Switching homoclinic networks

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November 27, 2009

Abstract

A heteroclinic network for an equivariant ordinary differential equation is called switching if each sequence of heteroclinic trajectories in it is shadowed by a nearby trajectory. It is called forward switching if this holds for positive trajectories. We provide an elementary example of a switching robust homoclinic network and a related example of a forward switching asymptotically stable robust homoclinic network. The examples are for five dimensional equivariant ordinary differential equations.

1 Introduction

In symmetric (i.e. equivariant) ordinary differential equations heteroclinic cycles can, in contrast to differential equations without symmetry, occur robustly (see [8] and references therein). A heteroclinic network, which we define as a connected component of the group orbit of a heteroclinic cycle, can thus be a robust invariant set. This brings the issue of determining the dynamics near a robust heteroclinic network. A trajectory that stays near a heteroclinic network will follow a sequence of heteroclinic trajectories, but the order in which this is possible may be restricted. Also, a heteroclinic network can be asymptotically stable, so that nearby forward trajectories are attracted to it [13]. Such a forward trajectory will then follow a sequence of heteroclinic trajectories, but again only selected such sequences may occur.

To make the discussion more precise, we introduce setting and notation. Consider an ordinary differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{1}$$

on a Euclidean space $\mathbb{R}^n$. A heteroclinic cycle for (1) consists of finitely many disjunct equilibria p_j, $j = 1, \ldots, \ell$, and heteroclinic trajectories $\gamma_j(t)$, $j = 1, \ldots, \ell$, so that

$$\lim_{t\to\infty} \gamma_j(t) = p_{j+1} = \lim_{t\to-\infty} \gamma_{j+1}(t), \qquad j = 1 \ldots, \ell$$

with the understanding that $\gamma_{\ell+1} = \gamma_1$ and $p_{\ell+1} = p_1$. In the context of this paper all equilibria are assumed to be saddle points.

We consider ordinary differential equations (1) that are equivariant under the representation of a discrete group G, i.e.

$$gf(\mathbf{x}) = f(g\mathbf{x}), \qquad \forall g \in G.$$

Here g is identified with its representation as linear operator on $\mathbb{R}^n$. Note that the fixed point space $\operatorname{Fix} H$ of a subgroup $H \subset G$, defined by

$$\operatorname{Fix} H = \{\mathbf{x} \in \mathbb{R}^n \mid g\mathbf{x} = \mathbf{x} \text{ for } g \in H\},$$

is invariant under the flow of f. Restricted to a fixed point space the stable and unstable manifolds of hyperbolic equilibria may intersect transversally. This is in particular true if a one-dimensional unstable manifold of an equibrium connects inside a fixed point space to an equilibrium that is a sink inside that fixed point space. This gives a robust heteroclinic trajectory, i.e. a small perturbation of the differential equation admits a nearby heteroclinic trajectory. A robust heteroclinic cycle consists of robust heteroclinic trajectories. Define a heteroclinic network as a connected component of the G-image of a heteroclinic cycle. A homoclinic network is a heteroclinic network in which all heteroclinic trajectories as well as all equilibria are symmetry related.

The simplest examples of homoclinic networks are found for equivariant differential equations in $\mathbb{R}^3$. If a homoclinic network in $\mathbb{R}^3$ is asymptotically stable, a trajectory that converges to it will actually converge to a heteroclinic cycle contained in it. We point out some of the examples that make clear that trajectories near a robust heteroclinic network may also form more intricate patterns. Kirk and Silber [12] study heteroclinic networks in $\mathbb{Z}_2^4$-equivariant differential equations in $\mathbb{R}^4$ containing two heteroclinic cycles that are not symmetry related but have a heteroclinic trajectory in common. Trajectories starting near one heteroclinic cycle may eventually converge to the other heteroclinic cycle. The heteroclinic network as a whole is not asymptotically stable but essentially asymptotically stable and thus contains the omega limit sets for large proportions of nearby points. An example of a heteroclinic network in $\mathbb{Z}_4 \ltimes \mathbb{Z}_2^3$-equivariant differential equations in $\mathbb{R}^5$ where trajectories eventually converge to a homoclinic network contained in it is described in [7]; the heteroclinic network is created in a transverse bifurcation from an asymptotically stable homoclinic network. Postlethwaite and Dawes [14] considered heteroclinic networks in $\mathbb{Z}_6 \ltimes \mathbb{Z}_2^6$-equivariant differential equations in $\mathbb{R}^6$ for which the equilibria lie on a single group orbit. They established the existence of trajectories that follow different heteroclinic trajectories in an irregular order, while converging to the heteroclinic network.

