Bifurcations of random differential equations with bounded noise

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1 Introduction

A large proportion of work on the topic of stochastic or random dynamics has focused on noise that is unbounded and, in particular, normally distributed. With such noise, the entire phase space is accessible (i.e. from any initial point any neighborhood may be reached with nonzero probability) and it follows that if the system has a stationary measure then it is unique and its support is the whole phase space. Typically, the density function for the stationary measure varies continuously with any parameter of the system. In light of these facts, Zeeman proposed that a bifurcation in a stochastic system be defined as a change in character of the density function as a parameter is varied [42, 43]. Such bifurcations have come to be known as phenomenological, or P-bifurcations. Arnold in his extensive work on Random Dynamical Systems (RDS) proposed two more definitions, namely *abstract* bifurcation when (local) topological conjugacy changes and *dynamical* bifurcation which is typically evidenced by a change of sign in one of the Lyapunov exponents of the dynamical system (see for example [3, 4, 20]). Many studies, starting with the work of I. Prigogine and his followers (see [28]), have addressed issues of bifurcations in stochastic systems from these perspectives, referring to one or the other of these bifurcations, nearly all in systems with unbounded noise (e.g. [34, 40]).

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Being dominated by the study of dynamical systems perturbed by Gaussian or other unbounded noises, much of the applied and mathematical literature on stochastic bifurcations has focused on the study of Langevin systems:

$$\dot{x} = A(x) + B(x)\xi_t$$

where the dependence on the noise is linear. Bounded noise in contrast may be much more general, but is less understood. In recent years the effects of bounded noise has received increasing attention for dynamical systems generated by both maps and differential equations. One type of bounded noise that has been of interest is Dichotomous Markov Noise (see the review article [11]). This type of noise is often accessible to analysis and arises naturally in various applications (e.g. [21, 37]).

In these pages we review aspects of dynamics and bifurcations in another type of bounded noise system, namely, random differential equations (RDEs) with bounded noise. We will consider random differential equations of the form

$$\dot{x} = f_\lambda(x, \xi_t), \tag{1}$$

depending on both a deterministic parameter λ and noise with realizations ξ_t that take values from some bounded ball in $\mathbb{R}^n$. The state x will belong to a compact, connected, smooth d dimensional manifold M.

A class of examples fitting into our context is constituted by certain degenerate Markov diffusion systems [6, 30] of the form

$$\begin{aligned} dx &= X_0(x)dt + \sum_{i=1}^{m} f_i(\eta) X_i(x)dt, \\ d\eta &= Y_0(\eta)dt + \sum_{j=1}^{l} Y_j(\eta) \circ dW_j, \end{aligned}$$

given by differential equations for the state space variable x, driven by a stochastic process η defined by a Stratonovich stochastic differential equation on a bounded manifold, see e.g. [31]. Another example is by random switching between solution curves of a finite number of ordinary differential equations [9], a generalization of dichotomous Markov noise. Under some conditions such noise is sufficiently rich to fit into the framework of this paper. Reference [15] also discusses some constructions of stochastic processes with bounded noise.

We will discuss the fact that under mild conditions on the noise, the RDEs admits a finite number of stationary measures with absolutely continuous densities. The stationary measures provide the eventual distributions of typical trajectories. Their supports are the regions accessible to typical trajectories in the long run. It is important to note that in the case of bounded noise, there may exist more than one stationary measure.

It was observed that under parameter variation, stationary measures of RDEs can experience dramatic changes, such as a change in the number of stationary measures or a discontinuous change in one of their supports. The RDEs we consider possess a finite number of absolutely continuous stationary measures. The stationary measures therefore have probability density functions. We distinguish the following changes in the density functions:

1. the density function of a stationary measure might change discontinuously (including the possibility that a stationary measure ceases to exist), or
2. the support of the density function of a stationary measure might change discontinuously.

A discontinuous change in the density function is with respect to the L^1 norm topology. A discontinuous change of the support of a stationary measure is with respect to the Hausdorff metric topology. It is appropriate to call such changes "hard" in reference to hard loss of stability in ordinary differential equations. In [8] a loss of stability of an invariant set is called hard if it involves a discontinuous change, in the Hausdorff topology, of the attractor. There is an obvious analogy with discontinuous changes in (supports of) density functions. The examples studied later show how adding a small amount of noise to a family of ordinary differential equations unfolding a bifurcation can lead to a hard bifurcation of density functions. We note that these hard bifurcations may not be captured by Arnold's notion of dynamical bifurcation.

Hard bifurcations are related to almost or near invariance in random dynamical systems, and the resulting effect of metastability. This phenomenon found renewed interest in [13, 14, 39]. In the context of control theory near invariance was studied in [16, 24] for RDEs and [17] for random diffeomorphisms. One approach, taken in [17, 44], to study near invariance is through bifurcation theory. It is then important to describe mechanisms that result in hard bifurcations.

The following sections will contain an overview of the theory of RDEs, in particular their bifurcations, along the lines of [12, 26, 27]. We do not touch on the similar theory for iterated random maps. Here are some pointers to the literature developing the parallel theory for randomly perturbed iterated maps. A description in terms of finitely many stationary measures can be found in [2, 44]. Aspects of bifurcation theory are considered in [33, 44, 45], see [25] for an application in climate dynamics. References [17, 23, 38, 44] consider quantitative aspects of bifurcations related to metastability and escape, we do not address such issues here.

2 Random differential equations

In this section we describe the precise setup of the random differential equations discussed in this chapter. Let M be a compact, connected, smooth d dimensional manifold and consider a smooth RDE

$$\dot{x} = f(x, \xi_t) \tag{2}$$

on M. The time-dependent perturbation ξ that will represent noise may be constructed in a number of ways. We consider ξ belonging to the space $\mathscr{U} = L^\infty(\mathbb{R}, \overline{B^n(\varepsilon)})$

of bounded functions with values in the closure $\overline{B^n(\varepsilon)}$ of the ε ball in $\mathbb{R}^n$. Give $\mathscr{U}$ the weak* topology, which makes it compact and metrizable (see [19, Lemma 4.2.1]). The flow defined by the shift:

$$\theta : \mathbb{R} \times \mathscr{U} \to \mathscr{U}, \qquad \theta^t(\xi(\cdot)) = \xi(\cdot + t),$$

is then a continuous dynamical system (see [19, Lemma 4.2.4]). Further, θ^t is a homeomorphism of $\mathscr{U}$ and θ^t is topologically mixing [19]. We refer to any random perturbation of this form as *noise of level* ε.

