Homoclinic and Heteroclinic Bifurcations in Vector Fields

Ale Jan Homburg
Korteweg–de Vries Institute for Mathematics
University of Amsterdam
1018 TV Amsterdam, The Netherlands

Björn Sandstede
Division of Applied Mathematics
Brown University
Providence, RI 02906, USA

June 17, 2010

Abstract

An overview of homoclinic and heteroclinic bifurcation theory for autonomous vector fields is given. Specifically, homoclinic and heteroclinic bifurcations of codimension one and two in generic, equivariant, reversible, and conservative systems are reviewed, and results pertaining to the existence of multi-round homoclinic and periodic orbits and of complicated dynamics such as suspended horseshoes and attractors are stated. Bifurcations of homoclinic orbits from equilibria in local bifurcations are also considered. The main analytic and geometric techniques such as Lin's method, Shil'nikov variables and homoclinic center manifolds for analyzing these bifurcations are discussed. Finally, a few related topics, such as topological moduli, numerical algorithms, variational methods, and extensions to singularly perturbed and infinite-dimensional systems, are reviewed briefly.

Contents

1 Introduction

Our goal in this paper is to review the existing literature on homoclinic and heteroclinic bifurcation theory for flows. More specifically, we shall focus on bifurcations from homoclinic and heteroclinic orbits between equilibria in autonomous ordinary differential equations (ODEs)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R}^m, \ t \in \mathbb{R}. \tag{1.1}$$

Throughout the entire survey, we shall assume that the nonlinearity f is sufficiently smooth for the results to hold (the precise requirements can be found in the cited references). We write $\mathcal{X}(\mathbb{R}^n)$ for the space of ODEs on $\mathbb{R}^n$ endowed with the C^∞ Whitney topology.

Equilibria p of (1.1) are time-independent solutions that therefore satisfy $f(p, \mu) = 0$. We say that a solution $h(t)$ of (1.1) is a heteroclinic orbit if $h(t) \to p_\pm$ as $t \to \pm\infty$ for equilibria $p_\pm \in \mathbb{R}^n$. If $p_- = p_+$, we say that $h(t)$ is a homoclinic orbit (assuming tacitly that $h(t)$ is not the equilibrium solution itself). We will also consider heteroclinic cycles which, by definition, consist of several heteroclinic orbits $h_j(t)$ labelled by the index $j = 1, \ldots, \ell$ so that

$$\lim_{t \to \infty} h_j(t) = p_{j+1} = \lim_{t \to -\infty} h_{j+1}(t), \qquad j = 1 \ldots, \ell$$

with the understanding that $h_{\ell+1} := h_1$ and $p_{\ell+1} = p_1$. Illustrations of homoclinic and heteroclinic orbits can be found in Figure 1.1.

Homoclinic and heteroclinic orbits play an important role in applications. For instance, we may be interested in modelling action potentials in nerve axons by an ODE of the form (1.1): in this case, we can think of u as representing the electric potential and certain ion concentrations in the nerve axon, with t being physical time. The rest state of the axon determines a natural equilibrium of (1.1), and action potentials in the axon correspond then to homoclinic orbits to this equilibrium. Heteroclinic orbits typically arise when a system can cycle through several different states. For instance, in certain Rayleigh–Bénard convection experiments, roll patterns may arise that can orient themselves at angles of 0, 120 or 240 degrees: as time progresses, the roll pattern cycles through this set of angles, but stays at each angle for a long time, followed by a fast transition to the next angle. Thus, in an appropriate ODE model with three equilibria corresponding to the rolls oriented at angles of 0, 120 and 240 degrees, heteroclinic orbits correspond to transitions from one equilibrium to another.

In the above examples, the independent variable t corresponds to physical time, whence we refer to such applications as describing *temporal dynamics*. Another class of applications, referred to as *spatial dynamics*, are problems where t represents a spatial direction. An important example in this respect are travelling-wave solutions of partial differential equations (PDEs) on unbounded domains. Consider, for instance, the reaction-diffusion system

$$U_t = D U_{xx} + F(u), \qquad U \in \mathbb{R}^N, \ x \in \mathbb{R}, \tag{1.2}$$

where t and x represent physical time and space, respectively. A travelling wave of (1.2) is a solution of the form $U(x, t) = U_*(x - ct)$ corresponding to a fixed profile U_* that travels to the right (for $c > 0$) or the left (for $c < 0$) as a function of time t. Upon substituting the ansatz $U(x, t) = U_*(x - ct)$ into (1.1) and using $\xi = x - ct$ as the new independent variable, we find that $(U, V) = (U_*, \partial_\xi U_*)$ must satisfy the ODE

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V \\ -D^{-1}(cV + F(U)) \end{pmatrix} \tag{1.3}$$
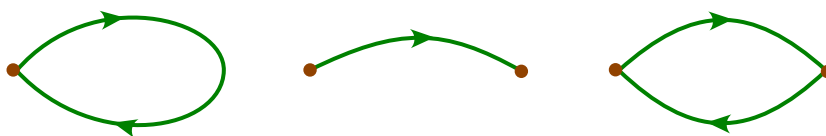

Figure 1.1: *The panels contain a homoclinic orbit, which connects an equilibrium to itself [left], a heteroclinic orbit that connects two different equilibria [center], and a heteroclinic cycle with two connecting orbits [right].*

which is of the form (1.1) with μ representing the wave speed c. Any solution of (1.3) gives a travelling wave with speed c of the PDE (1.2). In particular, homoclinic orbits of (1.3) correspond to pulses, i.e. to travelling waves that are localized in space, while heteroclinic orbits correspond to fronts which are waves that become constant as $x \to \pm\infty$. Pulses and fronts are of particular importance in applications. We remark that, once their existence is established, it is of interest to determine whether these structures are stable with respect to the PDE dynamics associated with (1.2): we refer the reader to [344] for a survey of this topic.

With these motivating examples in mind, we now turn to a discussion of the relevant issues and questions that surround homoclinic and heteroclinic orbits. One such issue is, of course, to establish the existence of these orbits in the first place, for instance by analytical or numerical techniques. The topic we shall focus on in this survey paper, however, is the dynamics of (1.1) near given homoclinic or heteroclinic orbits, particularly under changes of a systems parameter μ. This includes the persistence of homoclinic and heteroclinic orbits under parameter variations but also, more importantly, the characterization of all recurrent orbits, that is, of all solutions that stay in a fixed tubular neighborhood of a given homoclinic orbit or heteroclinic cycle for all times. Particularly interesting recurrent orbits are N-homoclinic and N-periodic orbits: these are solutions that follow the original homoclinic orbit or heteroclinic cycle N-times before closing up. In other words, these solutions have winding number N when considered as loops inside a tubular cylindrical neighborhood of a homoclinic orbit or a heteroclinic cycle.

In the spirit of local bifurcation theory, we are then interested in identifying bifurcation scenarios at which the recurrent dynamics near a set of connecting orbits changes qualitatively. At such bifurcation points, N-homoclinic orbits may spin off or complicated dynamics set in. An important characteristic feature of bifurcation points is their codimension which determines how many parameters we need to adjust before being able to observe a given bifurcation scenario. The codimension typically depends strongly on whether the underlying ODE has any additional structure such as respecting a group of symmetries or being time-reversible.

These are roughly the questions and topics that we wish to review and discuss in this survey paper. In §2, we shall review geometric properties of homoclinic orbits and introduce various hypotheses that will be used throughout the remainder of the paper. Section 3 contains a discussion of the analytical and geometric techniques that have been developed to investigate connecting orbits. Several common phenomena that arise in many different bifurcation scenarios are summarized in §4. The core of this paper is §5 where we give a catalogue of homoclinic and heteroclinic bifurcations for vector fields. We conclude with a brief discussion of related topics in §6.

References to original publications as well as books and review papers relevant to the topics of this article will be included in the sections below. Here, among recent books devoted to global bifurcation theory, we single out [374, 375] by LP Shil'nikov, AL Shil'nikov, Turaev and Chua and [202] by Il'yashenko and Li. Further recent books containing sections on global bifurcation theory include [5, 242] and, for equivariant systems, [87].

2 Homoclinic and heteroclinic orbits, and their geometry

This section serves as an introduction to homoclinic and heteroclinic orbits and to set up many of the hypotheses and assumptions that we shall refer to later when discussing homoclinic and heteroclinic bifurcations. Specifically, we will use this section to illustrate various geometric notions and how they are encoded and reflected analytically. To set the scene, consider again the ODE

$$\dot{u} = f(u, \mu), \qquad (2.1)$$

where $u \in \mathbb{R}^n$ and $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$.

2.1 Homoclinic orbits to hyperbolic equilibria

Assume that $h(t)$ is a given homoclinic orbit of (2.1), say for $\mu = 0$, which converges to the equilibrium p as $t \to \pm\infty$. One key assumption that we will often impose is that the equilibrium p itself does not undergo a local bifurcation at $\mu = 0$. A sufficient condition is hyperbolicity of p which, by definition, means that the matrix $f_u(p, 0)$, which is obtained by linearizing the right-hand side of (2.1) with respect to u at $u = p$ for $\mu = 0$, has no eigenvalues on the imaginary axis. We summarize this assumption as follows:

Hypothesis 2.1 (Hyperbolicity). *The equilibrium p is hyperbolic at $\mu = 0$, that is, the linearization $f_u(p, 0)$ has no eigenvalues on the imaginary axis.*

For the remainder of this section, we assume that p is a hyperbolic equilibrium of (2.1) and refer the reader to §2.2 for material on the geometry of the flow near homoclinic orbits to nonhyperbolic equilibria. The stable and unstable manifolds of a hyperbolic equilibrium p are defined by

$$
\begin{aligned}
W^{\mathrm{s}}(p, 0) &= \{u(0);\ u(t) \text{ satisfies (2.1) and } u(t) \to p \text{ as } t \to \infty\} \\
W^{\mathrm{u}}(p, 0) &= \{u(0);\ u(t) \text{ satisfies (2.1) and } u(t) \to p \text{ as } t \to -\infty\};
\end{aligned}
$$

see [394]. These sets turn out to be smooth immersed manifolds that are invariant under the flow, and the convergence towards p is, in fact, exponential in t.

If $h(t)$ is a homoclinic orbit to p for $\mu = 0$, then its entire orbit must lie in the intersection of stable and unstable manifolds of p so that $h(t) \in W^{\mathrm{s}}(p, 0) \cap W^{\mathrm{u}}(p, 0)$ for all t. In particular, the tangent spaces of stable and unstable manifolds evaluated at $u = h(t)$ intersect in an at least one-dimensional subspace that contains $\dot{h}(t)$. The analytical interpretation of the tangent spaces of stable and unstable manifolds is as follows. Consider the variational equation

$$
\dot{v} = f_u(h(t), 0)v \tag{2.2}
$$

obtained by linearizing (2.1) about $h(t)$. We then have

$$
\begin{aligned}
T_{h(t)}W^{\mathrm{s}}(p, 0) &= \{v(t);\ v(\cdot) \text{ satisfies (2.2) and } v(s) \to 0 \text{ as } s \to \infty\} \\
T_{h(t)}W^{\mathrm{u}}(p, 0) &= \{v(t);\ v(\cdot) \text{ satisfies (2.2) and } v(s) \to 0 \text{ as } s \to -\infty\}.
\end{aligned}
$$

In addition, we may consider the adjoint variational equation defined by

$$
\dot{w} = -f_u(h(t), 0)^* w \tag{2.3}
$$

where A^* denotes the transpose of a matrix A. If $\Phi(t, s)$ denotes the evolution of (2.2), then $\Phi(s, t)^*$ is the solution operator of (2.3) (just differentiate the equation $\Phi(t, s)\Phi(s, t) = \mathrm{id}$). Using the identity

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle v(t), w(t)\rangle = 0
$$

which holds for any solutions $v(t)$ of (2.2) and $w(t)$ of (2.3), we conclude that

$$
\begin{aligned}
\left[T_{h(t)}W^{\mathrm{s}}(p, 0)\right]^\perp &= \{w(t);\ w(\cdot) \text{ satisfies (2.3) and } w(s) \to 0 \text{ as } s \to \infty\} \\
\left[T_{h(t)}W^{\mathrm{u}}(p, 0)\right]^\perp &= \{w(t);\ w(\cdot) \text{ satisfies (2.3) and } w(s) \to 0 \text{ as } s \to -\infty\}.
\end{aligned}
$$

In particular, we obtain

$$
T_{h(t)}W^{\mathrm{s}}(p, 0) \cap T_{h(t)}W^{\mathrm{u}}(p, 0) = \{v(t);\ v(\cdot) \text{ satisfies (2.2) and } v(s) \to 0 \text{ as } |s| \to \infty\} \tag{2.4}
$$

$$
\left[T_{h(t)}W^{\mathrm{s}}(p, 0) + T_{h(t)}W^{\mathrm{u}}(p, 0)\right]^\perp = \{w(t);\ w(\cdot) \text{ satisfies (2.3) and } w(s) \to 0 \text{ as } |s| \to \infty\}, \tag{2.5}
$$

and both spaces have the same dimension. We shall assume that these spaces are one-dimensional. In particular, $\dot{h}(t)$ will span the intersection (2.4) of the stable and unstable tangent spaces, and we choose a solution $\psi(t)$ of (2.3) that spans the complement (2.5) of their sum.
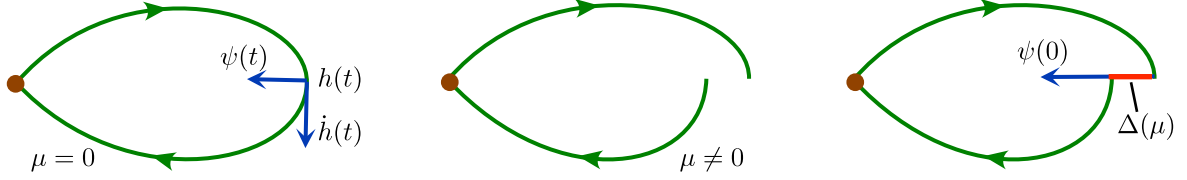
Figure 2.1: *Varying a parameter μ may result in nonintersecting stable and unstable manifolds, so that the homoclinic orbit ceases to exist: the distance between the nonintersecting manifolds is measured by the quantity $\Delta(\mu)$ defined in (2.6).*

Next, we discuss the persistence of the homoclinic orbit $h(t)$ if we change the parameter μ near $\mu = 0$. Firstly, since we assumed that p is hyperbolic, there will be a unique equilibrium $p(\mu)$ near p for all μ near zero, and $p(\mu)$ and its stable and unstable manifolds will depend smoothly on μ. However, $W^{\mathrm{s}}(p(\mu), \mu)$ and $W^{\mathrm{u}}(p(\mu), \mu)$ may no longer intersect; see Figure 2.1. To measure the distance between stable and unstable manifolds, we seek solutions near the homoclinic orbit in these manifolds that are closest to each other:

Lemma 2.1. *Assume that Hypothesis 2.1 is met and that $T_{h(t)}W^{\mathrm{s}}(p, 0) \cap T_{h(t)}W^{\mathrm{u}}(p, 0) = \mathbb{R}\dot{h}(t)$. For each μ close to zero, there are unique orbits $h^{\mathrm{s}}(\cdot; \mu) \in W^{\mathrm{s}}(p(\mu), \mu)$ and $h^{\mathrm{u}}(\cdot; \mu) \in W^{\mathrm{u}}(p(\mu), \mu)$ of (2.1) with $h^{\mathrm{s}}(0; 0) = h^{\mathrm{u}}(0; 0) = h(0)$ so that*

$$h^{\mathrm{u}}(0; \mu) - h^{\mathrm{s}}(0; \mu) \in \mathbb{R}\psi(0) \qquad \forall \mu.$$

The functions $h^{\mathrm{s}}(\cdot; \mu)$ and $h^{\mathrm{u}}(\cdot; \mu)$, considered with values in $C^0(\mathbb{R}^+, \mathbb{R}^n)$ and $C^0(\mathbb{R}^-, \mathbb{R}^n)$ respectively, are smooth in μ.

Proof. Let Σ be a small open ball centered at $h(0)$ in the hyperplane $h(0) + \dot{h}(0)^\perp$. The sets $\Sigma \cap [W^{\mathrm{s}}(p(\mu), \mu) \oplus \mathbb{R}\psi(0)]$ and $\Sigma \cap [W^{\mathrm{u}}(p(\mu), \mu) \oplus \mathbb{R}\psi(0)]$ are manifolds that intersect transversely along a line for all sufficiently small μ. This line intersects stable and unstable manifolds in unique points which are the initial conditions for the desired orbits. $\qquad \square$

In particular, the stable and unstable manifolds of $p(\mu)$ intersect near $h(0)$ for $\mu \approx 0$ if, and only if, $\langle \psi(0), h^{\mathrm{u}}(0; \mu) - h^{\mathrm{s}}(0; \mu) \rangle = 0$. It therefore makes sense to define the distance between stable and unstable manifolds near $h(0)$ as

$$\Delta(\mu) := \langle \psi(0), h^{\mathrm{u}}(0; \mu) - h^{\mathrm{s}}(0; \mu) \rangle.$$

Using the variation-of-constant formula, it can be shown that the distance function $\Delta(\mu)$ is given by

$$\Delta(\mu) = \langle \psi(0), h^{\mathrm{u}}(0; \mu) - h^{\mathrm{s}}(0; \mu) \rangle = \left[\int_{\mathbb{R}} \langle \psi(t), f_\mu(h(t), 0) \rangle \, \mathrm{d}t \right] \mu + \mathrm{O}(|\mu|^2) =: M\mu + \mathrm{O}(|\mu|^2). \qquad (2.6)$$

The quantity M is commonly referred to as the Melnikov integral: the homoclinic orbit will not persist if $M \neq 0$. In the following hypothesis, we summarize the assumption made in Lemma 2.1 together with the condition that $M \neq 0$.

Hypothesis 2.2 (Nondegeneracy)**.** *Consider the following nondegeneracy conditions:*

(i) Stable and unstable manifolds intersect as transversely as possible:

$$T_{h(0)}W^{\mathrm{s}}(p, 0) \cap T_{h(0)}W^{\mathrm{u}}(p, 0) = \mathbb{R}\dot{h}(0).$$

(ii) Stable and unstable manifolds unfold generically with respect to the parameter μ_1:

$$M := \int_{\mathbb{R}} \langle \psi(t), f_{\mu_1}(h(t), 0) \rangle \, \mathrm{d}t \neq 0.$$

It turns out that, for most of the bifurcation scenarios that will be discussed below, more detailed information about the asymptotic behavior of the homoclinic orbit $h(t)$ than mere convergence will be needed. The asymptotics of $h(t)$ for $|t| \gg 1$ are, to a large extent, determined by the linearization $\dot{v} = f_u(p, 0)v$ of (2.1) about the hyperbolic equilibrium p: in particular, the exponential decay rate of $\|h(t) - p\|$ will be given by the real part of one of the eigenvalues of the Jacobian $f_u(p, 0)$. The eigenvalues closest to the imaginary axis will typically dominate the asymptotics as they give the slowest possible exponential rates: we therefore call them the leading stable and unstable eigenvalues of $f_u(p, 0)$, denoted by ν^{s} and ν^{u}, respectively. More precisely, denote the eigenvalues of $f_u(p, 0)$ by ν_j with $j = 1, \ldots, n$, repeated with multiplicity and ordered by increasing real part so that

$$\mathrm{Re}\,\nu_1 \leq \mathrm{Re}\,\nu_2 \leq \cdots \leq \mathrm{Re}\,\nu_k < 0 < \mathrm{Re}\,\nu_{k+1} \leq \ldots \leq \mathrm{Re}\,\nu_{n-1} \leq \mathrm{Re}\,\nu_n. \tag{2.7}$$

The eigenvalues ν_j with $\mathrm{Re}\,\nu_j = \mathrm{Re}\,\nu_k$ are the leading stable eigenvalues, while the leading unstable eigenvalues ν_j are those that satisfy $\mathrm{Re}\,\nu_j = \mathrm{Re}\,\nu_{k+1}$. We expect that the leading eigenvalues are simple and unique (up to complex conjugation), which leads to the following assumptions on the leading eigenvalues that are often imposed:

Hypothesis 2.3 (Leading eigenvalues)**.** *Consider the following eigenvalue conditions:*

(i) *The unique leading unstable eigenvalue ν^{u} is real and simple, and we have $|\mathrm{Re}\,\nu^{\mathrm{s}}| > \nu^{\mathrm{u}}$.*

(ii) *The leading stable and unstable eigenvalues are unique, real and simple.*

(iii) *The leading unstable eigenvalues is unique, real and simple. There are precisely two leading stable eigenvalues ν^{s} and $\overline{\nu^{\mathrm{s}}}$, and these are complex[1] and simple.*

(iv) *The leading stable and unstable eigenvalues are unique (up to complex conjugation) and simple.*

The quotient $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ is often referred to as the saddle quantity. For future use, we shall also define real numbers λ^{ss} and λ^{uu} that separate the real parts of leading eigenvalues from those of the remaining strong eigenvalues so that

$$\mathrm{Re}\,\nu_1 \leq \cdots \leq \mathrm{Re}\,\nu_{k-n_{\mathrm{ls}}} < \lambda^{\mathrm{ss}} < \mathrm{Re}\,\nu_{k-n_{\mathrm{ls}}+1} = \cdots = \mathrm{Re}\,\nu_k < 0 \tag{2.8}$$

and analogously for the unstable eigenvalues.

Having defined the leading stable and unstable eigenvalues, we return to the asymptotic behavior of the homoclinic orbit and assume from now on that Hypothesis 2.3(iv) is met. Using the solutions $h^{\mathrm{s}}(t; \mu)$ and $h^{\mathrm{u}}(t; \mu)$ from Lemma 2.1, we define[2]

$$\begin{aligned} v^{\mathrm{s}}(\mu) &:= \lim_{t \to \infty} \mathrm{e}^{-\nu^{\mathrm{s}}(\mu)t}[h^{\mathrm{s}}(t; \mu) - p(\mu)] \\ v^{\mathrm{u}}(\mu) &:= \lim_{t \to -\infty} \mathrm{e}^{-\nu^{\mathrm{u}}(\mu)t}[h^{\mathrm{u}}(t; \mu) - p(\mu)], \end{aligned} \tag{2.9}$$

where $\nu^{\mathrm{s}}(\mu)$ and $\nu^{\mathrm{u}}(\mu)$ denote the leading eigenvalues of $f_u(p(\mu), \mu)$. It can be shown that these limits exist and are smooth in μ. Furthermore, $v^j(\mu)$ is a multiple of the eigenvector of $f_u(p(\mu), \mu)$ associated with $\nu^j(\mu)$ for $j = \mathrm{s}, \mathrm{u}$. In addition, there is a constant $\epsilon > 0$ so that

$$h^{\mathrm{s}}(t; \mu) = p(\mu) + \mathrm{e}^{\nu^{\mathrm{s}}(\mu)t}v^{\mathrm{s}}(\mu) + \mathrm{O}\left(\mathrm{e}^{(\mathrm{Re}\,\nu^{\mathrm{s}}(\mu)-\epsilon)t}\right), \qquad t \to \infty$$

and analogously for $h^{\mathrm{u}}(t; \mu)$. Thus, we can think of $v^{\mathrm{s}}(0)$ and $v^{\mathrm{u}}(0)$ as determining the effective dynamical components of $h(t)$ in the leading eigendirections. In particular, we see that $v^{\mathrm{s}}(0) = 0$ if, and only if, $h(t)$ lies in the strong stable manifold $W^{\mathrm{ss}}(p; 0)$ of the equilibrium p, which consists, by definition, of all solutions $u(t)$ to (2.1) that satisfy $\|u(t) - p\| = \mathrm{O}(\mathrm{e}^{\lambda^{\mathrm{ss}}t})$; see [394].

---

[1]Throughout the entire paper, we say that eigenvalues are complex if they are not real

[2]If the eigenvalues are not real, then consider the limits in (2.9) using complex coordinates
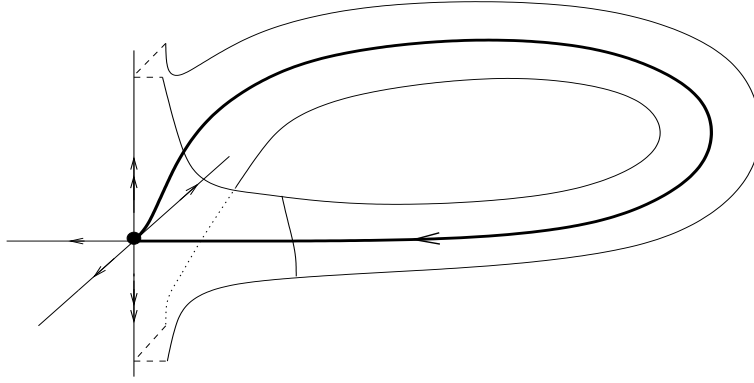
Figure 2.2: *Following the unstable manifold backwards along the homoclinic orbit, we expect that this manifold tends towards the strong unstable directions. This property is violated when the unstable manifold is in an inclination-flip configuration.*

Another important geometric property associated with a homoclinic orbit $h(t)$ are inclination properties of the stable and unstable manifolds when they are transported along the homoclinic orbit. As illustrated in Figure 2.2, we expect that the tangent space $T_{h(t)}W^{\mathrm{u}}(p,0)$ converges as $t \to \infty$ to the sum of $T_p W^{\mathrm{uu}}(p,0)$ and the eigendirection associated with the leading stable eigenvalue[3]: in this case, we say that the unstable manifold is not in an inclination-flip configuration. The asymptotic behavior as $t \to \infty$ of the unstable manifold along the homoclinic orbit can be encoded similarly to the way we encoded in $v^{\mathrm{u}}(0)$ the property that $h(t)$ should not lie in the strong unstable manifold of p: indeed, define

$$v_*^{\mathrm{s}} := \lim_{t \to -\infty} \mathrm{e}^{\nu^{\mathrm{s}} t} \psi(t), \qquad v_*^{\mathrm{u}} := \lim_{t \to \infty} \mathrm{e}^{\nu^{\mathrm{u}} t} \psi(t),$$

then v_*^{s} and v_*^{u} are multiples of the leading stable and unstable eigenvectors of the adjoint Jacobian $f_u(p,0)^*$. Furthermore, as desired, $v_*^{\mathrm{u}} \neq 0$ if, and only if, the unstable manifold is not in an inclination-flip configuration.

Generic homoclinic orbits do not lie in the strong stable or unstable manifolds, and the stable and unstable manifolds are not in inclination-flip configurations:

Hypothesis 2.4 (Inclination and orbit properties)**.** *The following conditions exclude inclination-flip and orbit-flip configurations:*

(i) *The stable manifold along the homoclinic orbit is not in an inclination-flip configuration, that is, $v_*^{\mathrm{s}} \neq 0$.*

(ii) *The homoclinic orbit is not in an orbit-flip configuration within the stable manifold, that is, it does not lie in the strong-stable manifold $W^{\mathrm{ss}}(p,0)$: $v^{\mathrm{s}} \neq 0$.*

(iii) *The unstable manifold along the homoclinic orbit is not in an inclination-flip configuration, that is, $v_*^{\mathrm{u}} \neq 0$.*

(iv) *The homoclinic orbit is not in an orbit-flip configuration within the unstable manifold, that is, $v^{\mathrm{u}} \neq 0$.*

Inclination and orbit flips have the following geometric interpretation. Assume again that the leading eigenvalues are real and simple. If Hypothesis 2.4 is met, then

$$\mathcal{O} := \operatorname{sign}\left[\langle v_*^{\mathrm{s}}, v^{\mathrm{s}}(0) \rangle \langle v_*^{\mathrm{u}}, v^{\mathrm{u}}(0) \rangle \right] \tag{2.10}$$

satisfies $\mathcal{O} = \pm 1$; in other words, Hypothesis 2.4 fails precisely when $\mathcal{O} = 0$. Geometrically, $\mathcal{O}$ is an orientation index that encodes the orientability of the two-dimensional homoclinic center manifold $W_{\mathrm{hom}}^{\mathrm{c}}(\mu)$ that we shall discuss in §3.4; see Figure 3.3 for an illustration. In particular, generically, the two-dimensional homoclinic center manifold changes at an inclination or orbit flip from orientable to nonorientable, or vice versa. We say that a homoclinic orbit is orientable if $\mathcal{O} = 1$ and call it nonorientable when $\mathcal{O} = -1$.

---

[3]We assume here that Hypothesis 2.3(iii) is met so that the leading eigenvalues are real and simple

Alternatively, inclination-flip conditions, Hypotheses 2.4(i) and 2.4(iii), can also be stated in the following slightly different geometric terms. First, invariant manifold theory provides invariant manifolds $W^{\mathrm{ls,u}}(p)$ near p whose tangent space at p consists of the generalized unstable eigenspace of $f_u(p,0)$ plus the leading stable eigendirections; similarly, there is an invariant manifold $W^{\mathrm{s,lu}}(p)$ whose tangent space consists of the generalized stable eigenspace plus the leading stable eigendirections. The smoothness of these manifolds depends on spectral gap conditions: in general, they are only continuously differentiable. Furthermore, even though their tangent space at p is unique, the manifolds themselves are not uniquely defined. The tangent bundle of $W^{\mathrm{s,lu}}(p)$ along the stable manifold is a smooth and uniquely defined vector bundle. Similarly, the tangent bundle of $W^{\mathrm{ls,u}}(p)$ along the unstable manifold is a smooth and uniquely defined vector bundle. We now have the following characterization:

Lemma 2.2. *The stable manifold along the homoclinic orbit is not in an inclination-flip configuration precisely if $W^{\mathrm{ls,u}}(p) \pitchfork W^{\mathrm{s}}(p)$ along the homoclinic orbit. The unstable manifold along the homoclinic orbit is not in an inclination-flip configuration precisely if $W^{\mathrm{s,lu}}(p) \pitchfork W^{\mathrm{u}}(p)$ along the homoclinic orbit.*

If v_*^{s} or v_*^{u} vanish, then we would like to characterize how the degenerate inclination of stable or unstable manifolds is unfolded as μ varies near zero. To this end, we need the following lemma whose proof is similar to that of Lemma 2.1.

Lemma 2.3. *Assume that Hypotheses 2.1 and 2.2(i) are met. For $j = \mathrm{s, u}$, there are unique solutions $\psi^j(\cdot\,;\mu)$ of*

$$\dot{w} = -f_u(h^j(t;\mu),\mu)^* w, \qquad j = \mathrm{s, u}$$

with $\|\psi^j(0;\mu)\| = 1$ and $\psi^j(t;\mu) \perp T_{h^j(t;\mu)} W^j(p(\mu),\mu)$ so that $\psi^{\mathrm{u}}(0;\mu) - \psi^{\mathrm{s}}(0;\mu) \in \mathbb{R}\dot{h}(0)$. The functions $\psi^{\mathrm{s}}(\cdot\,;\mu)$ and $\psi^{\mathrm{u}}(0;\mu)$, considered with values in $C^0(\mathbb{R}^+,\mathbb{R}^n)$ and $C^0(\mathbb{R}^-,\mathbb{R}^n)$ respectively, are smooth in μ.