The heteroclinic network studied by Aguiar, Castro and Labouriau in [1] contains suspended horseshoes in each tubular neighborhood of it. For their example, a symmetry reduction yields a quotient network with equilibria of different index and with complex eigenvalues reminiscent of heteroclinic cycles studied by Bykov [5]. There are suspended horseshoes in each neighborhood of the quotient heteroclinic cycle. We note that Worfolk [15] studies an inclination-flip bifurcation of a homoclinic cycle in which suspended horseshoes are created near the cycle. A recently studied example of complicated dynamics near a heteroclinic network is in [2] by Aguiar, Labouriau and Rodrigues; here the heteroclinic network connects not only equilibria but it contains equilibria as well as a periodic trajectory. Other constructions leading to switching dynamics, including the effect of noise and symmetry breaking, are in [3, 4, 11].

Trajectories that are near a heteroclinic network are near sequences of heteroclinic trajectories inside the heteroclinic network. To describe this we recall the following notions. The connectivity matrix $C = (c_{ij})$ of a heteroclinic network with heteroclinic trajectories γ_i, $i = 1, \dots, k$, is a 0-1 matrix, that is $c_{ij} \in \{0, 1\}$, where $c_{ij} = 1$ if and only if the endpoint (the ω-limit $\omega(\gamma_i)$) of the heteroclinic connection γ_i is equal to the starting point (the α-limit $\alpha(\gamma_j)$) of the heteroclinic connection γ_j. Further we introduce notation for topological Markov chains. Let

$$\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$$

denote the set of double infinite sequences $\kappa : \mathbb{Z} \to \{1, \dots, k\}$, $i \mapsto \kappa_i$, equipped with the product topology. Similarly we define

$$\Sigma_k^+ = \{1, \dots, k\}^{\mathbb{N}}.$$

Let $A = (a_{ij})_{i,j \in \{1,\dots,k\}}$ be a 0-1 matrix. By Σ_A we denote the topological Markov chain defined by A,

$$\Sigma_A = \{\kappa \in \Sigma_k \mid a_{\kappa_i \kappa_{i+1}} = 1\},$$

and correspondingly we define

$$\Sigma_A^+ = \{\kappa \in \Sigma_k^+ \mid a_{\kappa_i \kappa_{i+1}} = 1\}.$$

Let $\gamma_1, \dots, \gamma_k$ be heteroclinic trajectories that define a heteroclinic network Γ with connectivity matrix C, and let $\mathcal{U}$ be a tubular neighborhood of Γ. Take cross sections S_i transverse to γ_i and write Π for the first return map on the collection of cross sections $S := \cup_{j=1}^k S_j$. Let κ be a symbolic sequence in Σ_C. We call a trajectory $\mathcal{O}$ of (1) a *realization of* κ *in* $\mathcal{U}$, if $\mathcal{O} \subset \mathcal{U}$ and if there is an $x_\kappa \in \mathcal{O}$ such that $\Pi^i(x_\kappa) \in S_{\kappa_i}$, $i \in \mathbb{Z}$. The point x_κ we call *starting point* of the realization. In other words, a realization of a sequence κ is a trajectory that follows the heteroclinic trajectories γ_i in the order prescribed by κ. For $\kappa \in \Sigma_C^+$ there is a similar definition where the realization is a forward trajectory.

Following [1] we adopt the following definition.