Since $\xi \in \mathscr{U}$ is measurable, and f is smooth and bounded, the differential equation (2) has unique, global solutions $\Phi^t(x,\xi)$ in the sense of Caratheodory, i.e.:

$$\Phi^t(x,\xi) = x + \int_0^t f(\Phi^s(x,\xi),\xi_s)\,ds,$$

for any $\xi \in \mathscr{U}$ and all initial conditions x in M, and the solutions are absolutely continuous in t. Furthermore, solutions depend continuously on ξ in the space $\mathscr{U}$. By the assumptions, $\Phi^t(\cdot,\xi) : M \to M$ is a diffeomorphism for any ξ, and $t \geq 0$. Further, if ξ is continuous then Φ^t is a classical solution. We also consider the skew-product flow on $\mathscr{U} \times M$ given by $S^t \equiv \theta^t \times \Phi^t$.

We will suppose the following condition on the noise:

(H1) There exist $\delta_1 > 0$ and $t_1 > 0$ such that

$$\Phi^t(x,\mathscr{U}) \supset B(\Phi^t(x,0),\delta_1) \qquad \forall t > t_1, x \in M.$$

The assumption **(H1)** can be interpreted as guaranteeing that the perturbations are sufficiently robust.

We call a set $C \subset M$ a *forward invariant set* if

$$\Phi^t(C,\mathscr{U}) \subset C \tag{3}$$

for all $t \in \mathbb{R}^+$. There is a partial ordering on the collection of forward invariant sets by inclusion, i.e. $C' \preceq C$ if $C' \subset C$. We call C a *minimal forward invariant set*, abbreviated MFI set, if it is minimal with respect to the partial ordering $\preceq$.

Theorem 1 ([26]). *Let (2) be a random differential equation with ε-level noise whose flow satisfies* **(H1)** *on a compact manifold M. Then there are a finite number of MFI sets $E_1,\dots,E_k$ on M. Each MFI set is open and connected. The closures of different MFI sets are disjoint.*

We note that the concept of MFI set is the same as the concept of *invariant control sets* used in control theory [18, 19]. Up to this point the discussion is topological: MFI sets can be studied without assumptions about the noise realizations and can in particular be studied for differential inclusions [1, 33]. In deterministic systems, forward invariant sets are commonly called *trapping regions* and *attractors* are analogous to MFI sets. We will see later in the case of small noise, this relationship is more than analogy.

We continue with a discussion of stationary measures. For this we assume conditions on the distribution of transition probabilities. We suppose that $\mathscr{U}$ has a θ^t-invariant probability measure $\mathbb{P}$. Consider the evaluation operator $\pi^t : \mathscr{U} \to \overline{B^n(\varepsilon)}$ given by $\pi^t(\xi) = \xi_t$. Also consider the measure

$$\rho = \pi^t_* \mathbb{P}$$

on $\overline{B^n(\varepsilon)}$. Since $\mathbb{P}$ is θ^t invariant, it follows easily that ρ is independent of t. We call ρ the *distribution* of the noise. Let x be a point in M. We define the push-forward of $\mathbb{P}$ from $\mathscr{U}$ to M via Φ^t as the probability which acts on continuous functions $\psi : M \to \mathbb{R}$ by integration as:

$$(\Phi^t(x)_* \mathbb{P})\psi = \int_{\mathscr{U}} \psi\left(\Phi^t(x,\xi)\right) d\mathbb{P}(\xi).$$

The topological support of $\mathbb{P}$ may for instance be the continuous functions $C(\mathbb{R}, B^n(\varepsilon))$, the *cadlag* functions (see [3]), or even as in [29] the closure of the set of shifts of a specific function ξ. We will assume that θ^t is ergodic w.r.t. $\mathbb{P}$. Rather than $\mathscr{U}$ one may consider instead the topological support of $\mathbb{P}$ in $\mathscr{U}$.

(H2) There exists $t_2 > 0$ so that $\Phi^t(x)_* \mathbb{P}$ is absolutely continuous w.r.t. a Riemannian measure m on M for all $t > t_2$ and all $x \in M$.

Assumption **(H2)** requires that the noise not have "spikes". We remark that **(H1)** and **(H2)** may be replaced by conditions on the vector field and the noise.

A probability μ on M is said to be *stationary* if $\mathbb{P} \times \mu$ is S^t invariant, i.e. for any Borel set $A \subset \mathscr{U} \times M$:

$$\mathbb{P} \times \mu(S^t(A)) = \mathbb{P} \times \mu(A) \tag{4}$$

for all $t \in \mathbb{R}^+$. We say that a stationary measure μ is *ergodic* if $\mathbb{P} \times \mu$ is ergodic for the skew product flow S^t. Birkhoff's ergodic theorem then ensures that:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Psi(S^t(x,\xi))\, dt = \int_{M\times\mathscr{U}} \Psi\, d(\mathbb{P} \times \mu)$$

for $\mathbb{P} \times \mu$ almost every (ξ, x) and for every $\Psi \in C^0(\mathscr{U} \times M, \mathbb{R})$. In particular, if μ is ergodic, setting $\Psi = \psi \circ \pi_M$ for $\psi \in C^0(M, \mathbb{R})$ and the coordinate projection $\pi_M : M \times \mathscr{U} \to M$, we obtain:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(\Phi^t(x,\xi))dt = \int_{\mathscr{U}} \psi\, d\mu \tag{5}$$

for $\mathbb{P} \times \mu$-a.e. $(\xi, x) \in \mathscr{U} \times M$.