We then define

$$v_*^{\mathrm{s}}(\mu) := \lim_{t \to -\infty} \mathrm{e}^{\nu^{\mathrm{s}}(\mu)t} \psi^{\mathrm{s}}(t;\mu), \qquad v_*^{\mathrm{u}}(\mu) := \lim_{t \to \infty} \mathrm{e}^{\nu^{\mathrm{u}}(\mu)t} \psi^{\mathrm{u}}(t;\mu), \tag{2.11}$$

and it can again be shown that these limits exist and are smooth in μ. We may now impose that derivatives of $v_*^j(\mu)$ are nonzero at $\mu = 0$ whenever $v_*^j(0) = 0$ for $j = \mathrm{s}$ or $j = \mathrm{u}$.

2.2 Homoclinic orbits to nonhyperbolic equilibria

Like hyperbolic equilibria, nonhyperbolic equilibria may admit homoclinic solutions to it. In systems with additional structure, such as reversibility or a Hamiltonian structure, this may be typical or at least of low codimension. We now discuss various geometric notions that we shall use later when we review bifurcations of homoclinic orbits that converge to nonhyperbolic equilibria.

From now on, let p be a nonhyperbolic equilibrium of $\dot{u} = f(u,0)$, so that $f_u(p,0)$ has at least one eigenvalue on the imaginary axis. Center manifolds and normal forms can then be used to study the local bifurcations near the equilibrium p, and we refer the reader to [394, 412] for their properties and various examples.

First, consider the case where p is a saddle-node equilibrium: its linearization $f_u(p,0)$ has a simple real eigenvalue $\nu = 0$ and no further eigenvalues on the imaginary axis. In generic systems, homoclinic orbits to a saddle-node equilibrium occur as a codimension-one phenomenon. Define v_{c} and w_{c} to be the right and left eigenvectors of the eigenvalue 0 of $f_u(p,0)$.

Hypothesis 2.5 (Codimension-one saddle-node bifurcation)**.** *The following conditions define a generic saddle-node bifurcation:*

  (i) *The saddle-node equilibrium is not degenerate: $\langle w_{\mathrm{c}}, f_{uu}(p,0)[v_{\mathrm{c}}, v_{\mathrm{c}}]\rangle \neq 0$.*

  (ii) *The unfolding is generic: $\langle w_{\mathrm{c}}, f_\mu(p,0)\rangle \neq 0$.*

If this hypothesis is met, then the vector field on the one-dimensional center manifold can be brought into normal form $\dot{x}^{\mathrm{c}} = b(\mu) + a(\mu)(x^{\mathrm{c}})^2 + \mathrm{O}(|x^{\mathrm{c}}|^3)$, where $a(0) \neq 0$ and $b_\mu(0) \neq 0$. Associated with the saddle-node equilibrium p at $\mu = 0$ is its stable manifold $W^{\mathrm{s}}(p)$, which consists of orbits that converge towards p exponentially in t as $t \to \infty$, and its unstable manifold $W^{\mathrm{u}}(p)$, which consists of orbits that converge towards p exponentially in t as $t \to -\infty$. Invariant-manifold theory [394] also gives the existence of a center-stable manifold $W^{\mathrm{cs}}(p)$ and a center-unstable manifold $W^{\mathrm{cu}}(p)$ so that a center manifold is obtained as the transverse intersection $W^{\mathrm{c}}(p) = W^{\mathrm{cs}}(p) \pitchfork W^{\mathrm{cu}}(p)$. The stable set $\mathcal{M}^{\mathrm{s}}(p)$ that consists of orbits which converge (not necessarily exponentially) to p as $t \to \infty$ is a submanifold of $W^{\mathrm{cs}}(p)$ with boundary $W^{\mathrm{s}}(p)$. Similarly, the unstable set $\mathcal{M}^{\mathrm{u}}(p)$ that consists of orbits converging to p for $t \to -\infty$ is a manifold with boundary $W^{\mathrm{u}}(p)$ inside $W^{\mathrm{cu}}(p)$.

As for hyperbolic equilibria, we can classify the hyperbolic part of the spectrum of $f_u(p, 0)$ into different categories.

Hypothesis 2.6 (Leading hyperbolic eigenvalues)**.** *Consider the following conditions on the eigenvalues of $f_u(p, 0)$ with nonzero real part:*

(i) *The leading stable and unstable eigenvalues are unique, real and simple.*

(ii) *There are precisely two leading stable eigenvalues ν^{s} and $\overline{\nu^{\mathrm{s}}}$, and these are complex and simple.*

(iii) *The leading stable and unstable eigenvalues are unique (up to complex conjugation) and simple.*

A homoclinic orbit to p lies in the intersection $\mathcal{M}^{\mathrm{s}}(p) \cap \mathcal{M}^{\mathrm{u}}(p)$, and the following hypothesis excludes orbit-flip configurations.

Hypothesis 2.7 (Orbit properties)**.** *The following conditions exclude orbit-flip configurations:*

(i) *The homoclinic orbit is not in an orbit-flip configuration within the center-stable manifold, that is, it does not lie in the stable manifold.*

(ii) *The homoclinic orbit is not in an orbit-flip configuration within the center-unstable manifold, that is, it does not lie in the unstable manifold.*

Next, we discuss Hopf bifurcations of p, where the linearized vector field about the nonhyperbolic equilibrium p has complex conjugate eigenvalues on the imaginary axis when $\mu = 0$. Homoclinic orbits to a Hopf equilibrium occur, in generic systems, as a codimension-two phenomenon. This gives rise to the Shil'nikov–Hopf bifurcation, which we discuss in §5.1.10. We suppose that all eigenvalues of $f_u(p, 0)$ are away from the imaginary axis except for two simple eigenvalues $\nu^{\mathrm{c}}, \overline{\nu^{\mathrm{c}}}$ on the imaginary axis. Using the complex coordinate z on the two-dimensional local center manifold, we can write the vector field on the center manifold as $\dot{z} = \nu^{\mathrm{c}}(\mu)z + g(z, \bar{z}, \mu)$. Recall that a smooth coordinate change transforms this equation into the normal form $\dot{w} = \nu^{\mathrm{c}}(\mu)w + c_1(\mu)|w|^2 w + \mathrm{O}(|w|^5)$; see [394, 412].

Hypothesis 2.8 (Codimension-one Hopf bifurcation)**.** *The following conditions define a generically unfolded supercritical Hopf bifurcation:*

(i) *The Hopf equilibrium is not degenerate: $c_1(0) \neq 0$.*

(ii) *The unfolding is generic: $\nu_\mu^{\mathrm{c}}(0) \neq 0$.*

(iii) *The Hopf bifurcation is supercritical: $c_1(0) < 0$ and $\nu_\mu^{\mathrm{c}}(0) > 0$.*

2.3 Heteroclinic cycles with hyperbolic equilibria

A heteroclinic cycle is an invariant set for (2.1) that consists of disjoint equilibria $p_1, \dots, p_\ell$ and heteroclinic orbits $h_1(t), \dots, h_\ell(t)$ that connect p_i to p_{i+1} so that

$$\lim_{t \to -\infty} h_i(t) = p_i, \qquad \lim_{t \to \infty} h_i(t) = p_{i+1}$$

for $i = 1, \dots, \ell$, where $p_{\ell+1} = p_1$. A connected invariant set that can be written as a finite union of heteroclinic cycles (possibly including homoclinic loops) is called a polycycle.

As with homoclinic orbits we start with the key assumption that the equilibria do not undergo bifurcations, which is again guaranteed by a hyperbolicity condition.

Hypothesis 2.9 (Hyperbolicity). *The equilibria p_i are hyperbolic at $\mu = 0$ for $1 \le i \le \ell$.*

If the preceding assumption is met, then $h_i(t) \in W^{\mathrm{u}}(p_i) \cap W^{\mathrm{s}}(p_{i+1})$ for each i. We now explore different possible configurations that depend on Morse indices and leading eigenvalues at the equilibria. As the Morse index $\mathrm{ind}(p_i) := \dim W^{\mathrm{u}}(p_i)$ of p_i may differ from the index $\mathrm{ind}(p_{i+1})$ of p_{i+1}, heteroclinic orbits can occur robustly (or even in families) or may occur only if a sufficient number of parameters is varied. If $\mathrm{ind}(p_i) > \mathrm{ind}(p_{i+1})$, then transverse intersections of $W^{\mathrm{u}}(p_i)$ and $W^{\mathrm{s}}(p_{i+1})$ form a manifold of dimension $\mathrm{ind}(p_i) - \mathrm{ind}(p_{i+1})$. If $\mathrm{ind}(p_i) \le \mathrm{ind}(p_{i+1})$, the codimension of h_i typically equal to $d = \mathrm{ind}(p_{i+1}) - \mathrm{ind}(p_i) + 1$: for a family $\dot{u} = f(u, \mu)$ of vector fields with $\mu \in \mathbb{R}^d$, transverse intersections of $\cup_\mu (W^{\mathrm{u}}(p_i(\mu), \mu), \mu)$ and $\cup_\mu (W^{\mathrm{s}}(p_{i+1}(\mu), \mu), \mu)$ in $\mathbb{R}^n \times \mathbb{R}^d$ yield isolated heteroclinic orbits at isolated parameter values. Here, $p_i(\mu)$ is the continuation of p_i, and $W^{\mathrm{u}}(p_i(\mu), \mu)$ is the unstable manifold of $p_i(\mu)$ for $\dot{u} = f(u, \mu)$.

Following §2.1, consider the adjoint variational equation

$$\dot{w} = -f_u(h_i(t), 0)^* w. \tag{2.12}$$

Suppose $T_{h_i(0)} W^{\mathrm{u}}(p_i, 0) + T_{h_i(0)} W^{\mathrm{s}}(p_{i+1}, 0)$ has codimension d, then there are d linearly independent bounded solutions $\psi_i^1(t), \dots, \psi_i^d(t)$ to (2.12). We can now formulate nondegeneracy conditions akin to Hypothesis 2.2.

Hypothesis 2.10 (Nondegeneracy). *Consider the following nondegeneracy conditions on $h_i(t)$:*

(i) *The heteroclinic orbit h_i is of codimension $d_i \ge 0$: $\mathrm{ind}(p_{i+1}) - \mathrm{ind}(p_i) + 1 = d_i$, and the codimension of $T_{h_i(0)} W^{\mathrm{u}}(p_i, 0) + T_{h_i(0)} W^{\mathrm{s}}(p_{i+1}, 0)$ is d_i.*

(ii) *The heteroclinic orbit h_i is of codimension d_i, and stable and unstable manifolds unfold generically along h_i with respect to the parameter $\mu = (\mu_1, \dots, \mu_{d_i}) \in \mathbb{R}^{d_i}$ so that the matrix $M \in \mathbb{R}^{d_i \times d_i}$ with entries*

$$M_{kl} = \int_{\mathbb{R}} \langle \psi_i^k(t), f_{\mu_l}(h_i(t), 0) \rangle \, \mathrm{d}t$$

has full rank.

For $i = 1, \dots, \ell$, we define the cross sections Σ_i via

$$\Sigma_i = h_i(0) + Y_i, \qquad Y_i = f(h_i(0), 0)^\perp. \tag{2.13}$$

The cross section Σ_i is a hyperplane that intersects the orbit $h_i(t)$ transversally at $h_i(0)$. Let

$$Z_i = \left(T_{h_i(0)} W^{\mathrm{u}}(p_i, 0) + T_{h_i(0)} W^{\mathrm{s}}(p_{i+1}, 0) \right)^\perp \tag{2.14}$$

be the subspace of Y_i spanned by $\psi_i^1(0), \dots, \psi_i^{d_i}(0)$ and note that $h_i(0) + Z_i \subset \Sigma_i$. The argument used to prove Lemma 2.1 also gives the following corresponding lemma for heteroclinic orbits.

Lemma 2.4. *Suppose h_i is a heteroclinic orbit of codimension one (in particular, $\dim Z_i = 1$). For each μ close to 0, there are unique orbits $h_i^{\mathrm{s}}(\cdot; \mu) \in W^{\mathrm{s}}(p_{i+1}(\mu), \mu)$ and $h_i^{\mathrm{u}}(\cdot; \mu) \in W^{\mathrm{u}}(p_i(\mu), \mu)$ such that*

$$h_i^{\mathrm{u}}(0; \mu) - h_i^{\mathrm{s}}(0; \mu) \in Z_i, \qquad \forall \mu.$$

Next, we state conditions on the leading eigenvalues.

Hypothesis 2.11 (Leading eigenvalues). *Consider the following eigenvalue conditions:*

(i) *The unique leading unstable eigenvalue ν_i^{u} at p_i is real and simple, and we have $|\operatorname{Re}\nu_i^{\mathrm{s}}| > \nu^{\mathrm{u}}$.*

(ii) *The leading stable and unstable eigenvalues at p_i are unique, real and simple.*

(iii) *The leading stable eigenvalue of $f_u(p_{i+1}, 0)$ and unstable eigenvalues of $f_u(p_i, 0)$ are unique (up to complex conjugation) and simple.*

Assume that Hypotheses 2.11(iii) and 2.10(i) are met with $d_i = 1$. As in §2.1, it can be shown that

$$
\begin{aligned}
v_i^{\mathrm{s}}(\mu) &:= \lim_{t \to \infty} \mathrm{e}^{-\nu_{i+1}^{\mathrm{s}}(\mu)t}[h_i^{\mathrm{s}}(t; \mu) - p_{i+1}(\mu)] \\
v_i^{\mathrm{u}}(\mu) &:= \lim_{t \to -\infty} \mathrm{e}^{-\nu_i^{\mathrm{u}}(\mu)t}[h_i^{\mathrm{u}}(t; \mu) - p_i(\mu)]
\end{aligned}
\tag{2.15}
$$

exist and are smooth in μ, where $\nu_j^{\mathrm{s}}(\mu)$ and $\nu_j^{\mathrm{u}}(\mu)$ denote the leading eigenvalues of $f_u(p_j(\mu), \mu)$. Likewise, we define

$$
v_{i,*}^{\mathrm{s}} := \lim_{t \to -\infty} \mathrm{e}^{\nu_i^{\mathrm{s}}(0)t}\psi_i(t), \qquad v_{i,*}^{\mathrm{u}} := \lim_{t \to \infty} \mathrm{e}^{\nu_{i+1}^{\mathrm{u}}(0)t}\psi_i(t),
\tag{2.16}
$$

then $v_{i,*}^{\mathrm{s}}$ and $v_{i,*}^{\mathrm{u}}$ are multiples of the leading stable and unstable eigenvectors of the adjoint linearizations $f_u(p_i, 0)^*$ and $f_u(p_{i+1}, 0)^*$, respectively.

Hypothesis 2.12 (Inclination and orbit properties). *The following conditions exclude inclination-flip and orbit-flip configurations along a codimension-one heteroclinic orbit h_i:*

(i) *The stable manifold $W^{\mathrm{s}}(p_{i+1}, 0)$ along h_i is not in an inclination-flip configuration, that is, $v_{i,*}^{\mathrm{s}} \neq 0$.*

(ii) *h_i is not in an orbit-flip configuration within the stable manifold so that it does not lie in the strong-stable manifold $W^{\mathrm{ss}}(p_{i+1}, 0)$: $v_i^{\mathrm{s}}(0) \neq 0$.*

(iii) *The unstable manifold $W^{\mathrm{u}}(p_i, 0)$ along h_i is not in an inclination-flip configuration, that is, $v_{i,*}^{\mathrm{u}} \neq 0$.*

(iv) *h_i is not in an orbit-flip configuration within the unstable manifold, that is, $v_i^{\mathrm{u}}(0) \neq 0$.*

There is a geometric description of the inclination-flip property akin to Lemma 2.2. Recall the definition of the invariant manifolds $W^{\mathrm{ls,u}}(p_i)$ and $W^{\mathrm{s,lu}}(p_{i+1})$ from §2.1.

Lemma 2.5. *The stable manifold $W^{\mathrm{s}}(p_{i+1})$ along h_i is not in an inclination-flip configuration precisely if $W^{\mathrm{ls,u}}(p_i) \pitchfork W^{\mathrm{s}}(p_{i+1})$. The unstable manifold $W^{\mathrm{u}}(p_i)$ along h_i is not in an inclination-flip configuration precisely if $W^{\mathrm{s,lu}}(p_{i+1}) \pitchfork W^{\mathrm{u}}(p_i)$.*

3   Analytical and geometric approaches

The goal of homoclinic bifurcation theory is to investigate the recurrent dynamics near a homoclinic orbit $h(t)$. In other words, we are interested in finding all orbits that stay in a fixed tubular neighborhood of a given homoclinic orbit or heteroclinic cycle for all times.

A natural way for approaching this problem is to use Poincaré or first-return maps. Denote by Σ a cross section placed at $h(0)$ which, by definition, means that Σ is the ball of radius $\epsilon > 0$ centered at $h(0)$ in the hyperplane $h(0) + \dot{h}(0)^{\perp}$:

$$
\Sigma = B_\epsilon(h(0)) \subset h(0) + \dot{h}(0)^{\perp},
\tag{3.1}
$$

where $\epsilon > 0$ is chosen to be so small that the flow is transverse to Σ. Starting with an initial condition u_0 in Σ, we then follow $u(t)$ until it hits Σ again, say at time $t = T$, and define the first-return map Π via $\Pi(u_0) := u(T) \in \Sigma$. The main issue is that a solution that starts near $h(0)$ may not return to a
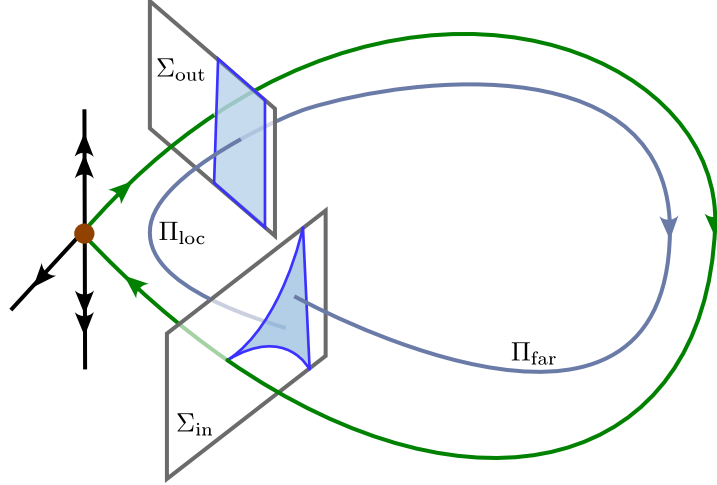
Figure 3.1: *The domain of the first-return map on a cross section does not contain an open neighborhood of $h(0)$ (one often has wedge shaped domains as shown here). For computational purposes, it is useful to consider two cross sections Σ_{in} and Σ_{out} and write the first-return map Π as a composition of transition maps between the cross sections.*

neighborhood of $h(0)$; see Figure 3.1. Thus, in general, the domain of the Poincaré map Π does not contain an open neighborhood of $h(0)$ in Σ. Furthermore, any solution that does return will spend a very long time near the equilibrium, thus spoiling most finite-time error estimates for Π from standard variation-of-constant formulae.

Several techniques have been developed to overcome these difficulties. We distinguish, to some extent artificially, two different approaches that treat homoclinic bifurcations from somewhat different viewpoints: The first approach computes the Poincaré map by writing it as a composition of a local transition map near the equilibrium with a global transition map; see Figure 3.1. The main difficulty here is to get expansions of the local transition map. The second approach is to seek orbits that stay near the homoclinic orbit as solution to some abstract functional-analytic system using Lyapunov–Schmidt reduction. Both of these methods can be used in conjunction with a geometric reduction to a low-dimensional invariant homoclinic center manifold that contains all recurrent dynamics. Each of these approaches has its own advantages and disadvantages that we shall comment on in the following when we discuss them in more detail.

3.1 Normal forms and linearizability

Given two cross sections Σ_{in} and Σ_{out} that are transverse to, respectively, the local stable and local unstable manifold of p, the local transition map Π_{loc} is the first-return map

$$\Pi_{\text{loc}} : \Sigma_{\text{in}} \longrightarrow \Sigma_{\text{out}}. \tag{3.2}$$

Understanding Π_{loc} requires solving

$$\dot{u} = f(u, \mu) = f_u(p, \mu)(u - p) + \mathrm{O}(\|u - p\|^2) \tag{3.3}$$

for u close to p. To calculate Π_{loc}, it is often advantageous to simplify the vector field on the right-hand side of (3.3). Of course, the best possible outcome is that the nonlinear terms can simply be transformed away which renders the vector field linear.

The Hartman–Grobman theorem states that there is a coordinate change, which is continuous in (u, μ), near each hyperbolic equilibrium p that transforms (3.3) into the linear system

$$\dot{v} = f_u(p, \mu)v. \tag{3.4}$$

Using such coordinates, the return map on a cross section is a homeomorphism but it is not clear how expansions in u can be derived that are needed for a bifurcation study. Belitskii [30] derived eigenvalue conditions which ensure the existence of a continuously differentiable coordinate change that linearizes a fixed flow. This linearization theorem can be applied to obtain statements on stability or the existence of hyperbolic sets near homoclinic solutions.

Theorem 3.1 ([30]). *Consider the equation $\dot{u} = f(u)$ on $\mathbb{R}^n$ near a hyperbolic equilibrium p and assume that f is of class C^2. With the eigenvalue ordering given in (2.7), if*

$$\operatorname{Re}\nu_i \neq \operatorname{Re}\nu_{j_1} + \operatorname{Re}\nu_{j_2} \tag{3.5}$$

for each i and all $1 \leq j_1 \leq \dim W^s(p) < j_2 \leq n$, then there is a local coordinate transformation of class C^1 that transforms the ODE into its linearization $\dot{v} = f_u(p)v$ near $u = p$.

For the study of bifurcations that involve periodic orbits, parameter-dependent versions and higher degrees of differentiability are needed. The next theorem, a parameter-dependent version of Sternberg's linearization result, gives conditions under which (3.3) can be transformed into (3.4) by an appropriate smooth coordinate transformation (see also §3.6.3 for a complementary linearization result). More generally, Takens [387] constructs partial linearizations near equilibria with center directions.

Theorem 3.2 ([352, 384, 387]). *Assume that $f(u, \mu)$ is C^∞ in (u, μ). Fix any $\ell \geq 1$, then there exist numbers $N = N(\ell, f_u(0,0)) \geq 1$ and $\epsilon > 0$ with the following property: if*

$$\nu_i \neq \sum_{j=1}^{n} N_j \nu_j \tag{3.6}$$

for each i and all natural numbers N_j with $2 \leq \sum_{j=1}^{n} N_j \leq N$, then we can C^ℓ-linearize (3.3) in $B_\epsilon(p)$ for $|\mu| < \epsilon$, i.e. there is a coordinate transformation of class C^ℓ in $u \in B_\epsilon(p)$ and $|\mu| < \epsilon$ that transforms (3.3) into (3.4).

If the system can be linearized, we can compute Π_{loc} explicitly once the transition time from Σ_{in} to Σ_{out} has been computed.

If the non-resonance condition (3.6) is not met, the vector field may still be transformed into normal form [80], and we refer the reader to [203] for an exposition of results on finitely smooth normal forms for families of vector fields and to [48] for analytic local normal forms for families of ODEs. Structure preserving normal forms (e.g. in the class of equivariant, conservative or reversible ODEs) are another important topic: we will not discuss this here but refer to [31] for Sternberg's theorem in an equivariant setting and to [47] for a general approach.

Useful expansions for Π_{loc} may be hard to obtain if the normal form is not linear though problems involving one-dimensional unstable separatrices are often tractable. Another approach in this situation is to use Shil'nikov variables which we shall discuss next.

3.2 Shil'nikov variables

Shil'nikov variables were introduced by Shil'nikov in 1968 to compute the local transition map near equilibria to leading order. Instead of solving an initial-value problem, solutions near the equilibrium are found using an appropriate boundary-value problem. The analysis of the resulting integral formulae leads to asymptotic expansions for the solutions. We concentrate on hyperbolic equilibria and refer the reader to [100] for results in the non-hyperbolic case. We remark that, in the case of a hyperbolic equilibrium with one-dimensional unstable directions, the approach leads to asymptotic expansions for solutions of an initial-value problem. Further information on Shil'nikov variables can be found, for instance, in the books [374, 375] where considerable space is devoted to their analysis and use.

Assume that $p = 0$ is a hyperbolic equilibrium for all μ. We may also assume that the stable and unstable eigenspaces of $f_u(0, \mu)$ do not depend on μ. We denote these spaces by E_0^{s} and E_0^{u}, respectively, and choose local coordinates $u = (x, y) \in E_0^{\mathrm{s}} \oplus E_0^{\mathrm{u}}$. In these coordinates, (3.3) becomes

$$
\begin{aligned}
\dot{x} &= A^{\mathrm{s}}(\mu)x + g^{\mathrm{s}}(x, y; \mu), & \operatorname{spec}(A^{\mathrm{s}}(\mu)) &= \operatorname{spec}(f_u(0, \mu)) \cap \{\operatorname{Re}\nu < 0\}, \\
\dot{y} &= A^{\mathrm{u}}(\mu)y + g^{\mathrm{u}}(x, y; \mu), & \operatorname{spec}(A^{\mathrm{u}}(\mu)) &= \operatorname{spec}(f_u(0, \mu)) \cap \{\operatorname{Re}\nu > 0\}.
\end{aligned}
\tag{3.7}
$$

For each fixed (x_0, y_1) close to zero and each $\tau \gg 1$, we seek solutions $(x, y)(t)$ of (3.7) that satisfy

$$
x(0) = x_0, \qquad y(\tau) = y_1.
\tag{3.8}
$$

We proceed as follows to construct these solutions. First, we separate eigenvalues of $f_u(0, 0)$ into leading and strong directions, and choose numbers λ^{ss} and λ^{uu} for $\mu = 0$ as in (2.8). Next, we write

$$
x = (x^{\mathrm{ls}}, x^{\mathrm{ss}}) \in E_0^{\mathrm{s}}, \qquad y = (y^{\mathrm{lu}}, y^{\mathrm{uu}}) \in E_0^{\mathrm{u}},
$$

where x^{ls} and x^{ss} lie in the generalized eigenspaces of $f_u(0, \mu)$ that belongs to stable eigenvalues of $f_u(0, \mu)$ whose real parts are, respectively, larger and smaller than λ^{ss}. Similarly, y^{lu} and y^{uu} lie in the generalized eigenspaces of $f_u(0, \mu)$ that belong to unstable eigenvalues of $f_u(0, \mu)$ whose real parts are, respectively, smaller and larger than λ^{uu}. Note that the leading stable and unstable eigenvalues may acquire slightly different real parts upon changing μ. An appropriate coordinate transformation [104, 374] brings (3.7) into the following normal form. More detailed normal forms may be obtained by similar methods if appropriate spectral conditions are met.

Proposition 3.1. *A smooth coordinate change brings the ODE near p into the form*

$$
\begin{pmatrix} \dot{x}^{\mathrm{ls}} \\ \dot{x}^{\mathrm{ss}} \\ \dot{y}^{\mathrm{lu}} \\ \dot{y}^{\mathrm{uu}} \end{pmatrix} = \begin{pmatrix} A^{\mathrm{ls}}(\mu) & 0 & 0 & 0 \\ 0 & A^{\mathrm{ss}}(\mu) & 0 & 0 \\ 0 & 0 & A^{\mathrm{lu}}(\mu) & 0 \\ 0 & 0 & 0 & A^{\mathrm{uu}}(\mu) \end{pmatrix} \begin{pmatrix} x^{\mathrm{ls}} \\ x^{\mathrm{ss}} \\ y^{\mathrm{lu}} \\ y^{\mathrm{uu}} \end{pmatrix} + \begin{pmatrix} \mathrm{O}((|x^{\mathrm{ls}}|^2 + |x^{\mathrm{ss}}|)|y|) \\ \mathrm{O}(|x^{\mathrm{ls}}|^2 + |x^{\mathrm{ss}}|(|x| + |y|)) \\ \mathrm{O}((|y^{\mathrm{lu}}|^2 + |y^{\mathrm{uu}}|)|x|) \\ \mathrm{O}(|y^{\mathrm{lu}}|^2 + |y^{\mathrm{uu}}|(|x| + |y|)) \end{pmatrix}.
$$

Proof. We may choose smooth coordinates $x = (x^{\mathrm{ss}}, x^{\mathrm{ls}}, y^{\mathrm{lu}}, y^{\mathrm{uu}})$ near p so that

$$
\begin{aligned}
W^{\mathrm{s}}(p) &= \{y = 0\}, & W^{\mathrm{u}}(p) &= \{x = 0\}, \\
W^{\mathrm{s,lu}}(p) \overline{\pitchfork}_{W^{\mathrm{s}}(\mathrm{p})} \{y^{\mathrm{uu}} = 0\}, & W^{\mathrm{ls,u}}(p) \overline{\pitchfork}_{W^{\mathrm{u}}(\mathrm{p})} \{x^{\mathrm{ss}} = 0\},
\end{aligned}
$$

where the notation $W \overline{\pitchfork}_q V$ means that W is tangent to V at q. This brings (3.7) into the form

$$
\begin{pmatrix} \dot{x}^{\mathrm{ls}} \\ \dot{x}^{\mathrm{ss}} \\ \dot{y}^{\mathrm{lu}} \\ \dot{y}^{\mathrm{uu}} \end{pmatrix} = \begin{pmatrix} A^{\mathrm{ls}}(\mu) & 0 & 0 & 0 \\ 0 & A^{\mathrm{ss}}(\mu) & 0 & 0 \\ 0 & 0 & A^{\mathrm{lu}}(\mu) & 0 \\ 0 & 0 & 0 & A^{\mathrm{uu}}(\mu) \end{pmatrix} \begin{pmatrix} x^{\mathrm{ls}} \\ x^{\mathrm{ss}} \\ y^{\mathrm{lu}} \\ y^{\mathrm{uu}} \end{pmatrix} + \begin{pmatrix} \mathrm{O}(|x^{\mathrm{s}}|(|x| + |y|)) \\ \mathrm{O}(|x^{\mathrm{ls}}|^2 + |x^{\mathrm{ss}}|(|x| + |y|)) \\ \mathrm{O}(|y^{\mathrm{u}}|(|x| + |y|)) \\ \mathrm{O}(|y^{\mathrm{lu}}|^2 + |y^{\mathrm{uu}}|(|x| + |y|)) \end{pmatrix}
$$

The remaining coordinate changes are described in detail in [299], and we give here only a brief overview. A polynomial coordinate change removes quadratic terms $x^{\mathrm{ls}}x^{\mathrm{lu}}$ from the differential equations for x^{ls}. Consider next a change of coordinates of the form

$$
\tilde{x}^{\mathrm{ls}} = x^{\mathrm{ls}} + p^{\mathrm{ls}}(y)x^{\mathrm{ls}}, \qquad \tilde{x}^{\mathrm{ss}} = x^{\mathrm{ss}}, \qquad \tilde{y}^{\mathrm{lu}} = y^{\mathrm{lu}}, \qquad \tilde{y}^{\mathrm{uu}} = y^{\mathrm{uu}}
\tag{3.9}
$$

for a function p^{ls} that vanishes along $y = 0$. Write the differential equation for x^{ls} in the new coordinates (skipping the tildes) as

$$
\dot{x}^{\mathrm{ls}} = A^{\mathrm{ls}}(\mu)x^{\mathrm{ls}} + P^{\mathrm{ls}}(x, y)x^{\mathrm{ls}} + g^{\mathrm{ls}}(x, y)x^{\mathrm{ss}}.
$$

Along the unstable manifold $x = 0$ we find

$$
P^{\mathrm{ls}}(0, y) = \dot{p}^{\mathrm{ls}} + p^{\mathrm{ls}}A^{\mathrm{ls}}(\mu) - A^{\mathrm{ls}}(\mu)p^{\mathrm{ls}},
$$

16

where the higher-order terms are of at least quadratic order. Consider $\dot{p}^{\mathrm{ls}}$ as a variable and construct the local unstable manifold tangent to $\{p^{\mathrm{ls}} = 0\}$ at the origin for the resulting differential equations

$$
\begin{aligned}
\dot{p}^{\mathrm{ls}} &= A^{\mathrm{ls}}(\mu)p^{\mathrm{ls}} - p^{\mathrm{ls}}A^{\mathrm{ls}}(\mu) + \mathrm{h.o.t.}, \\
\dot{y}^{\mathrm{lu}} &= A^{\mathrm{lu}}(\mu)y^{\mathrm{lu}} + \mathrm{h.o.t.}, \\
\dot{y}^{\mathrm{uu}} &= A^{\mathrm{uu}}(\mu)y^{\mathrm{uu}} + \mathrm{h.o.t.}
\end{aligned}
$$

along the unstable manifold $\{x = 0\}$. The resulting coordinate change will transform f^{ls} to a map with expansion

$$
f^{\mathrm{ls}}(x) = \mathrm{O}(|x^{\mathrm{ls}}|^2 + |x^{\mathrm{ss}}|(|x| + |y|)).
$$

As the coordinate change leaves y^{lu} unaltered, a similar coordinate change for f^{lu} can be performed. We have now achieved the following form:

$$
\begin{pmatrix} \dot{x}^{\mathrm{ls}} \\ \dot{x}^{\mathrm{ss}} \\ \dot{y}^{\mathrm{lu}} \\ \dot{y}^{\mathrm{uu}} \end{pmatrix} = \begin{pmatrix} A^{\mathrm{ls}}(\mu) & 0 & 0 & 0 \\ 0 & A^{\mathrm{ss}}(\mu) & 0 & 0 \\ 0 & 0 & A^{\mathrm{lu}}(\mu) & 0 \\ 0 & 0 & 0 & A^{\mathrm{uu}}(\mu) \end{pmatrix} \begin{pmatrix} x^{\mathrm{ls}} \\ x^{\mathrm{ss}} \\ y^{\mathrm{lu}} \\ y^{\mathrm{uu}} \end{pmatrix} + \begin{pmatrix} \mathrm{O}(|x^{\mathrm{ls}}|^2 + |x^{\mathrm{ss}}|(|x| + |y|)) \\ \mathrm{O}(|x^{\mathrm{ls}}|^2 + |x^{\mathrm{ss}}|(|x| + |y|)) \\ \mathrm{O}(|y^{\mathrm{lu}}|^2 + |y^{\mathrm{uu}}|(|x| + |y|)) \\ \mathrm{O}(|y^{\mathrm{lu}}|^2 + |y^{\mathrm{uu}}|(|x| + |y|)) \end{pmatrix}
$$

Within the stable manifold, there is a smooth strong stable foliation, which can be transformed into an affine foliation with leaves $\{x^{\mathrm{ss}} = \text{constant}\}$. The differential equation for x^{ls} in the set $\{y = 0\}$ depends only on x^{ls} and no longer on x^{ss}. Since eigenvalues of $A^{\mathrm{ls}}(\mu)$ have the same real part, one can smoothly linearize this differential equation. □

In the coordinates of the preceding proposition, there exists an $\epsilon > 0$ such that the boundary-value problem (3.8) has a unique solution $(x,y)(t)$ for each (x_0, y_1), μ and τ that satisfy $|x_0| + |y_0| < \epsilon$, $|\mu| < \epsilon$ and $\tau > 1/\epsilon$. This solution depends smoothly on the data (x_0, y_1, μ, τ).

