1)$ satisfies $f(a) = 1$.

The inequalities $f(0) > 1$ and $f(1) < 1$ imply that there is a point $a \in (0, 1)$ for which $f(a) = 1$. Let

$$\mathcal{R}(f)(x) \quad = \quad \frac{1}{a} f^2(ax).$$

The condition $f(0) > 1$ is equivalent to $p > -\epsilon(0)$. The rescaling factor a is such that $\mathcal{R}(f)$ has its critical point at $x = 1$, so that $\mathcal{R}(f) \in \mathfrak{C}^{2+\sigma}$.

Define the following sets of functions:

$$\begin{aligned} \mathrm{H}_k &= \{f \in \mathfrak{C}^{2+\sigma} : \ f^k(0) = 0 \text{ with } k \text{ minimal}\}, \\ \mathrm{HD}_k &= \{f \in \mathfrak{C}^{2+\sigma} : \ f(1) = 0, f^k(0) = 0 \text{ with } k \text{ minimal}\}, \\ \mathrm{S}_k &= \{f \in \mathfrak{C}^{2+\sigma} : \ f^k(1) = 1 \text{ with } k \text{ minimal}\}. \end{aligned}$$

Note that for $f \in \mathrm{HD}_k$, the periodic orbit $\mathcal{O}(0)$ goes through the critical point. For $f \in \mathrm{S}_k$, the critical point is periodic, in other words, f possesses a superstable periodic orbit.

The following theorem provides an isolated fixed point of the renormalization operator $\mathcal{R}$ and gives further properties relevant to universal scalings in the bifurcation diagram as will be discussed below. The proof of the theorem is contained in the next sections.

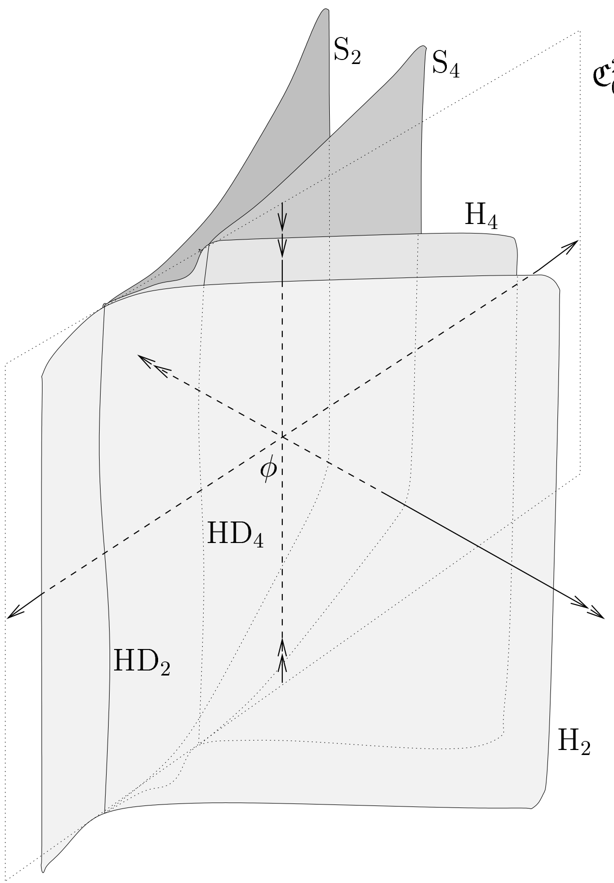

Figure 3: Bifurcation surfaces in the function space $\mathfrak{C}^{2+\sigma}$.

Theorem 2.1 *For $\beta > \frac{1}{2}$ and $\beta - \frac{1}{2}$ small, the renormalization operator $\mathcal{R}$ on $\mathfrak{C}^{2+\sigma}$ possesses an isolated fixed point ϕ. The function ϕ depends continuously on β and converges* *to $x \mapsto (1-\sqrt{x})^2$ as $\beta \to \frac{1}{2}$. The linearization $D\mathcal{R}$ at ϕ has two unstable eigenvalues δ_1, δ_2. They depend continuously on β and satisfy $\delta_1 \to 2$, $\delta_2 \to \infty$ as $\beta \to \frac{1}{2}$. The remainder of the spectrum of $D\mathcal{R}(\phi)$ is strictly inside the unit disc; the spectral projection to the eigenspace of the unstable eigenvalues is continuous. The unstable manifold of $\mathcal{R}$ is 2 dimensional. It intersects transversally the manifolds HD_2, H_2 and S_2, defined above. The one dimensional strong unstable manifold intersects H_2 and S_2 transversally.*

In the following sections we will prove the above theorem.

2.1 Existence of a fixed point for real analytic functions

In this section and the next section we start proving Theorem 2.1 by establishing the existence of a fixed point of $\mathcal{R}$. In the following sections we will prove hyperbolicity of the fixed point, identify the two unstable eigenvalues, and prove the transversality statements in Theorem 2.1. We consider first real analytic functions of the argument x^β. The results will be used in the next section for the study of $C^{2+\sigma}$ functions.

Let $\mathfrak{C}^\omega$ denote the set of functions $x \mapsto f(x)$, defined on a compact interval $[0, L]$, that are of the form $f(x) = g(x^\beta)$ for a real analytic function $u \mapsto g(u)$, and that are unimodal with a unique quadratic minimum at $x = 1$. We require that g extends to a bounded analytic function on a fixed complex neighborhood Ω of $[0, L^\beta]$. For $f \in \mathfrak{C}^\omega$ with $f(x) = g(x^\beta)$, let

$$\|f\|_0 \;=\; \sup_{u\in\Omega} |g(u)|.$$

Equipped with this norm, $\mathfrak{C}^\omega$ is a cone in a Banach space.

It is important to notice that $\mathcal{R}$ maps the set of functions in $\mathfrak{C}^\omega$ with critical value 0 into itself. Denote by $\mathfrak{C}_0^\omega$ this set of functions, i.e. $\mathfrak{C}_0^\omega$ consists of functions on a compact interval $[0, L]$ that can be written $r(x^\beta)(1-x^\beta)^2$ with $u \mapsto r(u)$ a bounded and positive analytic function on Ω. By a coordinate change $u = x^\beta$, we obtain a set of functions $\tilde{\mathfrak{C}}^\omega$ of the form $\tilde{r}(u)(1-u)^{2\beta}$ with $u \mapsto \tilde{r}(u)$ real analytic. The renormalization operator $\mathcal{R}$ induces an operator $\tilde{\mathcal{R}}$ on $\tilde{\mathfrak{C}}^\omega$. This operator is reminiscent of the renormalization operator studied in [ColEckLan80] for 2β somewhat larger then 1, however it is defined in a slightly different way. The renormalization in [ColEckLan80] rescales the second iterate on a small interval around the critical point of $f \in \tilde{\mathfrak{C}}^\omega$, whereas we rescale the second iterate on a small interval around the critical value 0. We can therefore not directly cite [ColEckLan80] to establish the existence of a fixed point of $\tilde{\mathcal{R}}$, but must re-examine their proof for the altered situation.

At this point, it is convenient to work with an alternative definition of a renormalization operator by choosing a different normalization of the functions. Let $\mathfrak{D}_0^\omega$ be the set of functions $f : [0,1] \to [0,1]$ of the form $f(x) = (1 - r(x^\beta)x^\beta)^2$, with $u \mapsto r(u)$ a positive and bounded analytic function on a fixed neighborhood Θ of $[0,1]$ in $\mathbb{C}$. The space $\mathfrak{D}_0^\omega$ will be equipped with the supnorm; if $f(x) = g(x^\beta)$, then $\|f\|_0 = sup_{u\in\Theta}|g(u)|$. Note that $f(0) = 1$ for each $f \in \mathfrak{D}_0^\omega$. The normalization condition used for functions in $\mathfrak{C}^\omega$ that the critical point sits at 1, is replaced by the requirement that 0 is mapped to 1. Observe further that $\mathfrak{D}_0^\omega$ is a submanifold of a Banach space, but not itself a (open subset of a) Banach space. The renormalization operator $\mathcal{S}$ on $\mathfrak{D}_0^\omega$ is defined by

$$\begin{aligned} \mathcal{S}(f)(x) &= \frac{1}{\lambda} f^2(\lambda(x)), \\ \lambda &= f(1). \end{aligned}$$

The domain of $\mathcal{S}$ is given by the functions $f \in \mathfrak{D}_0^\omega$ for which $0 \le f(1) \le 1$. Observe that the rescaling factor λ is explicitly defined, whereas the rescaling factor a in the definition of $\mathcal{R}$ must be solved from the equation $f(a) = 1$.