We say that a point $x \in M$ is μ-generic if (5) holds for every $\psi \in C^0(M, \mathbb{R})$ and for $\mathbb{P}$-a.e. $\xi \in \mathscr{U}$. The set of generic points of a stationary ergodic measure μ is called the *ergodic basin* of μ and will be denoted $E(\mu)$. An ergodic stationary probability measure whose basin has positive volume, $m(E(\mu)) > 0$, will be called a *physical measure*.

Theorem 2 ([22],[26]). *Let (2) be a random differential equation with ε-level noise whose flow satisfies* **(H1)** *and* **(H2)** *on a compact manifold M. Then there are a finite number of physical, absolutely continuous invariant probability measures $\mu_1, \ldots, \mu_k$ on M. Each μ_i is supported on the closure of a minimal forward invariant set E_i. Further, given any $x \in M$ and almost any $\xi \in \mathscr{U}$, there exists $t^* = t^*(x, \xi)$, such that $\Phi^t(x, \xi) \in E_i$ for some i and all $t > t^*$.*

We end this general introduction with a simple but important example of how MFI sets may occur. Suppose that the random differential equation (2) is a small perturbation of a deterministic system. In this case, attractors generally become minimal forward invariant sets. Consider a random differential equation:

$$\dot{x} = f(x, \varepsilon \xi_t) \tag{6}$$

where ε is a small parameter. For $\varepsilon = 0$ the system is deterministic.

Definition 1. A set A is called an *attractor* for (6) with $\varepsilon = 0$ if it is:

A1 Invariant and compact
A2 There is a neighborhood U of A such that for all $x \in \overline{U}$, $\Phi^t(x, 0) \in U$ for all $t \geq 0$ and $\Phi^t(x, 0) \to A$ as $t \to \infty$.
A3 (a) There is some $x \in U$ such that A is the ω limit set of x, or, (b) A contains a point with a dense orbit, or, (c) A is chain transitive.

If A satisfies A1 and A2 only, it is said to be *asymptotically attracting* or an *attracting set*. We call $\overline{U}$ a *trapping region*.

Theorem 3. *Suppose that (6) satisfies* **(H1)***,* **(H2)** *and that for $\varepsilon = 0$ it has an attractor A. Then for ε sufficiently small, (6) has a MFI set that is a small neighborhood of A. Suppose that U is a trapping region for A, then given ε small enough, this MFI set is unique in U.*

Proof. Since A is asymptotically stable for $\varepsilon = 0$, there exists a smooth Lyapunov function in a neighborhood of A, this Lyapunov function is strictly decreasing along solutions outside of A and these level sets enclose trapping regions [36]. Thus given any $\delta > 0$ there is a trapping region, which we denote by U_δ whose boundary is the δ level set of the Lyapunov function. For a fixed δ it follows that for ε small the Lyapunov function is decreasing along solutions at the boundary of U_δ. Thus U_δ is a forward invariant set for (2) for any ε sufficiently small. Thus U_δ must contain at least one minimal forward invariant set. Now consider any point $x \in U_\delta$. It follows easily that the set of all possible orbits of x, i.e.

$$O^+(x) = \cup_{t \geq 0} \Phi^t(x, \mathscr{U}) \subset U_\delta \tag{7}$$

is forward invariant [26]. Since A is asymptotically stable and x is inside its basin (for $\varepsilon = 0$) it follows from **(H1)** that $O^+(x)$ intersects A. Since A is an attractor, any of the conditions A3 (a), (b), or (c) with **(H1)** implies that $A \subset O^+(x)$. Since a forward invariant set must contain the forward orbits of all of its points, every forward invariant set in U_δ contains A. Therefore, there is only one MFI set in U_δ and it contains A.

Now consider a trapping region $U \supset A$. Suppose that δ is small enough that $U_\delta \subset U$ and ε_1 is small enough that the previous conclusion holds for U_δ. Note that $K = \overline{U} \setminus U_\delta$ is compact and that the Lyapunov function is strictly decreasing on it. Thus there exists ε_2 such the Lyapunov function is also decreasing for the perturbed flow on K for $\varepsilon \leq \varepsilon_2$. Thus there can be no forward invariant set in K for any ε less than the minimum of ε_1 and ε_2 and the conclusion holds. □

Corollary 1. *If x_0 is an asymptotically stable equilibrium for $\varepsilon = 0$, then for all sufficiently small $\varepsilon > 0$ the system has a small MFI set that contains x_0. If Γ is an asymptotically stable limit cycle for $\varepsilon = 0$, then for small $\varepsilon > 0$ the system has a MFI set that is a small tubular neighborhood of Γ.*

3 Random differential equations in one dimension

We discuss the simplest case of random differential equations on a circle. Consider a RDE

$$\dot{x} = f(x, \xi_t) \tag{8}$$

with x from the circle. In the context of bifurcations it is convenient to assume that the noise takes values from $\Delta = \overline{B^n(\varepsilon)} = [-\varepsilon, \varepsilon]$ and the following:

(H3) For each x the map $\Delta \to T_x M$ given by $v \mapsto f(x, v)$ is a diffeomorphism with a strictly convex image $D(x) = f(x, \Delta)$.

Definition 2. We say that a MFI set E is *isolated* or *attracting* if for any proper neighborhood U ($\overline{E} \subset U$) there is an open forward invariant set $F \subset U$ such that $\overline{E} \subset F$, F contains no other MFI set and $\overline{\Phi^t(F, \mathscr{U})} \subset F$ for all $t > 0$. Such a neighborhood F is called an *isolating neighborhood*.

Also note that under **(H3)** for each x, $f(x, \Delta)$ is a closed interval with endpoints $f(x, -\varepsilon)$ and $f(x, \varepsilon)$. Thus there is an envelope of all possible vector fields which are bounded below and above by $f(\cdot, -\varepsilon)$ and $f(\cdot, \varepsilon)$. Denote by $f_-(\cdot)$ and $f_+(\cdot)$ the upper and lower vector fields.