Definition 1.1. *A heteroclinic network* Γ *with connectivity matrix* C *is switching (forward switching) if for each sequence* $\kappa \in \Sigma_C$ *(*$\kappa \in \Sigma_C^+$*) and each tubular neighborhood* $\mathcal{U}$ *of* Γ*, there exists a realization of* κ *in* $\mathcal{U}$*.*

The heteroclinic network studied by Aguiar, Castro and Labouriau in [1] is switching. Our paper contains the following results.

- We provide an elementary example of an asymptotically stable robust homoclinic network that is forward switching (Theorem 2.1).
- We provide an elementary example of a robust homoclinic network that contains suspended horseshoes in each neighborhood of it and is switching (Theorem 2.2).

2 Examples of switching homoclinic networks

The examples we give of switching and forward switching robust homoclinic networks will be for ordinary differential equations on $\mathbb{R}^5$ that are equivariant with respect to a representation of a group $G = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$. One can easily provide variants of the construction. Define the representation of the group G on $\mathbb{R}^5$ as follows: $\mathbb{Z}_2 = \mathbb{Z}_2(g_0)$ is acting by

$$g_0(x, y_1, y_2, z_1, z_2) = (-x, z_1, z_2, y_1, y_2)$$

and $\mathbb{Z}_2^2 = \mathbb{Z}_2^2(g_1, g_2)$ is acting by

$$\begin{aligned} g_1(x, y_1, y_2, z_1, z_2) &= (x, -y_1, -y_2, z_1, z_2), \\ g_2(x, y_1, y_2, z_1, z_2) &= (x, y_1, y_2, -z_1, -z_2). \end{aligned}$$

Consider an ordinary differential equation (1) on $\mathbb{R}^5$ (here $\mathbf{x} = (x, y_1, y_2, z_1, z_2)$) that is G-equivariant.

Write P_i for the three dimensional fixed point space of $\langle g_i \rangle$, the subgroup generated by g_i, $i = 1, 2$. Thus P_1 is the (x, z_1, z_2)-space and P_2 is the (x, y_1, y_2)-space. Assume (1) has equilibria $p, g_0 p = -p$ on the x-axis with isotropy $\mathbb{Z}_2^2(g_1, g_2)$. The fixed point spaces P_1, P_2, $V^r = P_1 \cap P_2$ induce an invariant splitting for $Df(p)$. We make the following assumptions on the spectrum of $Df(p)$:

1. a real eigenvalue $\lambda^r < 0$ with eigenspace V^r (the x-axis)
2. a pair of complex conjugate eigenvalues $\lambda^c, \overline{\lambda^c}$, $\operatorname{Re} \lambda^c < 0$, with eigenspace $P_2 \ominus V^r$ (the (y_1, y_2)-plane),
3. a real eigenvalue $\lambda^e > 0$ and a real eigenvalue $\lambda^d < 0$ with eigenspaces contained in $P_1 \ominus V^r$ (the (z_1, z_2)-plane).

Assume further (1) has a heteroclinic trajectory γ connecting p to $g_0 p$, with isotropy $\langle g_1 \rangle$ and thus contained in P_1. Recall that the isotropy of a point is the subgroup leaving that point invariant. Along a trajectory of (1) the isotropy does not change.

Write Γ for the homoclinic network that is the closure of $G\gamma$: Γ consists of the two equilibria p and $g_0 p$ and four heteroclinic trajectories: $\gamma_1 := \gamma$ and $\gamma_2 := g_2 \gamma$ in P_1 connecting p to $g_0 p$, and their g_0-images $\gamma_3 := g_0 \gamma$ and $\gamma_4 := g_0 g_2 \gamma$, located in P_2 connecting $g_0 p$ to p. Figure 1 gives an impression, but note that the fixed point spaces are three dimensional.