Proposition 2.2 *For β slightly larger then $\frac{1}{2}$, $\mathcal{S}\Big|_{\mathfrak{D}_0^\omega}$ has a hyperbolic fixed point $\hat{\phi}$. The linearized operator $D\mathcal{S}(\hat{\phi})$ has an unstable eigenvalue δ_1, that depends continuously on β and approaches 2 as $\beta \to \frac{1}{2}$. The eigenspace $E(\delta_1)$ of δ_1 is one dimensional and converges to $\{k(1-\sqrt{x})^2,\ \ k \in \mathbb{R}\}$ as $\beta \to \frac{1}{2}$. The rest of the spectrum of $D\mathcal{S}(\hat{\phi})$ lies strictly inside the unit disc. The spectral projection on $E(\delta_1)$ is continuous.*

PROOF. Write $r(u) = 1 + \epsilon(u)$. We have

$$\begin{aligned} f^2(x) &= \left(1 - (1+\epsilon(f(x)^\beta))(1-(1+\epsilon(x^\beta))x^\beta)^{2\beta}\right)^2 \\ &= \left(1 - (1+\epsilon(f(x)^\beta))\left(1 - 2\beta(1+\epsilon(x^\beta))x^\beta + \mathcal{O}((2\beta-1)x^{2\beta})\right)\right)^2 \\ &= \left(\epsilon(f(x)^\beta) - (1+\epsilon(f(x)^\beta))\left(2\beta(1+\epsilon(x^\beta)) + \mathcal{O}((2\beta-1)x^\beta)\right)x^\beta\right)^2. \end{aligned}$$

Rescaling by a factor $\epsilon(1)^2$ and writing $\epsilon_f = \epsilon(f(\epsilon(1)^2 x)^\beta)$, $\epsilon = \epsilon(\epsilon(1)^{2\beta}x^\beta)$, we get

$$\mathcal{S}(f)(x) = \left(\frac{\epsilon_f}{\epsilon(1)} - (1+\epsilon_f)\epsilon(1)^{2\beta-1}\left(2\beta(1+\epsilon) + \mathcal{O}((2\beta-1)\epsilon(1)^{2\beta}x^\beta\right)x^\beta\right)^2.$$

To get an idea where to find a fixed point of $\mathcal{S}$, note the following. If $h(x) = (1-(1+\alpha)x^\beta)^2$ for some constant α, then

$$\mathcal{S}(h)(x) = \left(1 - 2\beta\alpha^{2\beta-1}(1+\alpha)^2 x^\beta + \mathcal{O}((2\beta-1)\alpha^{2\beta}x^{2\beta})\right)^2.$$

The order x^β terms in h and $\mathcal{S}(h)$ are equal if

$$1+\alpha = 2\beta\alpha^{2\beta-1}(1+\alpha)^2. \tag{1}$$

This defines α as a function of $2\beta - 1$. Each small α corresponds to a small value of $2\beta - 1$. In fact, $2\beta - 1 \sim \frac{-\alpha}{1+\ln\alpha}$. Observe that $2\beta - 1 \ll \alpha$. This computation suggests writing $\epsilon(x^\beta) = \alpha s(x^\beta)$ with α given by (1). Note that $\alpha^{2\beta-1} = \frac{1}{2\beta}\frac{1}{1+\alpha}$, we will use this identity below.

Denote by $\mathcal{T}$ the operator induced by $\mathcal{S}$ on the functions s. So if $f(x) = (1-(1+\alpha s(x^\beta))x^\beta)^2)$, then $\mathcal{S}(f)(x) = (1-(1+\alpha\mathcal{T}(s)(x^\beta))x^\beta)^2$. The operator $\mathcal{T}$ is given by the following expression, where $s_f = s(f(\alpha^2 s(1)^2 x)^\beta)$ and $s = s(\alpha^{2\beta}s(1)^{2\beta}x^\beta)$.

$$\begin{aligned}\mathcal{T}(s)(x^\beta) &= \frac{1}{\alpha}\left(-\frac{\alpha s_f - \alpha s(1)}{\alpha s(1)x^\beta} + 2\beta(1+\alpha s_f)(1+\alpha s)(\alpha s(1))^{2\beta-1} - 1\right)\\ &\quad +\mathcal{O}((2\beta-1)\alpha^{2\beta-1}s(1)^{2\beta}x^\beta).\end{aligned} \tag{2}$$

Observe that

$$\begin{aligned}\epsilon(f(\epsilon(1)^2x)^\beta) &= \epsilon(1-(1+\epsilon(\epsilon(1)^{2\beta}x^\beta))\epsilon(1)^{2\beta}x^\beta)^{2\beta})\\ &= \epsilon(1) - 2\beta\epsilon'(1)(1+\epsilon(0))\epsilon(1)^{2\beta}x^\beta + \mathcal{O}(\epsilon(1)^{4\beta}x^{2\beta}).\end{aligned}$$

Putting this in (2), gives

$$\begin{aligned}\mathcal{T}(s)(x^\beta) &= \frac{1}{\alpha}\left(2\beta(1+\alpha s(0))(\alpha s(1))^{2\beta-1}s'(1)\alpha + 2\beta(1+\alpha s_f)(1+\alpha s)(\alpha s(1))^{2\beta-1} - 1\right)\\ &\quad +\mathcal{O}((2\beta-1)\alpha^{2\beta-1}s(1)^{2\beta}x^\beta) + \mathcal{O}(\alpha^{4\beta-1}s(1)^{4\beta-1}x^\beta)\\ &= s(1)^{2\beta-1}\frac{1}{1+\alpha}s'(1)(1+\alpha s(0))\\ &\quad +\frac{1}{\alpha}\left((1+\alpha s(1))(1+\alpha s(0))\frac{1}{1+\alpha}s(1)^{2\beta-1} - 1\right)\\ &\quad +\mathcal{O}((2\beta-1)\alpha^{2\beta-1}s(1)^{2\beta}x^\beta) + \mathcal{O}(\alpha^{4\beta-1}s(1)^{4\beta-1}x^\beta).\end{aligned}$$

Let

$$\mathcal{T}_0 s(x) = s(1) + s(0) + s'(1) - 1.$$

Direct inspection gives that $\mathcal{T}(s) - \mathcal{T}_0 s$ and its derivatives with respect to s, converge to 0 as $\alpha \to 0$.

Define $\mathcal{R}_\epsilon = \mathcal{T} - \mathcal{T}_0$. Rewrite the equation $\mathcal{T}(s) = s$ as $\mathcal{R}_\epsilon(s) = (I - \mathcal{T}_0)\, s$. It follows from the identity $\mathcal{T}_0^2 s(x^\beta) = 2\mathcal{T}_0 s(x^\beta) - 1$, that

$$1 + (I - \mathcal{T}_0)\,\mathcal{R}_\epsilon(s)(x^\beta) = s(x^\beta). \tag{3}$$

The mapping given by the left hand side is a contraction in the supnorm on a neighborhood of $s \equiv 1$.

Observe that $\mathcal{T}_0 - 1$ is linear, its spectrum contains an eigenvalue 2 with the line of constant functions as the eigenspace. The rest of the spectrum of $\mathcal{T}_0 - 1$ is at the origin in the complex plane. Standard perturbation theory as found in e.g. [Kat76] and applied in a similar situation in renormalization theory in [ColEckLan80], proves the theorem. ∎

We will show how to obtain from Proposition 2.2 the corresponding result, i.e. the existence of a hyperbolic fixed point with appropriate spectral properties, for the renormalization operator $\mathcal{R}$. Let $\mathcal{U}_0^\omega$ be a small open neighborhood in $\mathfrak{C}_0^\omega$ of $x \mapsto (1-x^\beta)^2$. For $f \in \mathcal{U}_0^\omega$, let $h(f)$ be given by

$$h(f)(x) \;=\; \frac{1}{f(0)} f(f(0)x),$$

so that $h(f)(0) = 1$. Having fixed the region Ω on which $f \in \mathfrak{C}_0^\omega$ is a bounded analytic function of $u = x^\beta$, choose the region Θ in the definition of $\mathfrak{D}_0^\omega$ small enough so that $h(f) \in \mathfrak{D}_0^\omega$ for each $f \in \mathcal{U}_0^\omega$. Then h is smooth on $\mathcal{U}_0^\omega$ and h^{-1} is smooth on $h(\mathcal{U}_0^\omega)$. Note that, for β sufficiently near $\frac{1}{2}$, $\phi = h^{-1}(\hat{\phi}) \in \mathcal{U}_0^\omega$ and this ϕ is a fixed point of $\mathcal{R} = h^{-1} \circ \mathcal{S} \circ h$. To check hyperbolicity of ϕ, compute

$$D\mathcal{R}(\phi) \;=\; Dh^{-1}(\hat{\phi}) D\mathcal{S}(\hat{\phi}) Dh(\phi).$$

So, from $D\mathcal{S}(\hat{\phi})\hat{e}_1 = \delta_1 \hat{e}_1$, it follows that $D\mathcal{R}(\phi)e_1 = \delta_1 e_1$ for $e_1 = Dh^{-1}(\hat{\phi})\hat{e}_1$. Let $\hat{\pi}_1$ be the spectral projection onto $\hat{e}_1$ and define $\pi_1 = Dh^{-1}(\hat{\phi})\hat{\pi}_1 Dh(\phi)$. That the rest of the spectrum of $D\mathcal{R}(\phi)$ is strictly inside the unit circle in the complex plane, follows from

$$D\mathcal{R}^k(\phi) - \delta_1^k \pi_1 \;=\; Dh^{-1}(\hat{\phi}) \left(D\mathcal{S}^k(\hat{\phi}) - \delta_1^k \hat{\pi}_1 \right) Dh(\phi),$$

which is a contraction for k large enough.