Recall that MFI sets are invariant under forward solutions of the RDE, for all noise realizations and minimal with respect to set inclusion. For RDEs on the circle, the MFI sets are bounded open intervals or possibly the entire circle.

Proposition 1. *If (a, b) is a MFI set then for any $x \in (a, b)$,*

$$0 \in int(f(x, \Delta)).$$

Proof. If not, then there is an $x \in (a, b)$ such that either $f(x, \Delta) \leq 0$ or $f(x, \Delta) \geq 0$. In the first case the forward invariance of (a, b) implies that (a, x) is forward invariant. In the second case we obtain that (x, b) is forward invariant. Either case contradicts the minimality of (a, b). □

Proposition 2. *If* (a,b) *is a MFI set then*

$$f(a,\xi) \geq 0 \quad \text{and} \quad f(b,\xi) \leq 0 \tag{9}$$

for all $\xi \in \Delta = [-\varepsilon,\varepsilon]$ *and that* $f_-(a) = 0$ *and* $f_+(b) = 0$*. Further,* $f'_-(a) \leq 0$ *and* $f'_+(b) \leq 0$*.*

Proof. The inequalities (9) are necessary for a and b to be boundary points of a MFI set. The claim that $f_-(a) = f_+(b) = 0$ follows from **(H1)**. The final claim $f'_-(a) \leq 0$ and $f'_+(b) \leq 0$ then follows from the assumption that f is C^1. □

We can distinguish the following types for endpoints a and b based on the properties of f'. We say that a is *hyperbolic* if $f'_-(a) \neq 0$ and similarly for b. Otherwise, a or b is said to be non-hyperbolic. For one dimensional RDEs the following stability result is straightforward.

Proposition 3. *Given any* f *satisfying* **(H1)**, **(H2)**, **(H3)** *suppose that* (a,b) *is a MFI set with both a and b hyperbolic. Then* (a,b) *is isolated with some isolating neighborhood* W*. If* $\hat{f}$ *is sufficiently close to* f *in the* C^1 *topology, then* $\bar{f}$ *has a unique MFI set* $(\hat{a},\hat{b})$ *inside* W*. Further,* $\hat{a}$ *and* $\hat{b}$ *are close to a and b respectively and are each hyperbolic.*

Proof. If a is hyperbolic it follows that $f(x,\xi_t) > 0$ for all x in some neighborhood (c,a) and all $\xi \in \Delta$. Similarly, there is a neighborhood (b,d) on which $f(x,\xi_t) < 0$. It follows that $W = (c,d)$ is an isolating neighborhood for (a,b).

Now let $\delta > 0$ be sufficiently small so that $f'_-(x) > f'_-(a)/2$ for all $x \in [a-\delta, a+\delta]$. If $\hat{f}$ is within $f'_-(a)/2$ of f in the C^1 topology then the conclusion holds. □

We continue with families of RDEs and consider equations

$$\dot{x} = f_\lambda(x,\xi_t), \tag{10}$$

depending on both a deterministic parameter $\lambda \in \mathbb{R}$ and noise of level ε. For background on bifurcation theory in families of differential equations we recommend [32].

Definition 3. We say that a one-parameter family of vector fields $g_\lambda(x)$ *generically unfolds a quadratic saddle-node* point at x^*, if $g(x^*) = 0$, $g'(x^*) = 0$, $g''(x^*) \neq 0$ and $\partial g_\lambda(x^*)/\partial\lambda \neq 0$.

A one-dimensional RDE (10) *generically unfolds a quadratic saddle-node* at $x^* = a$ or b of a MFI set (a,b), if one of the extremal vector fields $f_\lambda(\cdot,\pm\varepsilon)$ generically unfolds a quadratic saddle-node at x^*.

Theorem 4 ([27]). *In a generic one-parameter family of one-dimensional bounded noise random differential equations (10) the only codimension one bifurcation of a MFI set is the generic unfolding of a quadratic saddle-node.*

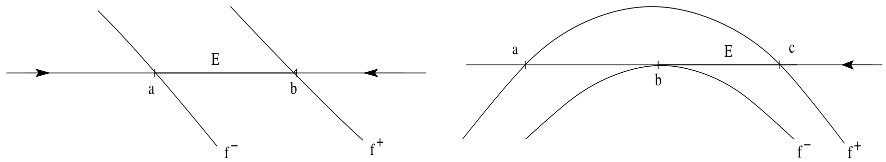

Fig. 1 (a) A stable one dimensional MFI set. Both endpoints of $E = (a,b)$ are hyperbolic. (b) A random saddle-node in one dimension. $E = (b,c)$ is minimal forward invariant.

Proof. By Proposition 3 a MFI set (a,b) is stable if a and b are both hyperbolic. Thus a bifurcation can occur only if hyperbolicity is violated at one of the endpoints. For codimension one hyperbolicity cannot be violated at both the endpoints simultaneously.

If the stationary point is odd, a standard argument shows that the bifurcation is not codimension one. If the stationary point is of even order ≥ 4, then standard arguments show that the family is not generic. □

4 Random differential equations on surfaces

We will consider bifurcations in a class of random differential equations

$$\dot{x} = f_\lambda(x, \xi_t) \tag{11}$$

as the parameter $\lambda \in \mathbb{R}$ is varied. Here x will belong to a smooth compact two-dimensional surface M. We treat such random differential equations with bounded noise where ξ_t takes values in a closed disk $\Delta \subset \mathbb{R}^2$. We will assume some regularity conditions on the way the noise enters the equations. In particular we assume that the range of vectors $f_\lambda(x,\Delta)$ is a convex set for each $x \in M$.

Let $\Delta \subset \mathbb{R}^2$ be the unit disc. We will assume that $f_\lambda(x,v)$ is a smooth vector field depending smoothly on parameters $\lambda \in \mathbb{R}$ and $v \in \Delta$, i.e. $(x,v,\lambda) \mapsto f_\lambda(x,v) \in TM$ is a C^∞ smooth function. When discussing properties of single vector fields, we suppress dependence of the RDE on λ from the notation.

