Heteroclinic bifurcations of Ω-stable vector fields on 3-manifolds.

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Abstract

We study one parameter families of vector fields that are defined on three dimensional manifolds and whose nonwandering sets are structurally stable. As families, these families may not be structurally stable; heteroclinic bifurcations that give rise to moduli can occur. Some but not all moduli are related to the geometry of stable and unstable manifolds. We study a notion of stability, weaker then structural stability, in which geometry and dynamics on stable and unstable manifolds are reflected. We classify the families from the above mentioned class of families that are stable in this sence.

1 Introduction

In this paper we are concerned with stability of families of vector fields. If for a family of vector fields the structure of the nonwandering set and how this varies with a parameter is the subject of interest, the notion of stability to be considered would be Ω-stability. This notion is too weak if one is also interested in, say, the geometry of basins of attraction. From simple examples it is clear that structural stability is too strong an equivalence relation for such a purpose; even an elementary bifurcation as a heteroclinic tangency between stable and unstable manifolds of two periodic orbits gives rise to invariants of structural stability ('moduli'). The geometry of stable and unstable manifolds may however not be affected, but behave similar for all nearby families. For this reason we introduce a notion of stability suited for studying the geometry and dynamics on stable and unstable sets. Then, for a class of one parameter families consisting of Ω-stable vector fields on a three dimensional compact manifold, we classify the stable ones with respect to this notion of stability.

Below we give precise definitions and a discussion. The paper is further organised as follows. In chapter 2 we prove a well known structural stability theorem (theorem 2.6) for single vector fields using methods which will prove useful in studying stability theorems for families. This then is pursued in chapter 3, where our main theorem (theorem 3.1) is stated and proved.

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1.1 Preliminaries

Let M be a compact connected 3 dimensional manifold and let $\mathfrak{X}$ be the class of smooth vector fields on M, equipped with the Whitney topology.

We recall a number of notions. For $X \in \mathfrak{X}$ and $x \in M$, the *stable set* $W^s(x)$ of x is defined by

$$W^s(x) = \{y \in M, \qquad X_t(y) \to X_t(x), t \to \infty\},$$

where X_t is the flow of X. For an orbit $\mathcal{O}(x)$,

$$W^s(\mathcal{O}(x)) = \bigcup_{t \in \mathbb{R}} W^s(X_t(x)).$$

Observe that $W^s(\mathcal{O}(x))$ is laminated by the stable sets $W^s(y)$, $y \in \mathcal{O}(x)$. Similarly, the unstable sets $W^u(x)$ and $W^u(\mathcal{O}(x))$ are defined, replacing X_t in the above definitions by X_{-t}.

The *nonwandering set* Ω of X is the union of points $x \in M$ so that, for every neighbourhood $\mathcal{U}$ of x, there are arbitrarily large $T > 0$ with $X_T(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The nonwandering set is a compact invariant set. A compact invariant set Λ is hyperbolic, if there is a DX_t invariant splitting $TM = E^s \oplus \langle X \rangle \oplus E^u$ along Λ, where $\langle X \rangle$ denotes the line bundle spanned by X and where further, for some $C_s, C_u > 0$, $\nu^s < 0$, $\nu^u > 0$,

$$\begin{aligned} \|DX_t|_{E^s}\| &\leq C_s e^{\nu^s t}, \;\; t > 0, && (1) \\ \|DX_t|_{E^u}\| &\leq C_u e^{\nu^u t}, \;\; t < 0. && (2) \end{aligned}$$

A vector field X is said to obey *axiom A*, if Ω is hyperbolic and the critical elements (singularities and periodic orbits) are dense in Ω. The relevance of this definition is explained by the spectral decomposition theorem:

Theorem 1.1 [Sma,1967], [PuSh,1970] *If X obeys axiom A, then the nonwandering set Ω of X is a finite union $\Omega_1 \cup \ldots \cup \Omega_N$ of components, each containing a dense orbit.*

Component has its usual meaning as relatively open and closed subset. These components are called basic sets. A basic set can be either a singularity, a periodic orbit, or a nontrivial basic set equal to the closure of a countable number of periodic orbits (and without singularities). Further, the stable sets of points and orbits in a basic set Ω_i are injectively immersed manifolds with $T^r W^s(x)$ and $T^r W^s(\mathcal{O}(x))$ depending continuously on $x \in \Omega_i$ for all r. Analogously for unstable sets.

There are a number of concepts expressing stability, that is robustness of certain aspects of the dynamics under perturbation of the vector field. We recall the notions of Ω-stability and structural stability. A vector field X is called *structurally stable*, if, for sufficiently small perturbations $\tilde{X}$ of X, there exists a near identity homeomorphism h on M, mapping orbits of X to orbits of $\tilde{X}$. Such a homeomorphism is called a *topological equivalence*. Observe that a topological equivalence maps stable and unstable sets of orbits to the stable and unstable sets of the corresponding orbits, but may not map stable and unstable sets of points to the stable and unstable sets of the corresponding points (as a conjugacy $h \circ X_t = \tilde{X}_t \circ h$ would). If a topological equivalence exists restricted to the nonwandering set of X, for all sufficiently small perturbations of X, we call X *Ω-stable*. We recall two stability theorems, providing sufficient conditions for Ω-stability and structural stability. Let X be a vector field obeying axiom A. A cycle is a sequence $\Omega_{i_1}, \ldots, \Omega_{i_n}, \Omega_{i_{n+1}} = \Omega_{i_1}$ of basic sets so that

$$(W^u(\Omega_{i_j}) - \Omega_{i_j}) \cap (W^s(\Omega_{i_{j+1}}) - \Omega_{i_{j+1}}) \neq \emptyset.$$

We say X satisfies the no-cycle condition, if there are no cycles. The vector field X is said to satisfy the strong transversality condition, if all intersections of stable and unstable manifolds of orbits in Ω are transversal.

Theorem 1.2 [PuSh,1970] *If X obeys axiom A and the no-cycle condition, then X is Ω-stable.*

Theorem 1.3 [Rob,1974], [Rob,1975] *If X obeys axiom A and the strong transversality condition, then X is structurally stable.*

The proof of theorem 1.3 in [Rob,1974], [Rob,1975] is by analytical methods; the desired topological equivalence between X and a sufficiently small perturbation of X is written as the solution of an equation which is then solved using the implicit mapping theorem. In section 2 we give a proof of theorem 1.3 for vector fields on three dimensional manifolds by geometric methods using 'compatible systems of invariant foliations'. Such a method was introduced by Palis [Pal,1969] to study structural stability of Morse-Smale systems, see also [PaSm,1970]. After finishing this paper we found the paper [Ver,1996] by J. Vera in which theorem 1.3 is proved, also by making use of compatible systems of invariant foliations. In [Me,1973], invariant foliations were used to show structural stability of two dimensional diffeomorphisms obeying axiom A and strong transversality.

The advantage of such a geometric proof is its possible extension to study stability of families, see chapter 3.

1.2 Families

For an interval I, let $C^\infty(I, \mathfrak{X})$ denote the set of smooth one parameter families, equipped with the uniform topology. Let $\{X_\mu\} \in C^\infty(I, \mathfrak{X})$. A parameter value $\bar{\mu}$ is called a bifurcation value of X_μ if $X_{\bar{\mu}}$ is not stable (depending on the type of stability considered). To compare different families undergoing bifurcations, one introduces notions of stability for families, corresponding to those for individual vector fields. A family $\{X_\mu\}$ is called structurally stable if, for any sufficiently nearby family $\{\tilde{X}_\mu\}$, there exists a homeomorphism $H : M \times I \to M \times I$ of the form

$$H(x, \mu) = (h_\mu(x), \eta(\mu)),$$

where h_μ is a topological equivalence between X_μ and $\tilde{X}_{\eta(\mu)}$ and where η is order preserving. One may instead demand H to be merely a bijection and η and $x \mapsto h_\mu(x)$ to be continuous, see [NPT,1983]. Similarly one defines Ω-stability, restricting h_μ to the nonwandering set.

We recall two results on structural stability of families of vector fields, as they hold on three dimensional manifolds. The cited papers actually contain results on n dimensional vector fields and also consider bifurcations where critical elements loose hyperbolicity.

Theorem 1.4 [PT,1983] *Generic one parameter families of gradient vector fields on compact three dimensional manifolds are structurally stable.*

Theorem 1.5 [LaPl,1993] *A generic one parameter family $\{X_\mu\}$ of vector fields on a compact three dimensional manifold whose nonwandering set consists of a finite number of hyperbolic critical elements and has no cycles is structurally stable, provided*

- *Stable and unstable manifolds of periodic orbits intersect transversally,*
- *If p is a singularity with one dimensional unstable manifold and complex conjugate stable eigenvalues, then $W^u(p)$ is contained in the stable manifold of an attracting critical element.*
  If p is a singularity with one dimensional stable manifold and complex conjugate unstable eigenvalues, then $W^s(p)$ is contained in the unstable manifold of a repelling critical element.

Let $\{X_\mu\}$ be a one parameter family of vector fields on M and let γ be an orbit of X_μ. The family $\{X_\mu\}$ is said to be structurally stable at (γ, μ) if for all small perturbations $\{Y_\mu\}$ of $\{X_\mu\}$, there exists a topological equivalence between $\{X_\mu\}$ and $\{Y_\mu\}$ defined on a neighbourhood of $(\overline{\gamma}, \mu)$ in $M \times \mathbb{R}$. The cases where γ is a heteroclinic orbit have been studied in [Str,1982], see also [Bel,1986]. In [Ver,1996] a sufficient condition is supplied for a family $\{X_\mu\}$ to be structurally stable, for parameter values near a specific parameter value μ, if it is known that X_μ possesses one heteroclinic orbit γ and $\{X_\mu\}$ is structurally stable at (γ, μ).

What strikes from these results is that the set of structurally stable families appears to be rather small; even an elementary bifurcation such as a tangency between the stable and unstable manifolds of two periodic orbits does not occur in a structurally stable family. So, if one is interested in the geometry of and dynamics on stable and unstable sets near bifurcations, structural stability is too strong an equivalence relation, whereas Ω-stability is too weak. We thus think it makes sense to consider other equivalence relations in studying families of vector fields. Specifically, we will study a notion of stability, which we call W-stability, in which we only incorporate the geometry and dynamics on stable and unstable manifolds. This will be pursued in the following section. We would at this point like to comment that seeing the gap between W-stability and structural stability (compare theorem 3.1 and the corollary following it with theorem 1.5 and results in [MeSt,1987], [NPT,1983], [Ver,1996]) also provides a better understanding of structural stability.

1.3 W-stability

We start again with single vector fields before treating families. To study the geometry of stable and unstable sets and the dynamics restricted to these sets, we propose the following equivalence relation. A vector field $X \in \mathfrak{X}$ is *W-stable*, if a neighbourhood $\mathcal{U}$ of X exists so that for all $\tilde{X} \in \mathcal{U}$, there is a bijection $h : M \to M$, mapping orbits of X to orbits of $\tilde{X}$, with h near the identity and with the additional property that h restricted to the stable set $W^s(\gamma)$ and to the unstable set $W^u(\gamma)$, for all orbits γ, is continuous.

For an orbit γ, the geometry of $W^s(\gamma)$ is preserved under h in the sense that if $x_n \in W^s(\gamma)$ converges to $x \in W^s(\gamma)$, then $h(x_n) \in W^s(h(\gamma))$ converges to $h(x) \in W^s(h(\gamma))$. Similarly for $W^u(\gamma)$ and for intersections $W^u(\gamma) \cap W^s(\eta)$, where η is a second orbit of X. The map h is not required to be continuous on the closure of stable and unstable sets.

Let I be an interval in $\mathbb{R}$ and write $C^\infty(I, \mathfrak{X})$ for the set of smooth one parameter families of vector fields on M, equipped with the uniform topology. A family $\{X_\mu\} \in C^\infty(I, \mathfrak{X})$ is W-stable if, for each family $\{\tilde{X}_\mu\}$, sufficiently near $\{X_\mu\}$, there exists a bijection $H : M \times I \to M \times I$ of the form

$$H(x, \mu) = (h_\mu(x), \eta(\mu)), \tag{3}$$

with η an order preserving homeomorphism and where h_μ gives, as above, an equivalence between X_μ and $\tilde{X}_{\eta(\mu)}$.