Proposition 3.2. *Assuming the normal form from Proposition 3.1, there is an $\eta > 0$ so that the solution $(x, y)(t)$ admits the expansion*

$$
\begin{aligned}
x^{\mathrm{ls}}(t) &= \mathrm{e}^{A^{\mathrm{ls}}(\mu)t}\left[x_0^{\mathrm{ls}} + \mathrm{O}(\mathrm{e}^{-\eta t})\right], & x^{\mathrm{ss}}(t) &= \mathrm{O}(\mathrm{e}^{(\mathrm{Re}\,\nu^{\mathrm{s}} - \eta)t}), \\
y^{\mathrm{lu}}(t) &= \mathrm{e}^{A^{\mathrm{lu}}(\mu)(t-\tau)}\left[y_0^{\mathrm{lu}} + \mathrm{O}(\mathrm{e}^{\eta(t-\tau)})\right], & y^{\mathrm{uu}}(t) &= \mathrm{O}(\mathrm{e}^{(\mathrm{Re}\,\nu^{\mathrm{u}} + \eta)(t-\tau)}).
\end{aligned}
\tag{3.10}
$$

These asymptotic expansions are derived by considering integral formulae for solutions obtained using variation of constants. Similar estimates also hold for the derivatives of the solution with respect to the data. Note the similarity with the expansions for linear differential equations. We refer the reader to [99, 104, 369, 374] for proofs and generalizations of this result.

3.3 Lin's method

In this section, we discuss a functional-analytic approach due to Lin [256]. For clarity of exposition, we assume that the equilibrium p of

$$
\dot{u} = f(u, \mu) \tag{3.11}
$$

is hyperbolic and that the homoclinic orbit $h(t)$ satisfies Hypothesis 2.1(i). We pick a nontrivial bounded solution $\psi(t)$ of the adjoint variational equation (2.3) about $h(t)$ and the cross section Σ from (3.1). The idea of Lin's method is to construct a sequence u_j of solutions to (3.11) that begin and end in Σ after spending a given number of time units near the homoclinic orbit. The key feature of these solutions is that the difference between the end point of the jth solution and the initial condition for the $(j+1)$th solution lies in $\mathbb{R}\psi(0)$: In particular, the individual solutions can be spliced together to form a solution of (3.11) if, and only if, the jumps in $\mathbb{R}\psi(0)$ vanish for all j. This is further illustrated in Figure 3.2.
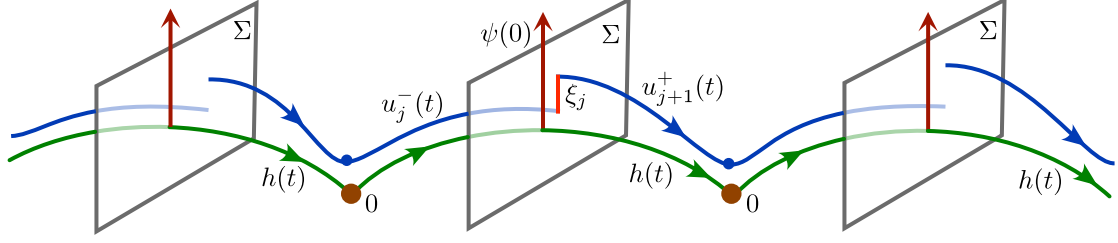
Figure 3.2: *Orbit pieces in Lin's method with jumps in a specified direction in the cross section.*

In detail, is has been shown in [256, 339] that there are constants $0 < \epsilon \ll 1$ and $T_* \gg 1$ such that the boundary-value problem

$$
\begin{aligned}
\dot{u}_j^- &= f(u_j^-, \mu) & t \in (-T_j, 0) \\
\dot{u}_j^+ &= f(u_j^+, \mu) & t \in (0, T_j) \\
u_j^\pm(0) &\in \Sigma \\
u_j^-(-T_j) &= u_j^+(T_j) \\
\|u_j^\pm(t) - h(t)\| &< \epsilon & t \in [-T_j, 0] \text{ and } [0, T_j], \text{ respectively} \\
u_j^-(0) - u_{j+1}^+(0) &\in \mathbb{R}\psi(0)
\end{aligned}
\tag{3.12}
$$

with $j \in \mathbb{Z}$ has a unique solution $\{u_j\}_{j \in \mathbb{Z}}$ for given data $\{T_j\}_{j \in \mathbb{Z}}$ and μ with $T_j > T_*$ and $|\mu| < \epsilon$, and the solution is smooth in those data. In particular, if the bifurcation functions

$$
\xi_j := \langle \psi(0), u_j^-(0) - u_{j+1}^+(0) \rangle
$$

all vanish, then the concatenation of the solutions u_j gives a solution u of (3.11) that follows the homoclinic orbit $h(t)$ for all times. Using the notation introduced in §2, the bifurcation functions ξ_j admit the expansion

$$
\begin{aligned}
\xi_j &= \langle \psi^{\mathrm{u}}(-T_j, \mu), h^{\mathrm{s}}(T_j, 0) - p(\mu) \rangle - \langle \psi^{\mathrm{s}}(T_{j+1}, \mu), h^{\mathrm{u}}(-T_{j+1}, 0) - p(\mu) \rangle + \Delta(\mu) \\
&\quad + \mathrm{O}\left(\mathrm{e}^{-(2\min(|\nu^{\mathrm{s}}|, \nu^{\mathrm{u}}) + \eta) \min_{k \in \mathbb{Z}} T_k} \right)
\end{aligned}
\tag{3.13}
$$

for some $\eta > 0$. We refer the reader to [344, §5.1.1] for a geometric explanation of the quantities that appear in the above expression. The error estimate in (3.13) can be greatly improved, and we refer to [256, 339] for details.

Example. If Hypothesis 2.3(i) is met, then we can use (2.6), (2.9) and (2.11) to simplify the expression (3.13) to get the bifurcation equations

$$
\langle v_*^{\mathrm{s}}(\mu), v^{\mathrm{s}}(\mu) \rangle \mathrm{e}^{2\nu^{\mathrm{s}} T_j} - \langle v_*^{\mathrm{u}}(\mu), v^{\mathrm{u}}(\mu) \rangle \mathrm{e}^{-2\nu^{\mathrm{u}} T_{j+1}} + M\mu + \mathrm{O}\left(\mathrm{e}^{-(2\min(|\nu^{\mathrm{s}}|, \nu^{\mathrm{u}}) + \eta) \min_{k \in \mathbb{Z}} T_k} + |\mu|^2 \right) = 0. \tag{3.14}
$$

∎

3.4 Homoclinic center manifolds

Instead of directly investigating the full n-dimensional ODE

$$
\dot{u} = f(u, \mu)
\tag{3.15}
$$

near a given homoclinic orbit $h(t)$, it may be desirable to first reduce the dimensionality of the system by constructing a locally invariant, normally hyperbolic manifold that contains the homoclinic orbit and all solutions staying close to it for all times. We refer to such a manifold as a *homoclinic center manifold*: normal hyperbolicity implies robustness under parameter perturbations, while the property that it contains all recurrent dynamics shows that it plays indeed a role similar to that of center manifolds in local bifurcation theory. The theorem stated below asserts that linear normal hyperbolicity along the homoclinic orbit implies the existence of a homoclinic center manifold.
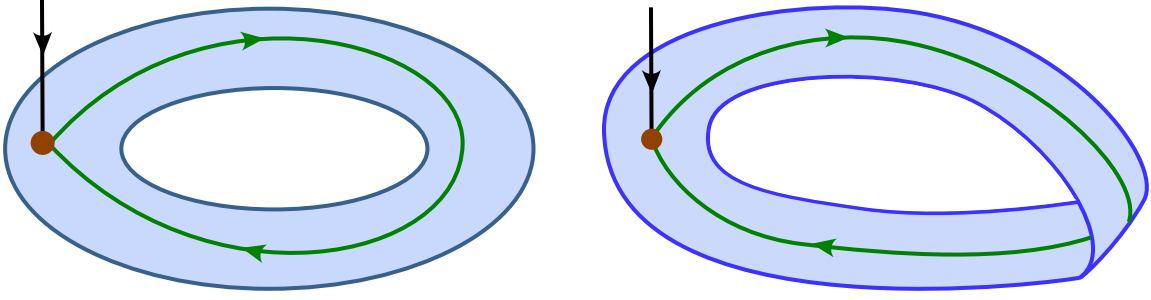
Figure 3.3: *If the bundle E^c is two-dimensional, a homoclinic center manifold is a two-dimensional surface that is diffeomorphic to either an annulus [left] or a Möbius band [right]. The associated orientation index $\mathcal{O}$ defined in (2.10) is $\mathcal{O} = 1$ [left] or $\mathcal{O} = -1$ [right].*

Hypothesis 3.1 (Linear normal hyperbolicity). *Assume that $h(t)$ is a homoclinic orbit of (3.15) for $\mu = 0$ that converges to the hyperbolic equilibrium p. Suppose further that*

$$\mathrm{spec}(f_u(p,0)) = \sigma^s \cup \sigma^c \cup \sigma^u, \qquad \max \mathrm{Re}\, \sigma^s < \min \mathrm{Re}\, \sigma^c, \qquad \max \mathrm{Re}\, \sigma^c < \min \mathrm{Re}\, \sigma^u,$$

and denote by $E_p^s \oplus E_p^c \oplus E_p^u$ the associated decomposition of $\mathbb{R}^n$ into generalized spectral eigenspaces of $f_u(p,0)$. In this setting, we assume that there are subspaces $E^j(t)$ of $\mathbb{R}^n$ for $j = s, c, u$, defined and continuous in $t \in \mathbb{R}$, so that $E^s(t) \oplus E^c(t) \oplus E^u(t) = \mathbb{R}^n$ for all $t \in \mathbb{R}$ and $E^j(t) \to E_p^j$ as $|t| \to \infty$ for each j, such that the evolution $\Phi(t,s)$ of

$$\dot{v} = f_u(h(t),0)v$$

maps $E^j(s)$ into $E^j(t)$ for all $t, s \in \mathbb{R}$ and each j. Lastly, we assume that $\dot{h}(t) \in E^c(t)$.

The following result has been proved in [182, 339, 343], see also [188, 331, 359, 400].

Theorem 3.3. *Assume that Hypothesis 3.1 is met. Pick any integer $l \geq 1$ and a number $\alpha \in (0,1)$ so that*

$$l + \alpha < \min \left\{ \frac{\min \mathrm{Re}\, \sigma^c}{\max \mathrm{Re}\, \sigma^s}, \frac{\min \mathrm{Re}\, \sigma^u}{\max \mathrm{Re}\, \sigma^c} \right\},$$

then there are a constant $\epsilon > 0$ and a locally invariant, normally hyperbolic homoclinic center manifold $W_{\mathrm{hom}}^c(\mu)$ associated with $h(t)$ and defined for $|\mu| < \epsilon$ with the following properties: $W_{\mathrm{hom}}^c(\mu)$ is of class $C^{l,\alpha}$ jointly in (u, μ) and has dimension equal to $\dim E_p^c$.

The tangent bundle $E^c(t) = T_{h(t)} W_{\mathrm{hom}}^c(0)$ along the homoclinic orbit is uniquely defined and, in fact, given by the intersection of $T_{h(t)} W^{s,lu}(p,0)$ and $T_{h(t)} W^{ls,u}(p,0)$.

If Hypotheses 2.2(i), 2.3(ii), and 2.4 are met, then $E^c(t)$ is a continuous bundle of planes that limit on the eigenspace of $f_u(p,0)$ associated with the leading eigenvalues, and a two-dimensional homoclinic center manifold therefore exists in this situation; see Figure 3.3. In this case, the orientability of $W_{\mathrm{hom}}^c(\mu)$ is determined by the index $\mathcal{O}$ defined in (2.10).

3.5 Stable foliations

For some global bifurcations, reductions to homoclinic center manifolds or other global center manifolds are not possible. In such situations, stable foliations may still provide reductions to semiflows on branched manifolds. This applies, in particular, to flows that contain Lorenz-like attractors, but also to studies of annihilation processes of suspended horseshoes through homoclinic bifurcations. Whether such reductions are helpful depends on the smoothness of the foliation (i.e. the smoothness of the holonomy map along the leaves of the foliation).

Geometric models for Lorenz-like attractors depend on the existence of a stable foliation $\mathcal{F}^{ss}$ with one-dimensional leaves for the flow. Stable foliations can be constructed by graph transform techniques. Their

smoothness, however, depends on spectral gap conditions at the equilibrium: if $\lambda^{\mathrm{ss}} < \lambda^{\mathrm{ls}}$ are the stable eigenvalues, and λ^{lu} is the unstable eigenvalue, then the stable foliation is C^l for $l < (\lambda^{\mathrm{ls}} - \lambda^{\mathrm{ss}})/\lambda^{\mathrm{lu}}$; see [324]. In particular, for open sets of eigenvalues, the stable foliation is merely continuous. A stable foliation $\mathcal{G}^{\mathrm{ss}}$ for a return map on a cross section is obtained by projecting $\mathcal{F}^{\mathrm{ss}}$ along flow lines into the section. This projection can be expected to increase smoothness and map $\mathcal{F}^{\mathrm{ss}}$ to a continuously differentiable foliation. A general result along these lines has been obtained by Homburg [182] in the context of bifurcations of singular horseshoes: we shall present it later in Proposition 4.3. Similar statements apply, for instance, to Lorenz-like attractors that occur in the unfolding of two homoclinic loops to an equilibrium with resonant eigenvalues; a continuous stable foliation for the flow projects to a continuously differentiable stable foliation for the return map on a cross section.

Alternatively, one could construct foliations directly for the return map defined on some cross section, rather than constructing them for flows and then projecting along flow lines. We shall now present a theorem that gives continuously differentiable stable foliations for return maps with prescribed asymptotic expansions. Consider a map $\Pi = (f, g)$ from $D = ([-1, 1] \setminus \{0\}) \times [-1, 1]^n \subset \mathbb{R} \times \mathbb{R}^n$ to itself of the form

$$
f(x, y) = \begin{cases} x_*^- + |x|^\alpha (A^- + \phi^-(x, y)), & x < 0, \\ x_*^+ + |x|^\alpha (A^+ + \phi^+(x, y)), & x > 0, \end{cases}
$$

$$
g(x, y) = \begin{cases} y_*^- + |x|^\alpha \psi^-(x, y), & x < 0, \\ y_*^+ + |x|^\alpha \psi^+(x, y), & x > 0. \end{cases}
$$

We make the following assumption.

Hypothesis 3.2 (Asymptotic expansions). *Assume that the following is true for some $\eta > 0$:*

(i) $A^-, A^+ \neq 0$.

(ii) $\phi^\pm, \psi^\pm$ *have continuous derivatives up to order two in D.*

(iii) *For some $\varepsilon > 0$, $\|\frac{\partial^{k+l}}{\partial x^k \partial y^l} \phi^\pm\| \leq \varepsilon |x|^{\eta-k}$ and $\|\frac{\partial^{k+l}}{\partial x^k \partial y^l} \psi^\pm\| \leq \varepsilon |x|^{\eta-k}$.*

The following result due to Shashkov and Shil'nikov is stated for maps with sufficiently small higher-order terms; sharper results can be found in [357].

Theorem 3.4 ([357]). *There exists an $\varepsilon_0 > 0$ so that, if Hypothesis 3.2 is met for $\varepsilon < \varepsilon_0$, then Π admits a stable C^1-smooth foliation on D with C^2 leaves that contains $\{x = 0\}$.*

The preceding result is similar to a result by Rychlik [334], who considered maps on D that are close to $(x, y) \mapsto (1 - c\,\mathrm{sign}(x)|x|^\alpha, 0)$ with $c \in (1, 2)$ and $c\alpha > 2$; closeness is expressed by estimates as in Hypothesis 3.2(iii) with $\alpha + \eta > 1$.

3.6 Case study: The creation of periodic orbits from a homoclinic orbit

We illustrate the different approaches to homoclinic bifurcation theory by applying them to a homoclinic bifurcation of codimension one with real leading eigenvalues: specifically, we assume that Hypotheses 2.1, 2.2, 2.3(iv) and 2.4 are all satisfied. First, we derive bifurcation equations that capture periodic orbits using, separately, Shil'nikov variables and Lin's method. Afterwards, we discuss how homoclinic center manifolds can be used to derive similar results. The resulting phase and bifurcation diagrams are summarized in Figure 3.4. We remark that this bifurcation is an example of a blue sky catastrophe where a periodic orbit disappears in a bifurcation at which its period goes to infinity.

3.6.1 Shil'nikov variables

For Shil'nikov variables, it is convenient to introduce a single cross section Σ that is transverse to the homoclinic solution h at $\mu = 0$. Proceeding in this way will also illuminate the differences and similarities between the approaches via Shil'nikov variables and Lin's method.
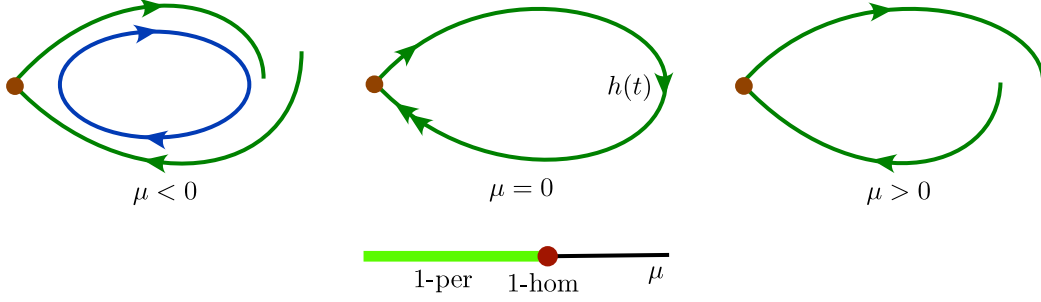
Figure 3.4: *The phase [top row] and bifurcation [bottom panel] diagrams of a nondegenerate homoclinic orbit $h(t)$ to an equilibrium with real leading eigenvalues $|\nu^{\mathrm{s}}| > \nu^{\mathrm{u}}$ is shown. A unique attracting single-round periodic orbit bifurcates to one side of $\mu = 0$ (for $\mu < 0$, say), and its period goes to infinity as μ tends to zero.*

Let $\Pi(\cdot, \mu) : \Sigma \to \Sigma$ be the first-return map given by the flow of $\dot{u} = f(u, \mu)$ and consider the dynamical system $x_{j+1} = \Pi(x_j, \mu)$ for $x_j \in \Sigma$. If both the stable and unstable manifolds of p along the homoclinic orbit are not in an inclination-flip configuration, then a parameter-dependent coordinate system $x = (x^{\mathrm{ss}}, x^{\mathrm{lu}}, x^{\mathrm{uu}})$ on Σ can be chosen so that

$$
\begin{aligned}
W^{\mathrm{s}}(p) \cap \Sigma & = & \{x^{\mathrm{lu}}, x^{\mathrm{uu}} = 0\}, \\
W^{\mathrm{u}}(p) \cap \Sigma & = & \{x^{\mathrm{ss}}, x^{\mathrm{lu}} = 0\}, \\
W^{\mathrm{s,lu}}(p) \cap \Sigma & \overline{\cap}_{W^{\mathrm{s}}(p) \cap \Sigma} & \{x^{\mathrm{uu}} = 0\}, \\
W^{\mathrm{ls,u}}(p) \cap \Sigma & \overline{\cap}_{W^{\mathrm{u}}(p) \cap \Sigma} & \{x^{\mathrm{ss}} = 0\},
\end{aligned}
$$

where the notation $W \overline{\cap}_q V$ means that W is tangent to V at q. The following result provides a normal form for the return map on Σ. Its proof combines Proposition 3.2 for solutions between the local cross sections Σ_{in} and Σ_{out} with expansions for the transition maps between Σ and Σ_{in} and between Σ_{out} and Σ; the transition time from Σ_{in} to Σ_{out} is solved for as a function of the initial data on these cross sections.

Proposition 3.3 ([188]). *Assume that Hypothesis 2.4 is met. In the coordinates constructed above, $x_{j+1} = \Pi(x_j, \mu)$ and x_j are related by*

$$
(x_{j+1}^{\mathrm{ss}}, x_{j+1}^{\mathrm{lu}}, x_j^{\mathrm{uu}}) = \tilde{\Pi}(x_j^{\mathrm{ss}}, x_j^{\mathrm{lu}}, x_{j+1}^{\mathrm{uu}}, \mu)
$$

for some map $\tilde{\Pi}$ with the asymptotics

$$
\begin{aligned}
x_{j+1}^{\mathrm{ss}} & = & \mathrm{O}(|x_j^{\mathrm{lu}}|^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}+\eta}), \\
x_{j+1}^{\mathrm{lu}} & = & a(\mu) + \varphi(\mu)[x_j^{\mathrm{lu}}]^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}} + \mathrm{O}(|x_j^{\mathrm{lu}}|^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}+\eta}), \\
x_j^{\mathrm{uu}} & = & \mathrm{O}(|x_{j+1}^{\mathrm{lu}}|^{1+\eta}),
\end{aligned}
\tag{3.16}
$$

where a and φ are smooth functions of μ.

Consider a sequence $\{x_j\}_{j\in\mathbb{Z}}$ in Σ. These points lie on the same orbit of $\Pi(\cdot, \mu)$ if, and only if,

$$
(x_{j+1}^{\mathrm{ss}}, x_{j+1}^{\mathrm{lu}}, x_j^{\mathrm{uu}}) - \tilde{\Pi}(x_j^{\mathrm{ss}}, x_j^{\mathrm{lu}}, x_{j+1}^{\mathrm{uu}}, \mu) = 0, \quad j \in \mathbb{Z}
\tag{3.17}
$$

or, more detailed,

$$
\begin{aligned}
x_{j+1}^{\mathrm{ss}} - \mathrm{O}(|x_j^{\mathrm{lu}}|^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}+\eta}) & = & 0, \\
x_{j+1}^{\mathrm{lu}} - a(\mu) - \varphi(\mu)[x_j^{\mathrm{lu}}]^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}} + \mathrm{O}(|x_j^{\mathrm{u}}|^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}+\eta}) & = & 0, \\
x_j^{\mathrm{uu}} - \mathrm{O}(|x_{j+1}^{\mathrm{u}}|^{1+\eta}) & = & 0.
\end{aligned}
\tag{3.18}
$$

If we denote by $l^\infty_{\mathbb{R}^N}$ be the space of bi-infinite sequences with entries in $\mathbb{R}^N$ equipped with the supremum norm, then equation (3.17) can be considered as an equation in $l^\infty_{\mathbb{R}^{n_{\mathrm{ss}}}} \times l^\infty_{\mathbb{R}} \times l^\infty_{\mathbb{R}^{n_{\mathrm{uu}}}}$. Since the first and third

equation in (3.18) depend smoothly on $(x^{\mathrm{ss}}, x^{\mathrm{uu}})$, we can use an implicit function theorem in l^∞, see [38] or [89], to solve the first and third equation in (3.18) for $(x^{\mathrm{ss}}, x^{\mathrm{uu}})$. Substitution into the second equation of (3.18) gives the reduced bifurcation equations

$$x_{j+1}^{\mathrm{lu}} - a(\mu) - \varphi(\mu)[x_j^{\mathrm{lu}}]^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}} + \mathrm{O}(|x^{\mathrm{u}}|^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}+\eta}) = 0. \tag{3.19}$$

Note that the higher-order terms depend on the entire sequence $x^{\mathrm{lu}} = \{x_j^{\mathrm{lu}}\}_{j\in\mathbb{Z}}$. Imposing periodicity on the entire sequence, we obtain N reduced bifurcation equations if we wish to investigate N-periodic orbits of $\Pi(\cdot, \mu)$ that have period N. In particular, the reduced bifurcation equation for a single-round periodic orbit takes the form

$$x^{\mathrm{lu}} - a(\mu) - \varphi(\mu)[x^{\mathrm{lu}}]^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}} - \mathrm{O}(|x^{\mathrm{lu}}|^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}+\eta}) = 0,$$

where $x^{\mathrm{lu}} \in \mathbb{R}$ with $|x^{\mathrm{lu}}| \ll 1$. Solving is straightforward: if $-\nu^{\mathrm{s}} > \nu^{\mathrm{u}}$, then the solution is $x^{\mathrm{lu}} = a'(0)\mu + \mathrm{o}(\mu)$ where μ is such that $a'(0)\mu > 0$; similarly, if $-\nu^{\mathrm{s}} < \nu^{\mathrm{u}}$, then the solution is $[x^{\mathrm{lu}}]^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}} = -a'(0)\mu/\varphi(0) + \mathrm{o}(\mu)$ where μ is such that $a'(0)\mu/\varphi(0) < 0$.

3.6.2 Lin's method

To apply the technique outlined in §3.3 to the bifurcation of single-round periodic orbits from homoclinic orbits, we assume $|\nu^{\mathrm{s}}| > \nu^{\mathrm{u}}$ and consider the bifurcation equations (3.14)

$$-\langle v_*^{\mathrm{u}}(\mu), v^{\mathrm{u}}(\mu)\rangle \mathrm{e}^{-2\nu^{\mathrm{u}} T_{j+1}} + M\mu + \mathrm{O}\left(\mathrm{e}^{-(2\nu^{\mathrm{u}}+\eta)\min_{k\in\mathbb{Z}} T_k} + |\mu|^2\right) = 0 \tag{3.20}$$

for $j \in \mathbb{Z}$ that describe the existence of solutions that follow the homoclinic orbit with prescribed return times $T_j > T_*$, where the Melnikov integral M has been defined in Hypothesis 2.2. To find single-round periodic orbits, we set $T_j = T$ for all j: With this choice, (3.20) is the same equation for all j due to uniqueness of solutions to (3.12), and (3.20) therefore reduces to

$$-\langle v_*^{\mathrm{u}}(\mu), v^{\mathrm{u}}(\mu)\rangle \mathrm{e}^{-2\nu^{\mathrm{u}} T} + M\mu + \mathrm{O}\left(\mathrm{e}^{-(2\nu^{\mathrm{s}}+\eta)T} + |\mu|^2\right) = 0. \tag{3.21}$$

For $M \neq 0$, this equation can be solved uniquely for μ as a function of T to get

$$\mu = \mu_*(T) = \frac{1}{M}\langle v_*^{\mathrm{u}}(0), v^{\mathrm{u}}(0)\rangle \mathrm{e}^{-2\nu^{\mathrm{u}} T} + \mathrm{O}\left(\mathrm{e}^{-(2\nu^{\mathrm{s}}+\eta)T}\right), \tag{3.22}$$

which proves that periodic orbits bifurcate from the homoclinic orbit and shows how period and system parameter are related for large periods.

3.6.3 Homoclinic center manifolds

As in local bifurcation theory, where Lyapunov–Schmidt reductions and center-manifold reductions each have their own advantages and disadvantages, analytic and geometric approaches to global bifurcation theory can complement each other: some aspects and questions are easier to investigate from a geometric viewpoint while, in other situations, it might be advantageous to use analytical techniques.

The assumptions we made at the beginning of §3.6 guarantee that the homoclinic center manifold given in Theorem 3.3 is two-dimensional and that it is of class $C^{1+\alpha}$ for some $\alpha > 0$ and depends in a $C^{1+\alpha}$ fashion on the parameter μ. The lack of smoothness of homoclinic center manifolds is often an obstacle for deriving and solving bifurcation equations near homoclinic orbits. On the other hand, the dimension reduction gives much insight into the geometry of the flow, which is helpful when studying stability and hyperbolicity. In our case, the existence of a two-dimensional homoclinic center manifold $W_{\mathrm{hom}}^{\mathrm{c}}$ immediately excludes the existence of N-periodic solutions for $N > 2$ if $W_{\mathrm{hom}}^{\mathrm{c}}$ is nonorientable, and for $N > 1$ for orientable $W_{\mathrm{hom}}^{\mathrm{c}}$.