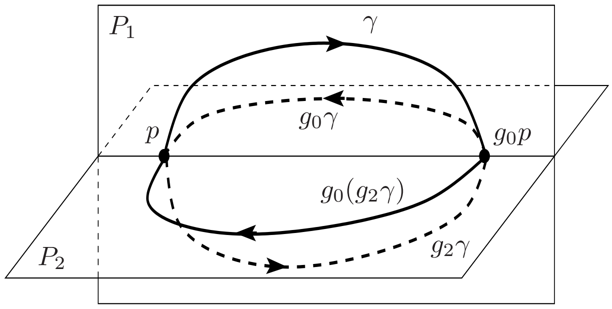

Figure 1: *The homoclinic network Γ contains two equilibria and four heteroclinic trajectories. The fixed point spaces are three dimensional, and eigenvalue conditions in Theorems 2.1 and 2.2 imply that the trajectories spiral towards the equilibria.*

With this notation the connectivity matrix C of Γ reads

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Factorizing with respect to g_0, i.e. identifying points x and $g_0 x$, means that in the factor space $\mathbb{R}^5/g_0$ trajectories and their g_0-image are identified. The reduced space $\mathbb{R}^5/g_0$ is isomorphic to the set $\{x \geq 0\}$ where the points $(0, y_1, y_2, z_1, z_2)$ and $(0, z_1, z_2, y_1, y_2)$ are identified. Hence the homoclinic network Γ (which is disjoint from $\mathrm{Fix}\, g_0$) appears in $\mathbb{R}^5/g_0$ as the closure of two homoclinic loops to an equilibrium p^r which we will refer to as the reduced homoclinic network Γ^r. The homoclinic loops of Γ^r we denote by γ_1^r and γ_2^r, where γ_1^r is the identification of γ_1 and γ_3, and γ_2^r represents γ_2 and γ_4. Note that $\gamma_2^r = g_2\gamma_1^r$. See Figure 2 for an impression.

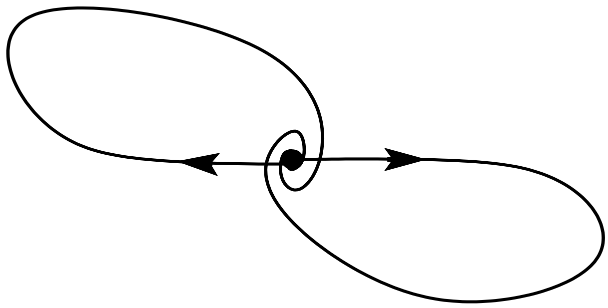

Figure 2: *The reduced homoclinic network Γ^r consists of two saddle-focus homoclinic trajectories.*

The connectivity matrix C^r of Γ^r reads

$$C^r = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Theorem 2.1. *Consider an ordinary differential equation (1) on $\mathbb{R}^5$ that is G-equivariant and possesses a homoclinic network Γ as above. Assume eigenvalue conditions $\lambda^d, \lambda^r < \mathrm{Re}\,\lambda^c < -\lambda^e$. Then Γ is a robust asymptotically stable homoclinic cycle which is forward switching.*

Proof. Noting that Γ contains two equilibria and the heteroclinic trajectories in Γ connect an equilibrium to the other equilibrium, it is clear that Γ is forward switching if and only if Γ^r is forward switching.

Take a cross section S_1 transverse to γ_1 that is symmetric under $\langle g_1 \rangle$: $g_1 S_1 = S_1$. Then $S = GS_1$ consists of four cross sections S_i, $1 \leq i \leq 4$, with S_i transverse to γ_i. We divide out the action of g_0, so that the homoclinic network Γ reduces to Γ^r, consisting of two saddle-focus homoclinic loops γ_1^r, γ_2^r. Write S_1^r, S_2^r for the two cross sections in $S^r = S/g_0$ that are transverse to γ_1^r and γ_2^r respectively. By Π_r we denote the first return map $S^r \to S^r$.