Definition 4. We will denote by R^∞ the space of bounded noise vector fields f satisfying **(H1)**, **(H2)**, **(H3)**. We will take as a norm on R^∞ the C^∞ norm on the vector fields $f : M \times \Delta \to TM$.

Definition 5. We will say that a MFI set E for f is *stable* if there is a neighborhood $U \supset E$ such that if $\tilde{f}$ is sufficiently close to f in R^∞ then $\tilde{f}$ has exactly one MFI set $\tilde{E} \subset U$ and $\tilde{E}$ is close to E in the Hausdorff metric. We will say that $f \in R^\infty$ is *stable* if all of its MFI sets $\{E_i\}$ are stable.

Definition 6. A *one-parameter family* of RDEs in R^∞ is a mapping from an interval $(0,1)$ given by $\lambda \mapsto f_\lambda$ that is smooth in λ in R^∞.

From **(H3)**, the vectors $f_\lambda(x,\xi)$ range over a strictly convex set $D^\lambda(x) \equiv f_\lambda(x,\Delta)$ that is diffeomorphic to a closed disk and has a smooth boundary, varying smoothly with x and λ. Define $K^\lambda(x)$ as the cone of positive multiples of vectors in $D^\lambda(x)$. Again, whenever we are concerned with single RDEs, we suppress dependence on the parameter λ.

Definition 7. A point $x \in M$ will be called *stationary* if $0 \in D(x)$, i.e. there is a possible vector field for which x is fixed.

If $0 \in \mathrm{int}D(x)$, then $K(x) = \mathbb{R}^2$. Outside the closed set $R = \{x \in M \mid 0 \in D(x)\}$, the cones $K(x)$ depend smoothly on x. By **(H3)** if $0 \in \partial D(x)$, then $K(x)$ is an open half-plane. Consider the direction fields E_i, $i = 1,2$, defined by the extremal half lines in the cones $\overline{K(x)}$ over the open set $P = M \setminus R$. By standard results we can integrate these two direction fields, obtaining two sets of smooth solution curves γ_i, $i = 1,2$ in P. Note that these two sets of curves each make a smooth foliation of P. We remark that the direction fields E_i are defined on the closure of P, but may give rise to nonunique solution curves at points in the boundary of P. Further, by the assumptions, the angle between the direction fields at any point is bounded below. However, at points on the boundary, the angle may be π, in which case the solution curves are tangent or coincide (but flow in opposite directions).

Definition 8. For each $x \in P$ denote by $\gamma_i(x)$, $i = 1,2$, the two local solution curves to the extremal direction fields. Denote by $\gamma_i^\pm$ the forward and backward portions of these curves.

We will build up a description of the possible boundary components of a MFI set. To begin, for a point on the boundary either (1) $K(x)$ is less than a half plane, or, (2) $K(x)$ is an open half plane, in which case x must a stationary point, i.e. $f(x,\xi) = 0$ for some $\xi \in \Delta$. We begin by classifying points of type (1).

Lemma 1. *If $x \in \partial E$ for a MFI set E and $K(x)$ is less than a half plane, then either:*

- *One of the local solution curves $\gamma_i(x)$ coincides locally with ∂E, or,*
- *Both backward solution curves $\gamma_i^-(x)$ belong to the boundary ∂E.*

Definition 9. We call a boundary point, x, of a MFI set, E, *regular* if one of $\gamma_i(x)$ coincides locally with ∂E. We call a segment of the boundary of E a *solution arc* if it consists of regular points. If both γ_i^- belong to ∂E locally, then we call x a *wedge* point.

The following theorem describes the geometry of MFI sets for typical RDEs on compact surfaces. Figure 2 depicts parts of the boundary and extremal flow lines near stationary and wedge points.

Theorem 5 ([27]). *There is an open and dense set $V \subset R^\infty$ so that for any random differential equation in V, a MFI set E has piecewise smooth boundary consisting of regular curves, a finite number of wedge points, and a finite number of hyperbolic points that belong to disks of stationary points inside E. Further, if a component γ is a periodic cycle, it has Floquet multiplier less than one. Any RDE in V is stable.*

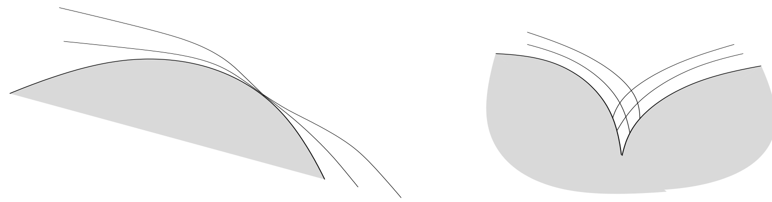

Fig. 2 Extremal flow lines near a stationary point (left picture) or wedge point (right picture) on the boundary of a MFI set.

Codimension one bifurcations in families of RDEs on compact surfaces are described in the following result.

Theorem 6 ([27]). *There exists an open dense set $\mathscr{O}$ of one-parameter families of RDEs in R^∞ such that the only bifurcations that occur are one of the following:*

1. *Two sets of stationary points collide at a stationary point on the boundary ∂E which undergoes a saddle-node bifurcation.*
2. *A MFI E collides with a set of stationary points outside E at a saddle-point p.*
3. *The Floquet multiplier of a non-isolated periodic cycle becomes one and then the cycle disappears.*

5 Random Hopf bifurcation

We will consider Hopf bifurcations in a class of random differential equations on the plane, as a case study of bifurcations in two dimensional RDEs.

Consider a smooth family of planar random differential equations

$$(\dot{x}, \dot{y}) = f_\lambda(x, y) + \varepsilon(u, v) \tag{12}$$

where $\lambda \in \mathbb{R}$ is a parameter and u, v are noise terms from $\Delta = \{u^2 + v^2 \leq 1\}$, representing radially symmetric noise. We consider noise such that hypotheses **(H1)** and **(H3)** are fulfilled. We assume that without the noise terms, i.e. for $\varepsilon = 0$, the family of differential equations unfolds a supercritical Hopf bifurcation at $\lambda = 0$ [32].