2.2 Existence of a fixed point for twice differentiable functions

A possible way to show that ϕ is an isolated fixed point of $\mathcal{R}$ in $\mathfrak{C}_0^{2+\sigma}$ goes as follows. Consider the mapping $1 + (I - \mathcal{T}_0)\mathcal{R}_\epsilon$ from the proof of Proposition 2.2 (see (3)), and show that there is a bounded ball around $s \equiv 1$ in $\mathfrak{C}_0^{2+\sigma}$ which is mapped into itself by $1 + (I - \mathcal{T}_0)\mathcal{R}_\epsilon$ and on which $1 + (I - \mathcal{T}_0)\mathcal{R}_\epsilon$ is a contraction in the supnorm. Since a closed bounded set in $\mathfrak{C}_0^{2+\sigma}$ is closed in the supnorm [Hen81], this would establish the existence of an isolated fixed point.

Instead, we will closely follow arguments from [Dav96] to show that ϕ is an isolated fixed point of $\mathcal{R}$ in $\mathfrak{C}_0^{2+\sigma}$ and establish spectral properties in $\mathfrak{C}_0^{2+\sigma}$.

Proposition 2.3 *For any $0 < \sigma < 1$ and for β slightly larger then $\frac{1}{2}$, ϕ is an isolated fixed point of $\mathcal{R}\Big|_{\mathfrak{C}_0^{2+\sigma}}$. The linearized operator $D\mathcal{R}(\phi)$ has an unstable eigenvalue δ_1, that depends continuously on β and approaches 2 as $\beta \to \frac{1}{2}$. The eigenspace $E(\delta_1)$ of δ_1 is one dimensional and converges to $\{k(1-\sqrt{x})^2,\ \ k \in \mathbb{R}\}$ as $\beta \to \frac{1}{2}$. The rest of the spectrum of $D\mathcal{R}(\phi)$ lies strictly inside the unit disc. The spectral projection to $E(\delta_1)$ is continuous.*

Since the proof closely follows the arguments in [Dav96], we will merely set up the proof and perform the calculations differing from those in [Dav96].

With ϕ the fixed point of $\mathcal{R}$, write $\psi(x^\beta) = \phi(x)$, so that $u \mapsto \psi(u)$ is a real analytic function. The fixed point equation $\frac{1}{a}\phi^2(ax) = \phi(x)$, $\phi(a) = 1$, for ψ becomes

$$\frac{1}{\lambda}\psi(\psi(\lambda^\beta u)^\beta) \;=\; \psi(u).$$

Let T be the corresponding operator on $C^{2+\sigma}$ functions:

$$\begin{aligned} T(g) &= \frac{1}{a} g(g(a^\beta u)^\beta), \\ g(a^\beta) &= 1. \end{aligned} \tag{4}$$

Calculate the formal derivative of T at g:

$$\begin{aligned} DT(g)h(u) &= \frac{1}{a}\left\{h(g(a^\beta u)^\beta) + g'(g(a^\beta u)^\beta)\beta g(a^\beta u)^{\beta-1} h(a^\beta u)\right\} \\ & \frac{h(a^\beta)}{a^{1+\beta}\beta g'(a^\beta)} g(g(a^\beta u)^\beta) - \frac{\beta h(a^\beta)}{a g'(a^\beta)} g'(g(a^\beta u)^\beta) g(a^\beta u)^{\beta-1} g'(a^\beta u) u. \end{aligned}$$

Write S for the operator DT at the fixed point ψ. Noting that

$$\psi'(u) \;=\; a^{\beta-1}\beta\psi'(\psi(a^\beta u)^\beta)\psi(a^\beta u)^{\beta-1}\psi'(a^\beta u),$$

we have

$$\begin{aligned} Sh(u) &= \frac{1}{a}\left\{h(\psi(a^\beta u)^\beta) + \beta\psi'(\psi(a^\beta u)^\beta)\psi(a^\beta u)^{\beta-1} h(a^\beta u)\right\} \\ &\quad + \frac{h(a^\beta)}{a^\beta \psi'(a^\beta)}\left\{\frac{1}{\beta}\psi(u) + u\psi'(u)\right\}. \end{aligned} \tag{5}$$

Following [Dav96], define

$$\begin{aligned} & S_\gamma h(u) = \\ & \quad \frac{1}{a}\left\{\left|a^\beta \beta\psi(a^\beta u)^{\beta-1}\psi'(a^\beta u)\right|^\gamma h(\psi(a^\beta u)^\beta) + a^{\beta\gamma}\left|\psi'(\psi(a^\beta u)^\beta)\beta\psi(a^\beta u)^{\beta-1}\right| h(a^\beta u)\right\} \\ &= a^{\beta\gamma-1}\left\{\left|\beta\psi(a^\beta u)^{\beta-1}\psi'(a^\beta u)\right|^\gamma h(\psi(a^\beta u)^\beta) + \left|\psi'(\psi(a^\beta u)^\beta)\beta\psi(a^\beta u)^{\beta-1}\right| h(a^\beta u)\right\}. \end{aligned} \tag{6}$$

Denote by $\rho(S_\gamma)$ the spectral radius of S_γ in the supnorm. We will need to estimate $\rho(S_\gamma)$. Note that for β close to $\frac{1}{2}$, the fixed point g is close to $u \mapsto (1-u)^2$. We obtain from this and (6) that $\|S_\gamma\| \le k a^{\beta\gamma-1}$, where $k \to 1$ as $\beta \to \frac{1}{2}$. Hence, $\|S_\gamma\|$ and therefore also $\rho(S_\gamma)$ is small if $\beta\gamma > 1$ and β close enough to $\frac{1}{2}$.

We cite the following lemma from [Dav96]:

Lemma 2.4 *Suppose $\gamma > 1$ and $\rho(S_\gamma) < \rho$, where $1 \leq \rho < \delta_1$. The spectral projection σ to the unstable eigenspace $E(\delta_1)$ of $\mathfrak{C}_0^\omega$ extends continuously to $\mathfrak{C}_0^\gamma$. Given $\epsilon > 0$, there is a positive integer $m > 0$ so that for all $h \in \mathfrak{C}_0^\gamma$,*

$$\|S^m h - \delta_1^m \sigma h\|_\gamma \;\leq\; \epsilon \rho^m \|h\|_\gamma$$

Choose $\gamma = 2 + \sigma$ with $0 < \sigma < 1$, so that in particular $\gamma > \frac{1}{\beta}$. We can then take $\rho = 1$. The conclusion of the above lemma implies that, apart from a single eigenvalue δ_1, the spectrum of S as an operator on $\mathfrak{C}_0^{2+\sigma}$ lies strictly inside the disc with center 0 and radius 1. ■

Note that the unstable manifold of ϕ in $\mathfrak{C}_0^{2+\sigma}$ is contained in $\mathfrak{C}_0^\omega$.

2.3 The spectrum at the fixed point in the full space

Recall that $\mathcal{R}$ leaves invariant the subset $\mathfrak{C}_0^{2+\sigma} \subset \mathfrak{C}^{2+\sigma}$ consisting of functions with critical value 0. The previous sections have shown the existence of a fixed point ϕ of $\mathcal{R}$ in $\mathfrak{C}_0^{2+\sigma}$ and have established spectral properties of $D\mathcal{R}(\phi)\Big|_{\mathfrak{C}_0^{2+\sigma}}$. In this section we calculate the remaining eigenvalue of $D\mathcal{R}(\phi)$.

Proposition 2.5 *For $\beta > \frac{1}{2}$ close to $\frac{1}{2}$, the spectrum of $D\mathcal{R}(\phi)$ consists of the spectrum of $D\mathcal{R}(\phi)\Big|_{\mathfrak{C}_0^{2+\sigma}}$ plus an unstable eigenvalue δ_2. This eigenvalue depends continuously on β and satisfies $\delta_2 \to \infty$ as $\beta \to \frac{1}{2}$. The eigenspace $E(\delta_2)$ of δ_2 is one dimensional.*

Proof. Note that there is a natural decomposition $\mathfrak{C}^{2+\sigma} = \mathfrak{C}_0^{2+\sigma} \oplus \mathbb{R}$, along with projections $\pi_1 : \mathfrak{C}^{2+\sigma} \to \mathfrak{C}_0^{2+\sigma}$ and $\pi_2 : \mathfrak{C}^{2+\sigma} \to \mathbb{R}$ given by:

$$\begin{aligned} \pi_1(f)(x) &= f(x) - f(1) \\ \pi_2(f)(x) &= f(1). \end{aligned}$$

The invariance of $\mathfrak{C}_0^{2+\sigma}$ implies that the partial derivative $\delta_2 = \frac{\partial}{\partial p}\mathcal{R}(f)(1)|_{\phi}$ is in the spectrum of $D\mathcal{R}(\phi)$. In the next section we show that δ_2 is an eigenvalue.