Even though the homoclinic center manifold is only $C^{1+\alpha}$, it is possible to find continuously differentiable linearizing coordinates of the flow on $W_{\mathrm{hom}}^{\mathrm{c}}(\mu)$ [182]. The argument to prove this statement utilizes the smoothness of the flow in $\mathbb{R}^n$ and differs as such from a parameter-dependent version of linearization results by Belitskii [30].
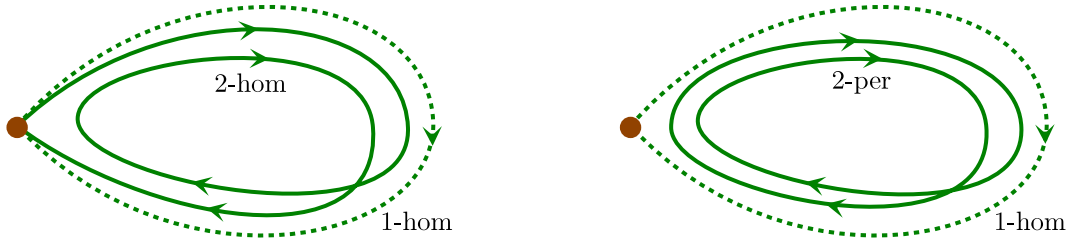
Figure 4.1: *Shown are a 2-round homoclinic orbit [left] and a 2-round periodic orbit [right], relative to a given primary homoclinic orbit. N-homoclinic and N-periodic orbits for larger N are defined analogously: these solutions make N rounds near the primary, or 1-homoclinic, orbit.*

Proposition 3.4. *There exist constants $\alpha > 0$ and $\varepsilon > 0$ so that, for $|\mu| < \varepsilon$, there are local $C^{1+\alpha}$ coordinates on $W_{\mathrm{hom}}^{\mathrm{c}}(\mu)$ near the equilibrium $p(\mu)$, which are also $C^{1+\alpha}$ in μ, for which the local transition map $\Pi_{\mathrm{loc}} : W_{\mathrm{hom}}^{\mathrm{c}}(\mu) \cap \Sigma_{\mathrm{in}} \to W_{\mathrm{hom}}^{\mathrm{c}}(\mu) \cap \Sigma_{\mathrm{out}}$ has the expression $\Pi_{\mathrm{loc}}(x, \mu) = x^{-\nu^{\mathrm{s}}(\mu)/\nu^{\mathrm{u}}(\mu)}$.*

Up to a reparameterization, the first-return map Π on $W_{\mathrm{hom}}^{\mathrm{c}}(\mu) \cap \Sigma_{\mathrm{in}}$ is therefore given by

$$\Pi(x, \mu) = \mu + x^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}}(a(\mu) + \mathrm{O}(x^\eta))$$

for some $\eta > 0$, and the bifurcation of single-round periodic orbits follows trivially from this expression. We mention again that the geometric approach via homoclinic center manifolds immediately excludes the existence of multi-round homoclinic and periodic solutions.

4 Phenomena

Although there are a great many types of homoclinic and heteroclinic bifurcations, as evidenced by the considerable list in §5, a number of features are common to several of them. In this section, we discuss such common features. We also survey a number of transitions through global bifurcations from Morse–Smale flows with finitely many critical elements (equilibria and periodic orbits) to complicated dynamics involving suspended transitive hyperbolic sets, singular hyperbolic attractors, and suspended Hénon-like attractors. Homoclinic-doubling cascades, the analogue of period-doubling cascades for homoclinic orbits, provide another mechanism for the transition from Morse–Smale to non Morse–Smale flows. Finally, we review the creation of intermittent time series through homoclinic bifurcations.

4.1 N-pulses and N-periodic orbits

The occurrence of multi-round homoclinic orbits is of central importance in applications to spatial dynamics, where they correspond to travelling or standing multi-pulses. A large collection of homoclinic bifurcation results have been derived with these applications in mind.

Multi-round homoclinic orbits may be created in bifurcations from a homoclinic orbit, which we refer to as the primary or single-round homoclinic orbit. The definition of a multi-round homoclinic orbit is given with reference to a tubular neighborhood of the primary homoclinic orbit. Let $\dot{u} = f(u, \mu)$ be a family of ODEs with a homoclinic orbit $h = \{h(t)\}_{t \in \mathbb{R}}$ at $\mu = 0$ and denote by by $\mathcal{U}$ a small tubular neighborhood of the closure of the homoclinic orbit h. If Σ is a cross section placed at $h(0)$ that is transverse to the orbit h when $\mu = 0$, then a homoclinic loop contained in $\mathcal{U}$ is called an N-homoclinic orbit or N-round homoclinic loop if it intersects Σ precisely N times. Multi-round periodic orbits are defined analogously by counting the number of intersections with Σ. Figure 4.1 illustrates a 2-round homoclinic orbit and a 2-round periodic orbit.

Near Shil'nikov saddle-focus homoclinic orbits, N-homoclinic orbits for all N appear in the unfolding provided the leading eigenvalues satisfy a certain condition (in this case, the homoclinic orbit is called a wild saddle-focus homoclinic orbit). Although perturbations from a codimension-one homoclinic bifurcation with real

leading eigenvalues do not give rise to multi-round homoclinic orbits, various codimension-two homoclinic bifurcations with real leading eigenvalues do. Consider a two-parameter family $\dot{u} = f(u, \mu)$ of ODEs that have a homoclinic solution $h(t)$ to a hyperbolic equilibrium with real leading eigenvalues when $\mu = 0$. As long as Hypotheses 2.1 and 2.2 hold, the homoclinic orbit persists along a curve in parameter space (see §6.1 for a topological continuation theory for homoclinic orbits). At isolated parameter points along such a curve, and assuming that the leading eigenvalues stay real, one of the conditions in Hypotheses 2.3 or 2.4 may be violated. At these points, multi-round homoclinic solutions may bifurcate from the curve of primary homoclinic solutions, and we refer to the bifurcation theorems in §5.1.5, §5.1.6, and §5.1.7 for the results. Subject to further conditions on spectrum and geometry of the flow, a double-round homoclinic orbit will be created in all these cases.

If the dynamics contains N-periodic orbits for all N, then the overall dynamics is often organized around suspended horseshoes: we say that the system has a suspended horseshoe if the first-return map Π to a cross section Σ of the homoclinic orbit admits a Smale's horsehoe, that is, a compact hyperbolic invariant set on which the dynamics coming from Π is conjugated to a shift on two symbols; see, for instance, [309] for further details. Suspended horseshoes are found in the unfoldings of various homoclinic bifurcations. Wild saddle-focus homoclinic orbits and bi-focus homoclinic orbits have suspended horseshoes in each neighborhood of the homoclinic orbit. Homoclinic orbits to equilibria with real leading eigenvalues can give rise to suspended horseshoes in cases of higher codimension. In particular, suspended horseshoes can be found in the unfolding of codimension-two inclination-flip and orbit-flip bifurcations, provided certain eigenvalue conditions are met. In all these scenarios, suspended horseshoes are created through homoclinic tangencies (akin to the Hénon family), which implies that it is impossible to give a complete description of the bifurcations in finitely many parameters: we refer to [44, 309] for an overview of the underlying theory.

4.2 Robust singular dynamics

Within the framework of perturbations in the C^1 topology, a comprehensive theory of generic dynamical properties of flows exists, at least for three-dimensional flows, which is still under active development. This program is in the spirit of Palis's papers [305, 306] in which conjectures for the dynamics of typical systems is sketched out. We will only skim through the relevant results, instead referring to the books [13, 44] and the review article [317] for further discussion.

The central technique is the C^1-connection lemma by Hayashi [174, 175]. For critical elements[4] σ, the connection lemma entails the following. Let σ be a hyperbolic critical element of a differential equation $\dot{u} = f(u)$ and suppose there exists a point $q \notin \sigma$ in $\left(\overline{W^s(\sigma) \cap W^u(\sigma)} \right) \cup \left(\overline{W^u(\sigma) \cap W^s(\sigma)} \right)$; such a point is called an almost homoclinic point. There is then a differential equation C^1 close to f that coincides with f in a neighborhood of σ and has a homoclinic solution to σ.

Making use of his connection lemma, Hayashi finished in [174, 175] the proof of the C^1 stability conjecture for flows, which states that a differential equation on a three-dimensional compact manifold is C^1 structurally stable if, and only if, it is uniformly hyperbolic and all stable and unstable manifolds are transverse.

There are, however, open sets of flows that are not C^1 structurally stable, so that the search for typical dynamical properties remains. Arroyo and Rodriguez-Hertz [24] proved that a differential equation on a three-dimensional compact manifold can be C^1 approximated either by a system that is uniformly hyperbolic or that has a homoclinic tangency between stable and unstable manifolds of periodic orbits or a singular cycle; a *singular cycle* is a heteroclinic cycle between critical elements among which is at least one equilibrium.

In this context, transitive but non-hyperbolic sets can exist. The primary example is the Lorenz attractor,

---

[4]A critical element is a periodic orbit or an equilibrium

which is the 'butterfly' attractor in the Lorenz equations [260, 383] given by

$$
\begin{aligned}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= \rho x - y - xz, \\
\dot{z} &= -\beta z + xy.
\end{aligned}
$$

To understand the geometry of the Lorenz attractor, geometric models have been developed [3, 162, 423]. The robust strange attractors in these models are called *geometric Lorenz attractors* or *Lorenz-like attractors*. Tucker provided a computer-assisted proof for the existence of a robust strange attractor in the Lorenz equations whose geometry is that of the robust strange attractors in geometric Lorenz models.

Theorem 4.1 ([397, 398]). *The Lorenz equations support a robust strange attractor for the classical parameter values $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$.*

It turns out that the relevant notion for a general theory of robust transitive sets is that of dominated splitting. A compact invariant set Λ of $\dot{u} = f(u, \mu)$ is a *partially hyperbolic set* if, up to time reversal, there is an invariant dominated splitting $T\Lambda = E^{\mathrm{s}} \oplus E^{\mathrm{c}}$: this means that there are positive constants K, λ such that

(i) E^{s} is contracting: $\|\partial_x \varphi_t|_{E^{\mathrm{s}}_x}\| \leq K e^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$;

(ii) E^{s} dominates E^{c}: $\|\partial_x \varphi_t|_{E^{\mathrm{s}}_x}\| \, \|\partial_x \varphi_t|_{E^{\mathrm{c}}_{\varphi_t(x)}}\| \leq K e^{\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

The central direction E^{c} of Λ is said to be volume expanding if the additional condition

$$
|\det(\partial_x \varphi_t|_{E^{\mathrm{c}}_x})| \geq K e^{\lambda t}
$$

holds for all $x \in \Lambda$ and $t > 0$. Let Λ be a compact invariant set of $\dot{u} = f(u)$ that contains at least one equilibrium, then Λ is called a *singular hyperbolic set* if it is partially hyperbolic with volume expanding central directions, and Λ is called a *singular hyperbolic attractor* if, in addition, it attracts all points in some open neighborhood.

A compact invariant set Λ for a flow φ_t of $\dot{u} = f(u)$ is robust transitive if it is the maximal invariant set $\cap_{t \in \mathbb{R}} \varphi_t(U)$ inside an open neighborhood U of Λ and, for the flow ψ_t of any differential equation $\dot{u} = g(u)$ with g C^1-close to f, $\cap_{t \in \mathbb{R}} \psi_t(U)$ is a nontrivial transitive set (nontrivial means that it does not consist of a critical element). Morales, Pacífico and Pujals demonstrated for three-dimensional flows that C^1 robust strange attractors which contain equilibria are singular hyperbolic sets. An invariant set is proper if it is not the whole manifold.

Theorem 4.2 ([282]). *A robust transitive set for a three-dimensional flow that contains an equilibrium is a proper singular hyperbolic attractor or repeller.*

There are ODEs in $\mathbb{R}^3$ that possess singular hyperbolic attractors with any number of equilibria, and §5.5.5 provides theorems that can be used to construct such examples; see also Figure 5.28. An example of a singular hyperbolic attractor from a fluid convection model that contains two equilibria can be found in [294]. We refer to [6, 12, 15, 262] for further results, addressing primarily ergodic properties, Lorenz-like and other singular hyperbolic attractors.

A different class of strange attractors that contain an equilibrium is found in contracting Lorenz models. Like the strange attractors encountered in the Hénon family, these strange attractors persist only in a measure theoretic sense. Contracting Lorenz models are geometric Lorenz models but with contracting instead of expanding central directions (the saddle quantity, i.e. the quotient $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ of the leading stable and unstable eigenvalues at the equilibrium, is larger than 1). Arneodo, Coullet and Tresser [19] noted that contracting Lorenz models contain dynamics that can be described by interval maps (in fact unimodal maps if one restricts to $\mathbb{Z}_2$-symmetric flows). Indeed, in geometric (contracting) Lorenz models, an invariant strong

stable foliation enables a reduction to a semiflow on a branched manifold and to interval maps through a first-return map. We recall a result by Rovella on the dynamics of contracting Lorenz models, focusing on strange attractors that contain an equilibrium.

Theorem 4.3 ([333]). *There exists a contracting Lorenz model $\dot{u} = f(u)$ in $\mathbb{R}^3$ with an attractor Λ containing a hyperbolic equilibrium so that the following properties hold:*

(i) *There exists a local basin of attraction $\mathcal{B}$ for Λ, a neighborhood $\mathcal{V}$ of f in the C^3 topology, and an open and dense subset $\mathcal{V}_0 \subset \mathcal{V}$ so that, for an ODE $\dot{u} = g(u)$ with $g \in \mathcal{V}_0$, the maximal invariant set in $\mathcal{B}$ consists of the equilibrium, one or two attracting periodic orbits, and a hyperbolic basic set.*

(ii) *In generic two-parameter families $\dot{u} = f(u, \mu)$ with $f(\cdot, 0) = f$, there is a set of positive measure containing $\mu = 0$ as a density point for which an attractor in $\mathcal{B}$ containing the equilibrium exists.*

The proof of the above result uses further eigenvalue conditions at the equilibrium to ensure C^3 strong stable foliations. Such strange attractors can occur in unfoldings of (symmetric) resonant homoclinic loops [284, 327]; see also §5.5.5.

Another type of strange attractors in three-dimensional flows that contain an equilibrium would be formed by strange attractors for which there is no strong stable foliation near the attractor and hence no reduction to a branched manifold. Apart from spiral attractors (see §5.1.2), these can be found in hooked Lorenz models [44]. Such attractors may also occur in unfoldings from certain homoclinic and heteroclinic bifurcations; see [228, 293]. For an analysis of the dynamics of interval maps with both singularities and critical points that yield approximate one-dimensional models for these attractors, see [110, 263, 264].

Examples of higher-dimensional robust strange attractors that contain equilibria are described in [46, 406].

4.3 Singular horseshoes

In this section we review the creation, or disappearance, of a suspended Smale horseshoe through sequences of homoclinic bifurcations; each periodic orbit in the horseshoe will be created in a homoclinic bifurcation. We will point out relations with bifurcation theory of singular cycles between an equilibrium and a periodic orbit. The disappearance of suspended horseshoes through homoclinic bifurcations has been shown to occur near the codimension-two bifurcations of homoclinic orbits in inclination-flip or orbit-flip configurations, which will be discussed in §5. The appearance in these bifurcations makes this phenomenon a common feature in the bifurcation diagrams of models of ODEs with two or more parameters.

Starting point is the disappearance through a homoclinic bifurcation of a periodic orbit that is part of a larger transitive invariant set. This can be expected to trigger additional bifurcations of orbits within the invariant set. The basic case illustrating this phenomenon is where the periodic orbit lies in the suspension of a hyperbolic horseshoe, and we review this scenario in detail below, following [182]. It is instructive to compare the bifurcation scenario with other scenarios in which horseshoes break up, in particular the involved scenarios triggered by homoclinic tangencies [309] or saddle-node bifurcations of periodic orbits [94, 96, 109, 431].

To fix ideas, consider a differential equation that admits an invariant set which is a suspended Smale horseshoe Λ. Orbits in Λ are coded by doubly infinite sequences $\{0, 1\}^{\mathbb{Z}}$. Write q for the periodic orbit that corresponds to the coding 0^{∞}. The stable manifold $W^{\mathrm{s}}(q)$ is a one-sided boundary leaf of the lamination $W^{\mathrm{s}}(\Lambda)$: near a point $x \in W^{\mathrm{s}}(q)$, other leaves in $W^{\mathrm{s}}(\Lambda)$ can be found only on one side of $W^{\mathrm{s}}(q)$. The unstable manifold $W^{\mathrm{u}}(q)$ is likewise a one-sided boundary leaf of the lamination $W^{\mathrm{u}}(\Lambda)$. Note that both $W^{\mathrm{s}}(q)$ and $W^{\mathrm{u}}(q)$ are orientable surfaces.

Bifurcations that possess a suspended horseshoe for $\mu < 0$ and undergo a homoclinic bifurcation of the periodic orbit $q(t)$ at $\mu = 0$ can be identified in the following set-up. Assume $\dot{u} = f(u, \mu)$ is a one-parameter family of ODEs on $\mathbb{R}^n$ that has a homoclinic solution $h(t)$ at $\mu = 0$ which satisfies Hypotheses 2.1, 2.2,
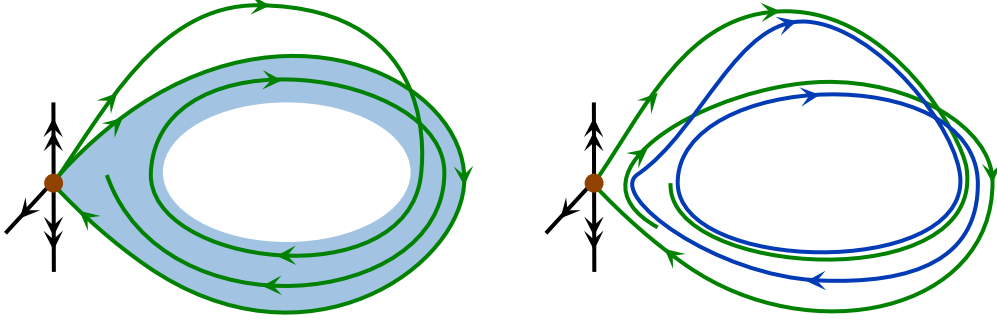
Figure 4.2: *A homoclinic orbit that coexists with a superhomoclinic orbit, which connects the equilibrium to the homoclinic orbit, implies the existence of a singular horseshoe [left]. Singular cycles between the equilibrium and a periodic orbit appear in the unfolding of singular horseshoes [right].*

2.3(ii), and 2.4. In particular, the linearization $f_u(p, 0)$ has unique real leading eigenvalues and is not in an inclination-flip or orbit-flip configuration.

Hypothesis 4.1. *Consider the following spectral and geometric conditions:*

(i) $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$.

(ii) The two-dimensional homoclinic center manifold $W_{\mathrm{hom}}^{\mathrm{c}}$ near h is an annulus.

Under these hypotheses one can identify a nonempty stable set of the homoclinic orbit.

Lemma 4.1. *If $\dot{u} = f(u, 0)$ has a homoclinic orbit h to an equilibrium p so that Hypotheses 2.1, 2.2, 2.3(ii), and 2.4 are met and, in addition, Hypothesis 4.1 is satisfied, then the stable set*

$$\mathcal{M}^{\mathrm{s}}(\overline{h}) = \{x \in \mathbb{R}^n; \ \omega(x) = \overline{h}\},$$

where $\overline{h} = h \cup \{p\}$ is the closure of h, is a manifold with boundary $W^{\mathrm{s}}(p)$, and $\dim \mathcal{M}^{\mathrm{s}}(\overline{h}) = 1 + \dim W^{\mathrm{s}}(p)$.

Under the above assumptions, so-called superhomoclinic orbits may exist which are transverse intersections of $\mathcal{M}^{\mathrm{s}}(\overline{h})$ and $W^{\mathrm{u}}(p)$ [182, 376, 401]. The following proposition connects the resulting bifurcations to bifurcations of singular cycles between an equilibrium and a periodic orbit as further illustrated in Figure 4.2.

Proposition 4.1. *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of ODEs on $\mathbb{R}^n$ with a homoclinic orbit h to an equilibrium p so that Hypotheses 2.1, 2.2, 2.3(ii), and 2.4 are met. If Hypothesis 4.1 is met and $\mathcal{M}^{\mathrm{s}}(\overline{h}) \pitchfork W^{\mathrm{u}}(p)$, then there are bifurcation values that accumulate onto $\mu = 0$ at which $\dot{u} = f(u, \mu)$ possesses a singular cycle between p and a saddle periodic orbit $q(t)$ such that $\dim W^{\mathrm{u}}(q) = \dim W^{\mathrm{u}}(p)$ and $W^{\mathrm{s}}(q) \pitchfork W^{\mathrm{u}}(p)$.*

In the reverse direction, if a singular cycle as stated exists, the λ-lemma implies that $W^{\mathrm{u}}(p)$ accumulates onto $W^{\mathrm{u}}(q)$, and small perturbations of the differential equation will create homoclinic orbits to p. Bifurcations from singular cycles form an interesting problem in their own right, and we review relevant material on this subject, which was initiated in [28], in §5.2.4. We note that the singular cycles in Proposition 4.1 are called expanding (and remark that the definitions in [28] consider the time reversed flow).

Let $\dot{u} = f(u, \mu)$ be a one-parameter family of ODEs on $\mathbb{R}^n$ with a homoclinic orbit h to an equilibrium p as in the above proposition, so that $\mathcal{M}^{\mathrm{s}}(\overline{h}) \pitchfork W^{\mathrm{u}}(p)$ along a solution $h_1(t)$. Identify a small neighborhood $\mathcal{U}$ of the closed set $\overline{h \cup h_1}$. We will formulate a bifurcation theorem for orbits in $\mathcal{U}$. Along the unstable separatrices of p one can identify a bundle of center directions E^{c} that are invariant under the variational equation and converge to the sum of the leading stable and leading unstable directions over p as the base point approaches p (see §3.4). This bundle yields a continuous plane bundle over $\overline{h \cup h_1}$.

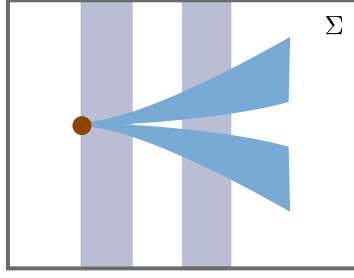**Hypothesis 4.2.** *Different orientations of the bundle of center directions E^{c} occur:*

Figure 4.3: *The return map for a singular horseshoe (pictured here in a two-dimensional section Σ) maps two wedge-shaped regions to vertical strips, roughly contracting horizontal directions and expanding vertical directions.*

(i) *The center bundle E^{c} along $\overline{h \cup h_1}$ is orientable.*

(ii) *The center bundle E^{c} along $\overline{h \cup h_1}$ is nonorientable.*

Theorem 4.4 ([182])**.** *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of ODEs on $\mathbb{R}^n$ with a homoclinic orbit h to the equilibrium p and a generalized homoclinic orbit h_1 in $\mathcal{M}^{\mathrm{s}}(\overline{h}) \pitchfork W^{\mathrm{u}}(p)$ so that Hypotheses 2.1, 2.2, 2.3(ii), and 2.4 are met. On one side of $\mu = 0$, say for $\mu < 0$, the invariant set in $\mathcal{U}$ is then a suspended horseshoe. For $\mu \geq 0$, the bifurcation set is a set of Lebesgue measure zero with the following properties:*

(i) *If Hypothesis 4.2(i) is met, then the bifurcation set is a Cantor set in which homoclinic bifurcation values lie dense.*

(ii) *If Hypothesis 4.2(ii) is met, then the bifurcation set is the union of a Cantor set in which homoclinic bifurcation values lie dense and infinitely many sequences of homoclinic bifurcation values that converge to points in the Cantor set.*

First, we describe the dynamics for $\mu < 0$. Let Σ be a small cross section transverse to the homoclinic orbit h. The return map on Σ maps two horizontal wedge-like strips that emerge from the same point to two vertical strips; see Figure 4.3. The invariant set of $\dot{u} = f(u, 0)$ in $\mathcal{U}$ is a singular hyperbolic set (as introduced in §4.2 for flows in $\mathbb{R}^3$ after reversing the direction of time) which we call a *singular horseshoe* (we note that our usage of this term differs from [246]).

We write Π^{-1} for the continuous extension of the inverse of the first-return map on Σ and remark that Π^{-1} has a fixed point at $W^{\mathrm{s}}(p) \cap \Sigma$; note that Π^{-1} is not invertible. Let Λ be the maximal invariant set of Π^{-1}, then we claim that iterates of Π^{-1} restricted to Λ are conjugate to a factor of a full shift on two symbols. To make this precise, let $\Omega = \{0, 1\}^{\mathbb{Z}}$ equipped, as usual, with the product topology. Let τ be the right shift $\tau s(i) = s(i - 1)$ on Ω. Define an equivalence relation on Ω by $s \sim s'$ if $s(i) = s'(i) = 0$ for all $i \leq 0$. Let $\Omega_* = \Omega/\sim$, and let τ_* be the map on Ω_* induced by τ; observe that τ_* is not invertible. Denote the two vertical strips in $\Pi(\Sigma)$ by V_0, V_1 with $W^{\mathrm{s}}(p) \cap \Sigma$ lying in the closure of V_0. For $x \in \Lambda$, let $h(x) \in \Omega_*$ be the itinerary of x under Π^{-1}: $h(x)(i) = j$ if $[\Pi^{-1}]^i(x) \in \overline{V_j}$, then h gives a conjugacy of Π^{-1} on Λ with τ_* on Ω_*, see [102].

Proposition 4.2. $h \circ \Pi^{-1} = \tau_* \circ h$.

The orbits in the suspended horseshoe, which exists for $\mu < 0$, disappear in bifurcations for $\mu > 0$ as stated in Theorem 4.4. The homoclinic bifurcations that occur for $\mu > 0$ come in two types, depending on the orientability of the homoclinic center manifolds. The homoclinic bifurcations of nonorientable homoclinic orbits are the locally isolated bifurcation values that appear in one of the two cases.

The analysis leading to Theorem 4.4 proceeds through dimension reductions using invariant manifolds and foliations. This reduction leads to interval maps, whose study then gives the theorem. We will continue with some details of the constructions. Choose a coordinate chart (x, y) on Σ so that $W^{\mathrm{s}}(p)$ intersects Σ in
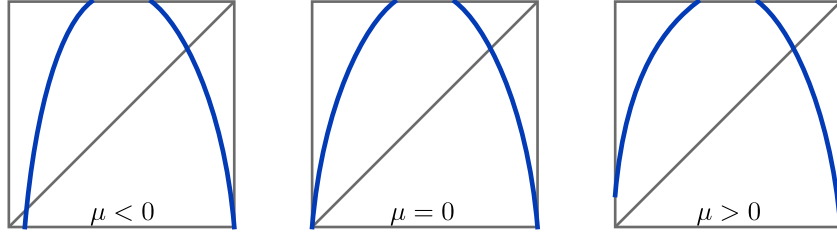
Figure 4.4: *The map π on the leaves of the unstable foliation is an expanding interval map for the case described in Hypothesis 4.2(ii).*

$(0,0)$ and $W^{\mathrm{u}}(p) \cap \Sigma = \{x = 0\}$. The first-return map Π on Σ admits a continuously differentiable unstable foliation that contains $\{x = 0\}$ as a leaf.

Proposition 4.3. *Under the conditions of Theorem 4.4, there exists a normally hyperbolic center-unstable manifold $W^{\mathrm{cu}}(\overline{h \cup h_1})$ in $\mathcal{U}$ that is $C^{1+\alpha}$ for some $\alpha > 0$ jointly in (u, μ). Moreover, there are $C^{1+\alpha}$ coordinates (x, y) on $W^{\mathrm{cu}}(\overline{h \cup h_1}) \cap \Sigma$ so that $\{x = constant\}$ defines an unstable foliation $\mathcal{F}^{\mathrm{u}}$ for the first-return map Π on Σ.*

The invariant manifolds and foliations are constructed by the usual graph transform techniques. We note that the center unstable manifold admits an invariant unstable foliation for the flow which is, in general, only continuous. Its projection along flow lines to the cross section Σ defines an invariant foliation for the first-return map that is continuously differentiable. The return map Π acts on the space of leaves of $\mathcal{F}^{\mathrm{u}}$ as a multi-valued interval map, while its inverse acts as a piecewise expanding interval map π. The domain of π is the union of two intervals I_1, I_2 with the left boundary I_1 equal to 0. In fact, one finds the following asymptotic expansions for π:

$$\pi(x) = \begin{cases} a(\mu) + b(\mu) x^{-\nu^{\mathrm{u}}/\nu^{\mathrm{s}}}(1 + \mathrm{O}(x^{\eta})), & x \in I_l, \\ d(\mu) x^{-\nu^{\mathrm{u}}/\nu^{\mathrm{s}}}(1 + \mathrm{O}(x^{\eta})), & x \in I_2, \end{cases}$$

for some $\eta > 0$; see Figure 4.4.

4.4   The boundary of Morse–Smale flows

Morse–Smale flows are flows with finitely many equilibria and periodic orbits that are all hyperbolic and whose stable and unstable manifolds intersect transversally. Morse–Smale flows are robust, and their dynamics can therefore only change when a family crosses the boundary of the set of all Morse–Smale flows.

Write $\mathcal{X}(M)$ for the set of smooth vector fields on a compact manifold M endowed with the Whitney topology. A bifurcation on the boundary of the set of Morse–Smale flows in $\mathcal{X}(M)$ is accessible if there exists a path $\chi : [0, 1] \mapsto \mathcal{X}(M)$ of Morse–Smale flows except for the bifurcation at the endpoint $\chi(1)$. We are interested in accessible bifurcation points in whose vicinity the Morse–Smale flows are not everywhere dense. In particular, we review bifurcations directly from Morse–Smale flows to flows with suspended horseshoes or strange attractors. Local bifurcations with such transitions are considered in §5.4.

Consider a one-parameter family of ODEs $\dot{u} = f(u, \mu)$ that unfold a saddle-node bifurcation of an equilibrium p occurring at $\mu = 0$. Shil'nikov established that the unfolding of a saddle-node bifurcation can create suspended horseshoes if there are multiple homoclinic solutions to the saddle-node equilibrium. The bifurcation is on the boundary of Morse–Smale flows and is accessible from the set of Morse–Smale flows.