For Ω-stable families we could demand a continuous dependence of h_μ on μ, by requiring $(x, \mu) \mapsto h_\mu(x)$ to be continuous restricted to $\bigcup_\mu (W^s(\gamma_\mu), \mu)$ and $\bigcup_\mu (W^u(\gamma_\mu), \mu)$, where γ_μ is the continuation of an orbit in the nonwandering set of X_μ.

We remark that in [PoTa,1993], for two nearby two dimensional diffeomorphisms possessing a homoclinic tangency (between stable and unstable manifolds of a hyperbolic fixed point), the existence of a conjugacy restricted to the unstable manifold of the fixed point is investigated. Compare further the definition of future stability in [Sma,1970] and the definition of weak-C^0-stability in [Tak,1974].

2 Stable vector fields

This section treats vector fields on M which satisfy axiom A and the strong transversality condition. Write

$$\mathfrak{X}_{str} = \{X \in \mathfrak{X},\ \ X \text{ satisfies axiom A + strong transversality}\}.$$

We construct stable and unstable foliations for vector fields in $\mathfrak{X}_{str}$ and then we use these to provide a proof of theorem 1.3 for three dimensional vector fields.

Let $X \in \mathfrak{X}_{str}$. The spectral decomposition theorem [Sma,1967], [PuSh,1970] gives that the nonwandering set Ω of X is a finite union $\Omega_1 \cup \ldots \cup \Omega_N$ of isolated critical elements and nontrivial basic sets. Let Ω_i be a nontrivial basic set. Define $\partial_s \Omega_i$ as the set of points $x \in \Omega_i$ so that $W^u_{loc}(x) \cap \Omega_i$ accumulates only from one side on x. Similarly, $\partial_u \Omega_i$ is defined. The following lemma follows from [NePa,1973], see also [PT,1993].

Lemma 2.1 *Let Ω_i be a nontrivial basic set. Then there is a finite number of periodic orbits $\gamma^s_j, 1 \le j \le n_s$, and $\gamma^u_j, 1 \le j \le n_u$, so that*

$$\begin{aligned} \partial_s \Omega_i &= \Omega_i \cap \bigcup_{j=1}^{n_s} W^s(\gamma^s_j), \\ \partial_u \Omega_i &= \Omega_i \cap \bigcup_{j=1}^{n_u} W^u(\gamma^u_j). \end{aligned}$$

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The sets $\partial_s \Omega_i$ and $\partial_u \Omega_i$ can still be empty. For example, if $\Omega = M$, then both $\partial_s \Omega_i = \emptyset$ and $\partial_u \Omega_i = \emptyset$. If Ω_i is a strange attractor, then $\partial_s \Omega_i = \emptyset$ and $\partial_u \Omega_i = \bigcup_{j=1}^{n_u} W^u(\gamma^u_j)$. Similarly, if Ω_i is a strange repeller, then $\partial_u \Omega_i = \emptyset$. If both $\partial_s \Omega_i \neq \emptyset$ and $\partial_u \Omega_i \neq \emptyset$, we call Ω_i a *(nontrivial) basic set of saddle type.*

Suppose Ω_i is a nontrivial basic set. A *Markov partition of size* $\alpha > 0$ consists of a finite collection of cross sections T_i containing boxes B_i, i.e. diffeomorphic images $\phi_i([-1,1]^2)$ of the square $[-1,1]^2$, with the following properties. Write

$$\begin{aligned} \partial_s B_i &= \phi_i(\{(x,y), |y| = 1\}), \\ \partial_u B_i &= \phi_i(\{(x,y), |x| = 1\}). \end{aligned}$$

Then

- diam $T_i \le \alpha$,
- $\overline{B}_i \subset T_i$,
- $T_i \cap T_j = \emptyset$, $i \neq j$.

- $\Omega_i \subset \bigcup_{0\le t\le\alpha} X_t(\bigcup_i B_i)$,
- $\partial_s B_i \subset W^s(\Omega_i)$, $\qquad \partial_u B_i \subset W^u(\Omega_i)$,
- $B_k \cap \bigcup_{t\ge 0} X_t(\partial_s B_j) \subset \partial_s B_k$, $\qquad B_k \cap \bigcup_{t\le 0} X_t(\partial_u B_j) \subset \partial_u B_k$.

Lemma 2.2 [Bow,1973] *Let Ω_i be a nontrivial basic set. Let $\alpha > 0$. Then a Markov partition of size α for Ω_i exists.* ∎

We remark that such Markov partitions can be constructed analogously as Markov partitions for basic sets of two dimensional diffeomorphisms [PT,1993]. Then, in case $\partial_s\Omega_i \neq \emptyset$, we can take $\partial_s B_i \subset \bigcup_{1\le j\le n_s} W^s(\gamma_j^s)$. Similarly, if $\partial_u\Omega_i \neq \emptyset$ we can take $\partial_u B_i \subset \bigcup_{1\le j\le n_u} W^u(\gamma_j^u)$.

By a continuous *foliation* $\mathfrak{F}$ (with leaves of dimension k) of an open subset $\mathcal{V}$ of M, we mean a disjoint decomposition of $\mathcal{V}$ into k dimensional embedded submanifolds such that $\mathcal{V}$ is covered by C^0 charts

$$\phi : D^k \times D^{n-k} \to \mathcal{V}$$

and $\phi(D^k \times \{x\}) \subset \mathfrak{F}_x$, where $\mathfrak{F}_x$ is the leaf of $\mathfrak{F}$ through x. The foliation is of class C^k if the charts ϕ can be chosen of class C^k. Sufficient for a foliation to be C^k is that the correspondence $x \mapsto T_x\mathfrak{F}_x$ is C^k. A C^k *lamination* $\mathfrak{F}$ is a continuous foliation with C^k leaves, such that $x \mapsto T_x^i\mathfrak{F}_x$ is continuous, $0 \le i \le k$. An *unstable foliation (lamination)* $\mathfrak{F}^u(\Omega_i)$ for X near a basic set Ω_i is a X_t invariant foliation (lamination) with $\mathfrak{F}^u_x(\Omega_i) = W^u(x)$, if $x \in \Omega_i$. Stable foliations (laminations) are defined similarly.

A construction we will use several times and therefore introduce here, is that of averaging foliations. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be two open sets on which foliations $\mathfrak{F}^1$, $\mathfrak{F}^2$ both with one dimensional leaves are defined, so that for $x \in \mathcal{U}_1 \cap \mathcal{U}_2$, $T_x\mathfrak{F}^1_x$ and $T_x\mathfrak{F}^2_x$ are close (say with an angle less then $\pi/2$ in some Riemannian structure). Take unit vector fields Y_1 on $\mathcal{U}_1$ and Y_2 on $\mathcal{U}_2$ so that integral curves of Y_1, Y_2 are leaves of $\mathfrak{F}^1$, $\mathfrak{F}^2$. The direction of Y_1, Y_2 should be so that for $x \in \mathcal{U}_1 \cap \mathcal{U}_2$, the angle between $Y_1(x)$ and $Y_2(x)$ in T_xM is less then $\pi/2$. Take a partition of one $\{\phi, 1-\phi\}$ subordinate to $\mathcal{U}_1$, $\mathcal{U}_2$. Replace the foliations $\mathfrak{F}^1$, $\mathfrak{F}^2$ by the averaged foliation $\mathfrak{F}$ of $\mathcal{U}_1 \cup \mathcal{U}_2$ whose leaves are integral curves of $\phi Y_1 + (1-\phi)Y_2$. Observe that $\mathfrak{F}_x = \mathfrak{F}^1_x = \mathfrak{F}^2_x$ if $\mathfrak{F}^1_x = \mathfrak{F}^2_x$.

When Ω_i is an invariant set of X and $W^s(\Omega_i)$ its stable set, a *fundamental domain* $D^s(\Omega_i)$ of $W^s(\Omega_i)$ is a set so that $W^s(\Omega_i) = \bigcup_{t\in\mathbb{R}} X_t(D^s(\Omega_i))$ and each orbit in $W^s(\Omega_i)$ intersects $D^s(\Omega_i)$ only once. A *fundamental neighbourhood* $N^s(\Omega_i)$ of $W^s(\Omega_i)$ is a set so that its saturation for positive time plus the local unstable set $W^u_{loc}(\Omega_i)$ provide a neighbourhood of Ω_i.

Lemma 2.3 *There exists an invariant unstable lamination $\mathfrak{F}^u(\Omega_i)$ near each basic set Ω_i.*

PROOF. If Ω_i is a repeller, $\mathfrak{F}^u_x(\Omega_i) = W^u(x)$ for $x \in \Omega_i$. Since the unstable manifolds of points in Ω_i fill a neighbourhood of Ω_i, $\mathfrak{F}^u(\Omega_i)$ is a foliation near Ω_i.

If Ω_i is an attracting critical element, $\mathfrak{F}^u_x(\Omega_i) = \{x\}$, x near Ω_i.

If Ω_i is a singularity of saddle type, take as a fundamental domain $D^s(\Omega_i)$ of $W^s(\Omega_i)$ a compact manifold of codimension one in $W^s(\Omega_i)$, transverse to the flow. Take a foliation $\{\mathfrak{F}^u_x, x \in D^s(\Omega_i)\}$ with leaves contained in a small neighbourhood of $D^s(\Omega_i)$ and transversally intersecting $W^s(\Omega_i)$. The saturation of $\mathfrak{F}^u$ for positive time gives, by the λ-lemma, a continuous unstable lamination near Ω_i.

If Ω_i is a periodic orbit of saddle type, take a cross section T_0 through Ω_i and a fundamental domain $D^s(\Omega_i) \subset W^s(\Omega_i) \cap T_0$ for the Poincaré return map on T_0; $D^s(\Omega_i)$ is an interval in $W^s(\Omega_i) \cap T_0$ which is open on one side and closed on the other side. Take a lamination $\mathfrak{F}^u$ on a neighbourhood $\mathcal{U}$ of $D^s(\Omega_i)$ with leaves transverse to $W^s(\Omega_i)$. By averaging we may assume that if $X_T(\mathfrak{F}^u)$, $T > 0$, and $\mathfrak{F}^u$ both define a foliation on some open set $\mathcal{U}$, then for $y \in \mathcal{U}$,

$$X_T(\mathfrak{F}^u_{X_{-T}(y)}) \cap \mathcal{U} \;\; = \;\; \mathfrak{F}^u_y \cap \mathcal{U}. \tag{4}$$

Take a fundamental neighbourhood $N^s(\Omega_i)$ extending $D^s(\Omega_i)$ and consisting of leaves of $\mathfrak{F}^u$. The saturation of $\mathfrak{F}^u$ defined on $N^s(\Omega_i)$, for positive time, gives, by the λ-lemma, a continuous unstable lamination near Ω_i.