Write $W^{\mathrm{s}}(p^r)$ for the stable manifold of p^r and $W^{\mathrm{s}}_{\mathrm{loc}}(p^r)$ for a local stable manifold (extended far enough along the heteroclinic trajectories to have an intersection with S_1^r and S_2^r). It is standard to compute that for a point $\mathbf{x} \in S^r$ at a distance d from $W^{\mathrm{s}}_{\mathrm{loc}}(p^r) \cap S^r$,

$$C_1 d^{-\mathrm{Re}\,\lambda^c/\lambda^e} \leq |\Pi_r(\mathbf{x}) - \Gamma^r \cap S^r| \leq C_2 d^{-\mathrm{Re}\,\lambda^c/\lambda^e} \tag{2}$$

for some positive constants $C_1 < C_2$. The reduced homoclinic network Γ^r is an attractor as $-\mathrm{Re}\,\lambda^c/\lambda^e > 1$; each small enough ball around $\Gamma^r \cap S^r$ is mapped into itself by the first return map Π_r. With Γ^r, the homoclinic network Γ is asymptotically stable.

It may be helpful to first consider the geometry for differential equations with two saddle-focus homoclinic loops in $\mathbb{R}^3$. Note that $S_j^r - W^{\mathrm{s}}_{\mathrm{loc}}(p^r)$ consists of two components; write $S_j^r - W^{\mathrm{s}}_{\mathrm{loc}}(p^r) = S_j^{r,1} \cup S_j^{r,2}$ so that $\Pi_r(S_j^{r,i}) \subset S_i^r$. Further, S_j^r is a union of strips $\mathcal{S}_k$ that are mapped by Π_r to arcs intersecting the local stable manifold of p^r in S_i^r in two connected components (see [9] for a computation with locally linear differential equations). As indicated in Figure 3, $\Pi_r(\mathcal{S}_k)$ thus intersects similar strips $\mathcal{S}_l, \mathcal{S}_m$ in both $S_j^{r,1}$ and $S_j^{r,2}$ closer to the local stable manifold. For each $\kappa \in \Sigma_C^+$, one can therefore construct decreasing compact subsets

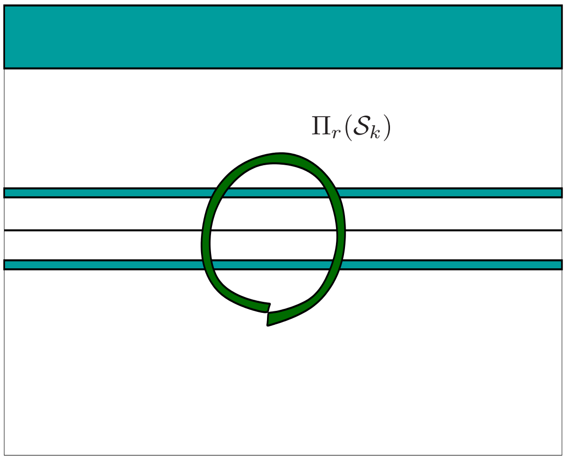

Figure 3: *The return map Π_r maps strips into spirals. Under eigenvalue conditions of Theorem 2.1 iterates of points under Π_r converge to $\Gamma^r \cap S^r$; the image $\Pi_r(\mathcal{S}_k)$ of a strip $\mathcal{S}_k$ is much closer to $\Gamma^r \cap S^r$.*

$D_\kappa^n \subset S_{\kappa_0}^r$ on which Π_r^j, $1 \leq j \leq n$, maps into $S_{\kappa_j}^r$. A point in $\cap_{n \geq 0} D_\kappa^n$ is an initial point of a realization of κ.

To see that this reasoning generalizes to our five dimensional setting we consider these strips as being foliated by curves which are part of curves intersecting $W^s_{loc}(p^r) \cap S^r$ transversally in S^r. Each such curve will be mapped by Π_r into a spiral emanating from $\Gamma^r \cap S^r$. Finally, the Π_r-images of the strips are composed of the corresponding parts of the Π_r-images of the addressed curves.