We will now estimate the partial derivative and establish in particular that it is a large positive number. Recall that $\mathcal{R}(f)(x) = \frac{1}{a} f^2(ax)$ with a given by $f(a) = 1$. Let us start by calculating the rescaling factor a. Writing $s = a^\beta$, we have $p + (1 + \epsilon(s))(1 - s)^2 = 1$, yielding

$$s \;=\; 1 - \sqrt{\frac{1-p}{1+\epsilon(s)}}.$$

Using this equation, s can be solved invoking the implicit function theorem. This gives s as a smooth function of p and ϵ. Note that $a^\beta \approx \frac{1}{2}(p + \epsilon(0))$ in the sense that

$$\frac{a^\beta}{p + \epsilon(0)} \;\to\; \frac{1}{2},$$

as $p,\epsilon \to 0$. Though a depends on f, we suppress this dependence from the notation.

The fixed point ϕ of $\mathcal{R}$ is of the form

$$\phi(x) \;=\; (1+\hat{\epsilon}(x^\beta))(1-x^\beta)^2,$$

for some C^ω function $u \mapsto \hat{\epsilon}(u)$. We will write $\hat{a}$ for the rescaling factor at ϕ; $\phi(\hat{a}) = 1$. Now

$$\frac{\partial}{\partial p}\mathcal{R}(f)(1) = \frac{\partial}{\partial p}\left(\frac{f^2(a)}{a}\right) = \frac{\partial}{\partial p}\left(\frac{p}{a}\right).$$

Hence

$$\frac{\partial}{\partial p}\mathcal{R}(f)(1)|_{f=\phi} = \frac{1}{\hat{a}} \approx \left(\frac{\hat{\epsilon}(0)}{2}\right)^{-\frac{1}{\beta}}.$$

This is a large number for β close to $\frac{1}{2}$, since then $\hat{\epsilon}$ is small. ■

2.4 Invariant manifolds of the fixed point

In this section we investigate properties of the unstable manifold of the fixed point ϕ of the renormalization operator $\mathcal{R}$. In particular we establish the transversality properties stated in Theorem 2.1.

Let us first discuss the statement that HD_2 intersects the unstable manifold $W^{u,uu}(\phi)$ of ϕ transversally. Since HD_2 is contained in $\mathfrak{C}_0^{2+\sigma}$, it suffices to consider $\mathcal{R}$ restricted to $\mathfrak{C}_0^{2+\sigma}$. The unstable manifold of $\mathcal{R}\Big|_{\mathfrak{C}_0^{2+\sigma}}$ converges to the line $\{k(1-\sqrt{x})^2,\;\; k\in\mathbb{R}\}$ as $\beta \downarrow \frac{1}{2}$ (compare [ColEckLan80]). The manifold HD_2 is given by $\{r(x^\beta)(1-x^\beta)^2,\;\; r(0)=1\}$. Transversality easily follows for β sufficiently close to $\frac{1}{2}$.

The next step is to show that the strong unstable manifold $W^{uu}(\phi)$ intersects H_2 transversally. For this, we perform some cone estimates in $\mathfrak{C}^\omega$.

For α real and positive, let $C_\alpha \subset \mathfrak{C}^\omega = \mathfrak{C}_0^\omega \times \mathbb{R}$ denote the cone

$$C_\alpha \;=\; \{\lambda(v,1) \in C_0^\omega \times \mathbb{R},\;\; \|v\| \le \alpha, \lambda \in \mathbb{R}\}.$$

The next lemma shows that there is an invariant cone field consisting of tiny (meaning α small) cones C_α.

Lemma 2.6 *For each $\alpha > 0$, there is a neighborhood $\mathcal{U}$ of $x \mapsto (1-x^\beta)^2$, so that for $f \in \mathcal{U}$ we have $D\mathcal{R}(f)C_\alpha \subset C_\alpha$. Moreover, $D\mathcal{R}(f)$ expands vectors in C_α.*

PROOF. Write

$$\begin{aligned} f^2(x) &= p + (1+\epsilon(f(x)^\beta)\left(1-\left(p+(1+\epsilon(x^\beta))(1-x^\beta)^2\right)^\beta\right)^2 \\ &= d_0 + d_1 x^\beta + d_2 x^{2\beta} + \mathcal{O}(x^{3\beta}). \end{aligned}$$

To calculate the coefficients d_0, d_1, d_2, it is convenient to treat $\epsilon(x^\beta)$ and $\epsilon(f(x)^\beta)$ as parameters. Under the assumption that $\epsilon = \epsilon(x^\beta)$ and $\epsilon_f = \epsilon(f(x)^\beta)$ are parameters, we will compute a series expansion $f^2(x) = \hat{d}_0 + \hat{d}_1 x^\beta + \hat{d}_2 x^{2\beta} + \mathcal{O}(x^{3\beta})$. By filling in, in this power series expansion of f^2, the actual expansions of ϵ and ϵ_f, the values of d_0, d_1, d_2 can be obtained. Using the identity $(1+p+\epsilon+h)^\beta = (1+p+\epsilon)^\beta + \beta(1+p+\epsilon)^{\beta-1}h + \frac{1}{2}\beta(\beta-1)(1+p+\epsilon)^{\beta-2}h^2 + \mathcal{O}(h^3)$, we get

$$\begin{aligned} f^2(x) &= p + (1+\epsilon_f)\left(1 - \left(1+p+\epsilon - 2(1+\epsilon)x^\beta + (1+\epsilon)x^{2\beta}\right)^\beta\right)^2 \\ &= p + (1+\epsilon_f)\Big(1 - (1+p+\epsilon)^\beta - 2(1+\epsilon)\beta(1+p+\epsilon)^{\beta-1}x^\beta + \\ &\quad \left\{(1+\epsilon)\beta(1+p+\epsilon)^{\beta-1} + 2(1+\epsilon)^2\beta(\beta-1)(1+p+\epsilon)^{\beta-2}\right\}x^{2\beta} \\ &\quad + \mathcal{O}(x^{3\beta})\Big)^2, \end{aligned}$$

so that

$$\begin{aligned} \hat{d}_0 &= p + (1+\epsilon_f)\left(1-(1+p+\epsilon)^\beta\right)^2, \\ \hat{d}_1 &= -4\beta(1+\epsilon_f)(1+\epsilon)(1+p+\epsilon)^{\beta-1}\left(1-(1+p+\epsilon)^\beta\right), \\ \hat{d}_2 &= 4(1+\epsilon_f)(1+\epsilon)^2\beta^2(1+p+\epsilon)^{2\beta-2} \\ &\quad +2(1+\epsilon_f)(1+\epsilon)\beta(1+p+\epsilon)^{\beta-1}\left(1-(1+p+\epsilon)^\beta\right) \\ &\quad +4(1+\epsilon_f)(1+\epsilon)^2\beta(\beta-1)(1+p+\epsilon)^{\beta-2}\left(1-(1+p+\epsilon)^\beta\right). \end{aligned}$$

So, e.g. the value of d_0 is $p + (1+\epsilon((1+p+\epsilon(0))^\beta))\left(1-(1+p+\epsilon(0))^\beta\right)^2$. The remainder term $f^2(x) - d_0 - d_1 x^\beta - d_2 x^{2\beta}$ is analytic as function of x^β and small of order $\mathcal{O}(x^{3\beta})$. Recall that the rescaling factor a is implicitly given by

$$a^\beta = 1 - \sqrt{\frac{1-p}{1+\epsilon(a^\beta)}},$$

so that $a^\beta \approx \frac{1}{2}(p+\epsilon(0))$ in the sense that

$$\frac{a^\beta}{p+\epsilon(0)} \to \frac{1}{2},$$

as $p, \epsilon \to 0$. Note that

$$\mathcal{R}(f)(x) = d_0 a^{-1} + d_1 a^{\beta-1} x^\beta + d_2 a^{2\beta-1} x^{2\beta} + \mathcal{O}(a^{3\beta-1}x^{3\beta}).$$

The remainder term $\mathcal{R}(f)(x) - d_0 a^{-1} + d_1 a^{\beta-1} x^\beta + d_2 a^{2\beta-1} x^{2\beta}$ is analytic in the argument x^β, it is small of order $\mathcal{O}(a^{3\beta-1}x^{3\beta})$ so that it converges to 0 uniformly in p, ϵ.