Suppose Ω_i is a nontrivial basic set of saddle type. Take a Markov partition with cross sections $T_1, \ldots, T_n$ and boxes $B_j \subset T_j$. We may choose

$$T_k \;\; \subset \;\; \mathfrak{H}_{p_k}, \tag{5}$$

for some $p_k \in \Omega_i$. Since Ω_i is of saddle type, choosing $T_j \backslash B_j$ small enough,

$$\Omega_i \cap T_j \;\; \subset \;\; B_j. \tag{6}$$

Take a Riemannian metric d on M so that the constant C_s in (1) equals 1 [HPPS,1970]. Consider curves $R_k \subset T_k$, disjoint from B_k and with $d(R_k, \partial_s B_k) = \varepsilon$ for some small $\varepsilon > 0$. Let S_k be the collection of curves $\cup_j X_{(0,\alpha]}(R_j) \cap T_k$. By the choice of Riemannian metric, $d(S_k, \partial_s B_k) < \varepsilon$. Let $N^s_k(\Omega_i)$ be the collection of those connected components of $T_k \backslash (R_k \cup S_k)$ that are bounded by $R_k \cup S_k$ and disjoint from $\Omega_i \cap T_k$. It is clear from the construction that $N^s(\Omega_i) = \bigcup_k N^s_k(\Omega_i)$ is a fundamental neighbourhood of Ω_i. Choose a foliation $\mathfrak{F}^u$ near $\cup_k R_k$, with leaves in $\cup_k T_k$ transverse to $W^s(\Omega_i)$. Integrating $\mathfrak{F}^u$ by the flow of X we obtain a foliation $\mathfrak{F}^u$ near $\cup_k S_k$. Now choose a foliation $\mathfrak{F}^{u,1}$ near $N^s(\Omega_i)$ and average $\mathfrak{F}^u$ near $\cup_k (R_k \cup S_k)$ and $\mathfrak{F}^{u,1}$. Call the obtained foliation $\mathfrak{F}^u$. Restrict $\mathfrak{F}^u$ to a fundamental neighbourhood near $N^s(\Omega_i)$. By (6), choosing $T_k \backslash B_k$ small enough, (4) will hold. The saturation for positive time of $\mathfrak{F}^u$ restricted to a fundamental neighbourhood near $N^s(\Omega_i)$, defines, by the generalized λ-lemma [Me,1973], [HPPS,1970], an unstable lamination $\mathfrak{F}^u(\Omega_i)$.

Finally suppose Ω_i is a strange attractor. Construct a foliation $\mathfrak{F}^u$ near $N^s(\Omega_i)$ as above. Since (6) does not hold, (4) may still not be satisfied. Denote by $\mathfrak{F}^{u,k}$ the foliation $\mathfrak{F}^u$ near $N^s_k(\Omega_i)$. By averaging $X_{(0,\alpha]}(\mathfrak{F}^u_k)$ and $\mathfrak{F}^{u,l}$ near $N^s_l(\Omega_i)$, if $X_{(0,\alpha]}(T_k)$ intersects T_l, $X_{(0,\alpha]}(\mathfrak{F}^u_k)$ equals $\mathfrak{F}^{u,l}$ on open sets near T_l where both foliations are defined. Since there is only a finite number of cross sections T_k, we can ensure (4). As above, the saturation for positive time of $\mathfrak{F}^u$ restricted to a fundamental neighbourhood near $N^s(\Omega_i)$, defines an unstable lamination $\mathfrak{F}^u(\Omega_i)$. ∎

Observe that the above proof provides an unstable foliation $\mathfrak{F}^u(\Omega_i)$ near $W^u(\Omega_i)$, not just near Ω_i. Stable foliations for ϕX are unstable foliations for $-\phi X$ and are thus constructed analogously.

For $X \in \mathfrak{X}_{str}$, $\Omega_1, \ldots, \Omega_N$ will denote the basic sets of X. Choose an ordering $\le$ of the basic sets $\Omega_1, \ldots, \Omega_N$ so that, if $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$, then $\Omega_i \le \Omega_j$. Possibly renumbering the basic sets, we may assume

$$\Omega_1 \le \ldots \le \Omega_N.$$

We will further write $\Omega_i \prec \Omega_j$, in case $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$ and $i \neq j$.

A collection of unstable foliations is called a *compatible system of unstable foliations*, if

$$\mathfrak{F}^u_x(\Omega_k) \quad \subset \quad \mathfrak{F}^u_x(\Omega_j), \tag{7}$$

for $\Omega_j \prec \Omega_k$ and x near Ω_k.

Lemma 2.4 *For $X \in \mathfrak{X}_{str}$, a compatible system of unstable laminations $\{\mathfrak{F}^u(\Omega_i)\}$ for X exists.*

PROOF. Notation is as in the proof of lemma 2.1. In particular, $N^s(\Omega_i)$ and $D^s(\Omega_i) = N^s(\Omega_i) \cap W^s(\Omega_i)$ denote a fundamental neighbourhood resp. a fundamental domain of $W^s(\Omega_i)$.

We construct the collection $\{\mathfrak{F}^u(\Omega_i)\}$ of unstable laminations inductively. We start by making an unstable lamination near the repellers of X. If Ω_i is a repeller, the leaves of $\mathfrak{F}^u(\Omega_i)$ are the unstable manifolds of points in Ω_i, filling a neighbourhood of Ω_i.

Suppose unstable foliations $\mathfrak{F}^u(\Omega_1), \ldots, \mathfrak{F}^u(\Omega_{i-1})$ have been constructed. To construct $\mathfrak{F}^u(\Omega_i)$, we use a second induction on the basic sets Ω_j with $\Omega_j \le \Omega_i$. We first define a foliation $\mathfrak{F}^u$ near $D^s(\Omega_i)$. Suppose $\Omega_j \prec \Omega_i$ (observe that Ω_i is not a repeller and Ω_j

is not an attractor; we may assume that Ω_i is not an attracting critical element, since then $\mathfrak{F}^u(\Omega_i)$ consists of points and compatibility with other unstable foliations is obvious). Consider the following situations.

1. Suppose there is no basic set Ω_k with $\Omega_j \prec \Omega_k \prec \Omega_i$. In this case, $W^u(\Omega_j) \cap \overline{D^s(\Omega_i)}$ is compact.

   If Ω_j is a repelling singularity, take a foliation $\mathfrak{F}^u$ near $W^u(\Omega_j) \cap D^s(\Omega_i)$ with leaves transverse to $W^s(\Omega_i)$.

   Suppose Ω_j is a repelling periodic orbit. Since $W^u(x), x \in \Omega_j$ intersects the orbit $\mathcal{O}(x)$ through x transversally, $W^u(x)$ intersects $W^s(\Omega_i)$ transversally. We can take a subfoliation $\mathfrak{F}^u$ of $\mathfrak{F}^u(\Omega_j)$ near $N^s(\Omega_i)$ with leaves transversally intersecting $W^s(\Omega_i)$.

   In all other cases, $\mathfrak{F}^u(\Omega_j)$ defines a foliation $\mathfrak{F}^u$ near $W^u(\Omega_j) \cap D^s(\Omega_i)$.

2. Suppose $\Omega_j \prec \Omega_{k_g}$, where Ω_{k_g}, $1 \leq g \leq m$ are the basic sets with $\Omega_{k_g} \prec \Omega_i$. Over a neighbourhood $\mathcal{T}$ in $D^s(\Omega_i)$ of the compact set $\bigcup_g W^u(\Omega_{h_g}) \cap D^s(\Omega_i)$, $\mathfrak{F}^u$ has already been constructed, compatible with $\mathfrak{F}^u(\Omega_j)$. Over a neighbourhood of the compact set $W^u(\Omega_j) \cap (D^s(\Omega_i) - \mathcal{T})$, a foliation $\mathfrak{F}^{u,1}$ is chosen as in 1. Average $\mathfrak{F}^u$, $\mathfrak{F}^{u,1}$ to obtain $\mathfrak{F}^u$ near $W^u(\Omega_j) \cap D^s(\Omega_i)$.

If $\Omega_i \subset \Omega - P$ we average foliations as in the proof of lemma 2.3 to ensure (4). $\mathfrak{F}^u(\Omega_i)$ is obtained by saturation of the leaves $\mathfrak{F}^u_x$ with x from a suitable fundamental neighbourhood of $W^s(\Omega_i)$. ■

Such compatible systems of foliations play a fundamental rôle in proving structural stability theorems, compare [Pal,1969], [PaSm,1970], [Me,1973].

The following lemma clarifies the structure of a fundamental domain of the stable manifold of a strange attractor and the foliation on it induced by the stable manifolds of orbits in it. By a Reeb component we mean a foliation of a cylinder or a Möbius band such that the boundary consists of leaves and all other leaves are diffeomorphic to $\mathbb{R}$.

Lemma 2.5 *For a fundamental domain $D^s(\Omega_i)$ of the stable manifold of a strange attractor Ω_i, one can take a compact manifold consisting of at most n_u connected components, which are tori and Klein bottles. The stable manifolds of orbits in Ω_i define a foliation of $D^s(\Omega_i)$ consisting of n_u circles and n_u Reeb components.*

PROOF. By (2.1), $W^s(\gamma^u_j)$ is an injectively immersed annulus. The unstable manifolds $W^u(\gamma^u_j)$ can be either immersed annuli or immersed Möbius bands (this last possibility can not occur if M is orientable). Write $V^u(\gamma_j^u)$ for a compact manifold of codimension one in $W^u(\gamma_j^u)$ and transverse to the flow. Consider, for some positive functions t_j,

$$K = \bigcup_{t \le t_j(x)} X_t(V^u(\gamma_j^u)).$$

We choose the functions t_j so that $t_j \ge T$ where T is a large number. For T large enough, since unstable manifolds are dense in Ω_i, we can further choose t_j so that for each component C_1 of ∂K, there exists another component C_2 contained in $\bigcup_{x \in C_1} W^s_{loc}(x)$. We can then easily perturb the union of K and the local stable manifolds connecting components of ∂K to obtain a fundamental domain $D^s(\Omega_i)$ of $W^s(\Omega_i)$. It is clear from the construction that each component of $D^s(\Omega_i)$ is either a torus or a Klein bottle (Klein bottles can only occur if M is nonorientable).

Stable manifolds of orbits in Ω_i intersect $D^s(\Omega_i)$ transversally because orbits intersect $D^s(\Omega_i)$ transversally. (Stable manifolds of points need not intersect $D^s(\Omega_i)$ transversally.) So stable manifolds of orbits in Ω_i define a foliation $\mathfrak{F}$ of $D^s(\Omega_i)$. Clearly, $W^{s,-}(\gamma_j^u) \cap D^s(\Omega_i)$ is a circle. By (2.1), these are the only compact leaves. ■

Theorem 2.6 *Vector fields from $\mathfrak{X}_{str}$ are structurally stable.*

Proof. Order the basic sets of X as in the proof of lemma 2.4. Since X is Ω-stable [PuSh,1970], for $\tilde{X}$ close to X the nonwandering set $\tilde{\Omega}$ of $\tilde{X}$ is a union $\tilde{\Omega}_1, \ldots, \tilde{\Omega}_N$ of basic sets with $\tilde{\Omega}_i$ close to Ω_i, while by strong transversality

$$\tilde{\Omega}_1 \le \ldots \le \tilde{\Omega}_N.$$

Let η be a completely integrable subbundle of TM defined over a neighbourhood of $\Omega - P$ and complementary to $\langle X \rangle$ (take e.g. in a Riemannian structure η equal to $\langle X \rangle^-$). Write $\mathfrak{H}$ for the foliation near $\Omega - P$ integrating η. An equivalence $g : \Omega \to \tilde{\Omega}$ exists by Ω-stability. We may take g so that

$$g(x) \ \in \ \mathfrak{H}_x \tag{8}$$

for $x \in \Omega - P$ [HPS,1977].

First we construct a smooth positive function ϕ on M with supp $(\phi - 1)$ contained in a small neighbourhood of the union of isolated periodic orbits and nontrivial basic sets of saddle type so that $\mathfrak{H}$ is locally $(\phi X)_t$ invariant near these basic sets. Near a periodic orbit, the choice of such a function ϕ is fairly standard [PaSm,1970]. Suppose Ω_i is a nontrivial basic set of saddle type. Let $\cup_j B_j$ be a Markov partition with $B_j \subset T_j$ as in lemma 2.2. By lemma 2.1 plus the remark following lemma 2.2,

$$\Omega_i \cap T_j \ \subset \ B_j \tag{9}$$

for $T_j \backslash B_j$ sufficiently small. We can further choose the cross sections T_j to be contained in leaves of $\mathfrak{H}$. By (9), we can take smooth positive functions ϕ_j defined near T_j with the following properties. On an open set $\mathcal{U}_j \subset$ supp $(\phi_j - 1)$, $(\phi_j X)_t$ leaves $\mathfrak{H}$ locally invariant, Ω_i is contained in $\bigcup_j \mathcal{U}_j$, and supp $(\phi_j - 1) \cap$supp $(\phi_k - 1) \cap$supp $(\phi_l - 1) \neq \emptyset$ for $j \neq k \neq l$. It is easily seen that we can alter ϕ_j and ϕ_k near supp $(\phi_j - 1) \cap$ supp $(\phi_k - 1)$, so that $\phi_j = \phi_k$ on $\mathcal{U}_j \cap \mathcal{U}_k$. Doing so for all pairs j, k, a function near Ω_i is obtained which can easily be extended to get a function ϕ on M as desired. We define a similar function $\tilde{\phi}$ near ϕ for $\tilde{X}$, so that $\mathfrak{H}$ is locally $(\tilde{\phi}\tilde{X})_t$ invariant near isolated periodic orbits and nontrivial basic sets of saddle type. Then g as defined in (8) is a conjugation on isolated periodic orbits and nontrivial basic sets of saddle type.