In a supercritical Hopf bifurcation taking place in (12) for $\varepsilon = 0$, a stable limit cycle appears in the bifurcation for $\lambda > 0$. For a fixed negative value of λ, the differential equations without noise posses a stable equilibrium and the RDE with small noise has a MFI set which is a disk around the equilibrium. Likewise, at a fixed positive value of λ for which (12) without noise possesses a stable limit cycle, small noise will give an annulus as MFI set. A bifurcation of stationary measures takes place when varying λ. We will prove the following bifurcation scenario for small $\varepsilon > 0$: the RDE (12) undergoes a hard bifurcation in which a globally attracting MFI set changes discontinuously, by suddenly developing a “hole”. This hard bifurcation takes place at a delayed parameter value $\lambda = \mathscr{O}(\varepsilon^{2/3})$ as described in Theorem 7 below.

For studies of Hopf bifurcations in stochastic differential equations (SDEs) we refer to [5, 7, 10, 41]. In such systems there is a unique stationary measure, with support equal to the entire state space. Bifurcations of supports of stationary measures, as arising in RDEs with bounded noise, do not arise in the context of SDEs.

Theorem 7 ([12]). *Consider a family of RDEs* (12) *depending on one parameter λ, that unfolds, when $\varepsilon = 0$, a supercritical Hopf bifurcation at $\lambda = 0$.*

For small $\varepsilon > 0$ and λ near 0, there is a unique MFI set E_λ. There is a single hard bifurcation at $\lambda_{\mathrm{bif}} = \mathscr{O}(\varepsilon^{2/3})$ as $\varepsilon \downarrow 0$. At $\lambda = \lambda_{\mathrm{bif}}$ the MFI set E_λ changes from a set diffeomorphic to a disk for $\lambda < \lambda_{\mathrm{bif}}$ to a set diffeomorphic to an annulus for $\lambda \geq \lambda_{\mathrm{bif}}$. At λ_{bif} the inner radius of this annulus is $r^* = O(\varepsilon^{1/3})$.

Figure 3 shows images taken from [12] of numerically computed invariant densities. For these images the RDEs are taken in normal form as

$$\begin{aligned} \dot{x} &= \lambda x - y - x(x^2+y^2) + \varepsilon u, \\ \dot{y} &= x + \lambda y - y(x^2+y^2) + \varepsilon v. \end{aligned} \tag{13}$$

The noise terms u and v are generated via the stochastic system:

$$\begin{aligned} du &= dW_1, \\ dv &= dW_2, \end{aligned} \tag{14}$$

where dW_1 and dW_2 are independent (of each other), normalized white noise processes. The equations (14) are interpreted in the usual way as Itō integral equations. In this setting in order to assure boundedness, (u,v) are restricted to the unit disk by imposing reflective boundary conditions.

The deterministic Hopf bifurcation involves the creation of a limit cycle. In the remaining part of this section we discuss the occurrence of attracting random cycles. Random cycles are closed curves that are invariant for the skew-product system and thus have a time dependent position in state space depending on the noise realization. The following material fits into the philosophy advocated by Arnold in [3] of studying random dynamical systems through a skew product dynamical systems approach, so as to capture dynamics with varying initial conditions. A comparison of bifurcations in both contexts of stationary measures and of invariant measures for the skew product system is contained in [45] in the context of random circle diffeomorphisms.

Random cycles are defined in analogy with random fixed points [3]. They are most elegantly treated in a framework of invertible flows, where the noise realizations ξ and the flow $\Phi^t_\lambda(x,\xi)$ are given for two sided time $t \in \mathbb{R}$. We henceforth consider the skew product flow

$$(x,\xi) \mapsto \left(\Phi^t_\lambda(x,\xi), \theta^t \xi\right)$$

with

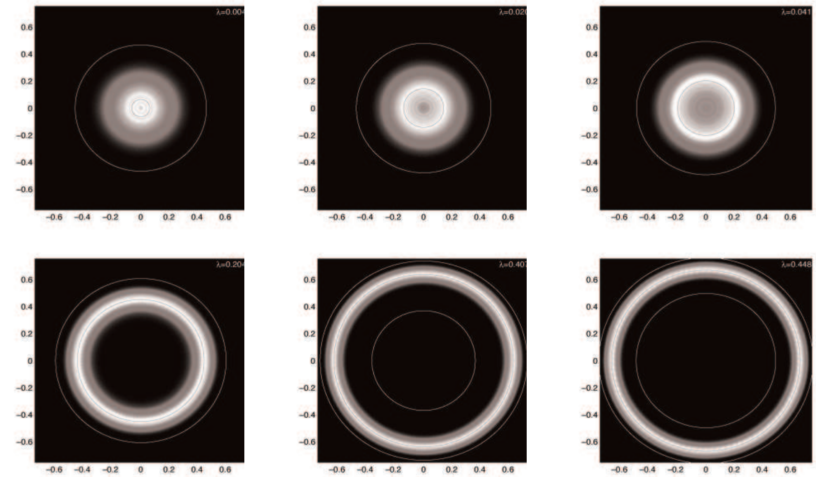

Fig. 3 Images of the invariant densities for system (13) with $\varepsilon = 0.1$ and increasing values of λ. From top left $\lambda = 0.004, 0.020, 0.041, 0.204, 0.407, 0.448$. The bottom middle plot, ($\lambda = 0.407$), is immediately after the hard bifurcation. In all six plots the circle exterior to the visible density is the outer boundary of the MFI set. In the last two plot the interior circle is the inner boundary of the MFI set. Figures taken from ref. [12], ©American Institute for Mathematical Sciences 2012.

$$\theta^t \xi_s = \xi_{t+s}$$

for $t, s \in \mathbb{R}$.