Take a curve

$$f_t(x) = p_t + (1+\epsilon_t(x^\beta))(1-x^\beta)^2$$

in $\mathfrak{C}^\omega$, for smooth functions $p_t = p_0 + t$ and $\epsilon_t = \epsilon_0 + tv + \mathcal{O}(t^2)$, $\|v\| \leq \alpha$, and with t from an interval containing 0. Denote by a_t the rescaling factor in $\mathcal{R}(f_t)$, i.e., $f_t(a_t) = 1$. Then

$$\frac{\partial}{\partial t} a_t|_{t=0} \quad \approx \quad \frac{1+v(0)}{\beta} \left(\frac{1}{2}(p_0 + \epsilon_0(0)) \right)^{\frac{1}{\beta}-1}, \tag{7}$$

as $p_0, \epsilon_0 \to 0$.

Write $\mathcal{R}(f_t)(x) = (\mathcal{R}(f_t)(x) - \mathcal{R}(f_t)(1)) + \mathcal{R}(f_t)(1)$. Observe that $x \mapsto \mathcal{R}(f_t)(x) - \mathcal{R}(f_t)(1)$ in contained in $\mathfrak{C}_0^\omega$. It follows from (7) that

$$\frac{\partial}{\partial t}(p_t a_t^{-1})|_{t=0} \quad = \quad a_0^{-1} - p_0 a_0^{-2} \frac{\partial}{\partial t} a_t|_{t=0}$$

is bounded from below by a constant times $(\frac{1}{2}(p_0 + \epsilon_0(0)))^{-1/\beta}$ for p_0, ϵ_0 small and $p_0 + \epsilon_0(0) > 0$ (which holds for f_0 from the domain of $\mathcal{R}$). Since $\mathcal{R}(f_t) = p_t a_t^{-1} + \mathcal{O}((p_t + \epsilon_t)^2 a_t^{-1}) + \mathcal{O}((p_t + \epsilon_t) a_t^{\beta-1}) + \mathcal{O}(a_t^{2\beta-1})$, also

$$\frac{\partial}{\partial t} \mathcal{R}(f_t)(1)|_{t=0} \quad \geq \quad c \left(\frac{1}{2}(p_0 + \epsilon_0(0)) \right)^{-\frac{1}{\beta}}$$

for some constant $c > 0$, for all small p_0, ϵ_0. Similarly,

$$\frac{\partial}{\partial t}(\mathcal{R}(f_t)(x) - \mathcal{R}(f_t)(1))|_{t=0} \quad \leq \quad C(p_0 + \epsilon_0(0))^{1-\frac{1}{\beta}}$$

for some $C > 0$. The lemma easily follows from these bounds. ■

The tangent spaces of $W^{uu}(\phi)$ are contained in the cones C_α from the above lemma. Hence, $W^{uu}(\phi)$ contains a curve from ϕ to $\{f \in \mathfrak{C}^\omega, \quad f(0) = 1\}$, the boundary of the domain of definition of $\mathcal{R}$. It also shows that $W^{uu}(\phi)$ intersects H_2 transversally, for β close to $\frac{1}{2}$. Indeed, H_2 is formed by the functions $f(x) = p + (1 + \epsilon(x^\beta))(1 - x^\beta)^2$ that satisfy $p + \epsilon(1 + p + \epsilon(0)) \left(1 - (1 + p + \epsilon(0))^\beta\right)^2 = 0$. This manifold is tangent to $\mathfrak{C}_0^{2+\sigma}$ along HD_2 and therefore it intersects $W^{uu}(\phi)$ transversally. Similarly one sees that S_2 intersects $W^{uu}(\phi)$ transversally.

3 Universal scalings

Theorem 2.1 provides an explanation of universal scalings in the bifurcation diagram of generic two parameter families of functions. We will discuss scaling structures for two parameter families in $\mathfrak{C}^\omega$, thus made up from real analytic functions. The renormalization operator $\mathcal{R}$ on $\mathfrak{C}^\omega$ is smooth. It follows from invariant manifold theory that $\mathcal{R}$ possesses local stable and unstable manifolds near ϕ, which are denoted by $W^s(\phi)$ and $W^{u,uu}(\phi)$, respectively. The local unstable manifold contains a strong unstable manifold $W^{uu}(\phi)$

and the weak unstable manifold $W^u(\phi) = W^{u,uu}(\phi) \cap \mathfrak{C}_0^\omega$. Note that $W^u(\phi)$ is a smooth manifold.

The next theorem is basic in finding scaling structures in bifurcation diagrams. Following its formulation, we provide a further discussion of its consequences. The theorem will be proved in section 3.1 below.

Theorem 3.1 *Consider a two parameter family* $\{f_{p,\epsilon};\quad (p,\epsilon) \in \mathbb{R}^2\}$ *of functions in* $\mathfrak{C}^\omega$ *with* $f_{0,\epsilon} \in \mathfrak{C}_0^\omega$*, that intersects the local stable manifold* $W^s(\phi)$ *transversally at some function* $f_{0,\bar{\epsilon}}$*. Let* (p_n, ϵ_n) *be a sequence of parameter values tending to* $(0,\bar{\epsilon})$*, such that* $\mathcal{R}^{2^n}(f_{p_n,\epsilon_n})$ *converges to a function* $\bar{f}$ *in the local unstable manifold* $W^{u,uu}(\phi)$*. Then*

$$\begin{aligned} \frac{p_{n+1}-p_n}{p_n-p_{n-1}} &\to \frac{1}{\delta_2} \quad \textit{if } \bar{f} \notin W^u(\phi), \\ \frac{\epsilon_{n+1}-\epsilon_n}{\epsilon_n-\epsilon_{n-1}} &\to \frac{1}{\delta_1} \quad \textit{if } \bar{f} \notin W^{uu}(\phi). \end{aligned}$$

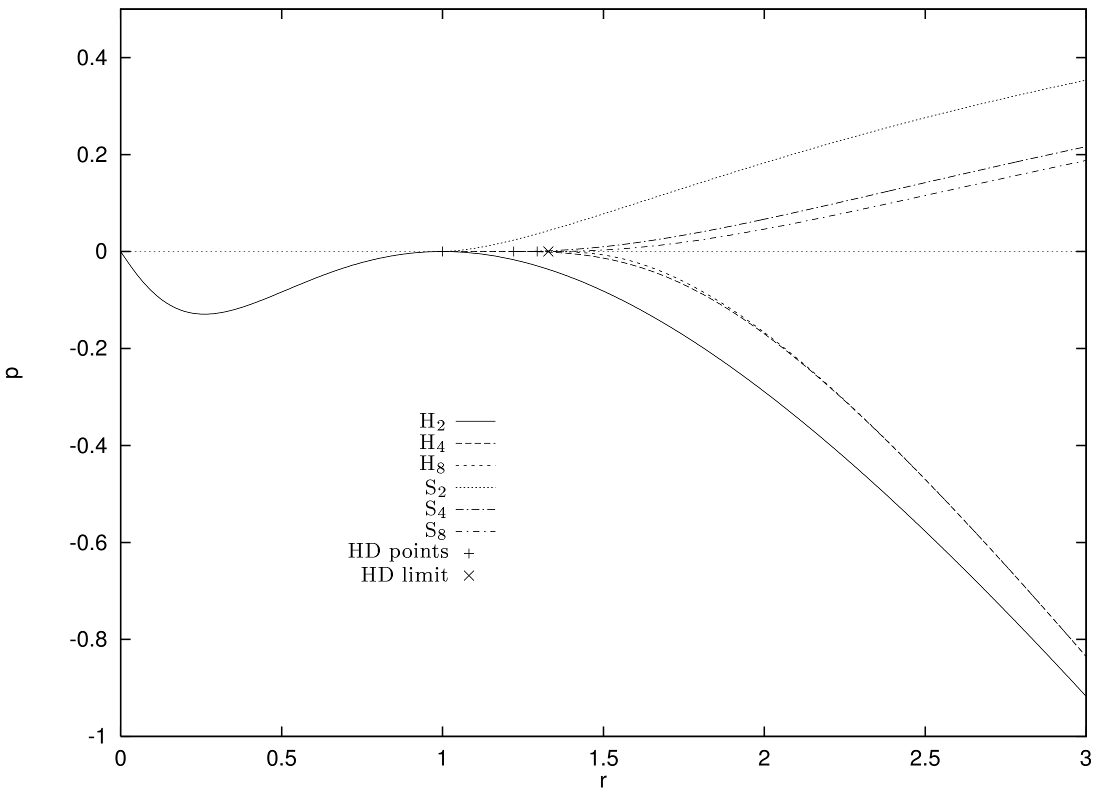

Figure 4: Bifurcation curves for $x \mapsto p + r(1-x^\beta)^2$ in the (p,r) parameter plane for $\beta = 0.6$.