We write $D^s(\Omega_i)$, $N^s(\Omega_i)$ for fundamental domains and fundamental neighbourhoods as in the proofs of lemma 2.3 and lemma 2.4; for $\Omega_i \subset \Omega - P$ we will take connected components of $D^s(\Omega_i)$ and $N^s(\Omega_i)$ within single leaves of $\mathfrak{H}$. Fundamental domains $D^s(\tilde{\Omega}_i)$ and fundamental neighbourhoods $N^s(\tilde{\Omega}_i)$ for $\tilde{X}$ can be chosen near those of X. For $\tilde{\Omega}_i \subset \tilde{\Omega} - \tilde{P}$, we can further take each connected component of $D^s(\tilde{\Omega}_i)$, $N^s(\tilde{\Omega}_i)$ in the same leaf of $\mathfrak{H}$ as the corresponding component of $D^s(\Omega_i)$, $N^s(\Omega_i)$.

Recall the construction of a compatible system of unstable laminations as in lemma 2.4. Choose a continuous positive function σ with supp $(\sigma - 1)$ outside a neighbourhood of Ω and smooth within unstable manifolds of orbits, so that, constructing a compatible system of unstable foliations $\{\mathfrak{F}^u(\Omega_i)\}$ for $\sigma\phi X$ as in the proof of lemma 2.4, this yields a system of laminations with

$$\mathfrak{F}^u_x(\Omega_i) \quad = \quad \mathfrak{F}^{cu}_x(\Omega_i) \cap \mathfrak{H}_x, \tag{10}$$

$x \in D^s(\Omega_i)$, $\Omega_i \subset \Omega - P$. Such a function σ can be made by induction on Ω_i, altering σ inductively on compact parts in $X_{(-\infty,0)}T_j$, where T_j is a cross section through Ω_i (for Ω_i an isolated periodic orbit) or a cross section from a Markov partition (for Ω_i a nontrivial basic set). Construct in an analogous manner a function $\tilde{\sigma}$ near σ and a nearby system of compatible laminations $\{\mathfrak{F}^u(\tilde{\Omega}_i)\}$ for $\tilde{\sigma}\tilde{\phi}\tilde{X}$; so

$$\mathfrak{F}^u_x(\tilde{\Omega}_i) \quad = \quad \mathfrak{F}^{cu}_x(\tilde{\Omega}_i) \cap \mathfrak{H}_x, \tag{11}$$

$x \in D^s(\tilde{\Omega}_i)$, $\tilde{\Omega}_i \subset \tilde{\Omega} - \tilde{P}$.

We construct the required homeomorphism $h : W^s(\Omega_i) \to W^s(\tilde{\Omega}_i)$ by induction on i. The induction starts by taking an equivalence $h = g$ between repellers of X and repellers of $\tilde{X}$. Suppose h is constructed on $\bigcup_{j=1}^{i-1} W^s(\Omega_j)$. When defining h on $W^s(\Omega_i)$, we have to arrange that h maps orbits in $W^u(\Omega_j) \cap W^s(\Omega_i)$ to orbits in $W^u(\tilde{\Omega}_j) \cap W^s(\tilde{\Omega}_i)$. Simultaneously we will alter $\tilde{\sigma}$ near supp $(\tilde{\sigma} - 1)$ so that following the construction of h below,

$$h(D^s(P)) \quad = \quad D^s(\tilde{P}) \tag{12}$$

and

$$h(x) \in \mathfrak{H}_x, \tag{13}$$

for $x \in D^s(\Omega - P)$. When changing $\tilde{\sigma}$, this will be done so that $\tilde{\sigma}$ will remain smooth within unstable manifolds; though altering $\tilde{\sigma}$ will alter $\{\mathfrak{F}^u(\tilde{\Omega}_i)\}$, this will still be a compatible system of unstable laminations.

We will first define h on $D^s(\Omega_i)$. A second induction on the basic sets Ω_j, $\Omega_j \leq \Omega_i$ will be performed. Suppose $\Omega_j \prec \Omega_i$.

1. Suppose there is no k with $\Omega_j \prec \Omega_k \prec \Omega_i$.

   First consider the case that Ω_i is a singularity of saddle type. Then $W^u(\Omega_j) \cap D^s(\Omega_i)$ is compact. Denote by $\pi : W^u(\Omega_j) \mapsto \Omega_j$ the projection $\pi(x) = \mathfrak{F}^u_x(\Omega_j) \cap \Omega_j$. Take a near identity homeomorphism h on $W^u(\Omega_j) \cap D^s(\Omega_i)$, so that

   $$h(x) \subset \mathfrak{F}^u_{h\circ\pi(x)}(\tilde{\Omega}_j). \tag{14}$$

   This is possible since $\mathfrak{F}^u(\tilde{\Omega}_j)$ is near $\mathfrak{F}^u(\Omega_j)$ and leaves of $\mathfrak{F}^u(\Omega_j)$ intersect $W^s(\Omega_i)$ transversally. Observe that h maps $W^u(\Omega_j) \cap D^s(\Omega_i)$ near $W^u(\tilde{\Omega}_j) \cap D^s(\tilde{\Omega}_i)$; alter $\tilde{\sigma}$ near $W^u(\tilde{\Omega}_j) \cap D^s(\tilde{\Omega}_i)$ so that $h(W^u(\Omega_j) \cap D^s(\Omega_i)) = W^u(\tilde{\Omega}_j) \cap D^s(\tilde{\Omega}_i)$.

   Suppose Ω_i is an attracting singularity. Here leaves of $\mathfrak{F}^u(\Omega_j)$ may intersect $D^s(\Omega_i)$ nontransversally. Take a near identity homeomorphism $\hat{h}$ defined on a neighbourhood of $W^u(\Omega_j) \cap D^s(\Omega_i)$ in $W^u(\Omega_j)$, so that (14) holds. Again this is possible since $\mathfrak{F}^u(\tilde{\Omega}_j)$ is near $\mathfrak{F}^u(\Omega_j)$. Let h be the restriction of $\hat{h}$ to $W^u(\Omega_j) \cap D^s(\Omega_i)$; by altering $\tilde{\sigma}$ near $W^u(\tilde{\Omega}_j) \cap D^s(\tilde{\Omega}_i)$, h maps $W^u(\Omega_j) \cap D^s(\Omega_i)$ into $D^s(\tilde{\Omega}_i)$.

   Next suppose that $\Omega_i \subset \Omega - P$ and Ω_i is not an attracting periodic orbit. Write $\partial D^s(\Omega_i) = \partial^{int} D^s(\Omega_i) \cup \partial^{ext} D^s(\Omega_i)$, where $\partial^{int} D^s(\Omega_i) = X_{(0,\infty)}(\partial D^s(\Omega_i)) \cap \partial D^s(\Omega_i)$. First define h on $D^s(\Omega_i) \cap W^u(\Omega_j)$ near $\partial^{ext} D^s(\Omega_i)$, subject to (14) and

   $$h(x) \subset g(W^s(\mathcal{O}(x))). \tag{15}$$

   This is possible since $\mathfrak{F}^u(\tilde{\Omega}_j)$ is near $\mathfrak{F}^u(\Omega_j)$ and leaves of $\mathfrak{F}^u(\Omega_j)$ intersect $W^s(\gamma)$ transversally. Extending h for positive time by conjugacy, h is defined on $D^s(\Omega_i) \cap W^u(\Omega_j)$ near $\partial^{int} D^s(\Omega_i)$. We can then extend h to $D^s(\Omega_i) \cap W^u(\Omega_j)$ so that (14) and (15) hold [NPT,1983]. Again, h maps $W^u(\Omega_j) \cap D^s(\Omega_i)$ near $W^u(\tilde{\Omega}_j) \cap D^s(\tilde{\Omega}_i)$. By altering $\tilde{\sigma}$ we obtain $h(x) \in \mathfrak{H}_x$ for $x \in W^u(\Omega_j) \cap D^s(\Omega_i)$.

   The case that Ω_i is an attracting periodic orbit can be treated similarly; take $\hat{h}$ on $W^u(\Omega_j)$ near $\partial^{ext} D^s(\Omega_i)$ so that (14) holds, $\hat{h}$ is defined on $W^u(\Omega_j)$ near $\partial^{int} D^s(\Omega_i)$ by conjugation. Extend $\hat{h}$ to $W^u(\Omega_j)$ near $D^s(\Omega_i)$ subject to (14). Let h be the restriction of $\hat{h}$ to $W^u(\Omega_j) \cap D^s(\Omega_i)$ and alter $\tilde{\sigma}$ as before.

2. Suppose $\Omega_j \prec \Omega_{k_1} \prec \ldots \prec \Omega_{k_m}$, where Ω_{k_g}, $1 \leq g \leq m$ are the critical elements with $\Omega_{k_g} \prec \Omega_i$. By induction h is constructed on $\bigcup_{g=1}^m W^u(\Omega_{k_g}) \cap \overline{D^s(\Omega_i)}$. Note that h is thus already constructed on a compact subset of $\overline{D^s(\Omega_i)}$.

   h has already been defined on $\bigcup_{g=1}^m W^s(\Omega_{k_g})$. Observe that Ω_{k_g} is of saddle type, so that $\mathfrak{F}^u(\Omega_{k_g})$ has one dimensional leaves. We require h to map $\mathfrak{F}^u(\Omega_{k_g})$ to $\mathfrak{F}^u(\tilde{\Omega}_{k_g})$: let π_g be a projection $\pi_g(x) = \mathfrak{F}^u_x(\Omega_{k_g}) \cap W^s(\Omega_{k_g})$ (there can be more then one point in this intersection; choose one) and define h near $W^u(\Omega_j) \cap W^s(\Omega_{k_g})$ as a near identity map with

$$h(x) \quad \subset \quad \mathfrak{F}^u_{h \circ \pi_g(x)}(\tilde{\Omega}_j). \tag{16}$$

   By compatibility of $\{\mathfrak{F}^u(\Omega_i)\}$ we can this way define h on $W^u(\Omega_j) \cap \mathcal{V}$, where $\mathcal{V}$ is a small neighbourhood of $\bigcup_{g=1}^m W^u(\Omega_{k_g}) \cap D^s(\Omega_i)$ in $D^s(\Omega_i)$.

   As above we define h on the complement $W^u(\Omega_j) \cap (D^s(\Omega_i) - \mathcal{V})$. Alter $\tilde{\sigma}$ as before.

Having defined h on $D^s(\Omega_i)$, we now extend h to $W^s(\Omega_i)$. We define h on $W^s(\Omega_i)$ by conjugation

$$h \circ (\sigma\phi X)_t \quad = \quad (\tilde{\sigma}\tilde{\phi}\tilde{X})_t \circ h. \tag{17}$$

Since h as defined on strange attractors and repellers is an equivalence and in general not a conjugation, we must alter h near strange attractors and repellers. Suppose Ω_i is a strange attractor. Let $E^s(\Omega_i)$ be a fundamental domain as in lemma 2.5 and write $F^s(\Omega_i) = X_\beta(E^s(\Omega_i))$ for some $\beta > 0$. Take $E^s(\Omega_i)$ sufficiently near Ω_i so that it is contained in the region foliated by $\mathfrak{H}$. It follows from the proof of lemma 2.5 that we may take $E^s(\Omega_i)$ so that leaves of $\mathfrak{H}$ intersect $E^s(\Omega_i)$, $F^s(\Omega_i)$ transversally. Multiplying $\tilde{\sigma}\tilde{\phi}\tilde{X}$ by a continuous positive function $\tilde{\psi}$ with supp $(\tilde{\psi} - 1)$ contained in $X_{(0,\beta)}(E^s(\Omega_i))$ and replacing $\tilde{\sigma}\tilde{\phi}\tilde{X}$ by $\tilde{\psi}\tilde{\sigma}\tilde{\phi}\tilde{X}$ we obtain $h(x) \in \mathfrak{H}_x$ for $x \in F^s(\Omega_i)$. Then extend h to $\bigcup_{t>0} X_t(F^s(\Omega_i))$ by letting h act as the identity on leaves of $\mathfrak{H}$. Near strange repellers, h is defined analogously.