Theorem 4.5 ([370]). *Consider a one-parameter family of ODEs that has a saddle-node equilibrium, which satisfies Hypothesis 2.5, so that $\mathcal{M}^{\mathrm{s}}(p)$ has transverse intersections with $\mathcal{M}^{\mathrm{u}}(p)$ along more than one orbit. On one side of the parameter $\mu = 0$, the family then has a hyperbolic transitive set with infinitely many periodic orbits.*
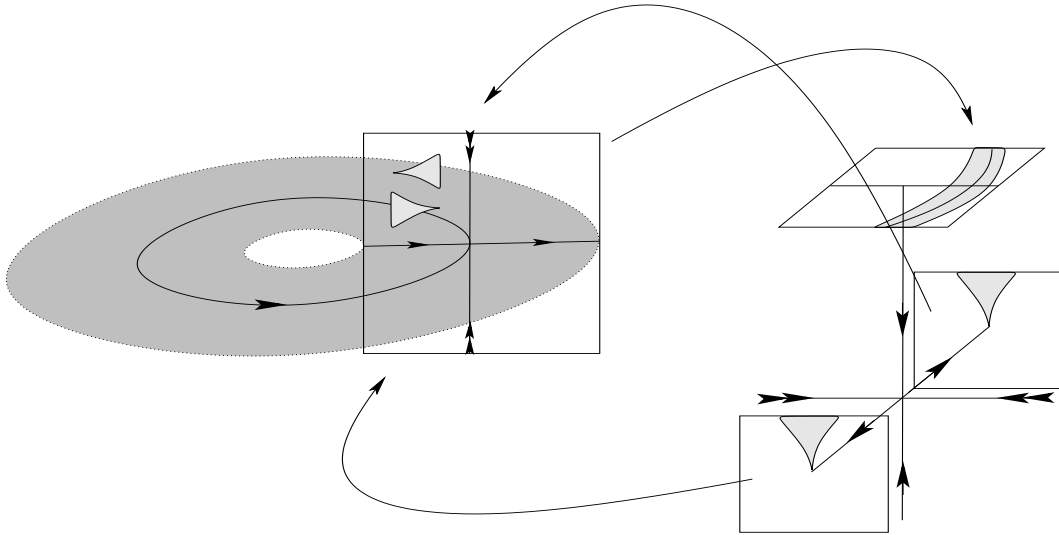
Figure 4.5: *Lorenz-like attractors are created directly from a Morse–Smale flow at the unfolding of certain singular cycles between a hyperbolic equilibrium and a saddle-node periodic orbit.*

There exist families as in the above theorem for which the periodic orbits in the hyperbolic transitive set span every possible knot and link type [148].

Later papers have produced several constructions of codimension-one bifurcations leading directly from a Morse–Smale flow to strange attractors of different types. Afraĭmovich, Chow and Liu [4] describe a codimension-one bifurcation from a Morse–Smale flow to Lorenz-like attractors, that involve singular cycles between a hyperbolic equilibrium p and a periodic solution q of saddle-node type: the equilibrium p is assumed to have two-dimensional stable and one-dimensional unstable directions and leading eigenvalues that satisfy $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} < 1$, while the periodic orbit q has an attracting normally hyperbolic direction. Its stable set $\mathcal{M}^{\mathrm{s}}(q)$ is thus an open region of $\mathbb{R}^3$ bounded by the stable manifold $W^{\mathrm{s}}(q)$ of q, and the unstable set $\mathcal{M}^{\mathrm{u}}(q)$ is a surface bounded by q. The differential equation at the moment of bifurcation satisfies $W^{\mathrm{u}}(p) \subset \mathcal{M}^{\mathrm{s}}(q)$ and admits a transverse intersection of $\mathcal{M}^{\mathrm{u}}(q)$ with $W^{\mathrm{s}}(p)$. Observe that both separatrices that form $W^{\mathrm{u}}(p)$ tend to the periodic orbit q. One-parameter unfoldings from an open set of vector fields with the given properties are shown to possess Lorenz-like attractors on one side of the bifurcation value $\mu = 0$; see Figure 4.5 for an illustration. The creation of Lorenz-like attractors from codimension-two homoclinic bifurcations in $\mathbb{Z}_2$-equivariant flows is considered in §5.5.5.

Additional analysis by Morales, Pacífico and Pujals [281] produced further examples of transitions from Morse–Smale flows, through bifurcations of singular cycles between a hyperbolic equilibrium and a saddle-node periodic orbit, to robust singular attractors. Their work contains examples were these last attractors are not Lorenz-like. In the example of Afraĭmovich, Chow and Liu, $\mathcal{M}^{\mathrm{u}}(q)$ intersects the strong stable foliation $\mathcal{F}^{\mathrm{ss}}$ of $\mathcal{M}^{\mathrm{s}}(q)$ transversally. In critical cycles, when a tangency of $\mathcal{M}^{\mathrm{u}}(q)$ with some leaf of $\mathcal{F}^{\mathrm{ss}}$ exists, transitions from Morse–Smale flows to flows with suspended Hénon-like attractors are possible [285]. We mention that Morales [277] has examples of direct transitions in one parameter families of ODEs from Lorenz-like attractors to suspended Plykin attractors, i.e. from a singular hyperbolic attractor to a uniformly hyperbolic attractor.

In two-parameter families of ODEs, one can study the creation of multiple homoclinic solutions to a saddle-node equilibrium through a quadratic tangency of $\mathcal{M}^{\mathrm{s}}(p)$ and $\mathcal{M}^{\mathrm{u}}(p)$. Let $\psi(t)$ be the unique bounded nonzero solution to the adjoint variational equation

$$\dot{w} = -f_u(h(t), 0)^* w$$

so that $\psi(t) \in (T_{h(t)}W^{\mathrm{cs}}(p) + T_{h(t)}W^{\mathrm{cu}})^{\perp}$ exists and decays to zero exponentially. Similar to Hypothesis 2.5,

we define the two vectors

$$M = \int_{-\infty} \langle \psi(t), f_\mu(h(t), 0) \rangle \, \mathrm{d}t, \qquad N = \langle w_c, f_\mu(p, 0) \rangle$$

in $\mathbb{R}^2$.

Hypothesis 4.3. *Consider the following nondegeneracy conditions:*

(i) $\langle w_c, f_{uu}(p, 0)[v_c, v_c] \rangle \neq 0$,

(ii) M *and* N *are linearly independent.*

The following result from [73, 278] establishes the existence of suspended Hénon-like strange attractors. arbitrarily close to the boundary of Morse–Smale systems in codimension-two unfoldings involving a saddle-node bifurcation of an equilibrium.

Theorem 4.6 ([73, 278])**.** *Assume Hypothesis 2.5 and* $-\nu^s/\nu^u > 1$. *A generic two-parameter family* $\dot{u} = f(u, \mu)$ *in* $\mathbb{R}^3$ *for which Hypothesis 4.3 is met then admits parameter values with suspended Hénon-like strange attractors.*

4.5 Homoclinic-doubling cascades

Cascades of period-doubling bifurcations in one-parameter families of ODEs have attracted much interest as it is one of the routes to onset of chaos. Scaling properties of the period-doubling bifurcations, universal in the sense that they do not depend on details of the family, are the most noticeable feature. Recall that the universal scalings are explained by renormalization theory and, in particular, by the existence of a fixed point for the renormalization operator with a single unstable eigenvalue. One can ask the general question on how this scenario can change if a second parameter is varied. One way involves the disappearance of the periodic orbits through homoclinic bifurcations and gives rise to cascades of homoclinic-doubling bifurcations. In this section, we review the relevant literature and discuss approximations by interval maps: universal scalings in the bifurcation diagram for the interval maps turn out to be related to the appearance of a fixed point of a renormalization operator with two unstable eigenvalues. An extensive numerical investigation of homoclinic-doubling cascades can be found in [298], and further numerical evidence for the existence of homoclinic-doubling cascades has been provided in the Shimizu–Morioka model

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= x - ay - xz, \\
\dot{z} &= -bz + x^2;
\end{aligned}
$$

see [375].

Under certain conditions, an orbit homoclinic to a hyperbolic equilibrium can undergo a homoclinic-doubling bifurcation that creates a double-round homoclinic orbit, see §5.1.5, §5.1.6, and §5.1.7. In a two-parameter family of ODEs, one can continue homoclinic solutions along curves in the parameter plane. A two-parameter family of ODEs is said to possess a cascade of homoclinic bifurcations if there exists a connected set of parameter values in the parameter plane, corresponding to homoclinic orbits, that contains a cascade $\{\mu_n\}_{n \in \mathbb{N}}$ of homoclinic-doubling bifurcations in which a 2^n-homoclinic orbit is created. The actual existence of this phenomenon in confirmed in [190], following earlier work in [226, 227].

Theorem 4.7 ([190])**.** *In the space of two-parameter families of smooth vector fields on* $\mathbb{R}^3$, *there is an open set consisting of families that possess a cascade of homoclinic-doubling bifurcations.*

Next, we outline the bifurcation theory of homoclinic-doubling cascades that occur in unfoldings of codimension-three homoclinic bifurcations which lead to strongly dissipative return maps that are small perturbations of interval maps.

Consider a three-parameter family $\dot{u} = f(u, \mu)$ of ODEs with $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$ that have a hyperbolic equilibrium at p. Write $\nu^{\mathrm{ss}} < \nu^{\mathrm{s}}$ for the two real negative eigenvalues at p and ν^{u} for the single real positive eigenvalue, and define

$$\alpha = -\nu^{\mathrm{ss}}/\nu^{\mathrm{u}}, \qquad \beta = -\nu^{\mathrm{s}}/\nu^{\mathrm{u}}.$$

Suppose the differential equation has a homoclinic orbit to p when $\mu_2 = 0$, which is an inclination-flip homoclinic orbit when $\mu_1 = \mu_2 = 0$. We let $\mu_3 = 2 - \frac{1}{\beta}$, so that $\beta = \frac{1}{2}$ when $\mu_3 = 0$. We assume that the eigenvalue index α will satisfy the open condition $\alpha > 1$. Furthermore, assume that the two-dimensional invariant manifold $W^{\mathrm{ls,u}}(p)$ of p has a quadratic tangency with the stable manifold $W^{\mathrm{s}}(p)$ along the homoclinic orbit when $\mu = 0$. Take a cross section Σ transverse to the homoclinic orbit.

Following [190, 191], there are coordinates $(x^{\mathrm{ss}}, x^{\mathrm{u}})$ on Σ so that the first-return map $\Pi(\cdot, \mu) : \Sigma \to \Sigma$ has the asymptotics

$$\Pi(x^{\mathrm{ss}}, x^{\mathrm{u}}, \mu) = \begin{pmatrix} a_0(\mu) + a_1(\mu)(x^{\mathrm{u}})^\beta + a_2(\mu)(x^{\mathrm{u}})^{2\beta} + \mathrm{O}((x^{\mathrm{u}})^{2\beta+\eta}) \\ \mu_2 + \mu_1(x^{\mathrm{u}})^\beta + a_3(\mu)(x^{\mathrm{u}})^{2\beta} + \mathrm{O}(|\mu_1|(x^{\mathrm{u}})^{\beta+\eta} + (x^{\mathrm{u}})^{2\beta+\eta}) \end{pmatrix} \tag{4.1}$$

for some $\eta > 0$ when μ varies near zero. The coefficients $a_j(\mu)$ depend smoothly on μ, and the higher-order terms can be differentiated for $x^{\mathrm{u}} > 0$.

Theorem 4.8. *Let $\dot{u} = f(u, \mu)$ be as above and assume that $a_3(0) > 1$. For each small fixed $\mu_3 > 0$, the resulting two-parameter family $\dot{u} = f(u, \mu)$ possesses a cascade of homoclinic-doubling bifurcations.*

Related scenarios near other codimension-three bifurcation points are treated in [191].

The proof of the preceding theorem utilizes return maps. First, a rescaling transforms the first-return map to a map that is a small perturbation of an interval map. Let Π be as in (4.1) and define rescaled coordinates $(\hat{x}^{\mathrm{ss}}, \hat{x}^{\mathrm{u}})$ by

$$x^{\mathrm{ss}} - a_0(\mu) = |\mu_1|\hat{x}^{\mathrm{ss}},$$
$$x^{\mathrm{u}} = \left| \frac{\mu_1}{2a_3(\mu)} \right|^{1/\beta} \hat{x}^{\mathrm{u}}.$$

The following proposition, which gives expansions for the first-return map in rescaled coordinates, is proved by a direct computation.

Proposition 4.4. *Let $\hat{\Pi}$ be the first-return map in the rescaled coordinates $(\hat{x}^{\mathrm{ss}}, \hat{x}^{\mathrm{u}})$ and write $b_2(\mu) = |\mu_1|^{\mu_3}/4a_3(\mu)$ and $b_1(\mu) = b_2(\mu)(4a_3(\mu)\mu_2|\mu_1|^{-2} - 1)$. For some $\eta > 0$, we then have*

$$\hat{\Pi}(\hat{x}^{\mathrm{ss}}, \hat{x}^{\mathrm{u}}) = \begin{pmatrix} \frac{a_1(\mu)}{2a_3(\mu)}(\hat{x}^{\mathrm{u}})^\beta + \mathrm{O}(|\mu_1|^\eta(\hat{x}^{\mathrm{u}})^\beta) \\ b_1(\mu) + b_2(\mu)(1 - (\hat{x}^{\mathrm{u}})^\beta)^2 + \mathrm{O}(|\mu_1|^\eta(\hat{x}^{\mathrm{u}})^\beta) \end{pmatrix}.$$

As $\mu_1 \to 0$, restricting $\hat{x}^{\mathrm{u}}$ to a compact interval and parameters (μ_1, μ_2, μ_3) to a chart near the origin on which $b = (b_1, b_2)$ is bounded, $\hat{\Pi}$ converges to a map that depends only on $\hat{x}^{\mathrm{u}}$ and is given by

$$\hat{x}^{\mathrm{u}} \mapsto f(\hat{x}^{\mathrm{u}}) = b_1 + b_2(1 - (\hat{x}^{\mathrm{u}})^\beta)^2. \tag{4.2}$$

Scaling properties that exist in the bifurcation diagram of f in the (b_1, b_2) parameter plane can be investigated with renormalization theory [194]. The renormalization scheme involves a renormalization operator whose fixed point has two unstable directions, in contrast to the single unstable direction one finds in the theory of period-doubling cascades. A renormalization theory for the actual differential equations is not available.

What follows is a concise version of the results, and we refer to [194] for precise and complete statements and a discussion of the implications for scaling properties of the bifurcation diagram. Consider the class of unimodal functions on a fixed interval $[0, R]$ for some $R > 1$ of the form $x \mapsto g(x^\beta)$ with g smooth and a minimal value $g(1) = 0$. The renormalization operator $\mathcal{R}$, defined on a subset of these functions, maps g to a rescaled version of the second iterate g^2.

Theorem 4.9 ([194]). *For $\beta > \frac{1}{2}$ with $|\beta - \frac{1}{2}| \ll 1$, the renormalization operator $\mathcal{R}$ possesses an isolated fixed point ϕ. The function ϕ depends continuously on β and converges to $x \mapsto (1 - \sqrt{x})^2$ as $\beta \to \frac{1}{2}$. The linearization $D\mathcal{R}$ at ϕ has two unstable eigenvalues ν_1, ν_2 which depend continuously on β and satisfy $\nu_1 \to 2$, $\nu_2 \to \infty$ as $\beta \to \frac{1}{2}$. The remainder of the spectrum of $D\mathcal{R}(\phi)$ is strictly inside the unit disc.*

4.6 Intermittency

Intermittency has been identified as one of the principal routes of the transition from a periodic state to chaos [37]. In this context, a time series is said to be intermittent if it is almost periodic apart from infrequent variations. Thus, intermittent time series consists of an almost periodic laminar phase and a chaotic burst or relaminarization phase. Saddle-node, Hopf or period-doubling bifurcations of periodic orbits can give rise to intermittency, which are labelled intermittency of type I, II, or III, respectively. Homoclinic bifurcations of homoclinic solutions to equilibria with real leading eigenvalues can also cause intermittent time series, and we review their characteristics in this section.

In equivariant systems, heteroclinic cycles can be robust under equivariant perturbations and attract orbits from an open neighborhood; see §5.5.1. Such stable heteroclinic cycles provide a mechanism for intermittency: a solution approaching the cycle spends long periods near equilibria and makes fast transitions from one equilibrium to the next. In a perfectly symmetric system, the return times increase monotonically and approach infinity, thus making the intermittent behavior uninteresting, though we remark that more complex switching phenomena can occur in more complicated heteroclinic networks. However, under small, symmetry-breaking perturbations, the cycling behavior persists even though there may no longer be a cycle, and the transition times no longer converge to infinity. Alternatively, stochastic perturbations lead to boundedness of transition times [17, 385]. Relevant references for these dynamical features include [84, 86, 181, 272].

Suppose $f(u, \mu)$ is a one-parameter family of vector fields that unfolds a homoclinic bifurcation with real leading eigenvalues at $\mu = 0$. We will consider codimension-one phenomena where a continuous bundle of center directions along the homoclinic orbit exists (see also §4.3).

We assume the unstable manifold of the equilibrium p is one-dimensional. It is therefore the union of two separatrices and the equilibrium p: one separatrix in $W^u(p)$ forms the homoclinic solution h, while we assume that the other separatrix converges to the homoclinic solution. This can only happen if $-\nu^s/\nu^u > 1$ at $\mu = 0$. Such flows appear in unfoldings of gluing bifurcations where two homoclinic orbits coexist; see §5.1.8. More precisely, we shall assume the following.

Hypothesis 4.4. *The unstable manifold $W^u(p)$ of the hyperbolic equilibrium p is one-dimensional and therefore equal to the union of p and two separatrices $W^u_\pm(p)$. The leading eigenvalues ν^s, ν^u are unique and real and satisfy $-\nu^s/\nu^u > 1$. We assume that there is a homoclinic orbit h to p that lies in $W^u_+(p)$. The bundle E^c of center directions is a continuous orientable plane bundle over the closure $\overline{h} = h \cup \{p\}$ of h.*

The above hypothesis implies that the homoclinic center manifold is an orientable annulus near the homoclinic orbit h. Due to the orientability of the plane bundle E^c and the condition on the saddle quantity $-\nu^s/\nu^u$, the stable set $\mathcal{M}^s(h)$ of initial data whose ω-limit set is $\overline{h}$ is an open set bounded by $W^s(p)$.

Hypothesis 4.5 (Existence of a superhomoclinic orbit). *The ω-limit set of points on the separatrix $h_1(t) \in W^u_-(p)$ equals $\overline{h}$ so that $W^u_-(p) \subset \mathcal{M}^s(\overline{h})$.*

We need additional information about the behavior around the superhomoclinic orbit $h_1(t)$. The plane bundle $E^c(h_1(t))$ along the superhomoclinic orbit $h_1(t)$ extends to a continuous plane bundle E^c over $\overline{h \cup h_1}$.

Hypothesis 4.6. *Consider the following properties for the plane bundle E^c over $\overline{h \cup h_1}$:*

  (i) *The bundle E^c over $\overline{h \cup h_1}$ is an orientable plane bundle.*

  (ii) *The bundle E^c over $\overline{h \cup h_1}$ is a nonorientable plane bundle.*

Theorem 4.10 ([182]). *Consider a one-parameter family of ODEs as above that satisfies Hypotheses 4.4 and 4.5.*

(i) *If Hypothesis 4.6(i) is met, then the bifurcation set in $\mu \geq 0$ is a Cantor set of zero Lebesgue measure: the one-sided boundary points of the Cantor set are homoclinic bifurcations, while there is a unique periodic attractor for parameters outside the bifurcation set.*

(ii) *If Hypothesis 4.6(ii) is met, then the bifurcation set is a cascade $\mu_n \downarrow 0$ of homoclinic bifurcations, while there are one or two periodic attractors for parameters outside the bifurcation set.*

The geometry of the flow near the separatrices of p becomes clear by the following center-manifold theorem, which is similar to the homoclinic center-manifold theorem [182, 400].

Theorem 4.11. *Assume that Hypotheses 4.4 and 4.5 are met, then there is a two-dimensional locally invariant, normally hyperbolic center manifold $W^{c}(W^{u}(p) \cup \{p\})$. The manifold is of class $C^{l,\alpha}$ jointly in (u, μ) for each $l \geq 1$ and $\alpha \in (0, 1)$ with*

$$l + \alpha < \min \left\{ \frac{\lambda^{ss}}{\nu^{s}}, \frac{\lambda^{uu}}{\nu^{u}} \right\}.$$

The manifold $W^{c}(W^{u}(p) \cup \{p\})$ is homeomorphic to either a torus with a hole when Hypothesis 4.6(i) is met or a Klein bottle with a hole when Hypothesis 4.6(ii) is met. We refer to [182] for a study of scaling properties of the bifurcation set; see also [265] for related results. The reduction result expressed by Theorem 4.11 entails that the dynamics is described by interval maps that occur as first-return maps on a cross section inside the center manifold. In the orientable case, this leads to the interval maps studied by Keener [212].

5 Catalogue of homoclinic and heteroclinic bifurcations

This section forms the core of this survey paper: it contains a catalogue of bifurcation results for homoclinic and heteroclinic orbits. We have subdivided the list into sections treating generic systems, conservative and reversible systems, and equivariant systems. Various homoclinic bifurcations, bifurcations from heteroclinic cycles, and the occurrence of homoclinic solutions from local bifurcations are reviewed.

We use the notation and the hypotheses that we introduced in §2. We consider differential equations of the form $\dot{u} = f(u, \mu)$ on $\mathbb{R}^n$ with parameters $\mu \in \mathbb{R}^d$ for some $d \geq 1$ that admit a homoclinic orbit h to an equilibrium p when $\mu = 0$; we restrict ourselves to three- or four-dimensional systems whenever results have only been proved for such lower-dimensional systems. A typical bifurcation result requires, apart from the defining conditions, a number of nondegeneracy and unfolding conditions that encapsulate a generic dependence on parameters.

5.1 Homoclinic orbits in generic systems

5.1.1 Creation of 1-periodic orbits

The problem of the birth of a limit cycle[5] from a homoclinic orbit to a hyperbolic equilibrium was solved for differential equations in the plane by Andronov and Leontovich [11]. Shil'nikov extended this work to differential equations on $\mathbb{R}^n$ [366, 369]. We state a general result on the creation of a limit cycle in homoclinic bifurcations with saddle quantity $-\operatorname{Re} \nu^{s}/\nu^{u}$ larger than one [375]. We also refer to §3.6 and Figure 3.4 for the case of real leading eigenvalues.

Theorem 5.1. *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of ODEs with a homoclinic solution $h(t)$ to a hyperbolic equilibrium p at $\mu = 0$. Assume that Hypotheses 2.2 and 2.3(i) are met, then a unique periodic*

---

[5]A limit cycle is a periodic orbit that is the ω- or α-limit set of an orbit other than itself
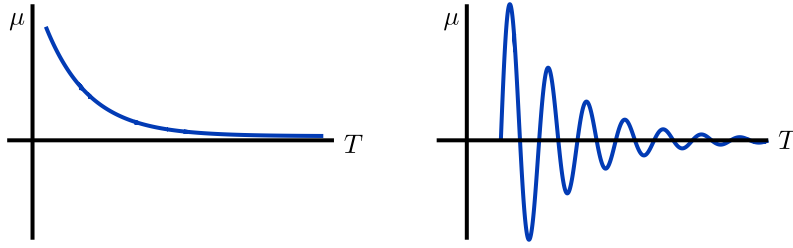
Figure 5.1: *Plotted are the period T versus the parameter μ for the 1-periodic orbits that bifurcate from a homoclinic orbit with real leading eigenvalue (Theorem 5.1) [left] and complex leading eigenvalue (Theorem 5.4) [right]. See (3.22) for the expression of period versus parameter in the first case; a similar formula, now with ν^{u} complex, holds in the latter case.*

solution $q(t, \mu)$ bifurcates from the homoclinic orbit; this periodic solution is hyperbolic, it bifurcates either for $\mu > 0$ or for $\mu < 0$, and $q(t, \mu)$ converges to $h(t)$ as $\mu \to 0$ for each fixed t. Furthermore, $W^{\mathrm{s}}(q) = \dim W^{\mathrm{s}}(p) + 1$.

Note that the periodic orbit $q(t, \mu)$ is stable if p has one-dimensional unstable manifold and that the period of $q(t, \mu)$ goes to infinity as $\mu \to 0$; see Figure 5.1, and also §3.6 and Figure 3.4. If the saddle quantity is less than one, we can simply apply the preceding theorem to the time-reversed vector field (noting that the conclusion about the dimension of the stable manifold is then for the unstable manifold in the original time variable).

In [307] the question was posed whether other bifurcations were possible in which a periodic orbit develops infinite period and disappears in a so-called *blue sky catastrophe*. This question was answered affirmatively by Medvedev [270, 271]; see also [49, 50, 202].

Theorem 5.2 ([270])**.** *There is a one-parameter family $\dot{u} = f(u, \mu)$ of vector fields on the Klein bottle that has a saddle-node bifurcation of a periodic orbit at $\mu = 0$ and an attracting periodic orbit for $\mu > 0$ with unbounded arclength as $\mu \downarrow 0$.*

The example given in [270] can be embedded in a family of ODEs on $\mathbb{R}^4$ that have a Klein bottle as normally attracting invariant manifold. For $\mu = 0$, the Klein bottle coincides with the set of homoclinic solutions of the saddle-node periodic orbit. A different example has been constructed by Shil'nikov and Turaev [377, 405]: again, a periodic attractor with unbounded period appears from a saddle-node bifurcation of a periodic orbit, but the unstable manifold of saddle-node periodic orbit now forms a tube that spirals back towards the saddle-node periodic orbit.

Theorem 5.3 ([377, 405])**.** *There is a one-parameter family $\dot{u} = f(u, \mu)$ of vector fields on $\mathbb{R}^3$ that has a saddle-node bifurcation of a periodic orbit at $\mu = 0$ and an attracting periodic orbit for each $\mu > 0$ with unbounded arclength as $\mu \downarrow 0$.*

A numerical analysis of the explicit family of ODEs

$$
\begin{aligned}
\dot{x} &= x[2 + \mu - b(x^2 + y^2)] + x^2 + y^2 + 2y, \\
\dot{y} &= -z^3 - (y + 1)(z^2 + y^2 + 2y) - 4x + \mu y, \\
\dot{z} &= z^2(y + 1) + x^2 - \varepsilon
\end{aligned}
$$

reveals the existence of the latter type of blue sky catastrophe in low order ODEs [144]. We also refer to [363, 365] for blue sky catastrophes in a singularly perturbed context and their role in explaining bursting phenomena in neuron models. Finally, we mention that the term blue sky catastrophe has also been used in reversible or conservative ODEs when sheets of periodic orbits are bounded by a homoclinic orbit; see Theorem 5.38 below.

5.1.2 Shil'nikov's saddle-focus homoclinic orbits

A systematic study of the dynamics near saddle-focus homoclinic orbits was pioneered by Shil'nikov since the mid 1960s. Under an eigenvalue condition that states that the real leading eigenvalue dominates the complex conjugate leading eigenvalues, infinitely many periodic orbits of saddle type were shown to occur in each neighborhood of the homoclinic orbit [367]. These periodic orbits are contained in suspended horseshoes that accumulate onto the homoclinic orbit [371]. In fact, the periodic orbits near two coexisting saddle-focus homoclinic orbits under the same eigenvalue condition are known to span every possible knot and link type [147]. Dynamical features beyond hyperbolic suspended horseshoes, including the existence of periodic and strange attractors accumulating onto the homoclinic orbit, were described in later papers [155, 185, 299].

Attractors with a spiral structure were envisaged to occur for perturbations of ODEs with two coexisting saddle-focus homoclinic orbits under suitable eigenvalue conditions [20]. Namely, a dissipative differential equation $\dot{u} = f(u)$ with two saddle-focus homoclinic orbits to the same equilibrium will admit an invariant tubular neighborhood $\mathcal{U}$ that contains both homoclinic orbits. Any sufficiently small perturbation of f will then have an attractor inside $\mathcal{U}$. There is some evidence, from the study of models of interval maps, that spiral attractors exist and are robust in the sense of measure [14, 302, 303].

In this section, we discuss mostly three-dimensional ODEs, as much of the existing literature treats three-dimensional ODEs and the most complete picture arises in three dimensions. Several of these results extend readily to higher-dimensional systems. We start with some general results in $\mathbb{R}^n$. Consider the differential equation $\dot{u} = f(u)$ with a hyperbolic equilibrium p that has a unique real leading unstable eigenvalue ν^{u} and unique complex conjugate leading stable eigenvalues $\nu^{\mathrm{s}}, \overline{\nu^{\mathrm{s}}}$. Assume the differential equation has a homoclinic solution $h(t)$ to p; see Figure 5.2 for an illustration. We will consider different conditions on the saddle quantity $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ as outlined in the following hypothesis.

Hypothesis 5.1 (Eigenvalue conditions). *Consider the following eigenvalue conditions:*

- *(i) The saddle-focus homoclinic orbit is tame:* $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$.

- *(ii) The saddle-focus homoclinic orbit is wild:* $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}} < 1$.

- *(iii) We have* $-2\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$ *(which, in $\mathbb{R}^3$, means that the differential equation is dissipative near p).*

Hypothesis 5.1(i) implies that an attracting periodic solution approaches the homoclinic solution h as μ goes to zero from one side and disappears there, a bifurcation that we already discussed in §5.1.1. Shil'nikov discovered that, when Hypothesis 5.1(ii) is met, the dynamics near the homoclinic solution involves infinitely many periodic orbits arbitrarily close to the homoclinic solution [367, 371]. A geometric explanation of the organization of these periodic orbits into infinitely many horseshoes has been given in [396]. We note that, for three-dimensional flows, Belitskii's linearization theorem [30], which we stated in §3.1, allows a C^1 linearization of the flow near the equilibrium, thus facilitating asymptotic expressions for the first-return map on a cross section. For higher-dimensional flows, the homoclinic center-manifold theorem [343] gives a three-dimensional homoclinic center manifold, provided the homoclinic orbit is not in a flip configuration. This allows a geometric reduction to the three-dimensional case; we note that, although the homoclinic center manifold is, in general, only continuously differentiable with Hölder continuous derivatives, we can still C^1 linearize. Shashkov and Turaev [360] treat the dynamics near the saddle-focus homoclinic orbit for C^1 vector fields.

Multi-round homoclinic orbits occur when unfolding the homoclinic orbit [126–128, 139, 140].

Theorem 5.4. *Assume that the system $\dot{u} = f(u, \mu)$ on $\mathbb{R}^n$ has a Shil'nikov saddle-focus homoclinic orbit for $\mu = 0$ which satisfies Hypotheses 2.2, 2.4, and 5.1(ii). At $\mu = 0$, there are infinitely many suspended Smale horseshoes in each neighborhood of the homoclinic solution. Furthermore, for each $N > 0$, N-homoclinic orbits exists for infinitely many parameter values which accumulate onto $\mu = 0$ from one side, say for $\mu > 0$. In fact, for $\rho > -\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$, in any interval $(0, \mu_+)$, there is a set of parameter values corresponding to*

N-homoclinic loops that are indexed by $\Gamma_N(\rho) = \{(k_1, \ldots, k_N) \in \mathbb{N}^N; \ \rho k_{i-1} < k_i < k_1/\rho\}$. *The parameter values corresponding to double-round homoclinic orbits form two sequences* $\mu_n^\alpha \downarrow 0$, $\alpha = 0, 1$ *with*

$$\lim_{n \to \infty} \left| \ln \frac{\mu_{n+1}^\alpha}{\mu_n^\alpha} \right| = 2\pi \left| \frac{\nu^{\mathrm{u}}}{\mathrm{Im}\,\nu^{\mathrm{s}}} \right|.$$

For multi-round homoclinic orbits, Feroe [127] has derived estimates for the time differences between successive rounds near the principal homoclinic orbit which, equivalently, gives the spacing between pulses in the time profiles of the homoclinic orbits.