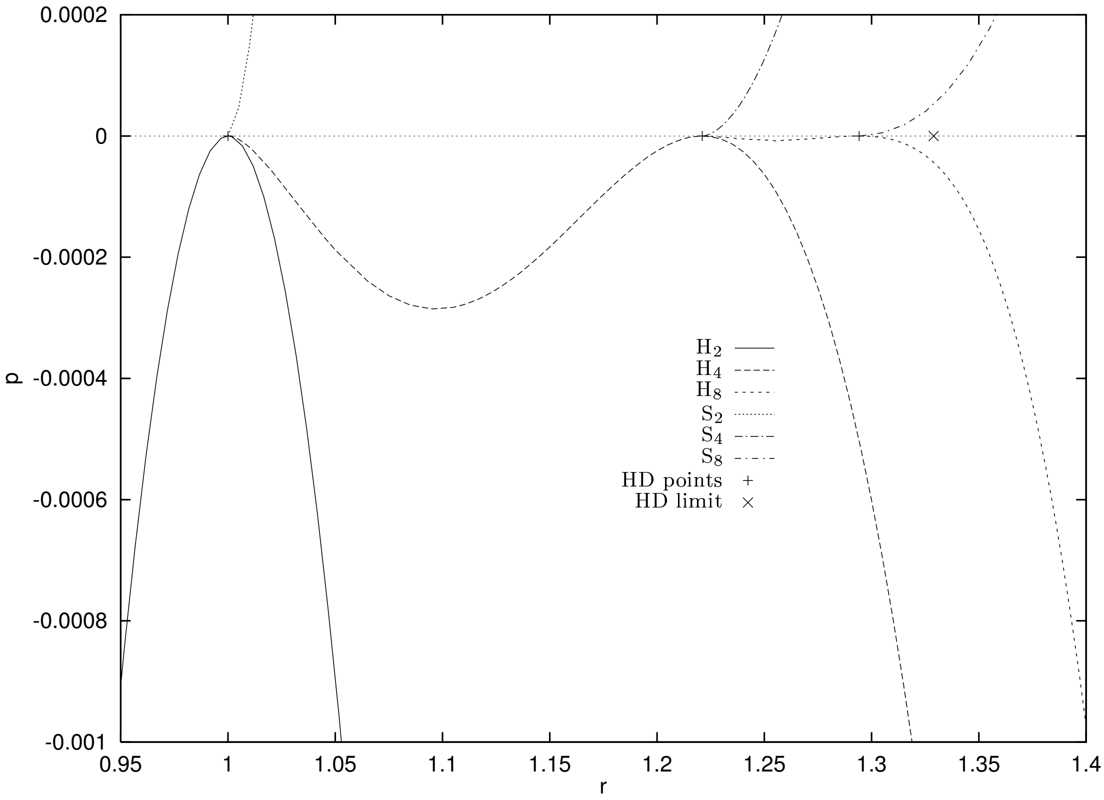

Figure 5: Blow up of figure 4.

The theorem can be applied to sequences of parameter values (p_n, ϵ_n) so that $\mathcal{R}^{2^n}(f_{p_n,\epsilon_n})$ is contained in a bifurcation surface (e.g. H_2, HD_2, or S_2) and converges to a function $\bar{f}$ in the intersection of this bifurcation surface with $W^{u,uu}(\phi)$. The theorem shows that scalings exist in bifurcation diagrams of generic two parameter families of functions in $\mathfrak{C}^{\omega}$, independent of details of the family, other than the value of β. It relates the unstable eigenvalues at the fixed point ϕ of the renormalization operator $\mathcal{R}$ to scalings in such bifurcation diagrams.

In figures 4 and 5, bifurcation curves for the family $x \mapsto p + r(1 - x^{\beta})^2$ with $\beta = 0.6$ are shown. These figures illustrate the scalings predicted by Theorem 3.1. In particular, the HD points converge exponentially at a rate δ_1 which is close to 2 for β close to $\frac{1}{2}$. As is illustrated in figure 5, the curve pieces in $\mathrm{H}_{2^{n+1}}$ between HD_{2^n} and $\mathrm{HD}_{2^{n+1}}$, accumulate on the line $\{p = 0\}$ exponentially fast at the strong rate δ_2.

It should be noted that Theorem 3.1 does not explain all scalings that exist in the bifurcation diagram. Define $\mathrm{H}_\infty = \lim_{k\to\infty} \mathrm{H}_{2^k}$, note that H_∞ is invariant under $\mathcal{R}$. Similarly, $\mathrm{P}_\infty = \lim_{k\to\infty} \mathrm{S}_{2^k}$ is invariant under $\mathcal{R}$. H_∞ and P_∞ constitute a 'boundary of chaos'. Theorem 3.1 does not address scalings of sequences of bifurcation values in S_{2^n}

converging to a point in P_∞ away from $W^s(\phi)$ (which are expected to scale exponentially with the usual Feigenbaum rate). Nor does it address scalings of sequences of bifurcation values in H_{2^n} converging to a point in H_∞ away from $W^s(\phi)$. We have no information on these scalings (however, [LyuPikZak89] and [Hom96] contain results on scalings of homoclinic bifurcations in similar situations).

The following proposition provides some knowledge on $\mathrm{H}_\infty^{u,uu} = \mathrm{H}_\infty \cap W^{u,uu}(\phi)$ and $\mathrm{P}_\infty^{u,uu} = \mathrm{P}_\infty \cap W^{u,uu}(\phi)$. Compare figure 4.

Proposition 3.2 $\mathrm{H}_\infty^{u,uu}$ *and* $\mathrm{P}_\infty^{u,uu}$ *are contained in a wedge* $\{(r,p) : |p| \le C(r-1)^{\delta_2/\delta_1}, r > 1\}$ *for some constant* $C > 0$. $H_\infty^{u,uu} \backslash \{\phi\} \subset \{f(1) < 0\}$ *and* $P_\infty^{u,uu} \backslash \{\phi\} \subset \{f(1) > 0\}$.

Proof. Because $W^{uu}(\phi)$ intersects H_2 and S_2 transversally, $\mathrm{H}_\infty^{u,uu}$ and $\mathrm{P}_\infty^{u,uu}$ are not contained in $W^{uu}(\phi)$. The fact that $\mathrm{H}_\infty^{u,uu}$, $\mathrm{P}_\infty^{u,uu}$ are contained in a wedge as claimed, is now standard.

To show that $\mathrm{H}_\infty^{u,uu} \backslash \phi \subset \{f(1) < 0\}$, first notice that $\mathrm{H}_\infty^{u,uu}$ is contained in $\{f(0) \le 0\}$ since homoclinic orbits cannot exist for positive critical values. To show that $\mathrm{H}_\infty^{u,uu}$ has no points in $\{f(1) = 0\}$ outside ϕ, consider fundamental domains of $\mathcal{R}$ restricted to $W^u(\phi) = W^{u,uu}(\phi) \cap \{f(1) = 0\}$. It suffices to establish the existence of such a fundamental domain on which the kneading sequences are different from that of the Feigenbaum map. Using the continuity of kneading sequences of functions with critical value 0 [MelStr93], and closeness of $W^u(\phi)$ to the line $\{k(1-\sqrt{x})^2, \quad k \in \mathbb{R}\}$, it is easy to provide such a fundamental domain, for β small enough. The analogous statement for $\mathrm{P}_\infty^{u,uu}$ is proved in the same manner. ∎

3.1 Proof of Theorem 3.1

Let us start with a comment on the proof in [ColEckLan80] where universal scalings in period doubling cascades are studied (see also [Pal83]). In [ColEckLan80], a differentiable stable foliation with codimension one leaves for the renormalization operator near the fixed point is constructed. The existence of a differentiable stable foliation (as opposed to a merely continuous foliation) is readily shown to imply the universal scalings observed in period doubling cascades. However, the operator $\mathcal{R}$ may not possess a differentiable stable foliation (with leaves of codimension two) near ϕ. We will therefore use alternative arguments to unravel the universal scalings in homoclinic doubling cascades.

By a smooth local coordinate change, we may assume that $W^s(\phi)$, $W^u(\phi)$ and $W^{uu}(\phi)$ are linear. We thus have smooth coordinates $x = (x_s, x_u, x_{uu})$ near ϕ on $\mathfrak{C}^\omega$, so that

$$\mathfrak{C}_0^\omega \quad = \quad \{x_{uu} = 0\}$$

and

$$W^s(\phi) \quad = \quad \{(x_u, x_{uu}) = (0,0)\},$$

$$\begin{aligned} W^u(\phi) &= \{(x_s, x_{uu}) = (\mathbf{0}, 0)\}, \\ W^{uu}(\phi) &= \{(x_s, x_u) = (\mathbf{0}, 0)\}. \end{aligned}$$

Smooth coordinate changes that leave $\mathfrak{C}_0^\omega$ invariant, do not effect the scalings we wish to show. Note that x_u and x_{uu} are both one dimensional coordinates. The renormalization operator $\mathcal{R}$ in these coordinates can be written as

$$\mathcal{R}(x_s, x_u, x_{uu}) = \begin{pmatrix} A_s x_s + x_s g_s(x) \\ \delta_1 x_u + x_u g_u^1(x) + x_{uu} g_u^2(x) \\ \delta_2 x_{uu} + x_{uu} g_{uu}(x) \end{pmatrix}, \tag{8}$$

where $g_s(x), g_u^1(x), g_u^2(x), g_{uu}(x) = \mathcal{O}(\|x\|)$.