Observe that by construction, $h(x) \in \mathfrak{H}_x$ for all x near $\Omega - P$. Note further that the equivalence h satisfies

$$h(\mathfrak{F}^u(\Omega_j)) \quad = \quad \mathfrak{F}^u(\tilde{\Omega}_j),$$

except near strange attractors and strange repellers. Near a strange attractor or a strange repeller Ω_j, h maps $\mathfrak{F}^{cu}(\Omega_j) \cap \mathfrak{H}$ to $\mathfrak{F}^{cu}(\tilde{\Omega}_j) \cap \mathfrak{H}$.

Remains to prove continuity of h. By construction, h is continuous when restricted to stable sets of basic sets. Suppose inductively that h restricted to $\bigcup_{i=j+1}^n W^s(\Omega_i)$ is continuous. We may assume that Ω_j is not a repelling critical element, for in that case continuity of h is clear. Let $\{x_n\}$ be a sequence of points converging to $x \in W^s(\Omega_j)$. Since

h is continuous restricted to $W^s(\Omega_j)$, we may assume that $x_n \in \bigcup_{i=j+1}^n W^s(\Omega_i)$ by taking a subsequence. Observe $h(x_n) \in \bigcup_{i=j+1}^n W^s(\tilde{\Omega}_i)$ and $h(x) \in W^s(\tilde{\Omega}_j)$. Continuity of h follows since h maps $\mathfrak{F}^u_{x_n}(\Omega_j)$ to $\mathfrak{F}^u_{h(x_n)}(\tilde{\Omega}_j)$ ($\mathfrak{F}^{cu}(\Omega_j) \cap \mathfrak{H}$ to $\mathfrak{F}^{cu}(\tilde{\Omega}_j) \cap \mathfrak{H}$ near a strange attractor or a strange repeller) and these leaves intersects $W^s(\Omega_j)$ resp. $W^s(\tilde{\Omega}_j)$ transversally. ■

3 Stable families of vector fields

Write

$$\mathfrak{X}_\Omega = \{X \in \mathfrak{X},\ \ X \text{ satisfies axiom A + no cycles}\}.$$

Consider, for a compact interval $I \subset \mathbb{R}$, the set $C^\infty(I, \mathfrak{X}_\Omega)$ of smooth one parameter families of vector fields in $\mathfrak{X}_\Omega$. In this section we investigate which of these families are W-stable. Recall that $\mathfrak{X}_{str}$ stands for the class of vector fields from $\mathfrak{X}_\Omega$ that satisfy the strong transversality condition.

Let $\{X_\mu\} \in C^\infty(I, \mathfrak{X}_\Omega)$. The nonwandering set of X_μ depends on μ, but we will suppress this dependence from the notation. An orbit of heteroclinic tangency is an orbit γ in $W^u(\Omega_p) \cap W^s(\Omega_q)$, where Ω_p, Ω_q are basic sets, so that

$$\dim(T_\gamma W^u(\Omega_p) \oplus T_\gamma W^s(\Omega_q)) < 3.$$

Suppose $\{X_\mu\}$ has an orbit of heteroclinic tangency at $\mu = \mu_0$. The orbit of heteroclinic tangency is said to unfold generically, if

1. $\dim(T_\gamma W^u(\Omega_p) \oplus T_\gamma W^s(\Omega_q)) = 2$.
2. The tangency of $W^u(\Omega_p)$ and $W^s(\Omega_q)$ along γ is quadratic [NPT,1983].
3. $\bigcup_\mu (W^u(\Omega_p), \mu)$ and $\bigcup_\mu (W^s(\Omega_q), \mu)$ intersect transversally along $\gamma \times \{\mu_0\} \subset M \times I$.

In case Ω_p is a critical element with one dimensional unstable manifold, or Ω_q is a critical element with one dimensional stable manifold, condition 2. is empty.

Suppose Ω_q is a singularity with dim $W^s(\Omega_q) = 2$ and the spectrum spec $DX_\mu(\Omega_q)$ consists of three different real numbers. Let $W^c(\Omega_q)$ denote a two dimensional centre unstable manifold. Although $W^c(\Omega_q)$ is not unique, any two centre unstable manifolds have the same tangent bundle along $W^u(\Omega_q)$. Similarly, for a singularity Ω_p with dim $W^u(\Omega_p) = 2$ and spec $DX_\mu(\Omega_q) \subset \mathbb{R}$, we write $W^c(\Omega_p)$ for a two dimensional centre stable manifold.

Theorem 3.1 *Let I be a compact interval in $\mathbb{R}$ and let $\{X_\mu\} \in C^\infty(I, \mathfrak{X}_\Omega)$ be such that for $\mu \in \partial I$, $X_\mu \in \mathfrak{X}_{str}$.*

Necessary and sufficient conditions for the family $\{X_\mu\}$ to be W-stable, are

1. *Each heteroclinic bifurcation of $\{X_\mu\}$ unfolds generically.*
2. *At each parameter value, $\{X_\mu\}$ has at most one orbit of heteroclinic tangency.*
3. *Suppose $\gamma \in W^u(\Omega_p) \cap W^s(\Omega_q)$ is an orbit of heteroclinic tangency, where Ω_p, Ω_q are basic sets.*

   If Ω_q is a singularity with dim $W^s(\Omega_q) = 1$, spec $DX_\mu(\Omega_q) \subset \mathbb{R}$ and further $W^u(\Omega_q)$ is not contained in the stable manifold of an attracting critical element, then

- *spec* $DX_\mu(\Omega_q)$ *consists of three different real numbers,*
- $W^c(\Omega_q)$ *intersects* $W^s(\Omega_p)$ *transversally along* γ *for* $\mu = \mu_0$*,*
- $W^{uu}(\Omega_q)$ *is contained in the stable manifold of an attracting critical element.*

If Ω_p *is a singularity with dim* $W^u(\Omega_p) = 1$*, spec* $DX_\mu(\Omega_p) \subset \mathbb{R}$*, and further* $W^s(\Omega_p)$ *is not contained in the unstable manifold of a repelling critical element, then*

- *spec* $DX_\mu(\Omega_p)$ *consists of three different real numbers,*
- $W^c(\Omega_p)$ *intersects* $W^u(\Omega_q)$ *transversally along* γ *for* $\mu = \mu_0$*,*
- $W^{ss}(\Omega_p)$ *is contained in the unstable manifold of a repelling critical element.*

4. *If* $\gamma \in W^u(\Omega_p) \cap W^s(\Omega_q)$ *is an orbit of heteroclinic tangency, where* Ω_p, Ω_q *are basic sets, then one of the following two conditions holds:*

   - Ω_q *is a critical element and* $W^s(\Omega_p)$ *is contained in the unstable manifold of a repelling critical element, or*
   - Ω_p *is a critical element and* $W^u(\Omega_q)$ *is contained in the stable manifold of an attracting critical element.*

We remark that the equivalence h_μ we will construct when the conditions in the statement of the theorem are satisfied, is such that $(x,\mu) \mapsto h_\mu(x)$ is continuous restricted to $\bigcup_\mu (W^s(\zeta_\mu), \mu)$ and $\bigcup_\mu (W^u(\zeta_\mu), \mu)$, for any orbit $\zeta_\mu \subset \Omega$.

Given a two dimensional manifold N, let $\mathfrak{A}_\Omega$ be the set of diffeomorphisms on N that satisfy axiom A and the no-cycle condition. Let $\mathfrak{A}_{str}$ be the set of diffeomorphisms from $\mathfrak{A}_\Omega$ that satisfy the strong transversality condition. Applying theorem 3.1 to the suspension of diffeomorphisms on N yields the following corollary, compare [MeSt,1987].

Corollary *Let* I *be a compact interval in* $\mathbb{R}$ *and let* $f_\mu \in C^\infty(I, \mathfrak{A}_\Omega)$ *be such that, for* $\mu \in \partial I$*,* $f_\mu \in \mathfrak{A}_{str}$*. Necessary and sufficient conditions for the family* f_μ *to be* W*-stable, are items 1, 2, 4 in the statement of theorem 3.1.* ■

Proof of sufficiency. By $\{\tilde{X}_\mu\}$ we denote a small perturbation of $\{X_\mu\}$. We wish to construct a map $(h_\mu, \eta) : M \times I \to M \times I$, where η is a homeomorphism and h_μ is a homeomorphism when restricted to stable and unstable manifolds, providing a topological equivalence along stable and unstable manifolds.

We start with the observation that it suffices to provide the proof for μ near a bifurcation value μ_0 [NPT,1983]; this follows by proving the theorem for families

$$N(\mu)X_\mu + (1 - N(\mu))\tilde{X}_\mu, \quad M(\mu)X_\mu + (1 - M(\mu))\tilde{X}_\mu,$$

for smooth functions $N, M : I \to [0,1]$, requiring $(h_\mu, \eta)(x,\mu) = (x,\mu)$ if $N(\mu) = M(\mu)$.

Because critical elements of X_μ are dense in Ω, it further suffices to restrict to values of μ near a bifurcation value μ_0 where the stable and unstable manifolds of two critical elements have a tangency. Write $\gamma \in W^u(\Omega_p) \cap W^s(\Omega_q)$ for the orbit of tangency at $\mu = \mu_0$. Taking into account the above considerations and up to reversing the direction of time we must study the following cases ($P \subset \Omega$ is the set of singularities).

1. $\Omega_p, \Omega_q \subset \Omega - P$.
2. $\Omega_p \subset P$, $\Omega_q \subset \Omega - P$.
3. $\Omega_p, \Omega_q \subset P$.

Section 3.1 contains the proof of W-stability for case 1, where at $\mu = \mu_0$ the stable and unstable manifold of two periodic orbits are tangent. Such a family is not structurally stable [Pal,1978], [NPT,1983]. Section 3.2 then contains the proof of theorem 3.1 in case 2 where in addition $DX_\mu(\Omega_p)$ possesses two complex conjugate stable eigenvalues and one real unstable eigenvalue. Also such a family is not structurally stable [Bel,1986]. The remaining cases give in fact structurally stable families and can be treated similarly, see also [LaPl,1993]. ■

Proof of necessity. It is clear that W-stable families have at most one orbit of heteroclinic tangency at each parameter value and have generically unfolding bifurcations. Also necessity of item 3 is easily recognised.

We prove that W-stable families satisfy item 4 in the statement of the theorem. We may assume that the tangency is between stable and unstable manifolds of critical elements, since these lie dense in the nonwandering set. We only consider the case where the stable and unstable manifold of two periodic orbits are tangent, the other possibilities are treated similarly. The proof is a simple counting argument, counting intersections of stable and unstable manifolds, and is simpler then the proof that families with a heteroclinic tangency are not structurally stable [Pal,1978], [NPT,1983], [MeSt,1987].