Recall that a random fixed point is a map $R : \mathscr{U} \to \mathbb{R}^2$ that is flow invariant,

$$\Phi^t_\lambda(R(\xi), \xi) = R(\theta^t \xi)$$

for $\mathbb{P}$ almost all ξ. A random cycle is defined as a continuous map $S : \mathscr{U} \times \mathbb{S}^1 \to \mathbb{R}^2$ that gives an embedding of a circle for $\mathbb{P}$ almost all $\xi \in \mathscr{U}$ and is flow invariant in the sense

$$\Phi^t_\lambda(S(\xi, \mathbb{S}^1), \xi) = S(\theta^t \xi, \mathbb{S}^1).$$

Different regularities of the embeddings $\mathbb{S}^1 \mapsto S(\xi, \mathbb{S}^1)$, such as Lipschitz continuity or some degree of differentiability, may be considered.

The random cycle is attracting if there is a neighborhood U_λ of the MFI set E_λ, so that for all $x \in U_\lambda$, the distance between $\Phi^t_\lambda(x, \xi)$ and $S(\theta^t \xi, \mathbb{S}^1)$ goes to zero as $t \to \infty$.

The following result establishes the occurrence of attracting random cycles following the hard bifurcation, for small noise amplitudes.

Theorem 8. *Consider a family of RDEs* (12) *depending on one parameter λ, that unfolds, when $\varepsilon = 0$, a supercritical Hopf bifurcation at $\lambda = 0$.*

For values of (λ,ε) with $\lambda > \lambda_{\mathrm{bif}}$ and ε sufficiently small, the MFI set E_λ is diffeomorphic to an annulus and the flow Φ_λ^t admits a Lipschitz continuous attracting random cycle $S : \mathscr{U} \times \mathbb{S}^1 \to \mathbb{R}^2$ inside E_λ.

Proof. The proof is an adaptation of the construction of limit cycles in the differential equations without noise. The boundedness of the noise allows one to replace the estimates by estimates that are uniform in the noise for small enough noise amplitudes. We indicate the steps in a proof, leaving details to the reader.

First note that we may replace the flow Φ_λ^t by its time one map, which is a diffeomorphism on the plane; this diffeomorphism and its derivatives depend continuously on the noise ξ. So, consider a map $z \mapsto f_\lambda(z;\xi)$ on the complex plane $\mathbb{C}$, unfolding a supercritical Neimark-Sacker bifurcation in λ, depending on bounded noise $\xi \in \mathscr{U}$ and on the parameter ε that multiplies the amplitude of the noise. Such maps without noise, i.e. with $\varepsilon = 0$, are known to possess invariant circles for small positive values of λ. We follow their construction as elaborated in [35]. With a normal form transformation, applied to the map without noise, a map

$$F_\lambda(z) = z(1+\lambda - f_1(\lambda)|z|^2)e^{i(\theta(\lambda)+f_3(\lambda)|z|^2)} + \mathscr{O}(|z|^5) \tag{15}$$

on the complex plane $\mathbb{C}$ is obtained. The reasoning in [35] continues with the following steps. Apply a rescaling and change to polar coordinates to write $z = \sqrt{\frac{\lambda}{f_1(\lambda)}}e^{i\varphi}(1+\sqrt{\lambda}u)$. Expressing F_λ in φ, u coordinates gives a map of the form

$$F_\lambda(\varphi,u) = (\varphi + \theta_1(\lambda) + \lambda^{3/2}K_\lambda(u,\varphi), (1-2\lambda)u + \lambda^{3/2}H_\lambda(u,\varphi)). \tag{16}$$

Next a graph transform is defined on a class of Lipschitz continuous graphs $\mathrm{Lip}_1(\mathbb{S}^1,[-1,1])$, with Lipschitz constant bounded by 1, equipped with the supnorm. It is determined by

$$\operatorname{graph} \mathscr{F}_\lambda(w) = F_\lambda(\operatorname{graph} w). \tag{17}$$

This is shown to be a contraction, leading to a unique fixed point which is the attracting invariant circle.

For ε small enough, this reasoning carries through to the random map as follows. First a graph transform depending on $\xi \in \mathscr{U}$ is defined. That is, F_λ from (15) (and (16)) gets replaced by a map $F_{\lambda,\xi}$ and the graph transform likewise by $\mathscr{F}_{\lambda,\xi}$. Iterates of $\mathscr{F}_{\lambda,\xi}$ are obtained as

$$\mathscr{F}^n_{\lambda,\xi} = \mathscr{F}_{\lambda,\theta^{n-1}\xi} \circ \ldots \circ \mathscr{F}_{\lambda,\theta^1\xi} \circ \mathscr{F}_{\lambda,\xi}. \tag{18}$$

The previous contraction argument is replaced by pull-back convergence:

$$S(\xi,\mathbb{S}^1) = \lim_{n\to\infty} \mathscr{F}^n_{\lambda,\theta^{-n}\xi}(w), \tag{19}$$

for any $w \in \mathrm{Lip}_1(\mathbb{S}^1, [-1,1])$. The graph of the limit function is called the pull-back attractor, its orbit under the flow Φ^t_λ is the random limit cycle. Note that this is the point where two sided time is needed.

The computations to check convergence in (19) are most easily carried out by writing $\varepsilon\xi = \varepsilon r e^{i\psi}$ for the noise and expanding $F_{\lambda,\xi}$ in ε for small ε: writing $F_\lambda = A e^{i\eta}$ and $\xi = r e^{i\psi}$ we get

$$F_{\lambda,\xi} = (A + \mathscr{O}(\varepsilon)) e^{i(\eta + \frac{1}{A}\mathscr{O}(\varepsilon))}. \qquad (20)$$

Then following the computations using the rescaling, assuming ε is sufficiently small for given λ, makes clear that the graph transform remains well defined, i.e. maps $\mathrm{Lip}_1(\mathbb{S}^1, [-1,1])$ into itself, and a contraction (for each fixed ξ).

Finally, the contraction properties of the graph transform, uniform in the random parameter, implies that the random cycle is attracting. □

We have confined ourselves with a statement on Lipschitz continuous random cycles, the graph transform techniques however allow establishing more smoothness [35]. The result does not discuss the dynamics on the random cycle, it is still possible to find an attracting random fixed point on it, compare [5, 7].