It suffices to consider x from a small box $\mathcal{B} = \{\|x_s\|, |x_u|, |x_{uu}| \leq \eta\}$ for some small $\eta > 0$. By a linear rescaling, we may assume $\eta = 1$. Then $s = \sup\{\|g_s\|, |g_u^1|, |g_u^2|, |g_{uu}|\}$, with the supremum taken over $\mathcal{B}$, is small.

To obtain convergence properties of $\{x_i\}$ as $N \to \infty$, we rely on a technique called Shil'nikov variables which amounts to formulating and solving a boundary value problem for orbits $\{x_i\}$ (see e.g. [Shi68, Den89]). Let $\xi = (\xi_s, \xi_u, \xi_{uu}) \in \mathcal{B}$. Let $\mathcal{F}$ be the map on the set of sequences $\mathbb{N} \cap [0, N] \to \mathfrak{C}^\omega$ defined by

$$\mathcal{F}(\{x_k\})_k = \begin{pmatrix} A_s^k \xi_s + \sum_{i=0}^{k-1} A_s^{k-i} x_{s,i} g_s(x_i) \\ \delta_1^{k-N} \xi_u + \sum_{i=0}^{N-k-1} \delta_1^{k-N+i} \left(x_{u,N-i} g_u^1(x_{N-i}) + x_{uu,N-i} g_u^2(x_{N-i}) \right) \\ \delta_2^{k-N} \xi_{uu} + \sum_{i=0}^{N-k-1} \delta_1^{k-N+i} x_{uu,N-i} g_{uu}(x_{N-i}) \end{pmatrix}, \tag{9}$$

where $x_k = (x_{s,k}, x_{u,k}, x_{uu,k})$. If $\{x_i\}$, $0 \leq i \leq N$, is an orbit of $\mathcal{R}$ with $x_{s,0} = \xi_s$, $x_{u,N} = \xi_u$, $x_{uu,N} = \xi_{uu}$, then $\{x_i\}$ is a fixed point of $\mathcal{F}$: $\mathcal{F}(\{x_i\}) = \{x_i\}$. Indeed, the right hand side of (9) is obtained from the variation of constants formula for orbits.

Let $\sigma < 1$ be larger then the spectral radius of A_s. With $\xi \in \mathcal{B}$ as before, let

$$\Sigma(\xi) = \left\{ \{x_i\}_{0 \leq i \leq N} \;\middle|\; \begin{array}{c} (x_{s,0}, x_{u,N}, x_{uu,N}) = (\xi_s, \xi_u, \xi_{uu}) \\ \|x_{s,i}\| \sigma^{-i}, |x_{u,i}| \delta_1^{N-i}, |x_{uu,i}| \delta_2^{N-i} < \infty \end{array} \right\}.$$

Equipped with the norm

$$\|\{x_i\}\| = \sup_{0 \leq i \leq N} \{\|x_{s,i}\| \sigma^{-i}, |x_{u,i-N}| \delta_1^{N-i}, |x_{uu,i-N}| \delta_2^{N-i}\},$$

$\Sigma(\xi)$ is a Banach space.

Lemma 3.3 *There is a bounded ball $\mathfrak{B}(\xi)$ in $\Sigma(\xi)$, so that for each positive integer N, $\mathcal{F}$ maps $\mathfrak{B}(\xi)$ into itself. Moreover, $\mathcal{F}$ is a contraction on $\mathfrak{B}(\xi)$.*

PROOF. We will show that there exists $M > 0$, so that for $\{x_i\} \in \Sigma(\xi_s, \xi_u, \xi_{uu})$ with $\|\{x_i\}\| \le M$, we have $\|\mathcal{F}(\{x_i\})\| \le M$. In the following estimates, C denotes a positive constant that may vary from line to line, but which is bounded uniformly in N.

$$
\begin{aligned}
\|x_{s,k}\| &\le C\sigma^k \|\xi_s\| + \sum_{i=0}^{k-1} C\sigma^{k-i} \|x_{s,i}\| \|g_s(x)\| \\
&\le C\sigma^k \|\xi_s\| + \sum_{i=0}^{k-1} CM\sigma^k \|g_s(x)\| \\
&\le C\sigma^k \|\xi_s\| + CsM\sigma^k, \\
|x_{u,k}| &\le \delta_1^{k-N} |\xi_u| + \sum_{i=0}^{N-k-1} \delta_1^{k-N+i} \left(|x_{u,N-i}||g_u^1(x)| + |x_{uu,N-i}||g_u^2(x)| \right) \\
&\le \delta_1^{k-N} |\xi_u| + \sum_{i=0}^{N-k-1} M\delta_1^{k-N} |g_u^1(x)| + M\delta_1^{k-N+i}\delta_2^{-i} |g_u^2(x)| \\
&\le \delta_1^{k-N} |\xi_u| + CsM\delta_1^{k-N}, \\
|x_{uu,k}| &\le \delta_2^{k-N} |\xi_{uu}| + \sum_{i=0}^{N-k-1} \delta_2^{k-N+i} |x_{uu,N-i}||g_{uu}(x)| \\
&\le \delta_2^{k-N} |\xi_{uu}| + \sum_{i=0}^{N-k-1} M\delta_2^{k-N} |g_{uu}(x)| \\
&\le \delta_2^{k-N} |\xi_{uu}| + CsM\delta_2^{k-N}.
\end{aligned}
$$

It follows from these estimates that $\mathcal{F}$ maps some bounded ball $\mathfrak{B}(\xi)$ in $\Sigma(\xi)$ into itself. Similar estimates show that $\mathcal{F}$ is a contraction on $\mathfrak{B}(\xi)$. ∎

The lemma shows that for each $\xi \in \mathcal{B}$, there is a unique orbit $\{x_i\}_{0 \le i \le N}$ contained in $\mathcal{B}$ and satisfying $x_{s,0} = \xi_s$, $x_{u,N} = \xi_u$, $x_{uu,N} = \xi_{uu}$. In fact, the computations in the proof of Lemma 3.3 can be easily adapted to show that

$$
\begin{aligned}
|x_{u,0}| &\sim \delta_1^{-N} |x_{u,N}| \text{ if } x_{u,N} \neq 0, \\
|x_{uu,0}| &\sim \delta_2^{-N} |x_{uu,N}| \text{ if } x_{uu,N} \neq 0.
\end{aligned}
$$

At this stage we have obtained a weak form of Theorem 3.1:

Proposition 3.4 *Let a two parameter family $f_{p,\epsilon}$ and a sequence of parameter values (p_n, ϵ_n) tending to $(0, \bar{\epsilon})$ be as in Theorem 3.1. So $\mathcal{R}^{2^n}(f_{p_n,\epsilon_n})$ tends to some function $\bar{f}$ as $n \to \infty$. Then*

$$
\begin{aligned}
-\frac{1}{n} \ln |p_n| &\to \delta_2 \text{ if } \bar{f} \notin W^u(\phi), \\
-\frac{1}{n} \ln |\epsilon_n - \bar{\epsilon}| &\to \delta_1 \text{ if } \bar{f} \notin W^{uu}(\phi).
\end{aligned}
$$

Note that this proposition relates scalings in the bifurcation diagram to the unstable eigenvalues of $D\mathcal{R}(\phi)$. The statement of the proposition however, is not as strong as the statement of Theorem 3.1. Indeed, it shows that $p_n \sim \delta_2^{-n}$ (where $a_n \sim b_n$ means that a_n/b_n is bounded and bounded away from 0, uniformly in n). The statement of Theorem 3.1 implies that $p_n\delta_2^n$ converges to some constant c as $n \to \infty$. Similarly for ϵ_n.

We will proceed with the proof of Theorem 3.1. For this, we look at the Shil'nikov variables approach in more detail. First we apply an additional smooth local coordinate change. The proof of the following lemma will be postponed.

Lemma 3.5 *There are smooth coordinates $x = (x_s, x_u, x_{uu})$ near ϕ, in which $\mathcal{R}$ has the following expression.*

$$\mathcal{R}(x_s, x_u, x_{uu}) = \begin{pmatrix} A_s x_s + x_s g_s(x) \\ \delta_1 x_u + x_u^2 g_u^1(x) + x_{uu} g_u^2(x) \\ \delta_2 x_{uu} + x_{uu} x_u g_{uu}^1(x) + x_{uu}^2 g_{uu}^2(x) \end{pmatrix},$$

where $g_s(x) = \mathcal{O}(\|x\|)$ and $g_u^1(x), g_u^2(x), g_{uu}^1(x), g_{uu}^2(x)$ are all of order $\mathcal{O}(\|x_s\|)$.