Write ζ, ξ for the periodic orbits in Ω_p resp. Ω_q with the orbit of heteroclinic tangency $\gamma \in W^u(\zeta) \cap W^s(\xi)$ at $\mu = \mu_0$. Let ν denote the positive characteristic multiplier of the periodic orbit ζ ($\nu = \lim_{t\to\infty} \frac{1}{t} \frac{\ln \|D(X_\mu)_t(x)v\|}{\|v\|}$, $x \in \zeta, v \in T_x W^u(x)$) and let λ denote the negative characteristic multiplier of ξ. If item 4 does not hold, there exist orbits α, β in Ω with

$$\begin{aligned} W^u(\alpha) \cap W^s(\Omega_p) &\neq \emptyset, \\ W^s(\beta) \cap W^u(\Omega_q) &\neq \emptyset. \end{aligned}$$

These intersections are transverse and therefore, by the λ-lemma, $W^u(\alpha)$ accumulates on $W^u(\Omega_p)$ and $W^s(\beta)$ accumulates on $W^s(\Omega_q)$. If $W^s(\xi)$ is an immersed Möbius band, let T_ξ denote twice the period of ξ, otherwise let T_ξ denote the period of ξ. Similarly, if $W^u(\zeta)$ is an immersed Möbius band, let T_ζ denote twice the period of ζ, otherwise let T_ζ denote the period of ζ. Let Σ be a cross section transverse to γ. Take coordinates (x, y) on Σ with

$$W^s(\xi) \cap \Sigma \;=\; \{y = 0\}.$$

We may assume that Σ is part of a cross section S transverse to ξ. By a small perturbation of $\{X_\mu\}$, we may assume that the time T_ξ map $(X_\mu)_{T_\xi}$ maps S into itself (actually into a larger section extending S) and is linear in suitable smooth coordinates (compare figure 2). It is now easily seen that components B_n of $W^s(\beta) \cap \Sigma$ exist so that, in a metric d on Σ,

$$d(B_n, W^s(\xi) \cap \Sigma) \;\sim\; e^{-nT_\zeta \nu}, \tag{18}$$

where $\sim$ means equal up to a positive factor that may depend on n, μ but is bounded and bounded away from zero uniformly in n and μ. By the definition of T_ξ, the eigenvalues of $(X_\mu)_{T_\xi}$ restricted to S are positive. Therefore, the components B_n are all on one side of $W^s(\xi) \cap \Sigma$. Similarly, components A_n of $W^u(\alpha) \cap \Sigma$ exist so that

$$d(A_n, W^u(\zeta) \cap \Sigma) \;\sim\; e^{nT_\xi \lambda}. \tag{19}$$

By the choice of T_ζ, all components A_n are on one side of $W^u(\zeta) \cap \Sigma$. Write α_n for the tops of the parabolas A_n (viewing A_n as the graph $\{(x, y_n(x))\}$ of a map y_n). If one of $W^s(\xi)$, $W^u(\zeta)$ is an immersed Möbius band, we can choose components A_n, B_n so that α_n and B_n are contained in one connected component of $\{y \neq 0\}$. We may assume this component to be $\{y > 0\}$, then. If both $W^s(\xi)$, $W^u(\zeta)$ are immersed annuli, it may happen that the points α_n are in the other component of $\{y \neq 0\}$ then the curves B_n.

Consider first the case where both $B_n, \alpha_n \subset \{y > 0\}$. For each $i \in \mathbb{N}$, let $n(i)$ be the smallest integer so that $B_{n(i)}$ is below α_i, see figure 1. Then

$$e^{iT_\xi \lambda} \;\sim\; e^{-n(i)T_\zeta \nu}.$$

So, choosing two components A_i, A_j, $i < j$, the number $n(j) - n(i)$ of components B_n intersecting A_i but not A_j satisfies

$$e^{(j-i)T_\xi \lambda} \;\sim\; e^{-(n(j)-n(i))T_\zeta \nu}$$

Taking logarithms and letting $j - i \to \infty$, this yields

$$\frac{n(j) - n(i)}{j - i} \;\to\; \frac{-T_\zeta \nu}{T_\xi \lambda}. \tag{20}$$

Since an equivalence h_μ (at $\mu = \mu_0$) has to preserve the quotient $(n(j) - n(i))/(j - i)$, it follows that $-T_\zeta\nu/T_\xi\lambda$ is an invariant ('modulus') of W-stability. Observe that the occurence of this modulus is directly related to the geometry of intersections of stable and unstable manifolds, at the bifurcation value $\mu = \mu_0$.

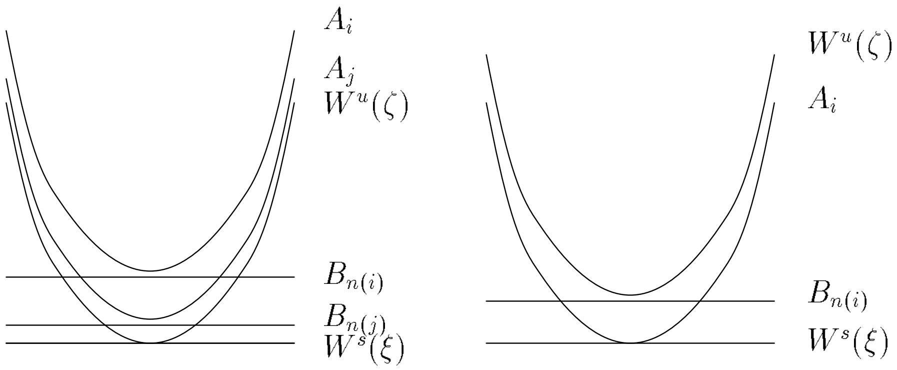

Figure 1: Stable and unstable manifolds intersecting Σ. In both pictures, $B_n \subset \{y > 0\}$. In the left picture, the tops of the parabolas A_n are contained in $\{y > 0\}$ when $\mu = \mu_0$. In the right picture, where $\mu = \nu_i$, the tops of the parabolas A_n are, for $\mu = \mu_0$, contained in $\{y < 0\}$.

Now suppose the tops of the parabolas A_n and the curves B_n are, for $\mu = \mu_0$, in different connected components of $\{y \neq 0\}$, for definiteness, say $B_n \subset \{y > 0\}$ and $\alpha_n \subset \{y < 0\}$. Here we will find a modulus of W-stability by varying μ. Let ν_i be the bifurcation value where A_i is tangent to $W^s(\zeta)$. Let $n(i)$ be the smallest integer so that $B_{n(i)}$ is below α_0 at $\mu = \nu_i$, see figure 1. Then, at $\mu = \nu_i$,

$$e^{iT_\xi\lambda} \sim e^{-n(i)T_\zeta\nu}.$$

Letting $i \to \infty$,

$$\frac{n(i)}{i} \to \frac{-T_\zeta\nu}{T_\xi\lambda}. \tag{21}$$

Again it follows that $-T_\zeta\nu/T_\xi\lambda$ is an invariant of W-stability. ■

3.1 Heteroclinic orbits between periodic orbits

Here we prove W-stability of families $\{X_\mu\}$ satisfying the conditions of theorem 3.1, for μ near a bifurcation value μ_0 where the stable and unstable manifolds of two periodic orbits in basic sets Ω_p, Ω_q have a tangency. By reversing the direction of time if necessary, we may assume that Ω_q is an isolated periodic orbit and $W^u(\Omega_q) \subset W^s(\Omega_r)$ for an attracting critical element Ω_r. Write γ for the orbit of tangency in $W^u(\Omega_p) \cap W^s(\Omega_q)$ at $\mu = \mu_0$. The perturbed family $\{\tilde{X}_\mu\}$ has an orbit $\tilde{\gamma}$ of heteroclinic tangency in $W^u(\tilde{\Omega}_p) \cap W^s(\tilde{\Omega}_q)$ near γ at a parameter value $\tilde{\mu_0}$ near μ_0.

Choose an ordering $\leq$ of the basic sets $\Omega_1, \ldots, \Omega_N$ of $\{X_\mu\}$ so that for $\mu = \mu_0$, if $\overline{W^u(\Omega_i)} \cap \overline{W^s(\Omega_j)} \neq \emptyset$, then $\Omega_i \leq \Omega_j$. Possibly renumbering the basic sets, we may assume

$$\Omega_1 \leq \ldots \leq \Omega_N,$$

for all values of $\mu \in I$. The perturbed family $\{\tilde{X}_\mu\}$ has nearby basic sets $\tilde{\Omega}_i$ with $\tilde{\Omega}_1 \leq \ldots \leq \tilde{\Omega}_N$.

Let $\mathfrak{H}$ be a codimension one foliation near $\Omega - P$ with leaves transverse to the flow, as in the proof of theorem 2.6. Like in that proof, we replace $\{X_\mu\}$ by $\{\phi_\mu X_\mu\}$, where ϕ_μ is a smooth positive function with supp ϕ_μ contained in a small neighbourhood of the union of isolated periodic orbits and nontrivial basic sets of saddle type, so that $\mathfrak{H}$ is locally $(\phi_\mu X_\mu)_t$ invariant near these basic sets. Similarly define a function $\tilde{\phi}_\mu$ near ϕ_μ so that $\mathfrak{H}$ is locally $(\tilde{\phi}_\mu \tilde{X}_\mu)_t$ invariant near isolated periodic orbits and nontrivial basic sets of saddle type, with further $\phi_\mu = \tilde{\phi}_\mu$ if $X_\mu = \tilde{X}_\mu$.

We want to construct compatible systems of unstable laminations. For some function σ_μ we construct a compatible system of unstable laminations $\{\mathfrak{F}^u(\Omega_i)\}_{i \neq q,r}$ for $\{\sigma_\mu \phi_\mu X_\mu\}$. The function σ_μ will be a positive continuous function, $(x, \mu) \mapsto \sigma_\mu(x)$ will be smooth when restricted to unstable manifolds $\bigcup_\mu (W^u(\zeta_\mu), \mu)$, ζ_μ an orbit in Ω, and supp $(\sigma_\mu - 1)$ is outside a neighbourhood of Ω. As before we can choose σ_μ and construct $\{\mathfrak{F}^u(\Omega_i)\}_{i \neq q,r}$ so that $\mathfrak{F}^u_x(\Omega_i) \subset \mathfrak{H}_x$, $x \in D^s(\Omega_i)$, for $\Omega_i \subset \Omega - P$. A foliation $\mathfrak{F}(\Omega_q)$, compatible with $\{\mathfrak{F}^u(\Omega_i)\}_{i \neq q,r}$, is obtained as follows. Choose $Q \in \Omega_q$. Let $D^s(\Omega_q)$ be a fundamental domain for the Poincaré return map on $W^s(\Omega_q) \cap \mathfrak{H}_Q$. Take $D^s(\Omega_q)$ so that $\gamma \cap \mathfrak{H}_Q \not\subset \partial D^s(\Omega_q)$, see figure 2. Near $\gamma \cap D^s(\Omega_q)$ let $\mathfrak{F}^1 = \mathfrak{F}^u(\Omega_p)$. Near $D^s(\Omega_q)$ but outside a neighbourhood of $\gamma \cap D^s(\Omega_q)$, a foliation $\mathfrak{F}^2$ is defined as in the proof of lemma 2.4. Average $\mathfrak{F}^1, \mathfrak{F}^2$ to get a foliation $\mathfrak{F}$ near $D^s(\Omega_q)$. By altering σ_μ we may assume $\mathfrak{F}_x \subset \mathfrak{H}_Q$, $x \in D^s(\Omega_q)$. Again we take σ_μ restricted to unstable manifolds of orbits to be a smooth function. Saturate for positive time the leaves of $\mathfrak{F}$ in $\mathfrak{H}_Q$ to obtain $\mathfrak{F}(\Omega_q)$. Note that leaves of $\mathfrak{F}(\Omega_q)$ are tangent to $W^s(\Omega_q)$ along γ at $\mu = \mu_0$. Finally, leaves of the unstable foliation $\mathfrak{F}^u(\Omega_r)$ are single points, since Ω_r is an attracting critical element. Observe that the system $\{\mathfrak{F}^u(\Omega_i), \mathfrak{F}(\Omega_q)\}_{i \neq q}$ is compatible. A continuous positive function $\tilde{\sigma}_\mu$ and a compatible system of foliations $\{\mathfrak{F}^u(\tilde{\Omega}_i), \mathfrak{F}(\tilde{\Omega}_q)\}_{i \neq q}$ for $\{\tilde{\sigma}_\mu \tilde{\phi}_\mu \tilde{X}_\mu\}$ is constructed analogously, with $\tilde{\sigma}_\mu = \sigma_\mu$ and $\mathfrak{F}^u(\tilde{\Omega}_i) = \mathfrak{F}^u(\Omega_i)$, $\mathfrak{F}(\tilde{\Omega}_q) = \mathfrak{F}(\Omega_q)$ if $\tilde{X}_\mu = X_\mu$.