The dynamics near saddle-focus homoclinic solutions is considerably more complex than hinted at by the existence of infinitely many horseshoes. To outline the complexity, we restrict ourselves to three-dimensional ODEs, though several results have straightforward generalizations to more dimensions (see e.g. [103]). Dynamics and bifurcations are studied through the first-return map Π on a cross section Σ for which expressions can be derived by transforming the differential equation near the equilibrium into normal form and deriving expansions for solutions near the equilibrium by estimating integral formulae; see §3.2 on Shil'nikov variables. This procedure gives:

Proposition 5.1. *Assume Hypothesis 5.1(ii) is met, then there is a cross section Σ and smooth coordinates (θ, z) on Σ so that $\Pi(\cdot, \mu)$ has the following asymptotic expansion:*

$$\Pi(\theta, z, \mu) = \begin{pmatrix} a + \phi_1(\theta) z^{\frac{-\operatorname{Re}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}}} \sin\left(-\frac{\operatorname{Im}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}} \ln z\right) + \phi_2(\theta) z^{\frac{-\operatorname{Re}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}}} \cos\left(-\frac{\operatorname{Im}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}} \ln z\right) + R_1(\theta, z, \mu) \\ b + \phi_3(\theta) z^{\frac{-\operatorname{Re}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}}} \sin\left(-\frac{\operatorname{Im}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}} \ln z\right) + \phi_4(\theta) z^{\frac{-\operatorname{Re}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}}} \cos\left(-\frac{\operatorname{Im}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}} \ln z\right) + R_2(\theta, z, \mu) \end{pmatrix}.$$

The functions a, b, ϕ_i, $i = 1, 2, 3, 4$, are smooth functions of θ and μ (the dependence on μ is suppressed in the notation) and satisfy $\det \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} \neq 0$. *Furthermore, there exist $\eta > 0$ and positive constants C_i so that*

$$\left| \frac{\partial^{k+l+m}}{\partial \theta^k \partial z^l \partial \mu^m} R_i(\theta, z, \mu) \right| \leq C_{k+l+m} z^{\frac{-\operatorname{Re}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}} + \eta}.$$

A topological invariant or a modulus is a function of the vector field that is invariant under topological equivalence; see also §6.2. The saddle quantity, whether larger than one or not, is always a topological invariant of saddle-focus homoclinic orbits [21, 67, 116, 395]. The proof of Togawa [395] uses link types of period orbits for saddle quantities smaller than one. Dufraine [116] proved that the absolute value of $\mathrm{Im}\,\nu^{\mathrm{s}}$ is also a modulus.

Theorem 5.5. *Suppose $f \in \mathcal{X}(\mathbb{R}^3)$ has a saddle-focus homoclinic orbit as above, then $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ and $|\mathrm{Im}\,\nu^{\mathrm{s}}|$ are topological invariants.*

The topological invariance of the saddle quantity reflects the dependence of the global configuration of the stable manifold of p on the saddle quantity. This fact has further implications for the bifurcation diagrams of bifurcations for two coexisting saddle-focus homoclinic orbits solutions [356, 358, 375]. It also suggests that it is natural to consider two-parameter families $\dot{u} = f(u, \mu)$ with $\mu \in \mathbb{R}^2$ to unfold the homoclinic bifurcation.

Hypothesis 5.2 (Generic unfolding of saddle quantities). *The saddle quantity unfolds generically with respect to the parameter μ_2:*

$$\frac{\partial}{\partial \mu_2} \left(\frac{-\operatorname{Re}\nu^{\mathrm{s}}}{\nu^{\mathrm{u}}} \right) \neq 0.$$

If Hypothesis 2.2(ii) is met, we may assume that $f(u, \mu)$ has homoclinic orbits when $\mu_1 = 0$. Setting $\mu_1 = 0$ and varying μ_2 allows us to study variations in the dynamics when keeping the homoclinic orbit but changing eigenvalues. In [142, 150, 155], generic two-parameter families of differential equations near a Shil'nikov saddle-focus are studied, in particular with regard to curves of multi-round homoclinic orbits: they show that, in between the curves corresponding to double-round homoclinic orbits which, as stated in Theorem 5.4, occur along curves that accumulate onto $\mu_1 = 0$, there are curves of N-homoclinic orbits that fold back in μ_2.
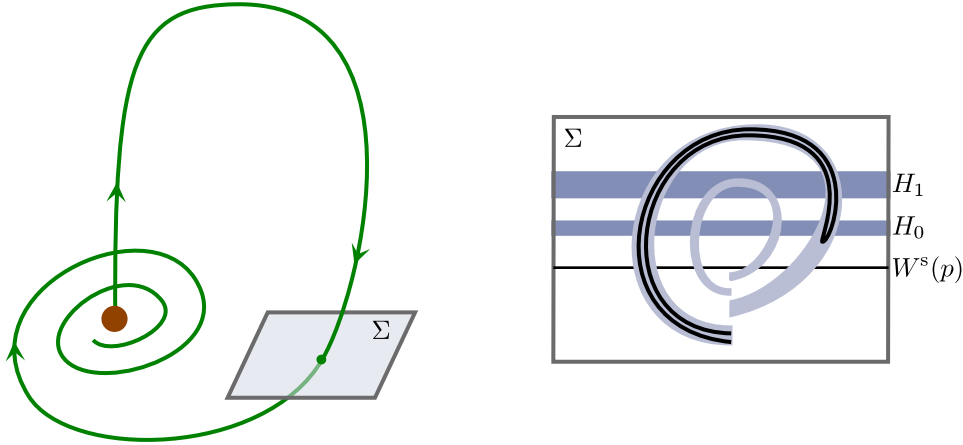
Figure 5.2: *A Shil'nikov saddle-focus homoclinic orbit is shown in the left panel. If the complex eigenvalues are closest to the imaginary axis, then the return map $\Pi : \Sigma \to \Sigma$ contains infinitely many horseshoe maps obtained by restriction to strips H_i. The second iterate Π^2 restricted to a union $H_i \cup H_j$ may contain nonhyperbolic dynamics: this is illustrated in the right panel where the horseshoe-shaped images of two strips under Π and, with a darker shade, part of the image under Π^2 of the union of the strips are shown.*

Theorem 5.6 ([155]). *Assume that $\dot{u} = f(u, \mu)$ with $(u, \mu) \in \mathbb{R}^3 \times \mathbb{R}^2$ has a Shil'nikov saddle-focus homoclinic orbit for $\mu_1 = 0$ that satisfies Hypotheses 2.2, 2.4, and 5.1(ii), 5.2. There is then a dense set of μ_2 values near $\mu_2 = 0$ for which a curve of homoclinic orbits of N-homoclinic orbits with $N \geq 3$ is tangent to the line $\mu_2 = \text{constant}$ in the parameter plane.*

The horseshoes from Theorem 5.4 are not isolated in the recurrence set. The existence of infinitely many suspended horseshoes and the nonhyperbolic dynamics around it, which we shall present below, is reminiscent of the unfolding of homoclinic tangencies. Ovsyannikov and Shil'nikov [299] established that the set of equations with nonhyperbolic dynamics is dense in the space of ODEs with Shil'nikov homoclinic orbits. More precisely, write $\mathcal{X}_H(\mathbb{R}^3)$ for the space of ODEs in $\mathcal{X}(\mathbb{R}^3)$ with a Shil'nikov homoclinic solution, then, for each $\varepsilon > 0$, the set of ODEs in $\mathcal{X}_H(\mathbb{R}^3)$ that admit

(i) a saddle-node bifurcation of a periodic orbit, and

(ii) a period-doubling bifurcation of a periodic orbit, and

(iii) a homoclinic tangency to a hyperbolic periodic orbit

inside an ε-tubular neighborhood of the homoclinic solution is dense in $\mathcal{X}_H(\mathbb{R}^3)$.

For dissipative vector fields, the homoclinic tangencies give rise to suspended Hénon-like strange attractors. In fact, combining [299] and [93] (see also [318, 319]) yields the following result.

Theorem 5.7. *For each $\epsilon > 0$, there is a dense subset $\mathcal{D}$ in the set of ODEs in $\mathcal{X}_H(\mathbb{R}^3)$ for which Hypothesis 5.1(ii)-(iii) holds with the following property: for all $f \in \mathcal{D}$, $\dot{u} = f(u)$ has infinitely many coexisting strange attractors in an ϵ-neighborhood of $\overline{h}$.*

For ODEs with wild saddle-focus homoclinic orbits and $-2 \operatorname{Re} \nu^s / \nu^u = 1$ (on the boundary of the class of ODEs where Hypothesis 5.1(iii) is met), renormalizations of a return map yield near area-preserving maps [41].

Next, we formulate a bifurcation result that makes the existence of attractors in parameterized families more precise [185]. To prove this result, one analyzes the first-return map Π; the additional eigenvalue condition $-2 \operatorname{Re} \nu^s / \nu^u > 1$ assumed below allows for an improved asymptotic expansion.

Theorem 5.8 ([185]). *Consider a one-parameter family $\dot{u} = f(u, 0, \mu_2)$ of ODEs in $\mathbb{R}^3$ so that each differential equation has a saddle-focus homoclinic orbit. Suppose Hypothesis 5.2(ii) is met and that Hypothesis 5.1(ii)-(iii) holds for each μ_2. Let Σ be a cross section transverse to h and denote by $\mathcal{U}_n$ a decreasing sequence of tubular neighborhoods of $\overline{h}$.*

Let $P_{2,n}$ be the set of parameter values μ_2 for which f has an attracting 2-periodic orbit in $\mathcal{U}_n$, then, for n large enough, $P_{2,n}$ is open and dense in I and $\lim_{n \to \infty} |P_{2,n}| = 0$, where $|\cdot|$ denotes Lebesgue measure. Furthermore, the set of parameter values $\gamma \in I$ for which X_γ has infinitely many 2-periodic attractors is dense in I, but has zero measure.

Let $A_{2,n}$ be the set of parameter values in $\gamma \in I$ for which X_γ has a Hénon-like strange attractor in $\mathcal{U}_n$ that intersects Σ in 2 connected components. For n large enough, $A_{2,n}$ has positive measure and is dense in I, and $\lim_{n \to \infty} |A_{2,n}| = 0$.

The complicated dynamics and the large number of attractors do not trap most orbits in a neighborhood of the homoclinic orbit:

Theorem 5.9 ([185]). *Let $\dot{u} = f(u)$ be a differential equation in $\mathbb{R}^3$ with a saddle-focus homoclinic orbit and assume that Hypothesis 5.1(ii)-(iii) is met. Let Σ be a cross section transverse to the homoclinic orbit h and $\mathcal{U}_n$ be a decreasing sequence of tubular neighborhoods of the homoclinic orbit. If D_n is the set of points x in $\Sigma \cap \mathcal{U}_n$ whose forward orbit leaves $\mathcal{U}_n$, then $|D_n|/|\Sigma \cap \mathcal{U}_n| \to 1$ as $n \to \infty$.*

5.1.3 Bi-focus homoclinic orbits

Homoclinic orbits to an equilibrium at which both the leading stable and the leading unstable eigenvalues are complex conjugate give dynamics similar to that near wild saddle-focus homoclinic orbits that we discussed in the previous section. However, apart from a nonresonance condition, no eigenvalue conditions are needed. Also, in an unfolding, subsidiary homoclinic orbits typically accumulate on the primary bifurcation value from both sides.

Starting point is a one-parameter family $\dot{u} = f(u, \mu)$ on $\mathbb{R}^n$ with a homoclinic orbit to a hyperbolic equilibrium p for $\mu = 0$.

Hypothesis 5.3 (Leading eigenvalues). *The leading stable eigenvalues of $f_u(p, 0)$ are two simple complex eigenvalues $\nu^s, \overline{\nu^s}$; similarly, the leading unstable eigenvalues are two simple complex eigenvalues $\nu^u, \overline{\nu^u}$.*

A homoclinic orbit to an equilibrium that satisfies the preceding eigenvalue conditions is called a bi-focus homoclinic orbit. The existence of infinitely many suspended horseshoes accumulating onto a bi-focus homoclinic orbit has been proved by Shil'nikov [371], while bifurcations to subsidiary homoclinic orbits were studied in [149]. A discussion of the geometry of first-return maps for four-dimensional flows can be found in [135].

Theorem 5.10. *Assume that $\dot{u} = f(u, \mu)$ with $(u, \mu) \in \mathbb{R}^n \times \mathbb{R}$ has a Shil'nikov bi-focus homoclinic orbit for $\mu = 0$ that satisfies Hypotheses 2.2, 5.3, and 2.4. For $\mu = 0$, there are infinitely many suspended Smale horseshoes in each neighborhood of the homoclinic solution. If $\operatorname{Im} \nu^s / \operatorname{Im} \nu^u \notin \mathbb{Q}$, then, for any $N > 0$, there is an infinite number of parameter values near $\mu = 0$ at which an N-homoclinic orbit exists and these parameter values accumulate from both sides onto $\mu = 0$.*

As for Shil'nikov saddle-focus homoclinic orbits discussed in §5.1.2, the horseshoes from Theorem 5.10 are not isolated in the recurrence set. Ovsyannikov and Shil'nikov [300] established that, in the space of ODEs with a Shil'nikov bi-focus homoclinic orbit in $\mathbb{R}^4$, one finds again dense subsets with nonhyperbolic dynamics. Write $\mathcal{X}_H(\mathbb{R}^4)$ for the space of ODEs in $\mathcal{X}(\mathbb{R}^4)$ with a nondegenerate Shil'nikov bi-focus homoclinic solution, then, for each $\varepsilon > 0$, the set of ODEs in $\mathcal{X}_H(\mathbb{R}^4)$ for which

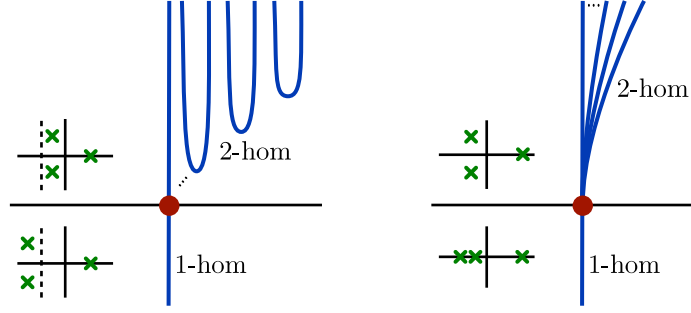(i) a saddle-node bifurcation of a periodic orbit, and

Figure 5.3: *Shown are the bifurcation diagrams of 2-homoclinic orbits for the Belyakov transitions described in Theorem 5.11 [left] and Theorem 5.12 [right]; the eigenvalues at the equilibrium p are shown as insets. The bifurcation curves for N-homoclinic orbits for any $N \geq 2$ look similar.*

(ii) a period-doubling bifurcation of a periodic orbit, and

(iii) a homoclinic tangency to a hyperbolic periodic orbit

occurs within an ε-tubular neighborhood of the homoclinic solution is dense in $\mathcal{X}_H(\mathbb{R}^4)$.

5.1.4 Belyakov transitions

Two different codimension-two homoclinic bifurcations are commonly referred to as Belyakov transitions [34]: the first involves an equilibrium with two real eigenvalues that collide and become complex [33], while the second [34] refers to the transition from tame to wild saddle-focus homoclinic orbits (see §5.1.2). Although we formulate conditions for two-parameter flows $\dot{u} = f(u, \mu)$ with $(u, \mu) \in \mathbb{R}^n \times \mathbb{R}^2$, we state the results for three-dimensional flows as in Belyakov's papers.

Hypothesis 5.4 (Transition from tame to wild homoclinic loops)**.** *At $\mu = 0$, we have $\operatorname{Re} \nu^{\mathrm{s}} + \nu^{\mathrm{u}} = 0$ and $[\operatorname{Re} \nu^{\mathrm{s}} + \nu^{\mathrm{u}}]_{\mu_2} \neq 0$.*

Theorem 5.11 ([34])**.** *Let $\dot{u} = f(u, \mu)$ with $(u, \mu) \in \mathbb{R}^3 \times \mathbb{R}^2$ be a two-parameter family of ODEs on $\mathbb{R}^3$ with a homoclinic solution $h(t)$ to a hyperbolic equilibrium p at $\mu = 0$. Suppose that Hypotheses 2.2, 2.4, and 5.4 are met. Upon changing parameters, we may also assume that the primary homoclinic orbit exists for $\mu_1 = 0$ and that $[\operatorname{Re} \nu^{\mathrm{s}} + \nu^{\mathrm{u}}]_{\mu_2} > 0$ so that the primary homoclinic orbit is a wild saddle-focus homoclinic orbit for $\{\mu_1 = 0, \mu_2 > 0\}$. There is then a countable set of bifurcation curves for double-round homoclinic orbits that accumulate onto $\{\mu_1 = 0, \mu_2 > 0\}$.*

Belyakov [34] has further statements on curves of 3-round homoclinic orbits and curves of saddle-node bifurcations of periodic orbits, and it can also be shown, for instance using Lin's method, that N-homoclinic orbits bifurcate for each $N \geq 2$.

In the second Belyakov transition, two eigenvalues of the linearized vector field at the equilibrium collide on the real axis and become complex. Recall that the imaginary parts of the leading stable eigenvalues are nonzero if the discriminant $\Delta(\mu)$ of $f_u(p, \mu)$ restricted to the leading stable directions is negative.

Hypothesis 5.5 (Non-semisimple leading eigenvalues)**.** *At $\mu = 0$, the real leading stable eigenvalue ν^{s} has geometric multiplicity one and algebraic multiplicity two with $\partial_{\mu_2} \Delta(0) \neq 0$ and $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} < 1$. Furthermore,*

$$\lim_{t \to \infty} \frac{1}{|t|} \mathrm{e}^{|\nu^{\mathrm{s}} t|} \|h(t)\| \neq 0, \qquad \lim_{t \to -\infty} \frac{1}{|t|} \mathrm{e}^{|\nu^{\mathrm{s}} t|} \|\psi(t)\| \neq 0$$

at $\mu = 0$.

Theorem 5.12 ([33, 243])**.** *Let $\dot{u} = f(u, \mu)$ with $(u, \mu) \in \mathbb{R}^3 \times \mathbb{R}^2$ be a two-parameter family of ODEs on $\mathbb{R}^3$ with a homoclinic solution $h(t)$ to the hyperbolic equilibrium p at $\mu = 0$. Suppose Hypotheses 2.2,*
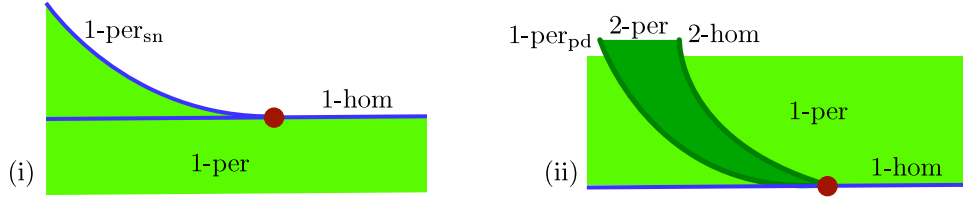
Figure 5.4: *Shown are the bifurcation diagrams at a resonant bifurcation described in Theorem 5.13 for orientable (i) and nonorientable (ii) homoclinic orbits.*

2.4(iii)-(iv), and 5.5 are met. We may assume that the primary homoclinic orbit exists for $\mu_1 = 0$ and that $\partial_{\mu_2}\Delta(0) < 0$ so that complex conjugate leading stable eigenvalues occur for $\mu_2 > 0$. There are then infinitely many one-sided curves of 2-homoclinic orbits in the half plane $\mu_1 > 0$ that emerge from $\mu = 0$, are tangent to $\{\mu_1 = 0\}$ at $\mu = (0,0)$ and accumulate onto $\mu_1 = 0$ from one side. Furthermore, there are infinitely many one-sided curves of saddle-node bifurcations and of period-doubling bifurcations of periodic orbits in $\mu_1 > 0$ that emerge from $\mu = 0$, are tangent to $\{\mu_1 = 0\}$ at $\mu = (0,0)$, and accumulate onto $\mu_1 = 0$ from both sides.

In this situation, it can again be shown that N-homoclinic orbits bifurcate for each $N \geq 2$. See Figure 5.3 for sketches of the bifurcation diagrams for the Belyakov transitions.

5.1.5   Resonant homoclinic orbits

The prototype homoclinic bifurcation theorem for homoclinic orbits with real leading eigenvalues, Theorem 5.1, requires that the saddle quantity is not equal to one. A homoclinic orbit to an equilibrium with real leading eigenvalues for which the saddle quantity is equal to one is called a homoclinic orbit at resonance. Bifurcations from homoclinic orbits at resonance for planar vector fields have been investigated by Leontovich [254] and Nozdracheva [296]. Chow, Deng and Fiedler [89] have treated the general bifurcation problem in $\mathbb{R}^n$. We review these results here and remark that the case of complex conjugate eigenvalues that are at resonance is discussed in the previous section. We will consider two-parameter families $\dot{u} = f(u, \mu)$ with $\mu = (\mu_1, \mu_2)$ to unfold a homoclinic orbit at resonance.

Hypothesis 5.6 (Resonance condition)**.** *We assume that the leading eigenvalues satisfy $\nu^{\mathrm{s}}(0) = \nu^{\mathrm{u}}(0)$ and $\partial_{\mu_2}\nu^{\mathrm{s}}(0) \neq \partial_{\mu_2}\nu^{\mathrm{u}}(0)$.*

We shall also assume that the homoclinic orbit is not in an inclination-flip and or orbit-flip configuration: this implies the existence of a two-dimensional homoclinic center manifold as in §3.4. It turns out that there are two cases, with different bifurcation diagrams, that depend on the orientability of the homoclinic center manifold. Under Hypothesis 2.4 there is a continuous bundle of planes E^{c} along the homoclinic orbit, invariant under the variational equation, so that $\lim_{t \to \pm\infty} E^{\mathrm{c}}(h(t))$ is the sum of the eigenspaces associated with the leading eigenvalues.

Hypothesis 5.7 (Generic separatrix value)**.** *For $\mu = 0$, the separatrix quantity*

$$\mathrm{e}^{\int_{-\infty}^{\infty} \mathrm{div}_2(h(t))\,\mathrm{d}t} \neq 1,$$

is not equal to one, where div_2 denotes the rate of change of area within the plane field $E^{\mathrm{c}}(h(t))$.

Theorem 5.13 ([89])**.** *Let $\dot{u} = f(u, \mu)$ with $\mu = (\mu_1, \mu_2)$ be a two-parameter family of ODEs on $\mathbb{R}^n$ with a homoclinic solution $h(t)$ to the hyperbolic equilibrium p at $\mu = 0$. Suppose Hypotheses 2.2, 2.3(ii), and 2.4 are met. Furthermore, assume that $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} = 1$ at $\mu = 0$ and that Hypotheses 5.6, 5.8 hold. If the homoclinic center manifold $W_{\mathrm{hom}}^{\mathrm{c}}$ is orientable, then the bifurcation diagram is as shown in Figure 5.4(i): a one-sided curve of saddle-node bifurcations of periodic orbits emerges from the curve of homoclinic orbits at $\mu = 0$ in the parameter plane. If the homoclinic center manifold $W_{\mathrm{hom}}^{\mathrm{c}}$ is orientable, then the bifurcation diagram is as*

41

shown in Figure 5.4(ii): a one-sided curve of period-doubling bifurcations of periodic orbits and a one-sided curve of 2-homoclinic orbits emerge from the curve of primary homoclinic orbits at $\mu = 0$ in the parameter plane. All bifurcation curves are exponentially flat to the curve of primary homoclinic orbits at $\mu = 0$.

The bifurcation equations for recurrent orbits take the form

$$x_{i+1} = \mu_1 + a x_i^{1+\mu_2} + R(\{x_i\}_{i \in \mathbb{Z}}, \mu)$$

with $R(\{x_i\}_{i \in \mathbb{Z}}, \mu) = O(\|\{x_i\}_{i \in \mathbb{Z}}\|^{1+\eta})$ for some $\eta > 0$. In fact, $|a|$ equals the separatrix quantity defined in Hypothesis 5.7, and the sign of a reflects the orientation of the homoclinic center manifold [325, 326]. The bifurcation equations can be readily solved for N-homoclinic orbits and N-periodic orbits with $N = 1, 2$; see also [339]. Note that the occurrence of N-homoclinic orbits and N-periodic orbits with $N > 2$ is excluded by the existence of a two-dimensional homoclinic center manifold.

Lorenz-like attractors can be found near differential equations with two homoclinic orbits to the same equilibrium at resonance. This has been researched by Robinson [325–327] whose results we review in Theorem 5.80 in §5.5.5. Other bifurcation phenomena that occur in resonant bifurcations from two homoclinic orbits are described in [146, 186].

Higher codimension bifurcations of ODEs in the plane have been considered by Leontovich [254] and Roussarie [329]. In these studies, bounds are derived on the number of limit cycles that may appear in unfoldings. The following account follows [329]. Let $\dot{u} = f(u, \mu)$ be a smooth family of ODEs on $\mathbb{R}^2$ with a saddle type equilibrium for $\mu = 0$ whose saddle quantity $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ is equal to one. For μ small and fixed $\ell \in \mathbb{N}$, the family near the equilibrium is then C^ℓ-equivalent to

$$
\begin{aligned}
\dot{x}_{\mathrm{s}} &= -x_{\mathrm{s}} + \sum_{i=0}^{N(\ell)} \alpha_i(\mu)(x_{\mathrm{s}} x_{\mathrm{u}})^i x_{\mathrm{u}}, \\
\dot{x}_{\mathrm{u}} &= x_{\mathrm{u}},
\end{aligned}
$$

where $\alpha_i(\mu)$ are smooth functions of μ. The C^ℓ-equivalence is by rescaling time in an x-dependent C^ℓ-smooth fashion and conjugating with diffeomorphisms of class C^ℓ in $(x_{\mathrm{s}}, x_{\mathrm{u}}, \mu)$. Introduce the function

$$\omega(x, \varepsilon) = \frac{x^{-\varepsilon} - 1}{\varepsilon}. \tag{5.1}$$

Note that, for each $k > 0$, $x^k \omega$ tends to $-x^k \ln(x)$ as $\varepsilon \to 0$, uniformly for $x \in [0, X]$ for any $X > 0$. A Dulac map is a local transition map from a cross section transverse to the local stable manifold to a cross section transverse to the local unstable manifold. We may assume that these cross sections are given by $\{x_{\mathrm{s}} = 1\}$ and $\{x_{\mathrm{u}} = 1\}$. The Dulac map $D(x, \mu)$ now has the following expansion:

$$D(x, \mu) = x + \alpha_1 (x\omega + \cdots) + \alpha_2 (x^2 \omega + \cdots) + \cdots + \alpha_{N+1} (x^{N+1} \omega + \cdots) + \psi_\ell(x, \mu), \tag{5.2}$$

for a C^ℓ function ψ_ℓ of (x, μ) which is ℓ-flat for $x = 0$. Each function between brackets is a finite combination of terms $x^i \omega^j$ with $0 \le j \le i$ in an increasing order ($x^i \omega^j < x^{i'} \omega^{j'}$ precisely if $i < i'$ or $i = i'$ and $j < j'$). For $\mu = 0$, the Dulac map is equivalent to either $x \mapsto \beta_k x^k$ or $x \mapsto \alpha_{k+1} x^{k+1} \ln(x)$. Using this expansion, the following estimate on the number of limit cycles that appear in an unfolding from the homoclinic loop has been proved.

Theorem 5.14 ([329]). *Let $\dot{u} = f(u, \mu)$ be a family of ODEs on the plane with a homoclinic loop to the hyperbolic equilibrium for $\mu = 0$. Suppose that the saddle quantity $\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ is equal to one at $\mu = 0$ and that the Dulac map for $\mu = 0$ is not flat. If the Dulac map for $\mu = 0$ is equivalent to $x \mapsto \beta_k x^k$, then $\dot{u} = f(u, \mu)$ has at most $2k$ limit cycles for small μ. If the Dulac map for $\mu = 0$ is equivalent to $x \mapsto \alpha_{k+1} x^{k+1} \ln(x)$, then $\dot{u} = f(u, \mu)$ has at most $2k + 1$ limit cycles for small μ.*

We remark that, for analytic ODEs on the plane, the Dulac map is known to be nonflat [200], so that a uniformly bounded number of limit cycles appears in the unfolding of a resonant homoclinic loop.

A homoclinic center manifold reduces the general case of ODEs in $\mathbb{R}^n$ to a two-dimensional flow. As the degree of differentiability of the homoclinic center manifold is limited by gap conditions on the spectrum at the equilibrium, this reduction is of limited use for the study of bifurcations of periodic orbits. Bifurcations of higher codimension for higher-dimensional flows are considered in [164, 165, 331], and we remark that more complicated dynamics may occur: as stated in the next theorem, Turaev found an example of codimension three in which infinitely many limit cycles appear under perturbations in the non-planar case.

Theorem 5.15 ([402])**.** *Let $\dot{u} = f(u)$ be an ODE on $\mathbb{R}^n$ for $n \geq 4$ with a homoclinic solution $h(t)$ to the hyperbolic equilibrium p. Suppose that Hypotheses 2.2(i), 2.3(ii), and 2.4 are met and that $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} = 1$. Assume furthermore that the separatrix quantity is equal to one and that the stable eigenvalue closest to ν^{s} is complex. Under generic conditions, the homoclinic orbit is then the limit of infinitely many isolated periodic orbits.*

5.1.6 Inclination flips

Yanagida [428] realized that 2-homoclinic solutions might appear in the unfolding of codimension-two bifurcations of homoclinic orbits to equilibria with real leading eigenvalues. He considered three different scenarios that are related to the three nondegeneracy conditions that we introduced in Hypotheses 2.3 and 2.4. The first of these scenarios, homoclinic orbits at resonance, has been discussed in §5.1.5. The remaining two bifurcations, which are concerned with homoclinic orbits that are in inclination-flip or orbit-flip configurations, are discussed in this and the following section. Depending on eigenvalue conditions, 2-homoclinic orbits may appear in the unfolding or complicated dynamics may set in, involving N-homoclinic orbits for all N and suspended Hénon-like attractors.

We consider two-parameter families $\dot{u} = f(u, \mu)$ of three-dimensional vector fields with $(u, \mu) \in \mathbb{R}^3 \times \mathbb{R}^2$. Throughout, we assume that the eigenvalues of $f_u(p, 0)$ at the equilibrium p satisfy $\nu^{\mathrm{ss}} < \nu^{\mathrm{s}} < 0 < \nu^{\mathrm{u}}$ and that $h(t)$ is an orbit homoclinic to p for $\mu = 0$.

Hypothesis 5.8 (Inclination flip)**.** *Using the notation from Hypothesis 2.4, we assume that $v^{\mathrm{s}}, v^{\mathrm{u}}, v_*^{\mathrm{u}} \neq 0$ at $\mu = 0$. Furthermore, we assume that $v_*^{\mathrm{s}} = 0$ at $\mu = 0$ with $\partial_{\mu_2} v_*^{\mathrm{s}}(\mu)|_{\mu=0} \neq 0$.*

As outlined in §2.1, the orientation of the two-dimensional homoclinic center manifold changes at an inclination flip. Define

$$\alpha = -\nu^{\mathrm{ss}}/\nu^{\mathrm{u}}, \qquad \beta = -\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$$

and observe $\alpha > \beta > 0$. We distinguish the following three cases:

$$
\begin{array}{ll}
\text{Type A:} & \beta > 1; \\
\text{Type B:} & \alpha > 1 \text{ and } \frac{1}{2} < \beta < 1; \\
\text{Type C:} & \alpha < 1 \text{ or } \beta < \frac{1}{2},
\end{array}
$$

and impose the following additional nondegeneracy condition for type C bifurcations:

Hypothesis 5.9 (Nondegeneracy conditions for type C)**.** *For type C, we assume that*

(i) $\beta \neq \frac{1}{2}\alpha$.