Proposition 3.6 *Take coordinates near ϕ given by Lemma 3.5. Let $\{x_i\}$, $0 \le i \le N$, be an orbit of $\mathcal{R}$ contained in $\mathcal{B}$. Then, for some $\kappa > 1$,*

$$\begin{aligned} x_{u,0} &= x_{u,N}\delta_1^{-N} + \mathcal{O}((\delta_1\kappa)^{-N}), \\ x_{uu,0} &= x_{uu,N}\delta_2^{-N} + \mathcal{O}((\delta_2\kappa)^{-N}), \end{aligned}$$

as $N \to \infty$.

Proof. Define $y_{u,k} = x_{u,k}\delta_1^{N-k}$ and $y_{uu,k} = x_{uu,k}\delta_2^{N-k}$. Then the fixed point equation $\mathcal{F}(\{x_i\}) = \{x_i\}$, for the x_u, x_{uu} coordinates can be written as

$$\begin{aligned} y_{u,k} &= \xi_u + \sum_{i=0}^{N-k-1} \left(\delta_1^{-i} y_{u,N-i}^2 g_u^1(x_{N-i}) + \left(\frac{\delta_2}{\delta_1}\right)^{-i} y_{uu,N-i} g_u^2(x_{N-i}) \right), \\ y_{uu,k} &= \xi_{uu} + \sum_{i=0}^{N-k-1} \left(\delta_1^{-i} y_{uu,N-i} y_{u,N-i} g_{uu}^1(x_{N-i}) + \delta_2^{-i} y_{uu,N-i}^2 g_{uu}^2(x_{N-i}) \right). \end{aligned}$$

Take $k = 0$ in the above. Since $\|x_{s,i}\| \le K\sigma^i$ for some $K > 0$, we can estimate that for some $C > 0$ (independent of N),

$$\left| \sum_{i=0}^{N-1} \left(\delta_1^{-i} y_{u,N-i}^2 g_u^1(x_{N-i}) + \left(\frac{\delta_2}{\delta_1}\right)^{-i} y_{uu,N-i} g_u^2(x_{N-i}) \right) \right| \le Cs \left(\sigma^N + \delta_1^{-N} + \left(\frac{\delta_2}{\delta_1}\right)^{-N} \right),$$

and

$$\left| \sum_{i=0}^{N-1} \left(\delta_1^{-i} y_{uu,N-i} y_{u,N-i} g_{uu}^1(x_{N-i}) + \delta_2^{-i} y_{uu,N-i}^2 g_{uu}^2(x_{N-i}) \right) \right| \le Cs \left(\sigma^N + \delta_1^{-N} + \delta_2^{-N} \right).$$

Both sums converge to 0 exponentially fast in N. The proposition follows. ■

PROOF OF THEOREM 3.1. This is an easy corollary of the previous proposition. ■

It remains to prove Lemma 3.5. As a first step, we smoothly linearize $\mathcal{R}$ restricted to the local unstable manifold. Despite possible rational dependence of δ_1, δ_2 (which would obstruct in general the existence of smoothly linearizing coordinates), this is possible because $W^u(\phi)$ is a smooth manifold.

Lemma 3.7 *There are smooth local coordinates in $\mathfrak{C}^\omega$ near ϕ, in which $\mathcal{R}$ restricted to $W^{u,uu}(\phi)$ is linear.*

PROOF. It suffices to construct a smooth invariant foliation $\mathfrak{F}^u$ of $W^{u,uu}(\phi)$ with one dimensional leaves that includes $W^u(\phi)$. Indeed, by [HirPugShu77], there exists a smooth strong unstable foliation $\mathfrak{F}^{uu}$ of $W^{u,uu}(\phi)$ including $W^{uu}(\phi)$. The existence of a smooth foliation $\mathfrak{F}^u$ means that a smooth coordinate change separates the weak unstable and strong unstable coordinate on the local unstable manifold. One dimensional maps with a hyperbolic fixed point can be smoothly linearized, so that the lemma would be proven.

Let $(x_u, x_{uu}) \mapsto R(x_u, x_{uu})$ denote the renormalization operator restricted to $W^{u,uu}(\phi)$; R is defined on an open neighborhood $\mathcal{U}$ of $(0,0)$. To construct $\mathfrak{F}^u$, we construct a DR invariant line bundle which gives $\mathfrak{F}^u$ by integration. Let $R^{(1)}$ be the induced mapping on $\mathcal{U} \times \mathcal{L}(\mathbb{R}, \mathbb{R})$:

$$\begin{aligned} R^{(1)}(x_u, x_{uu}, \alpha) &= (R(x_u, x_{uu}), \beta), \\ \text{graph } \beta &= DR(x_u, x_{uu}) \text{graph } \alpha. \end{aligned}$$

Writing

$$DR^{(1)}(x_u, x_{uu}) = \begin{pmatrix} a(x_u, x_{uu}) & b(x_u, x_{uu}) \\ c(x_u, x_{uu}) & d(x_u, x_{uu}) \end{pmatrix},$$

one has

$$R^{(1)}(x_u, x_{uu}, \alpha) = \left(R(x_u, x_{uu}), \frac{a(x_u, x_{uu})\alpha + b(x_u, x_{uu})}{c(x_u, x_{uu})\alpha + d(x_u, x_{uu})} \right).$$

It is now easily seen that $R^{(1)}$ has a hyperbolic singularity at $(0,0,0)$ and

$$DR^{(1)}(0,0,0) = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ * & * & \delta_2/\delta_1 \end{pmatrix}.$$

Observe that $\delta_1 < \delta_2/\delta_1 < \delta_2$. The sought for invariant line bundle is given by a smooth invariant manifold F that projects injectively to $\mathbb{R}^2 \times \{0\}$ by the coordinate projection. This manifold is obtained as follows: let $\mathfrak{G}^{uu}$ be a strong unstable foliation for $\mathcal{R}^{(1)}$ with one dimensional leaves and let $W^u(0,0,0)$ be the weak unstable manifold $\{(x_u,0,0);\ \ (x_u,0)\in\mathcal{U}\}$. Then

$$F \;=\; \bigcup_{x\in W^u(0,0,0)} \mathfrak{G}^{uu}_x ,$$

where $\mathfrak{G}^{uu}_x$ is the leaf of $\mathfrak{G}^{uu}$ that contains the point x. Since $W^u(0,0,0)$ is a smooth manifold and $\mathfrak{G}^{uu}$ a smooth foliation, F provides the sought for invariant line bundle and thus, by integrating, the foliation $\mathfrak{F}^u$. ∎

Proof of Lemma 3.5. Applying the coordinate change from Lemma 3.7, we may assume that in (8), $g_u^1(x), g_u^2(x), g_{uu}(x) = \mathcal{O}(\|x_s\|)$. Consider a smooth local coordinate change $(x_s,x_u,x_{uu})\mapsto(y_s,y_u,y_{uu})$ of the form

$$\begin{aligned} y_s &= x_s,\\ y_u &= x_u + p_u(x_s)x_u,\\ y_{uu} &= x_{uu}+p_{uu}(x_s)x_{uu}, \end{aligned}$$

for smooth functions p_u, p_{uu} which vanish at $x_s=\mathbf{0}$. Write $\mathcal{R}$ in the new coordinates as

$$\tilde{\mathcal{R}}(y_s,y_u,y_{uu}) \;=\; \begin{pmatrix} A_s y_s + y_s h_s(y) \\ \delta_1 y_u + h_u^1(y_s,y_u,y_{uu})y_u + h_u^2(y_s,y_u,y_{uu})y_{uu} \\ \delta_2 y_{uu} + h_{uu}(y_s,y_u,y_{uu})y_{uu} \end{pmatrix}.$$

We seek coordinates $y=(y_s,y_u,y_{uu})$, so that $h_u^1(y_s,0,0)=0$ and $h_{uu}(y_s,0,0)=0$.

Write $\mathcal{R}=(\mathcal{R}_s,\mathcal{R}_u,\mathcal{R}_{uu})$ and let $\mathcal{P}(y_s,p_u,p_{uu}) = (p_u\circ\mathcal{R}_s(y_s,0,0), p_{uu}\circ\mathcal{R}_s(y_s,0,0))$. Treating p_u and p_{uu} as variables, the demands that $h_u^1(y_s,0,0)=0$ and $h_{uu}(y_s,0,0)=0$, yield

$$\mathcal{P}(y_s,p_u,p_{uu}) \;=\; \begin{pmatrix} p_u - \frac{1}{\delta_1}g_u^1(y_s,0,0) + h.o.t. \\ p_{uu} - \frac{1}{\delta_2}g_{uu}(y_s,0,0) + h.o.t. \end{pmatrix},$$

where the higher order terms are quadratic and higher order in (y_s,p_u,p_{uu}). One obtains (p_u,p_{uu}) as functions of y_s by constructing the strong stable manifold for the mapping $(y_s,p_u,p_{uu})\mapsto(\mathcal{R}_s(y_s,0,0),\mathcal{P}(y_s,p_u,p_{uu}))$. ∎

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