Now we define the reparametrization $\eta : I \to I$. Let $\Sigma \subset \mathfrak{H}_Q$ be a small neighbourhood in $\mathfrak{H}_Q$ of $\gamma \cap \mathfrak{H}_Q$. Take coordinates (x_1, x_2) on Σ in which $D^s(\Omega_q) = \{x_2 = 0\}$ and write

$$\mathfrak{F}^u(\Omega_p) = \{(x_1, f_\mu(x_1, x_2))\},$$

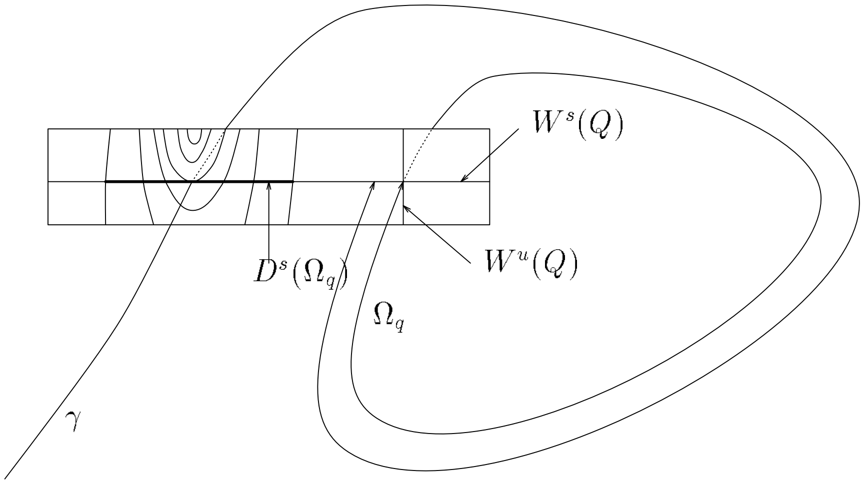

Figure 2: The foliation $\mathfrak{F}(\Omega_q)$ near the fundamental domain $D^s(\Omega_q)$.

where for each x_2 an $\eta \subset \Omega$ exists so that graph $(x_1, \mu) \mapsto f_\mu(x_1, x_2) \subset \bigcup_\mu (W^u(\eta), \mu) \subset M \times I$. We claim that the function $f_\mu(x_1, x_2)$ can be chosen with the following properties:

$$(x_1, \mu) \mapsto f_\mu(x_1, x_2) \text{ is smooth,} \tag{22}$$

$$(x_1, x_2, \mu) \mapsto \left(\frac{\partial}{\partial x_1}\right)^i f_\mu(x_1, x_2) \text{ is continuous for all } i, \tag{23}$$

$$\frac{\partial}{\partial x_1} f_{\mu_0}(0,0) = 0, \tag{24}$$

$$\left(\frac{\partial}{\partial x_1}\right)^2 f_{\mu_0}(0,0) \neq 0, \tag{25}$$

$$\frac{\partial}{\partial \mu} f_{\mu_0}(0,0) \neq 0. \tag{26}$$

The first item follows since $\mathfrak{F}^u(\Omega_i)$ defines a smooth foliation of unstable manifolds $\bigcup_\mu (W^u(\zeta_\mu), \mu)$, ζ_μ an orbit in Ω [HPS,1977]. The second item holds because $\mathfrak{F}^u(\Omega_p)$ is a lamination, while the other items follow from the fact that we have a generically unfolding heteroclinic tangency. Let $x_1 = c_\mu(x_2)$ be the curve at which $\frac{\partial}{\partial x_1} f_\mu(x_1, x_2) = 0$, c_μ is a continuous function by (23), (24), (25). Let $\tilde{\Sigma} \subset \mathfrak{H}_Q$ be a small neighbourhood of $\tilde{\gamma} \cap D^s(\tilde{\Omega}_q)$. Take coordinates $(\tilde{x}_1, \tilde{x}_2)$ on $\tilde{\Sigma}$ with $W^s(\tilde{\Omega}_q) \cap \tilde{\Sigma} = \{\tilde{x}_2 = 0\}$ and write

$$\mathfrak{F}(\Omega_q) = \{(\tilde{x}_1, \tilde{f}_\mu(\tilde{x}_1, \tilde{x}_2))\},$$

where the function $\tilde{f}_\mu$ satisfies similar properties as f_μ. Let $\tilde{x}_1 = \tilde{c}_\mu(\tilde{x}_2)$ be the continuous curve at which $\frac{\partial}{\partial \tilde{x}_1} \tilde{f}_\mu(\tilde{x}_1, \tilde{x}_2) = 0$. By (26) we can define a homeomorphism $\eta : I \to I$ so that $c_\mu(0) \subset W^u(\zeta)$ for some orbit $\zeta \subset \Omega$ implies $\tilde{c}_{\eta(\mu)}(0) \subset W^u(g(\zeta))$ -where g is a topological equivalence between Ω and $\tilde{\Omega}$-. We can define η so that $\eta(\mu) = \mu$ if $X_\mu = \tilde{X}_\mu$. Note that if $W^u(\Omega_i)$ and $W^s(\Omega_q)$, $\Omega_i \le \Omega_p$, have a tangency for $\mu = \nu$, then $W^u(\tilde{\Omega}_i)$ and $W^s(\tilde{\Omega}_q)$ have a tangency for $\mu = \eta(\nu)$.

Now we construct an equivalence between the families $\{X_\mu\}$ and $\tilde{X}_{\eta(\mu)}$. Inductively, we define h_μ on the stable manifolds $W^s(\Omega_j)$, $j \neq q, r$, as in the proof of theorem 2.6. In this construction, $\tilde{\sigma}_\mu$ is altered so as to get

$$h_\mu(x) \in \mathfrak{H}_x, \tag{27}$$

for $x \in D^s(\Omega_j)$ in case $\Omega_j \subset \Omega - P$, or

$$h_\mu(x) \subset D^s(\tilde{\Omega}_j), \tag{28}$$

for $x \in D^s(\Omega_j)$, if $\Omega_j \subset P$. Next we define h_μ on $D^s(\Omega_q)$. By the demand that h_μ maps $\{\mathfrak{F}^u(\Omega_j)\}$ to $\{\mathfrak{F}^u(\tilde{\Omega}_j)\}$, since h_μ has already been defined on the space $W^s_{loc}(\Omega_p)$ of leaves $\mathfrak{F}^u(\Omega_p)$, h_μ is defined on a small neighbourhood $\mathcal{U}$ of $\gamma \cap D^s(\Omega_q)$ in $D^s(\Omega_q)$. Alter $\tilde{\sigma}_\mu$ near $\mathcal{U}$ to get (27) for $x \in \mathcal{U}$. On $D^s(\Omega_q) - \mathcal{U}$, h_μ is defined as in the proof of theorem 2.6. Alter $\tilde{\sigma}_\mu$ as before to get (27) for $x \in D^s(\Omega_q)$ and extend h_μ to $W^s(\Omega_q)$ by conjugacy. Finally we define h_μ on $W^s(\Omega_r)$. Take a homeomorphism i_μ on $N^s(\Omega_q)$ –a fundamental neighbourhood in $\mathfrak{H}_Q$ extending $D^s(\Omega_q)$– that extends h_μ on $D^s(\Omega_q)$, maps $\mathfrak{F}^u(\Omega_p)$ to $\mathfrak{F}^u(\tilde{\Omega}_p)$ and satisfies (27) for $x \in N^s(\Omega_q)$. Let h_μ on a neighbourhood of $W^u(\Omega_q) \cap D^s(\Omega_r)$ be defined by $h_\mu((\sigma_\mu\phi_\mu X_\mu)_t(x)) = (\tilde{\sigma}_\mu\tilde{\phi}_\mu\tilde{X}_\mu)_t(x)$, for $x \in N^s(\Omega_q)$, $(\sigma_\mu\phi_\mu X_\mu)_t(x) \in D^s(\Omega_r)$. This defines h_μ on a neighbourhood $\mathcal{U}$ of $W^u(\Omega_q) \cap D^s(\Omega_r)$ in $D^s(\Omega_r)$. By altering $\tilde{\sigma}_\mu$, (27) resp. (28) holds with $j = r$, if $\Omega_r \subset \Omega - P$ resp. $\Omega_r \subset P$. Extend h_μ to $D^s(\Omega_r) - \mathcal{U}$ as in the proof of theorem 2.6; alter $\tilde{\sigma}_\mu$ so that (27) resp. (28) holds. The remainder of the construction of h_μ is again as in the proof of theorem 2.6.

By the same argument as before, h_μ is continuous outside orbits of heteroclinic tangency. It is then clear from the construction of h_μ near an orbit of heteroclinic tangency that h_μ is continuous restricted to stable and unstable manifolds. ■

3.2 Spiral-like invariant manifolds

Suppose $\{X_\mu\}$ has an orbit $\gamma \subset W^u(\Omega_p) \cap W^s(\Omega_q)$ of heteroclinic tangency at $\mu = \mu_0$, where Ω_p is a singularity and Ω_q is a periodic orbit or a nontrivial basic set. In this section we further assume that dim $W^u(\Omega_p) = 1$ and spec $DX_\mu(\Omega_q)$ consists of one positive real eigenvalue and two complex conjugate eigenvalues with negative real part. By [Str,1982], [Bel,1986], $\{X_\mu\}$ is a structurally unstable family. We prove in this section that, provided the conditions of theorem 3.1 are satisfied, $\{X_\mu\}$ is a W-stable family. We consider the case where $W^u(\Omega_q) \subset W^s(\Omega_r)$ for an attracting critical element Ω_r. The case where $W^s(\Omega_p)$ is in the unstable manifold of a repelling critical element is treated analogously.

Replace $\{X_\mu\}$, $\{\tilde{X}_\mu\}$ by $\{\phi_\mu X_\mu\}$, $\{\tilde{\phi}_\mu \tilde{X}_\mu\}$ as before. Construct a compatible system of unstable foliations $\{\mathfrak{F}^u(\Omega_i)\}_{i\neq q,r}$ for $\{\sigma_\mu \phi_\mu X_\mu\}$, for a continuous positive function σ_μ with supp $(\sigma_\mu - 1)$ outside a neighbourhood of Ω, as in the previous section. Similarly, $\tilde{\phi}_\mu \tilde{X}_\mu$ is replaced by $\tilde{\sigma}_\mu \tilde{\phi}_\mu \tilde{X}_\mu$ and a compatible system of unstable laminations $\{\mathfrak{F}^u(\tilde{\Omega}_i)\}_{i\neq q,r}$ for $\{\tilde{\sigma}_\mu \tilde{\phi}_\mu \tilde{X}_\mu\}$ is constructed. Let $\mathcal{U} \subset D^s(\Omega_q)$ be a small interval containing $W^u(\Omega_p) \cap D^s(\Omega_q)$ in its interior for $\mu = \mu_0$. Through $\mathcal{U}$ we define a foliation $\mathfrak{F}(\Omega_q)$ to be equal to $\mathfrak{F}^u(\Omega_p)$. Through $D^s(\Omega_q) - \mathcal{U}$, $\mathfrak{F}(\Omega_q)$ is defined as before. Observe that leaves of $\mathfrak{F}(\Omega_q)$ are not transverse to $W^s(\Omega_q)$ along γ. A foliation $\mathfrak{F}(\tilde{\Omega}_q)$ is defined similarly.

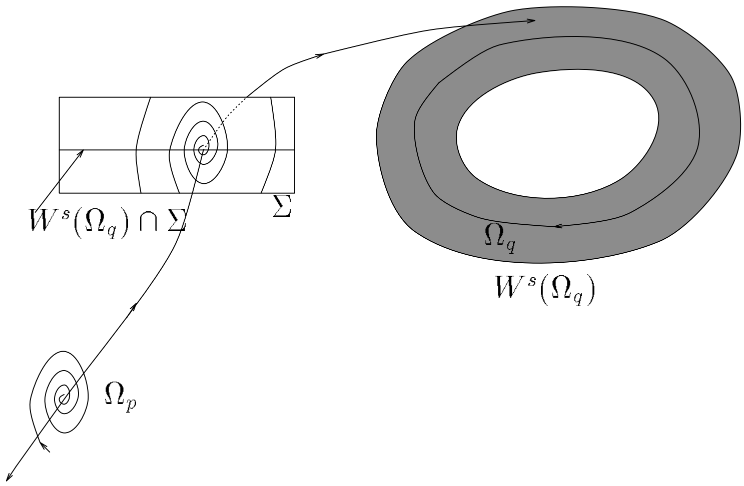

Figure 3: The spirals $\mathfrak{S}(\Omega_p)$ in Σ.