(ii) If $\beta < \frac{1}{2}\alpha$, the homoclinic orbit does not lie in the unique smooth leading stable manifold $W^{\mathrm{ls}}(p)$.

(iii) If $\beta > \frac{1}{2}\alpha$, $W^{\mathrm{ls,u}}(p)$ has a nondegenerate quadratic tangency with $W^{\mathrm{s}}(p)$ along the homoclinic orbit.

To expand on the above hypothesis: If $\beta < \frac{1}{2}\alpha$, a typical one-dimensional leading stable manifold is C^1 but not C^2, but there exists a unique leading stable manifold that is smooth. If $\beta > \frac{1}{2}\alpha$, then $W^{\mathrm{ls,u}}(p)$ is a C^2 manifold, and Hypothesis 5.9(iii) is well defined.
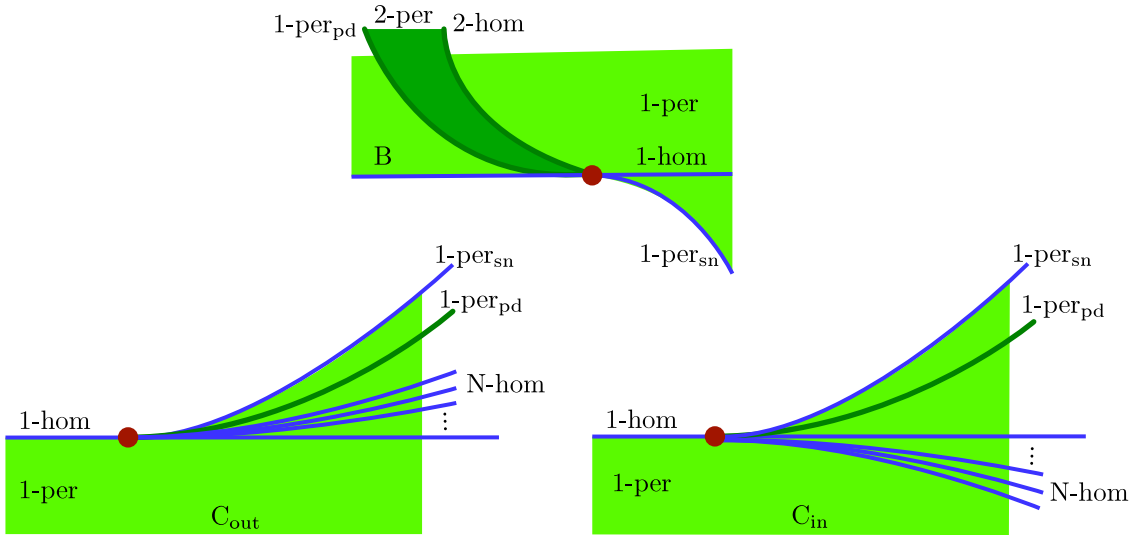
Figure 5.5: *The bifurcation diagrams at inclination- and orbit-flip bifurcations are shown for homoclinic-doubling bifurcations (type B) and the more involved type C bifurcations. For type C bifurcations, there are infinitely many branches of N-homoclinic orbits for each fixed $N \geq 2$.*

For inclination flips of type A, we refer to Theorem 5.1 in §5.1.1: a single periodic solution is created when crossing the curve of homoclinic orbits in the parameter plane; this result is also true in $\mathbb{R}^n$ and does not require that ν^s is real. The bifurcation diagrams in the remaining cases are shown in Figure 5.5. The unfolding for case B was treated in [217], see also [190, 218, 362, 364, 375], for vector fields in $\mathbb{R}^n$: it leads to homoclinic doubling, and the result holds also when ν^{ss} is not simple or complex (in this case, α is defined using Re ν^{ss} in place of ν^{ss}).

Theorem 5.16. *Assume that Hypotheses 2.2, 2.3(ii) and 5.8 are met. Suppose further that the inclination flip is of type B, then the bifurcation diagram is as shown in Figure 5.5 with one-sided curves of saddle-node and period-doubling bifurcations of periodic orbits and a one-sided curve of 2-homoclinic orbits that emerge from the inclination-flip point at $\mu = 0$ on the branch of primary homoclinic orbits.*

Finally, the unfolding for case C gives rise to N-homoclinic orbits for all N that are created through the unfolding of a singular horseshoe [189] (see also §4.3): There are two cases that differ in the global geometry of the stable and unstable manifolds of the equilibrium, and the existing proofs are limited to three-dimensional vector fields.

Theorem 5.17. *Assume that Hypotheses 2.2, 2.3(ii) and 5.8 are met. Suppose further that the inclination flip is of type C and that Hypothesis 5.9 is met. Depending on a global condition on the stable and unstable manifolds, the bifurcation diagram is then given by one of the two cases shown in Figure 5.5. In either case, infinitely many one-sided curves of N-homoclinic orbits emerge for each $N \geq 2$ from the inclination-flip point at $\mu = 0$ on the branch of primary homoclinic orbits.*

Naudot [290] proved the existence of suspended Hénon-like attractors in the unfolding of type-C inclination-flip homoclinic orbits in $\mathbb{R}^3$. Results on inclination-flips in $\mathbb{Z}_2$-equivariant ODEs, in which Lorenz-like strange attractors appear, can be found in Theorems 5.61 and 5.81 below. Applications in which inclination-flips appear include travelling waves in FitzHugh–Nagumo equation [240], 1 : 2 spatial resonances in systems with broken $O(2)$ symmetry [313], and models for instabilities in thermal convection [293].

5.1.7 Orbit flips

At an orbit-flip bifurcation, the homoclinic orbit approaches the equilibrium along a strong stable or strong unstable direction. The most comprehensive study of the homoclinic orbit-flip bifurcation can be found in

[339], and the bifurcation diagrams closely resemble those at inclination flips. Geometrically, the orientation of the two-dimensional homoclinic center manifold changes at an orbit flip; see §2.1.

We consider two-parameter families $\dot{u} = f(u, \mu)$ of vector fields with $(u, \mu) \in \mathbb{R}^n \times \mathbb{R}^2$. Throughout, we assume that the leading eigenvalues of $f_u(p, 0)$ at the hyperbolic equilibrium p are real and simple, and that $h(t)$ is a homoclinic orbit to p for $\mu = 0$. We assume an orbit-flip configuration within the stable manifold $W^{\mathrm{s}}(p)$.

Hypothesis 5.10 (Orbit flip)**.** *We assume that $v_*^{\mathrm{s}}, v^{\mathrm{u}}, v_*^{\mathrm{u}} \neq 0$ at $\mu = 0$, while $v^{\mathrm{s}}(0) = 0$ and $\partial_{\mu_2} v^{\mathrm{s}}(0) \neq 0$.*

We write λ^{ss} for the largest real part of the stable eigenvalues that lie strictly to the left of the leading eigenvalue ν^{s}, and define

$$\alpha = -\lambda^{\mathrm{ss}}/\nu^{\mathrm{u}}, \qquad \beta = -\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$$

so that $\alpha > \beta > 0$. We distinguish the cases

$$
\begin{array}{ll}
\text{Type A:} & \beta > 1; \\
\text{Type B:} & \beta < 1 \text{ and } \alpha > 1; \\
\text{Type C:} & \alpha < 1.
\end{array}
$$

If the eigenvalues are of type C, we need the following additional genericity assumption, which is similar to Hypothesis 5.9(ii):

Hypothesis 5.11 (Nondegeneracy condition for type C)**.** *There exists a unique eigenvalue ν^{ss} of $f_u(p, 0)$ with $\mathrm{Re}\, \nu^{\mathrm{ss}} = \lambda^{\mathrm{ss}}$ (hence, $\nu^{\mathrm{ss}} = \lambda^{\mathrm{ss}}$ is real and simple), and the homoclinic orbit $h(t)$ satisfies $\lim_{t \to \infty} \mathrm{e}^{-\nu^{\mathrm{ss}} t} h(t) \neq 0$.*

For orbit flips of type A, we refer to Theorem 5.1 in §5.1.1: a single periodic solution is created when crossing the curve of homoclinic orbits in the parameter plane. Results for orbit flips of type B and C are stated in the following two theorems and summarized in Figure 5.5.

Theorem 5.18 ([339])**.** *Assume that Hypotheses 2.2, 2.3(ii) and 5.10 are met. If the orbit flip is of type B, then the bifurcation diagram is as shown in Figure 5.5 with one-sided curves of saddle-node and period-doubling bifurcations of periodic orbits and a one-sided curve of 2-homoclinic orbits that emerge from the inclination-flip point at $\mu = 0$ on the branch of primary homoclinic orbits.*

Theorem 5.19 ([339])**.** *Assume that Hypotheses 2.2, 2.3(ii) and 5.10 are met. If the orbit flip is of type C and that Hypothesis 5.11 is met. Depending on a global condition on the stable and unstable manifolds, the bifurcation diagram is then given by one of the two cases shown in Figure 5.5. In particular, infinitely many one-sided curves of N-homoclinic orbits emerge for each $N \geq 2$ from the inclination-flip point at $\mu = 0$ on the branch of primary homoclinic orbits.*

In [280], the occurrence of inclination-flips in perturbations from orbit-flips is discussed. We refer to §5.3.6 for orbit-flip bifurcations in conservative or reversible ODEs.

5.1.8 Coexisting homoclinic orbits

In this section, we review the unfolding of two-parameter families $\dot{u} = f(u, \mu)$ on $\mathbb{R}^n$ with a hyperbolic equilibrium at p. We assume that there are two homoclinic orbits h_0 and h_1 to p for $\mu = 0$.

Periodic orbits near the two coexisting homoclinic orbits can often be described completely by symbolic codes. Let $q(t)$ be a periodic orbit that bifurcates in a gluing bifurcation, then we define its itinerary $\chi(q)$ by listing the index i of the unstable separatrix h_i it follows in consecutive loops. More precisely, pick two cross sections Σ_0, Σ_1 transverse to h_0 and h_1, respectively, and list the sequence of indices $0, 1$ that corresponds to which section the periodic orbit intersects at each return: this sequence is denoted by $\chi(q)$.
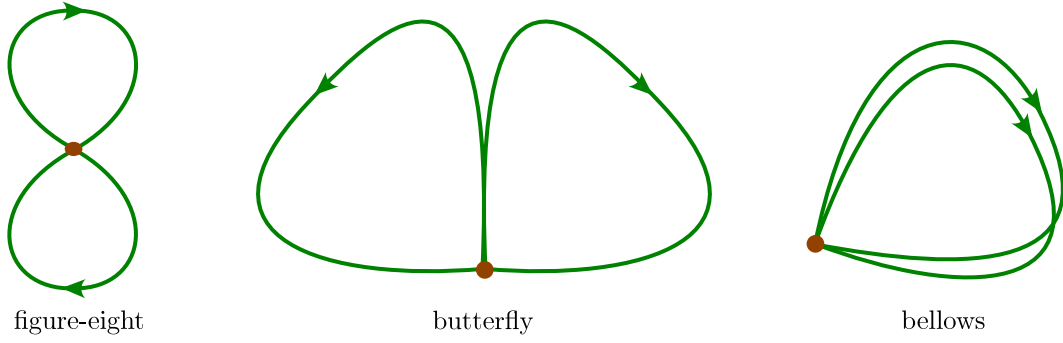
Figure 5.6: *Figure-eight, butterfly, and bellows configurations of two coexisting homoclinic orbits are illustrated (note that bellows can only exist in dimension four or greater).*

If the eigenvalue condition $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$ is met, the bifurcation involving two homoclinic orbits is often referred to as a gluing bifurcation. It turns out that the symbolic codes of periodic orbits that can be created in gluing bifurcations and the codes of periodic points for rigid rotations on the circle are closely related. Consider a circle rotation $R_\alpha(x) = x + \alpha \mod 1$ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We divide the circle in two intervals $I_0 = [0, \alpha)$ and $I_1 = [\alpha, 1)$ and introduce a symbolic itinerary $\chi_\alpha \in \{0, 1\}^{\mathbb{Z}}$ via $\chi_\alpha(i) = j$ if $R_\alpha^i \in I_j$. Any sequence that occurs as an itinerary for some α is called a rotation compatible sequence. A given rotation compatible sequence defines α uniquely as the frequency with which the symbol zero occurs in it: in other words, α is the *rotation number* of the itinerary.

Theorem 5.20 ([137, 138]). *We assume that the unstable manifold of p is one-dimensional and that there are two homoclinic orbits h_0 and h_1 to p for $\mu = 0$. Furthermore, suppose that $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$. The itinerary of any periodic orbit created for nearby parameter values is then necessarily rotation compatible. Furthermore, each bifurcating periodic orbit is asymptotically stable, and there are at most two periodic orbits for each given parameter value. If two periodic orbits exist for the same parameter value, then the rotation numbers of their itineraries are Farey neighbors[6].*

In the following, we assume that Hypothesis 2.3(ii) holds so that the leading eigenvalues $\nu^{\mathrm{s}}, \nu^{\mathrm{u}}$ at p are unique and real. Recall the definition of the vectors $v_i^{\mathrm{s}}(\mu)$ and $v_i^{\mathrm{u}}(\mu)$ with $i = 0, 1$ from (2.9), where the subscript i indicates that these vectors are computed for the homoclinic orbit h_i. Different geometric configurations can now be distinguished as follows; see also Figure 5.6.

Hypothesis 5.12 (Geometric configurations). *Assume that p has unique real leading eigenvalues $\nu^{\mathrm{s}}, \nu^{\mathrm{u}}$.*

(i) *Figure eight:* $v_0^{\mathrm{s}}(0) = -v_1^{\mathrm{s}}(0)$, $v_0^{\mathrm{u}}(0) = -v_1^{\mathrm{u}}(0)$.

(ii) *Expanding butterfly:* $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} < 1$ *and* $v_0^{\mathrm{s}}(0) = v_1^{\mathrm{s}}(0)$, $v_0^{\mathrm{u}}(0) = -v_1^{\mathrm{u}}(0)$.

(iii) *Contracting butterfly:* $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$ *and* $v_0^{\mathrm{s}}(0) = v_1^{\mathrm{s}}(0)$, $v_0^{\mathrm{u}}(0) = -v_1^{\mathrm{u}}(0)$.

(iv) *Bellows:* $v_0^{\mathrm{s}}(0) = v_1^{\mathrm{s}}(0)$, $v_0^{\mathrm{u}}(0) = v_1^{\mathrm{u}}(0)$.

Bifurcations from two coexisting homoclinic solutions have been investigated by Turaev [399, 403]. Near differential equations with two homoclinic orbits in the expanding butterfly configuration or the bellows configuration, one finds differential equations with suspended horseshoes (see Theorem 5.79 in §5.5.5).

We now present the bifurcation diagrams of the remaining cases, where the flow remains Morse–Smale outside bifurcation curves, and refer to [375] for a more detailed description, including symbolic codings of orbits. We shall refer to a geometric configuration of two homoclinic orbits as orientable, semi-orientable, or nonorientable if respectively both, one, or none of the homoclinic center manifolds $W^{\mathrm{c}}(h_i)$ with $i = 1, 2$ are annuli. If a primary homoclinic orbit admits an orientable homoclinic center manifold (and thus has nonempty stable set), then one encounters the intermittency phenomenon surveyed in §4.6 along any curve in parameter plane that crosses the branch of primary homoclinic orbits transversally.

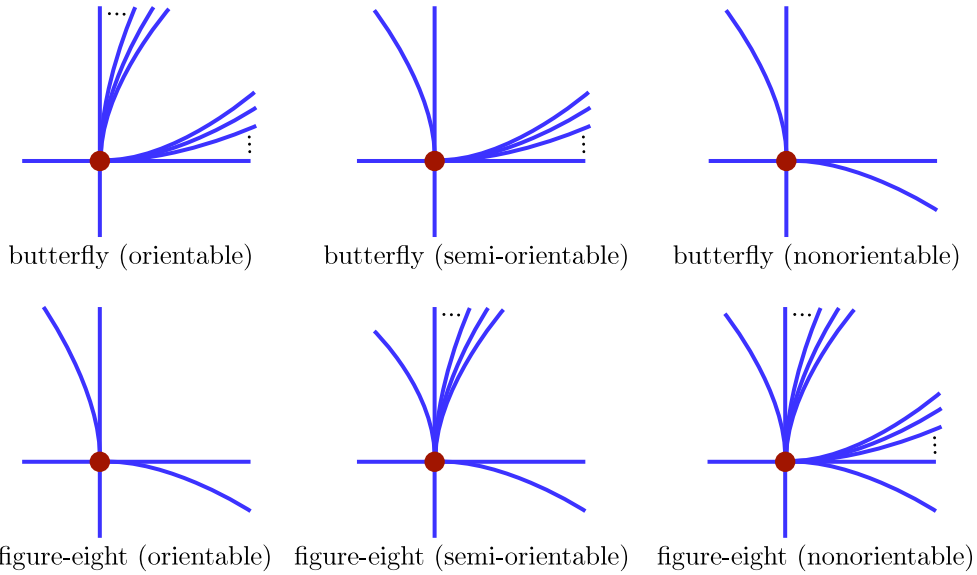[6]Recall that two rational numbers p/q and r/s are Farey neighbors if $|ps - qr| = 1$

Figure 5.7: *Curves of homoclinic bifurcations in unfoldings of flows with two coexisting homoclinic solutions to an equilibrium with real leading eigenvalues: single one-sided curves correspond to 2-round homoclinic orbits, while sequences of one-sided curves correspond to multi-round homoclinic orbits.*
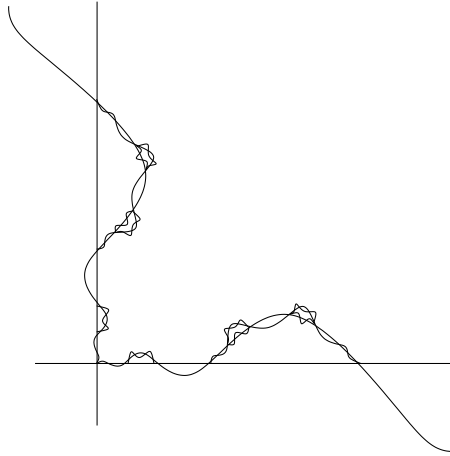

Figure 5.8: *Curves of homoclinic bifurcations in unfoldings of flows with two coexisting tame saddle-focus homoclinic solutions.*

Theorem 5.21 ([399, 403])**.** *Assume that the equilibrium p is hyperbolic and that Hypothesis 2.3(ii) is met with $-\nu^s/\nu^u > 1$. Suppose furthermore that there are distinct homoclinic orbits $h_i(t)$ to p for $i = 1, 2$ which exist when $\mu_i = 0$, satisfy Hypotheses 2.2(i) and 2.4, and are unfolded generically with respect to the parameter μ_i. The bifurcation diagrams for the figure-eight configuration (Hypothesis 5.12(i)) and for the contracting butterfly (Hypothesis 5.12(iii)) are then as shown in Figure 5.7. Homoclinic orbits with arbitrarily large arclength are found in the orientable and semi-orientable butterfly and the nonorientable figure eight. For each parameter value, there exist at most two periodic orbits.*

The next theorem treats the case when the leading stable eigenvalues are complex and the homoclinic orbits are tame. See also [180] for a description of the dynamics near two tame saddle-focus homoclinic orbits.

Theorem 5.22 ([356, 358])**.** *Assume that the equilibrium p is hyperbolic and that Hypothesis 2.3(iii) is met with $-\operatorname{Re}\nu^s/\nu^u > 1$. Suppose furthermore that there are distinct homoclinic orbits $h_i(t)$ to p for $i = 1, 2$ which exist when $\mu_i = 0$, satisfy Hypotheses 2.2(i) and 2.4, and are unfolded generically with respect to the parameter μ_i. The bifurcation diagram is then as shown in Figure 5.8 and, for each parameter value, there are at most two periodic orbits.*

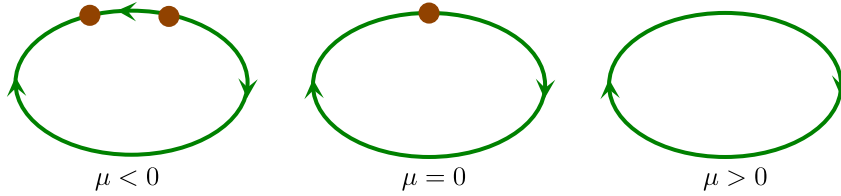$\mu < 0$ $\mu = 0$ $\mu > 0$

Figure 5.9: *The unfolding of a generic homoclinic orbit to a saddle-node equilibrium as described by Theorem 5.24.*

Due to the moduli for topological equivalence (see Theorem 5.5), the bifurcation diagrams for different values of the saddle quantity $-\operatorname{Re}\nu^{\mathrm{s}}/\nu^{\mathrm{u}}$ differ from each other.

5.1.9 Degenerate homoclinic orbits

A homoclinic orbit to a hyperbolic equilibrium p is called degenerate if the tangent spaces of the stable and unstable manifolds $W^{\mathrm{s}}(p)$ and $W^{\mathrm{u}}(p)$ along the homoclinic orbit $h(t)$ have more than the vector field direction in common. Typically, this defines a homoclinic bifurcation of codimension three at which $T_{h(0)}W^{\mathrm{s}}(p) \cap T_{h(0)}W^{\mathrm{s}}(p)$ is two-dimensional. We consider an equation $\dot{u} = f(u, \mu)$ in $\mathbb{R}^n$ with $\mu \in \mathbb{R}^3$ for which a homoclinic orbit $h(t)$ to a hyperbolic equilibrium p exists at $\mu = 0$.

Hypothesis 5.13. *Assume that $T_{h(0)}W^{\mathrm{s}}(p,0) \cap T_{h(0)}W^{\mathrm{s}}(p,0)$ is two-dimensional and that the stable and unstable manifolds $W^{\mathrm{s}}(p,\mu)$ and $W^{\mathrm{u}}(p,\mu)$ of p for $\dot{u} = f(u,\mu)$ intersect transversally along $h \times \{0\}$ in the product $\mathbb{R}^n \times \mathbb{R}^3$ of state and parameter space.*

Note that this requires $n \geq 4$. We have the following result.

Theorem 5.23 ([413]). *Suppose that h is a homoclinic orbit to the hyperbolic equilibrium p and assume that Hypothesis 5.13 is met, then the set of parameter values for which a single-round homoclinic orbits exists forms a Whitney umbrella in parameter space.*

To visualize the geometry, consider a degenerate homoclinic orbit $h(t)$ in $\mathbb{R}^4$ to an equilibrium with stable and unstable manifolds that are two-dimensional. Take a cross section Σ transverse to $h(t)$ at $t = 0$, then $W^{\mathrm{s}}(p,\mu)$ and $W^{\mathrm{u}}(p,\mu)$ intersect Σ along curves $W^{\mathrm{s}}_{\Sigma}(p,\mu)$ and $W^{\mathrm{u}}_{\Sigma}(p,\mu)$. Suppose that these curves intersect in a single point when $\mu = 0$ and have a quadratic tangency at this point then, in a generic three-parameter family, the set of parameter values for which $W^{\mathrm{s}}_{\Sigma}(p,\mu)$ and $W^{\mathrm{u}}_{\Sigma}(p,\mu)$ intersect forms a Whitney umbrella.

5.1.10 Homoclinic orbits to nonhyperbolic equilibria

In this section, we discuss homoclinic orbits to nonhyperbolic equilibria and restrict ourselves to bifurcations of overall codimension at most two. In particular, we consider only local bifurcations of codimension at most two and remark that homoclinic orbits to nonhyperbolic equilibria can be robust under perturbations that do not unfold the local bifurcation at the equilibrium: examples are provided by homoclinic orbits to saddle-node (codimension one) or Hopf/saddle-node (codimension two) equilibria. Additional degeneracies of the homoclinic orbit may, of course, further increase the codimension. We do not consider homoclinic orbits to pitchfork equilibria, which have been discussed in [100].

We start with homoclinic orbits to saddle-node equilibria. Consider a one-parameter family $\dot{u} = f(u,\mu)$ on $\mathbb{R}^n$ unfolding a saddle-node equilibrium p for $\mu = 0$. The geometry of the flow near p is clarified in §2.2, and we assume that Hypothesis 2.5 is met. Thus, the dimensions of the stable and unstable sets $\mathcal{M}^{\mathrm{s}}(p)$ and $\mathcal{M}^{\mathrm{u}}(p)$ of p add up to $n + 1$: these sets can therefore intersect transversally to give an isolated homoclinic orbit.
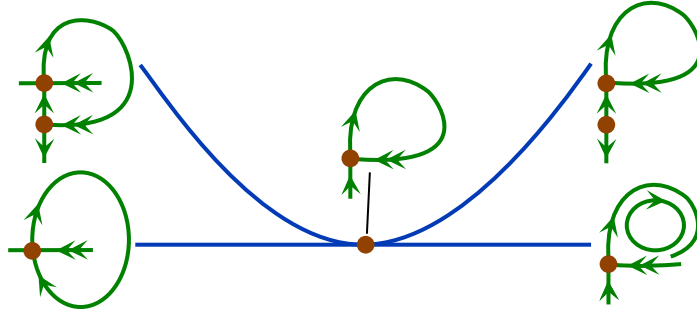
Figure 5.10: *The unfolding of a nongeneric homoclinic orbit to a saddle-node equilibrium as described by Theorem 5.25.*

Theorem 5.24 ([11, 368]). *In the above setup, assume that Hypotheses 2.5 and 2.7 are met, then the homoclinic orbit and the equilibrium p form a normally hyperbolic set diffeomorphic to a circle, which therefore persists for μ near zero.*

On the persistent invariant circle, the dynamics changes from two hyperbolic equilibria with heteroclinic connections between them for $\mu < 0$, say, to a single periodic orbit for $\mu > 0$, with the homoclinic loop to the saddle-node equilibrium forming the boundary at $\mu = 0$; see Figure 5.9.

If the sets $\mathcal{M}^s(p)$ and $\mathcal{M}^u(p)$ intersect simultaneously and transversally at several distinct homoclinic orbits or have a tangency at a single homoclinic orbit, then the resulting ODE can lie on the boundary of Morse–Smale systems; see §4.4 for details. In particular, coexisting transverse intersections of $\mathcal{M}^s(p)$ and $\mathcal{M}^u(p)$ lead to the creation of suspended horseshoes in an appropriate unfolding; see Theorem 4.5.

Next, we consider a codimension-two bifurcation of homoclinic orbits to saddle-node equilibria where Hypothesis 2.7 is violated. By reversing the direction of time if necessary, we may assume that Hypothesis 2.7(i) is violated. Thus, consider a two-parameter family $\dot{u} = f(u, \mu)$ on $\mathbb{R}^n$ with a saddle-node equilibrium p at $\mu_1 = 0$. We need the following unfolding condition for the homoclinic connection when varying μ_2.

Hypothesis 5.14. *Upon setting $\mu_1 = 0$ and varying only μ_2, the manifolds $W^{cu}(p, \mu)$ and $W^s(p, \mu)$ intersect transversally in $\mathbb{R}^n \times \mathbb{R}$ along $h \times \{0\}$.*

Theorem 5.25 ([92, 100, 261, 349]). *In the above two-parameter setting, assume that p is a saddle-node equilibrium when $\mu_1 = 0$. We assume that Hypothesis 2.5 is met if μ_1 is varied and that there is a homoclinic orbit $h(t)$ at $\mu = 0$ for which Hypothesis 2.7(ii) is met but Hypothesis 2.7(i) is violated. Suppose further that the unfolding condition Hypothesis 5.14 is met. The bifurcation diagram then contains a one-sided curve that emerges from $\mu = 0$ tangent to the μ_2-axis along which a homoclinic orbit to a hyperbolic equilibrium exists. Heteroclinic connections and periodic orbits bifurcate also; see Figure 5.10.*

The codimension-three bifurcation of a homoclinic orbit to a Bogdanov–Takens equilibrium[7] has been treated in [122]. As the unfolding of a local Bogdanov–Takens bifurcation gives rise to small homoclinic orbits, both small and large homoclinic orbits occur here. A codimension-three bifurcation of a homoclinic orbit to a saddle-node equilibrium in which both Hypothesis 2.7(i) and 2.7(ii) are violated occurs in a model for solitary pulses in an excitable reaction-diffusion medium [433]: this bifurcation is dynamically more complicated and leads to inclination-flip homoclinic orbits and cascades of T-point bifurcations [193].

Finally, we consider homoclinic loops to an equilibrium that undergoes a supercritical Hopf bifurcation. This bifurcation, which is commonly referred to as a Shil'nikov–Hopf bifurcation, has codimension two, and we therefore treat two-parameter families $\dot{u} = f(u, \mu)$. Suppose that the Hopf bifurcation of the equilibrium p occurs at $\mu_1 = 0$ and unfolds generically in the parameter μ_1 so that Hypothesis 2.8 is met. Assume also a generic unfolding that breaks the intersection of the center-stable with the unstable manifold of p when varying μ_2 for $\mu_1 = 0$.

---

[7]The local bifurcation of codimension two with two non-semisimple eigenvalues at zero; see §5.4.1
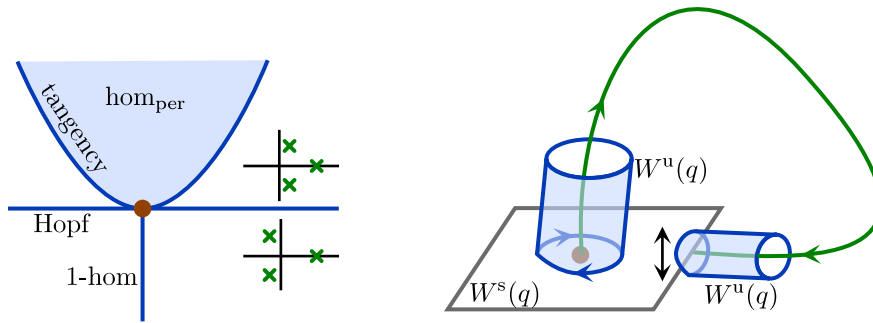
Figure 5.11: *The unfolding of a Shil'nikov–Hopf homoclinic orbit described in Theorem 5.26 is sketched in the left panel, where the insets show the eigenvalues at the equilibrium: a wild saddle-focus homoclinic orbit exists along the curve 1-hom, while the stable and unstable manifolds of the bifurcating small periodic orbits q have precisely two transverse intersections above the parabola which disappear through a tangency along the parabola. The mechanism that creates and destroys transverse homoclinic orbits of the small periodic orbits q for fixed $\mu_2 > 0$ as μ_1 varies is illustrated in the right panel, where the arrows indicate how the unstable manifolds of equilibrium and periodic orbits move as μ_1 varies.*

Hypothesis 5.15 (Generic unfolding). *For $\mu_1 = 0$, the intersection of $W^{cs}(p, \mu)$ and $W^u(p, \mu)$ unfolds generically with respect to the parameter μ_2.*

Shil'nikov–Hopf bifurcations lead to complicated dynamics such as suspended horseshoes via two different mechanisms: first, their unfolding may contain homoclinic tangencies of the stable and unstable manifolds of the small periodic orbits that are created in the Hopf bifurcation; secondly, wild saddle-focus homoclinic orbits are created at such bifurcations. Hirschberg and Knobloch [178] considered three-dimensional flows and study first-return maps, while Deng and Sakamoto [105] derived bifurcation equations in higher-dimensional systems. In higher-dimensional systems, additional conditions such as a nondegeneracy condition akin to Hypothesis 2.2(i) and the absence of inclination-flip and orbit-flip configurations akin to Hypothesis 2.4, are needed. We formulate here a bifurcation result in $\mathbb{R}^3$ and refer to [105] for the higher-dimensional results.