To make these arguments more precise, we consider certain invariant manifolds, for which we now introduce notation. Write $W^{\mathrm{s}}(p)$ for the codimension one stable manifold of p and $W^{\mathrm{u}}(p)$ for its one dimensional unstable manifold. The conditions $\lambda^d, \lambda^r < \mathrm{Re}\,\lambda^c$ express that $P_2 \ominus V^r$ are leading stable directions. There exist non-unique three dimensional invariant manifolds (leading stable/unstable manifolds) $W^{\mathrm{ls,u}}(p)$ that contain $W^{\mathrm{u}}(p)$ and are tangent to the direct sum of the leading stable directions and unstable directions. The tangent bundle of $W^{\mathrm{ls,u}}(p)$ along $W^{\mathrm{u}}(p)$ is unique. By invariance of P_1, $W^{\mathrm{ls,u}}(p)$ is transverse to P_1. It is therefore also transverse to $W^{\mathrm{s}}(g_0 p)$:

$$W^{\mathrm{ls,u}}(p) \pitchfork_\gamma W^{\mathrm{s}}(g_0 p). \tag{3}$$

This condition is reminiscent of a non-inclination flip condition. For the reduced differential equation we get invariant manifolds as well and (3) stipulates

$$W^{\mathrm{ls,u}}(p^r) \pitchfork_{\gamma_1^r \cup \gamma_2^r} W^{\mathrm{s}}(p^r), \tag{4}$$

the usual non-inclination flip condition for homoclinic loops (see e.g. [10]).

Since the leading stable eigenvalues of $Df(p^r)$ are complex conjugate we have that a trajectory in the stable manifold of p^r, away from the strong stable manifold, will spiral towards p^r (such a trajectory lies inside a two dimensional leading stable manifold). Take a curve in S^r that intersects $W^{\mathrm{s}}_{\mathrm{loc}}(p^r) \cap S^r$ transversally away from the strong stable manifold of p^r. Let C be a component of this curve in $S^r - W^{\mathrm{s}}_{\mathrm{loc}}(p^r)$ so that C ends at $W^{\mathrm{s}}_{\mathrm{loc}}(p^r)$. By the λ-lemma, the positive flow of C converges to $W^{\mathrm{u}}(p^r)$. Using [6] the positive flow of C traces out a spiral in S^r, i.e. C is mapped to a spiral in S^r by Π_r.

As Γ^r is not contained in the strong stable manifold of p^r, a small neighborhood U of $\Gamma^r \cap S^r$ will be away from the strong stable manifold of p^r. Write V for a component of $U - W^{\mathrm{s}}_{\mathrm{loc}}(p^r)$. Take of foliation $\mathcal{F}$ of U consisting of two dimensional surfaces so that each surface in it is the intersection of some leading stable/unstable manifold $W^{\mathrm{ls,u}}(p^r)$ with S^r. This defines a foliation with surfaces of V, i.e. to one side of $W^{\mathrm{s}}_{\mathrm{loc}}(p^r) \cap S^r$, by restriction. We can refine the foliation $\mathcal{F}$ to a foliation $\mathcal{G}$ of curves transverse to $W^{\mathrm{s}}_{\mathrm{loc}}(p^r)$. The image under Π_r of any curve C in $\mathcal{G}$ is a spiraling curve in S^r. As C lies inside a leaf of $\mathcal{F}$, $\Pi_r(C)$ lies inside a leading stable/unstable manifold. The image $\Pi_r(V)$ is a thickened spiral (a union of spiraling curves). Now by (2) and (4), the two dimensional picture sketched above applies to our setting as well: there is a sequence of strips, i.e. subsets of S^r whose distance to the local stable manifold lies between two positive numbers, whose image under Π_r intersects strips closer to the local stable manifold and on both sides of it. □

Theorem 2.2. *Consider an ordinary differential equation* (1) *on* $\mathbb{R}^5$ *that is G-equivariant and possesses a homoclinic network* Γ *as above. Assume* $\lambda^d, \lambda^r < -\lambda^e < Re\,\lambda^c$. *Then* Γ *is a robust homoclinic cycle which is switching.*

Proof. The proof runs along the same lines as the one of the above theorem. Under the present assumptions the reduced system has a suspended horseshoe in each neighborhood $\mathcal{U}^r$ of the reduced homoclinic network Γ^r [6]. □

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