Now we define the reparametrization $\eta : I \to I$. Let Σ be a small cross section extending $\mathcal{U}$. Consider the foliation $\mathfrak{L}(\Omega_p)$ of a neighbourhood of $W^u(\Omega_p)$ (excluding $W^u(\Omega_p)$) with leaves

$$\mathfrak{L}_x(\Omega_p) = \bigcup_{t\geq 0} (X_\mu)_t \mathfrak{F}^u_x(\Omega_p), \qquad x \in D^s(\Omega_p).$$

This induces a foliation $\mathfrak{S}(\Omega_p)$ of $\Sigma - (W^u(\Omega_p) \cap \Sigma)$ with spiral-like leaves

$$\mathfrak{S}_x(\Omega_p) = \mathfrak{L}_x \cap \Sigma, \qquad x \in \Sigma,$$

see figure 3. Take coordinates (x_1, x_2) on Σ in which $\mathcal{U} = \{x_2 = 0\}$ and $\gamma \cap \Sigma = (0,0)$. Parametrize the spirals $\mathfrak{S}_x(\Omega_p)$ by $(r,t,\mu) \mapsto f_\mu(r,t)$, where for each r an $\eta \subset \Omega$ exists so that $\{(f_\mu(r,t),\mu),\ \ \mu \in I, t \in [0,\infty)\} \subset \bigcup_\mu (W^u(\eta),\mu) \subset M \times I$. We claim that f_μ can be chosen with the following properties.

$$(t,\mu) \ \ \mapsto \ \ f_\mu(r,t) \text{ is smooth,} \tag{29}$$

$$
\begin{aligned}
(r,t,\mu) &\mapsto \left(\frac{\partial}{\partial t}\right)^i f_\mu(r,t) \text{ is continuous for all } i, && (30)\\
\frac{\partial}{\partial t}\theta(f_\mu(r,t)) &\neq 0, && (31)\\
& \text{where } \theta(x) \text{ is the angle of the vector } x \text{ with the positive } x_1\text{-axis.}\\
\lim_{t\to\infty}\kappa_\mu(r,t) &\to \infty, && (32)\\
& \text{where } \kappa_\mu(r,t) \text{ is the curvature of } \mathfrak{S}_x(\Omega_p) \text{ at } x=f_\mu(r,t).\\
\frac{\partial}{\partial \mu} f_0(r,t) &\neq 0. && (33)
\end{aligned}
$$

The verification of these properties is postponed to lemma 3.2. Let $x_1 = c_\mu(x_2)$ be the curve at which $\frac{\partial}{\partial x_1} f_\mu(x_1,x_2) = 0$. By (30), (31), (32), c_μ is a continuous function. For $\{\tilde{X}_\mu\}$ we take a cross section $\tilde{\Sigma}$ near Σ and maps $\tilde{f}_\mu$ parametrizing the spirals $\mathfrak{S}(\tilde{\Omega}_p)$ possessing similar properties as f_μ. Let $\tilde{x}_1 = \tilde{c}_\mu(\tilde{x}_2)$ be the continuous curve at which $\frac{\partial}{\partial \tilde{x}_1}\tilde{f}_\mu(\tilde{x}_1,\tilde{x}_2) = 0$. By (33) we can define a homeomorphism $\eta : I \to I$ so that if $c_\mu(0) \subset W^u(\zeta)$ for some orbit $\zeta \subset \Omega$, then $\tilde{c}_{\eta(\mu)}(0) \subset W^u(g(\zeta))$ –where g is a topological equivalence between Ω and $\tilde{\Omega}$–.

Finally an equivalence h_μ between X_μ and $\tilde{X}_{\eta(\mu)}$ must be constructed. Inductively, we define h_μ on $\bigcup_{j\neq q,r} W^s(\Omega_j)$ just as in theorem 2.6. The definition of h_μ on $D^s(\Omega_q)$ proceeds as in the previous section; near $\gamma \cap D^s(\Omega_q)$ h_μ is defined by the demand that h_μ maps $\mathfrak{F}^u(\Omega_p)$ to $\mathfrak{F}^u(\tilde{\Omega}_p)$ (we know h_μ on the space $W^s_{loc}(\Omega_p)$ of leaves of $\mathfrak{F}^u(\Omega_p)$) and $D^s(\Omega_q)$ into $D^s(\tilde{\Omega}_q)$. Having defined h_μ on $D^s(\Omega_q)$, extend h_μ to $W^s(\Omega_q)$ by conjugation. To define h_μ on $D^s(\Omega_r)$, we proceed as follows. Extend h_μ as defined on $D^s(\Omega_q)$ to a map i_μ on a fundamental neighbourhood extending $D^s(\Omega_q)$, mapping maps $\mathfrak{F}(\Omega_q)$ to $\mathfrak{F}(\tilde{\Omega}_q)$ and satisfying (27) with $j = q$. By conjugacy this defines a homeomorphism h_μ on a neighbourhood $\mathcal{U}$ of $W^u(\Omega_q) \cap D^s(\Omega_r)$ in $D^s(\Omega_r)$. The remaining part of the construction of h_μ goes as in the previous section. ■

Lemma 3.2 *(29),…,(33) above hold.*

Proof. Properties (29), (30), (33) follow since $\mathfrak{S}$ is a lamination, unstable manifolds vary smoothly in μ, and the heteroclinic bifurcation at $\mu = \mu_0$ unfolds generically. Take a coordinate chart (x,y,z) near Ω_p so that

$$
\begin{aligned}
\{z=0\} &\subset W^s(\Omega_p), && (34)\\
\{(x,y)=0\} &\subset W^u(\Omega_p). && (35)
\end{aligned}
$$

Multiplying $\{X_\mu\}$ with a smooth positive function, we may assume that in cylindric coordinates (r,θ,z),

$$
X_\mu = F(r,\theta,z)\frac{\partial}{\partial r} + \omega\frac{\partial}{\partial \theta} + G(r,\theta,z)\frac{\partial}{\partial z}. \quad (36)
$$

This implies (31). Consider the time $2\pi/\omega$ flow $f = (X_\mu)_{2\pi/\omega}$ of X_μ. Write $\lambda \pm i\omega$ and ν for the eigenvalues of $DX_\mu(\mathbf{0})$, so $\lambda < 0$, $\nu > 0$. We have

$$Df(\mathbf{0}) = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & N \end{pmatrix},$$

where $0 < \Lambda = e^{\lambda 2\pi/\omega} < 1$, $N = e^{\nu 2\pi/\omega} > 1$. A formal computation, which we leave to the reader, shows that we may take smooth coordinates near $\mathbf{0}$ in $\mathbb{R}^3$ so that in fact

$$f(x,y,z) \;=\; \begin{pmatrix} \Lambda x + xs(x,y,z) \\ \Lambda y + ys(x,y,z) \\ Nz + t(z) + zr(x,y,z) \end{pmatrix}, \tag{37}$$

with $s, r = \mathcal{O}(\|(x,y,z)\|^2)$, $t = \mathcal{O}(|z|^2)$. Lift f to $f^{(1)}$ on $\mathbb{R}^3 \times \mathfrak{G}_2(\mathbb{R}^3)$, where $\mathfrak{G}_2(\mathbb{R}^3)$ is the Grassmannian manifold of two-planes in $\mathbb{R}^3$, by

$$f^{(1)}(x,\alpha) \;=\; (f(x), Df(x)\alpha). \tag{38}$$

For any $\alpha \supset T_{\mathbf{0}}W^u(\mathbf{0})$,

$$f^{(1)}(\mathbf{0},\alpha) \;=\; (\mathbf{0},\alpha). \tag{39}$$

With $T_\alpha \mathfrak{G}_2(\mathbb{R}^3) = \mathcal{L}(\alpha,\alpha^-) \cong \mathcal{L}(\mathbb{R}^2,\mathbb{R})$,

$$\begin{aligned} Df^{(1)}(\mathbf{0},\alpha)|_{\{\mathbf{0}\}\times\mathcal{L}(\alpha,\alpha^-)}v &= (Df(\mathbf{0})|_{\alpha^-}) \circ v \circ (Df(\mathbf{0})|_\alpha)^{-1} \\ &= (\Lambda) \circ v \circ \begin{pmatrix} 1/\Lambda & 0 \\ 0 & 1/N \end{pmatrix}. \end{aligned}$$

Thus one sees that, in natural coordinates $T_{(\mathbf{0},\alpha)}(\mathbb{R}^3 \times \mathfrak{G}_2(\mathbb{R}^3)) \cong \mathbb{R}^3 \times \mathcal{L}(\mathbb{R}^2,\mathbb{R}) \cong \mathbb{R}^5$,

$$Df^{(1)}(\mathbf{0},\alpha) \;=\; \begin{pmatrix} Df(\mathbf{0}) & & \mathbf{0} \\ \ast & 1 & 0 \\ & 0 & \Lambda/N \end{pmatrix}$$

We claim that $Df^{(1)}(\mathbf{0},\alpha)$ is in fact diagonal (so $\ast = \mathbf{0}$). To see this we may take α equal to the x,z-plane. Then, writing $V = V_1u_1 + V_2u_2$, $V_1, V_2 \in \mathbb{R}$,

$$Df(x,y,z)\begin{pmatrix} u_1 \\ V \\ u_2 \end{pmatrix} = \begin{pmatrix} \Lambda + s + x\frac{\partial s}{\partial x} & x\frac{\partial s}{\partial y} & x\frac{\partial s}{\partial z} \\ y\frac{\partial s}{\partial x} & \Lambda + s + y\frac{\partial s}{\partial y} & y\frac{\partial s}{\partial z} \\ z\frac{\partial r}{\partial x} & z\frac{\partial r}{\partial y} & N + r + z\frac{\partial r}{\partial z} + \frac{\partial t}{\partial z} \end{pmatrix}\begin{pmatrix} u_1 \\ V \\ u_2 \end{pmatrix}.$$

By (37), the off-diagonal terms of $Df(\mathbf{0})$ are of second order; $Df^{(1)}(\mathbf{0},\alpha)$ is therefore diagonal.

Similarly, lift f to the induced map $f^{(2)}$ on the space of 2-jets of two dimensional manifolds. In a chart near $(x, \alpha) \in \mathbb{R}^3 \times \mathfrak{G}_2(\mathbb{R}^3)$, this is the space

$$\mathbb{R}^3 \times \mathcal{L}(\alpha, \alpha^-) \times \mathcal{L}^2_{sym}(\alpha, \alpha^-),$$

where $\mathcal{L}^2_{sym}(\alpha, \alpha^-) \subset \mathcal{L}(\alpha, \alpha^- \times \mathcal{L}(\alpha, \alpha^-))$ is the space of the symmetric quadratic forms. A computation as above shows that we can write

$$Df^{(2)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = \begin{pmatrix} Df^{(1)}(\mathbf{0}) & \mathbf{0} \\ * & \begin{matrix} \Lambda/N^2 & 0 & 0 & 0 \\ 0 & 1/N & 0 & 0 \\ 0 & 0 & 1/N & 0 \\ 0 & 0 & 0 & 1/\Lambda \end{matrix} \end{pmatrix}.$$

Note $0 < \Lambda/N^2, 1/N < 1$ and $1 < 1/\Lambda$. The following can now easily be deduced (see [DFN,1984]): the principal curvatures of $\mathfrak{L}_x(\Omega_p)$ at a point x near $W^u(\Omega_p)$ consist of one large and one small positive number, the principal directions are almost perpendicular. The principal direction corresponding to the larger principal curvature is almost perpendicular to the z-axis, the principal direction corresponding to the smaller principal curvature is almost parallel to the z-axis. (32) follows. ■

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