References

[1] J. P. Aubin and A. Cellina, *Differential inclusions* (Springer-Verlag, Berlin, 1984).
[2] V. Araújo, Ann. Inst. Henri Poincaré, Analyse non linéaire **17**, 307 (2000).
[3] L. Arnold, *Random Dynamical Systems* (Springer-Verlag, Berlin, 1998).
[4] L. Arnold, in *IUTAM Symposium on Nonlinearity and Stochastic Structural Dynamics* (Madras, 1999), Solid Mech. Appl. Vol 85, 15 (Kluwer Acad. Publ., Dordrecht 2001).
[5] L. Arnold, G. Bleckert, and K. R. Schenk-Hoppé, in *Stochastic dynamics (Bremen, 1997)*, 71 (Springer-Verlag, Berlin 1999).
[6] L. Arnold and W. Kliemann, On unique ergodicity for degenerate diffusions, Stochastics **21**, 41 (1987).
[7] L. Arnold, N. Sri Namachchivaya, and K. R. Schenk-Hoppé, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **6**, 1947 (1996).
[8] V. I. Arnold, V. S. Afraimovich, Yu. S. Il'yashenko, and L. P. Shil'nikov, *Bifurcation theory and catastrophe theory*, (Springer-Verlag, Berlin, 1999).
[9] Y. Bakhtin and T. Hurth, ArXiv:1203.5744v1 [math.DS] (2012) (unpublished).
[10] I. Bashkirtseva, L. Ryashko, and H. Schurz, Chaos Solitons Fractals **39**, 72 (2009).
[11] I. Bena, Internat. J. Modern Phys. B **20**, 2825 (2006).
[12] R. T. Botts, A. J. Homburg, and T. R. Young, Discrete Contin. Dyn. Syst., Ser. A **32**, 2997 (2012).

[13] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein, J. Eur. Math. Society **6**, 399 (2004).
[14] A. Bovier, V. Gayrard, and M. Klein, J. Eur. Math. Society **7**, 69 (2005).
[15] G. Colombo, P. D. Pra, V. Křivan, and I. Vrkoč, Math. Control Signals Systems **16**, 95 (2003).
[16] F. Colonius, T. Gayer, and W. Kliemann, SIAM J. Appl. Dyn. Syst. **7**, 79 (2007).
[17] F. Colonius, A. J. Homburg, and W. Kliemann, J. Difference Equ. Appl. **16**, 127 (2010).
[18] F. Colonius and W. Kliemann, in *Stochastic Dynamics*, edited by H. Crauel and M. Gundlach, (Springer-Verlag, Berlin, 1999).
[19] F. Colonius and W. Kliemann, *The dynamics of control*, (Birkhauser Boston, Inc., Boston, 2000).
[20] H. Crauel, P. Imkeller, and M. Steinkamp, in *Stochastic Dynamics* edited by H. Crauel and M. Gundlach, (Springer-Verlag, Berlin, 1999).
[21] A. d'Onofrio, A. Gandolfi, and S. Gatton, Physica A: Statistical Mechanics and its Applications (to be published).
[22] J. L. Doob, *Stochastic Processes*, (Wiley, New York, 1953).
[23] G. Froyland and O. Stancevic, ArXiv:1106.1954v2 [math.DS] (2011) (unpublished).
[24] T. Gayer, J. Diff. Equations **201**, 177 (2004).
[25] M. Ghil, M. D. Chekroun, and E. Simonnet, Phys. D **237**, 2111 (2008).
[26] A. J. Homburg and T. R. Young, Regular & Chaotic Dynam. **11**, 247 (2006).
[27] A. J. Homburg and T. R. Young, Topol. Methods Nonlin. Anal. **35**, 77 (2010).
[28] W. Horsthemke and R. Lefever, *Noise-induced transitions. Theory and applications in physics, chemistry, and biology*, Springer Series in Synergetics, Vol. 15, (Springer-Verlag, Berlin, 1984).
[29] R. Johnson, in *Stochastic Dynamics*, edited by H. Crauel and M. Gundlach, (Springer-Verlag, Berlin, 1999).
[30] W. Kliemann, Ann. Probab. **15**, 690 (1987).
[31] H. Kunita, *Stochastic flows and stochastic differential equations*, (Cambridge University Press, Cambridge, England 1990).
[32] Yu. A. Kuznetsov, *Elements of applied bifurcation theory*, (Springer-Verlag, Berlin, 2004).
[33] J. S. W. Lamb, M. Rasmussen, and C. S. Rodrigues, ArXiv:1105.5018v1 [math.DS] (2011) (unpublished).
[34] K. Mallick and P. Marcq, Eur. Phys. J. B **36** 119 (2003).
[35] J. E. Marsden and M. McCracken, *The Hopf bifurcation and its applications*, (Springer-Verlag, Berlin, 1976).
[36] T. Nadzieja, Czechoslovak Math. J. **40**, 195 (1990).
[37] L. Ridolfi, P. DOdorico and F. Laio, *Noise-Induced Phenomena in the Environmental Sciences*, (Cambridge University Press, Cambridge, England, 2011).
[38] C. S. Rodrigues, C. Grebogi, and A. P. S. de Moura, Phys. Rev. E **82**, 046217 (2010).

[39] C. Schütte, W. Huisinga, and S. Meyn, Ann. Appl. Probab. **14**, 419 (2004).
[40] T. Tateno, Phys. Rev. E **65**, 021901 (2002).
[41] S. Wieczorek, Phys. Rev. E **79**, 036209 (2009).
[42] E. C. Zeeman, Nonlinearity **1**, 115 (1988).
[43] E. C. Zeeman, Bull. London Math. Soc. **20**, 545 (1988).
[44] H. Zmarrou, A. J. Homburg, Ergod. Th. Dyn. Systems **27**, 1651 (2007).
[45] H. Zmarrou, A. J. Homburg, Discrete Contin. Dyn. Syst., Ser. B **10**, 719 (2008).