Theorem 5.26 ([105, 178]). *Suppose $\dot{u} = f(u, \mu)$ is a two-parameter family in $\mathbb{R}^3$ for which a homoclinic orbit to a Hopf equilibrium exists when $\mu = 0$. If Hypotheses 2.8(iii) and 5.15 are met, then there is a one-sided curve in parameter space that emerges from $\mu = 0$ and is transverse to the curve of Hopf bifurcations along which a generic wild saddle-focus homoclinic orbit exists. Furthermore, there is a curve tangent to the curve of Hopf bifurcations along which homoclinic tangencies of the stable and unstable manifolds of the small periodic orbits created in the Hopf bifurcation occur; see Figure 5.11.*

We finish with a few remarks on the interaction of homoclinic and Hopf/saddle-node bifurcations. Local unfoldings of Hopf/saddle-node equilibria are discussed in §5.4.2, and we remark that small Shil'nikov homoclinic orbits occur in certain cases of the local unfolding. With a global homoclinic orbit present, global Shil'nikov homoclinic orbits will also occur. These bifurcations of a homoclinic orbit to an equilibrium at a Hopf/saddle-node bifurcation arise in models of semiconductor lasers with optical injection [233, 434], and we refer also to [234] for a numerical investigation of such global Shil'nikov homoclinic orbits with one or more global excursions.

5.2 Heteroclinic cycles in generic systems

In this section, we discuss primarily bifurcations from heteroclinic cycles that connect different equilibria, though cycles that involve a periodic orbit instead of an equilibrium are considered briefly in §5.2.4. Thus, we consider the system $\dot{u} = f(u)$ in $\mathbb{R}^n$ with a heteroclinic cycle that consists of disjoint equilibria p_i and heteroclinic orbits $h_i(t)$ with $h_i(t) \in W^u(p_i) \cap W^s(p_{i+1})$ for each i with $1 \le i \le \ell$, where indices are taken modulo ℓ; see §2.3. We will mostly consider heteroclinic cycles with two heteroclinic orbits as the
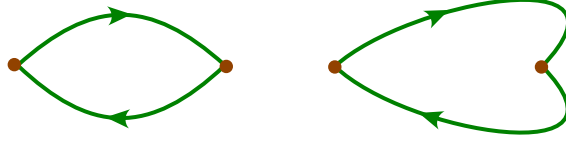
Figure 5.12: *Planar heteroclinic cycles can be nontwisted [left] or doubly twisted [right].*

codimension gets too large otherwise. We recall that the Morse index of an equilibrium p is defined via $\mathrm{ind}(p) := \dim W^{\mathrm{u}}(p)$.

5.2.1 Heteroclinic cycles with saddles of identical Morse index

We discuss heteroclinic cycles built from codimension-one heteroclinic connections that occur when the equilibria involved have the same Morse index. One observation is that homoclinic orbits can appear in unfoldings of heteroclinic bifurcations. A prototype result is Theorem 5.27 below by Chow, Deng, and Terman. We start with results on heteroclinic cycles between two equilibria with real leading eigenvalues. Bifurcations that involve equilibria with complex conjugate leading eigenvalues exhibit more complicated features and will be discussed briefly afterwards.

Let $\dot{u} = f(u, \mu)$ be a two-parameter family in $\mathbb{R}^n$ that has a heteroclinic cycle for $\mu = 0$ with heteroclinic orbits h_1 and h_2 that connect the hyperbolic equilibria p_1 and p_2 with identical Morse index: thus, we assume that Hypothesis 2.10(i) is met at $\mu = 0$ with codimension $d_i = 1$ for $i = 1, 2$. Furthermore, we assume that the leading eigenvalues at both equilibria are unique and real (Hypothesis 2.11(ii)) and that Hypothesis 2.12 is met along both heteroclinic orbits, so that neither is in either orbit- or inclination-flip configuration. As a consequence, there exists a two-dimensional continuous plane bundle E^{c} along the heteroclinic cycle $h_1 \cup h_2 \cup p_1 \cup p_2$ that is invariant under the variational equations along h_1 and h_2. The plane $E^{\mathrm{c}}_{p_i}$ is spanned by two eigenvectors e^{s}_i and e^{u}_i that belong to the leading eigenvalues and which we pick according to

$$e^{\mathrm{s}}_i = \lim_{t \to \infty} \dot{h}_{i-1}(t)/\|\dot{h}_{i-1}(t)\|, \qquad e^{\mathrm{u}}_i = \lim_{t \to -\infty} \dot{h}_i(t)/\|\dot{h}_i(t)\|, \tag{5.3}$$

with indices taken modulo 2. We choose orientations on $E^{\mathrm{c}}_{p_i}$ so that the bases $\{e^{\mathrm{s}}_i, e^{\mathrm{u}}_i\}$ are both positively oriented. By continuity, this induces an orientation on the plane bundle $E^{\mathrm{c}}_i(t)$ along $h_i(t)$ by continuing the orientation from $t = -\infty$. We define the orientation index $\mathcal{O}_i = \pm 1$ as follows:

$$\mathcal{O}_i = \begin{cases} 1 & \text{if the orientation } E^{\mathrm{c}}_i(t) \text{ along } h_i(t) \text{ matches the orientation of } E^{\mathrm{c}}_{p_i} \text{ when } t \to \infty, \\ -1 & \text{otherwise} \end{cases} \tag{5.4}$$

and refer to Figure 5.12 for an illustration.

Hypothesis 5.16 (Twist conditions)**.** *Consider the following twist conditions along the heteroclinic cycle:*

(i) *nontwisted:* $\mathcal{O}_1, \mathcal{O}_2 = 1$,

(ii) *single twisted:* $\mathcal{O}_1 \mathcal{O}_2 = -1$,

(iii) *doubly twisted:* $\mathcal{O}_1, \mathcal{O}_2 = -1$.

Note that, in the spirit of §3.4, there exists a two-dimensional center manifold near the heteroclinic cycle which will be orientable if the cycle is nontwisted or doubly twisted, and nonorientable otherwise.

Theorem 5.27 ([90, 91, 101, 225, 342])**.** *Let $\dot{u} = f(u, \mu)$ be a two-parameter family that admits a heteroclinic cycle with heteroclinic orbits h_1, h_2 to two hyperbolic equilibria p_1, p_2 with identical Morse index when $\mu = 0$.*

(i) *Suppose that Hypotheses 2.10 with $d_i = 1$, 2.11(i) at p_i, and 2.12(ii),(iv) for h_i are met for $i = 1, 2$. In the parameter plane, there are then two curves of heteroclinic orbits that intersect at $\mu = 0$. Branching off $\mu = 0$ and tangent to the curves of the heteroclinic orbit h_i are two one-sided curves of homoclinic orbits. Other solutions may bifurcate as well.*
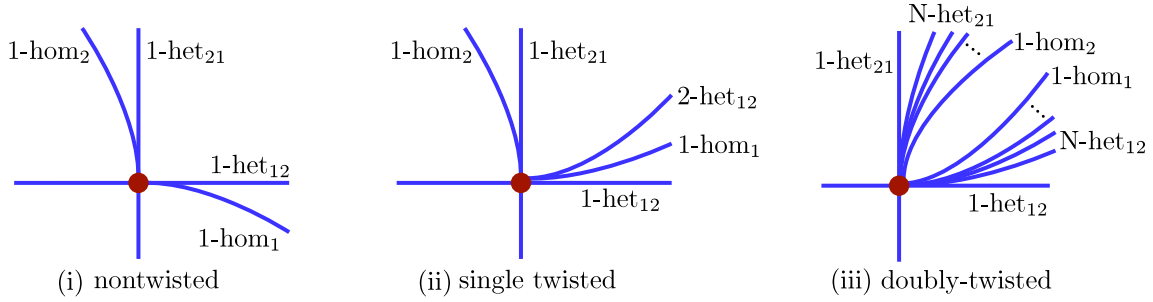
Figure 5.13: *Bifurcation diagrams for Theorem 5.27: subscripts denote to which equilibria the solution connects. In the right panel, there is a unique curve of N-heteroclinic orbits that connect p_1 to p_2 for each $N \geq 2$, and these curves accumulate onto the branch of homoclinic orbits to p_2.*

(ii) Suppose that Hypotheses 2.10 with $d_i = 1$, 2.11(ii) with[8] $-\nu_i^s/\nu_i^u < 1$, and 2.12 for h_i hold for $i = 1, 2$. If Hypothesis 5.16(i) is also met, then the complete bifurcation diagram is given in Figure 5.13(i). If Hypothesis 5.16(ii) is met, the complete bifurcation set given in Figure 5.13(ii) which contains, in addition, a one-sided curve of 2-heteroclinic orbits that makes two passages near the twisted heteroclinic orbit. If Hypothesis 5.16(iii) is met, then the complete bifurcation set, given by Figure 5.13(iii), contains, for each $N \geq 1$, a unique one-sided curve of N-heteroclinic orbits between p_1 and p_2 that make $N+1$ passages near h_1 and N near h_2 as well as another branch of N-heteroclinic orbits between p_2 and p_1 that make N passages near h_1 and $N+1$ near h_2.

The geometric picture is as follows. There is a two-dimensional normally hyperbolic center manifold that contains the heteroclinic cycle for $\mu = 0$ (see §3.4 and [342]). Take two cross sections Σ_1 and Σ_2 in the center manifold transverse to the heteroclinic orbits h_1 and h_2, respectively, then there are C^1 coordinates on these cross sections in which the transition maps are of the form $x \mapsto b_i(\mu) + a_i(\mu)x^{-\nu_i^s/\nu_i^u} + \mathrm{o}(x^{-\nu_i^s/\nu_i^u})$ for $x > 0$; see §3.6.3. If $-\nu_i^s/\nu_i^u > 1$ for both $i = 1, 2$, then the return map restricted to the center manifold is a contraction and, consequently, there can be at most one periodic orbit (if $-\nu_i^s/\nu_i^u < 1$, then the return map is an expansion, and the conclusion still holds).

The above results assumed, apart from real simple leading eigenvalues, that the saddle quantities are either both larger or both smaller than one: thus, we assumed that we are in the first of the two cases distinguished below:

Hypothesis 5.17 (Saddle quantities). *We distinguish two cases for the saddle quantities $\lambda_i = -\nu_i^s/\nu_i^u$:*

(i) $\lambda_1, \lambda_2 > 1$ *(the case $\lambda_1, \lambda_2 < 1$ is brought to this case by chancing the direction of time);*

(ii) $\lambda_1\lambda_2 < 1$.

Heteroclinic cycles under Hypothesis 5.17(ii) have been considered by Shashkov [353–355], and we refer to these references for the bifurcation diagrams. Depending on the orientability of the primary heteroclinic orbits, the two-parameter bifurcation diagrams show, in addition to homoclinic and heteroclinic bifurcation curves, bifurcation curves of saddle-node or periodic-doubling bifurcations of periodic orbits.

Bifurcations from heteroclinic cycles where at least one of the equilibria has complex conjugate leading eigenvalues lead to great complexity in bifurcation diagram and dynamics. Various cases have been studied in [64, 65, 358], and we refer to these references and also to [375] for an overview of the resulting dynamics. Apart from determining bifurcation curves in an unfolding, one can also look for prevalent dynamics as parameters are varied. We present a result in this direction due to San Martín, involving a heteroclinic cycle

---

[8]We assume here that the saddle quantities are both smaller than one: if they are both larger than one, the result holds for the time-reversed system; note that this changes the definition of 1-het$_{12}$ to 1-het$_{21}$, and vice versa, in Figure 5.13, but not that of 1-hom$_j$.

in $\mathbb{R}^3$ between an equilibrium with real leading stable eigenvalues and a second equilibrium with complex conjugate stable eigenvalues.

Consider a two-parameter family $\dot{u} = f(u, \mu)$ on $\mathbb{R}^3$ that has a heteroclinic cycle consisting of connecting orbits h_1 and h_2 and hyperbolic equilibria p_1 and p_2 of index 1 when $\mu = 0$.

Hypothesis 5.18 (Heteroclinic cycle with one saddle focus). *Suppose that p_1 has real eigenvalues $\nu_1^{\mathrm{ss}} < \nu_1^{\mathrm{s}} < 0 < \nu_1^{\mathrm{u}}$, while p_2 has a real eigenvalue $\nu_2^{\mathrm{u}} > 0$ and complex eigenvalues $\nu_2^{\mathrm{s}}, \overline{\nu_2^{\mathrm{s}}}$ with $\mathrm{Re}\,\nu_2^{\mathrm{s}} < 0$. For $\mu = 0$, there is a heteroclinic cycle with $h_1 \in W^{\mathrm{u}}(p_1) \cap W^{\mathrm{s}}(p_2)$ and $h_2 \in W^{\mathrm{u}}(p_2) \cap W^{\mathrm{s}}(p_1)$.*

Hypothesis 5.19 (Expanding heteroclinic cycle). *The heteroclinic cycle is expanding: $\mathrm{Re}\,\nu_2^{\mathrm{s}} \nu_1^{\mathrm{s}} < \nu_2^{\mathrm{u}} \nu_1^{\mathrm{u}}$.*

The following result shows that hyperbolic dynamics is prevalent in an unfolding of an expanding heteroclinic cycle that involves a saddle focus.

Theorem 5.28 ([337]). *Consider a two-parameter family of ODEs on $\mathbb{R}^3$ that satisfies Hypotheses 5.18, 5.19, and 2.10(ii). Furthermore, assume that h_2 satisfies Hypothesis 2.12(ii), and h_1 satisfies Hypothesis 2.12(i). We also assume the generic condition that, for μ near 0, there are C^2 linearizing coordinates near p_1 and p_2. Last, we assume that the heteroclinic cycle is the locally maximal invariant set in a small open neighborhood $\mathcal{U}$ of itself when $\mu = 0$. Let P be the set of parameter values μ near zero for which $\dot{u} = f(u, \mu)$ has a chain recurrent set in $\mathcal{U}$ that is equal to the two equilibria $p_j(\mu)$ and at most one nontrivial hyperbolic basic set, then*

$$\lim_{\varepsilon \downarrow 0} \frac{|B_\varepsilon(0) \cap P|}{|B_\varepsilon(0)|} = 1,$$

where $B_\varepsilon(0)$ is a disc of radius ε around zero, and $|\cdot|$ denote two-dimensional Lebesgue measure.

Similarly hyperbolic dynamics is prevalent for contracting heteroclinic cycles where the inequality in Hypothesis 5.19 is reversed: in this case, the single hyperbolic basic set in the definition of P is replaced by an attracting periodic orbit; see [337].

A central question in the study of heteroclinic cycles of planar ODEs has been to estimate its cyclicity, that is, to find bounds on the number of cycles that can appear in unfoldings[9]. In §5.1.5, this problem is discussed for the unfolding of planar homoclinic loops near resonant eigenvalues; see, in particular, Theorem 5.14. Here, we briefly discuss general heteroclinic cycles and begin with the flow near a hyperbolic equilibrium with eigenvalues that are not necessarily at resonance. Let $\dot{u} = f(u, \mu)$ be a smooth family of ODEs on $\mathbb{R}^2$ with a saddle type equilibrium at $\mu = 0$. For μ small and fixed $\ell \in \mathbb{N}$, the vector field near the equilibrium is then C^ℓ-equivalent to

$$
\begin{aligned}
\dot{x_{\mathrm{s}}} &= \nu^{\mathrm{s}} x_{\mathrm{s}} + \sum_{i=0}^{N(\ell)} \alpha_i(\mu) (x_{\mathrm{s}} x_{\mathrm{u}})^i x_{\mathrm{u}}, \\
\dot{x_{\mathrm{u}}} &= \nu^{\mathrm{u}} x_{\mathrm{u}},
\end{aligned}
$$

where $\alpha_i(\mu)$ are smooth functions of μ. The C^ℓ-equivalence is by rescaling time in an x-dependent C^ℓ-smooth fashion and conjugating with diffeomorphisms of class C^ℓ in (x_s, x_u, μ); see [330] and use [47] for the parameter-dependent case. Recall that a Dulac map is a local transition map from a cross section transverse to the local stable manifold to a cross section transverse to the local unstable manifold. We may assume that these cross sections are given by $\{x_{\mathrm{s}} = 1\}$ and $\{x_{\mathrm{u}} = 1\}$. Mourtada proves the following expansion for the Dulac map.

Theorem 5.29 ([286]). *For each fixed $k \in \mathbb{N}$, there is a neighborhood U_k of $\mu = 0$ in $\mathbb{R}$ so that the Dulac map $D(x, \mu)$ has the asymptotic expansion*

$$D(x, \mu) = x^{-\nu^{\mathrm{s}}/\nu^{\mathrm{u}}} \left[a(\mu) + \psi(x, \mu) \right]$$

---

[9]This question has its origin in Hilbert's 16th problem; see [201] for a review
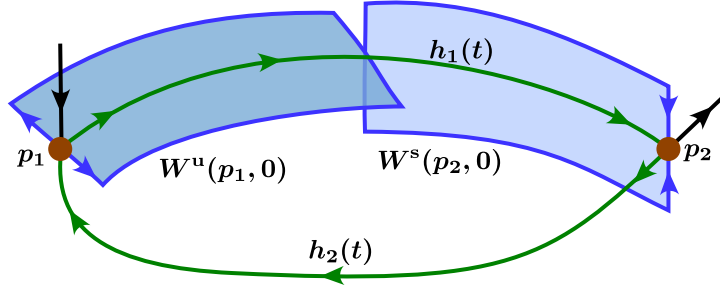
Figure 5.14: *The geometric configuration of heteroclinic orbits at a T-point is illustrated.*

for $\mu \in U_k$, where a is a smooth function of μ and

$$\lim_{x \to 0} x^j \partial_x^j \psi(x, \mu) = 0$$

for each $j \leq k$.

A *hyperbolic polycycle* is a polycycle for a planar ODE that consists of hyperbolic equilibria and connecting heteroclinic orbits. The above expansion for the Dulac map near a hyperbolic equilibrium can now be used to bound the cyclicity of hyperbolic polycycles. Note that a return map on a cross section transverse to a hyperbolic polycycle is a composition of local diffeomorphisms and Dulac maps. Mourtada derived the following bound for the number of limit cycles that can appear in the unfolding of hyperbolic polycycles. We remark that Kaloshin [210] obtained an explicit bound for the cyclicity of more general polycycles, which we state as Theorem 5.36 further below.

Theorem 5.30 ([286, 287]). *There is a function $C : \mathbb{N} \to \mathbb{N}$ so that the following holds. Let $\dot{u} = f(u, \mu)$ be a generic family of planar ODEs depending on a parameter $\mu \in \mathbb{R}^k$ that has a hyperbolic polycycle that contains k equilibria when $\mu = 0$, then the number of limit cycles that exist near the polycycle for small values of μ is bounded by $C(k)$.*

We mention that $C(2) = 2$, $C(3) = 3$, $C(4) = 5$. The cyclicity of hyperbolic polycycles with two equilibria was studied in [81, 124, 288].

5.2.2 T-points: Heteroclinic cycles with saddles of different index

T-points are codimension-two bifurcations of heteroclinic cycles that involve equilibria with different Morse indices. We consider

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^3 \times \mathbb{R}^2 \tag{5.5}$$

and assume that p_i with $i = 1, 2$ are hyperbolic equilibria of (5.5) for all μ.

Hypothesis 5.20. *The linearization $f_u(p_1, 0)$ has simple eigenvalues $\nu_1^{\mathrm{s}}, \nu_1^{\mathrm{u}}, \tilde{\nu}_1^{\mathrm{u}}$ with $\nu_1^{\mathrm{s}} < 0 < \mathrm{Re}\, \nu_1^{\mathrm{u}} \leq \mathrm{Re}\, \tilde{\nu}_1^{\mathrm{u}}$, while the linearization $f_u(p_2, 0)$ has simple eigenvalues $\tilde{\nu}_2^{\mathrm{s}}, \nu_2^{\mathrm{s}}, \nu_2^{\mathrm{u}}$ with $\mathrm{Re}\, \tilde{\nu}_2^{\mathrm{s}} \leq \mathrm{Re}\, \mathrm{Re}\, \nu_2^{\mathrm{s}} < 0 < \nu_2^{\mathrm{u}}$. We assume that the saddle quantities*

$$\lambda_1 := -\frac{\mathrm{Re}\, \nu_1^{\mathrm{u}}}{\nu_1^{\mathrm{s}}}, \qquad \lambda_2 := -\frac{\nu_2^{\mathrm{u}}}{\mathrm{Re}\, \nu_2^{\mathrm{s}}}$$

satisfy $\lambda_j \neq 1$.

Thus, the manifolds $W^{\mathrm{s}}(p_1, \mu)$ and $W^{\mathrm{u}}(p_2, \mu)$ are one-dimensional, while $W^{\mathrm{u}}(p_1, \mu)$ and $W^{\mathrm{s}}(p_2, \mu)$ are two-dimensional. We assume that (5.5) has a heteroclinic cycle for $\mu = 0$ that consists of a transversely constructed heteroclinic orbit $h_1(t) \in W^{\mathrm{u}}(p_1, 0) \cap W^{\mathrm{s}}(p_2, 0)$ and a codimension-two connection $h_2(t) \in W^{\mathrm{u}}(p_2, 0) \cap W^{\mathrm{s}}(p_1, 0)$, as indicated in Figure 5.14. We also need to make several genericity assumptions.
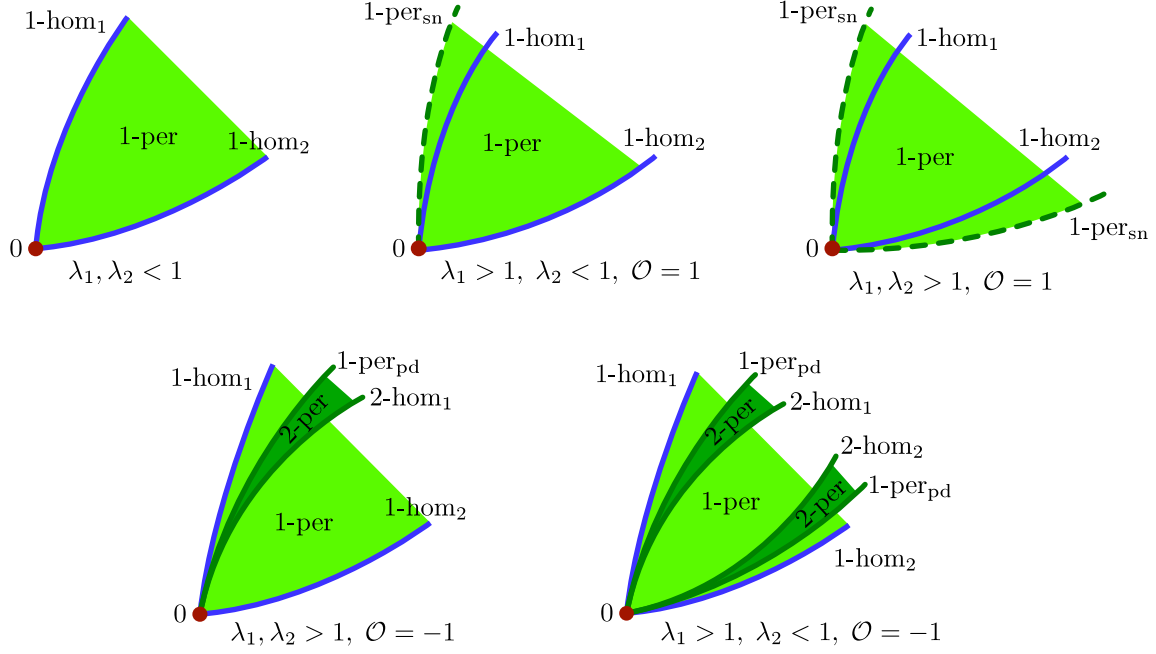
Hypothesis 5.21. *We assume the following:*

Figure 5.15: *Shown are the bifurcation diagrams in the parameter plane $\mu \in \mathbb{R}^2$ near a generic T-point at $\mu = 0$ that involves two hyperbolic equilibria with real and simple eigenvalues: the diagrams depend on the saddle quantities λ_j defined in Hypothesis 5.20 and the orientation index $\mathcal{O}$ from Hypothesis 5.21(iii).*

(i) *The heteroclinic orbit $h_1(t) \in W^{\mathrm{u}}(p_1, 0) \cap W^{\mathrm{s}}(p_2, 0)$ satisfies Hypothesis 2.10 with $d = 0$, while the orbit $h_2(t) \in W^{\mathrm{u}}(p_2, 0) \cap W^{\mathrm{s}}(p_1, 0)$ satisfies Hypothesis 2.10 with $d = 2$.*

(ii) *Neither h_1 nor h_2 are in an orbit-flip configuration (that is, h_j obeys Hypothesis 2.12(ii) and (iv) for $j = 1, 2$).*

(iii) *If the eigenvalues of p_1 and p_2 are all real, we assume in addition that the heteroclinic cycle is not in an inclination-flip configuration: more precisely, we assume that the closure of $W^{\mathrm{s}}(p_2, 0)$ is homeomorphic to a cylinder ($\mathcal{O} := 1$) or to a Möbius band ($\mathcal{O} := -1$); in this case, the closure of $W^{\mathrm{u}}(p_1, 0)$ is also homeomorphic to a cylinder if $\mathcal{O} = 1$ and to a Möbius band if $\mathcal{O} = -1$.*

The following theorem summarizes the different bifurcation diagrams at T-points when the eigenvalues of the equilibria p_j are real.

Theorem 5.31 ([61]). *Assume that Hypotheses 5.20 and 5.21(i)-(iii) are met. If the eigenvalues of both equilibria p_1 and p_2 are all real, then the bifurcation diagrams near $\mu = 0$ are as shown in Figure 5.15.*

Next, we consider the case where one of the equilibria (say p_1) has complex eigenvalues, while the other equilibrium (p_2) has only real eigenvalues.

Theorem 5.32 ([61, 150, 151]). *Assume that Hypotheses 5.20 and 5.21(i)-(ii) are met. Assume furthermore that the eigenvalues of p_2 are real, while p_1 is a focus so that $\operatorname{Im} \nu_1^{\mathrm{u}} = -\operatorname{Im} \tilde{\nu}_1^{\mathrm{u}} > 0$. The bifurcation diagram of heteroclinic orbits and 1-homoclinic orbits is shown in Figure 5.16(i). In addition, for each μ close to zero, there are infinitely many hyperbolic periodic orbits near the heteroclinic cycle and the dynamics contains a shift of two symbols.*

Finally, we consider the situation where both p_1 and p_2 are foci. In coordinate systems in which the vector fields near $p - 1$ and p_2 are linearized, we pick a number $T \gg 1$ and place small sections at $h_2(-T)$ near p_2 and at $h_2(T)$ near p_1. Without loss of generality, the return map Π along h_2 that maps these section into each other satisfies

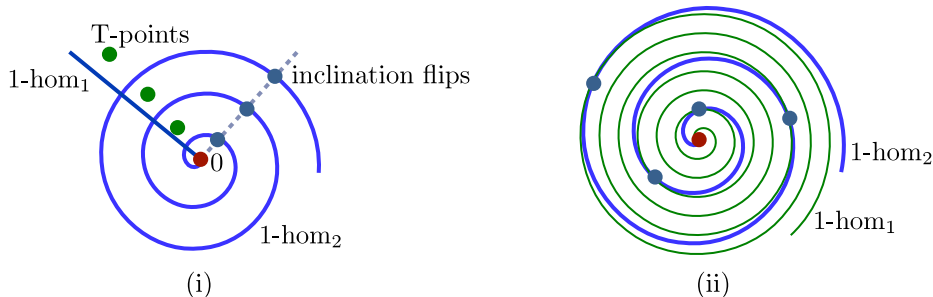$$\Pi_u(h_2(-T)) = \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix}.$$

Figure 5.16: *Shown are the partial bifurcation diagrams in the parameter plane $\mu \in \mathbb{R}^2$ near a generic T-point at $\mu = 0$. Panel (i) is for the case where the unstable eigenvalues of p_1 are not real, while the eigenvalues of p_2 are all real: There is a sequence of parameter values that converge to $\mu = 0$ and correspond to T-points of (5.5); furthermore, the homoclinic orbits to the saddle p_1 undergo a sequence of inclination flips that accumulate at $\mu = 0$. Panel (ii) shows the bifurcation diagram of 1-homoclinic orbits when both equilibria have complex leading eigenvalues.*

We set

$$\gamma := \frac{1}{2}\left(d^2 + \frac{1}{d^2}\right) \geq 1, \qquad \Gamma := \left(\frac{\lambda_1}{\operatorname{Im}\nu_1^{\mathrm{u}}}\right)^2 + \left(\frac{\lambda_2}{\operatorname{Im}\nu_2^{\mathrm{s}}}\right)^2 - 2\gamma\frac{\lambda_1\lambda_2}{|\operatorname{Im}\nu_1^{\mathrm{u}}|\,|\operatorname{Im}\nu_2^{\mathrm{s}}|} + 1 - \gamma^2.$$

We can now state the following theorem.

Theorem 5.33 ([64, 65]). *Assume that Hypotheses 5.20 and 5.21(i)-(ii) are met. Assume furthermore that the unstable eigenvalues of p_1 and the stable eigenvalues of p_2 are complex and that $\lambda_1 \operatorname{Im}\nu_2^{\mathrm{s}} \neq \lambda_2 \operatorname{Im}\nu_1^{\mathrm{u}}$ and $\Gamma \neq 0$. Under these assumptions, the following is true:*

First, the manifolds $W^{\mathrm{u}}(p_1, 0)$ and $W^{\mathrm{s}}(p_2, 0)$ intersect transversely at infinitely many different heteroclinic orbits. If $\Gamma < 0$, the class of systems for which there is a heteroclinic orbit along which $W^{\mathrm{u}}(p_1, 0)$ and $W^{\mathrm{s}}(p_2, 0)$ intersect non-transversely is dense is the class of systems with generic T-points.

Second, the bifurcation diagram of 1-homoclinic orbits is as shown in Figure 5.16(ii): in particular, there exists an infinite sequence of parameter values that accumulate at $\mu = 0$ so that (5.5) has a pair of coexisting homoclinic orbits to p_1 and p_2: the two homoclinic bifurcation curves intersect transversely in the parameter plane when $\Gamma > 0$, while near each system for which $\Gamma < 0$ there is a system that satisfy the assumptions stated above for which one of these intersections is not transverse.

Depending on the values of the saddle quantities λ_i, the dynamics of (5.5) can contain attractors and repellors near parameter values at which homoclinic orbits to the foci p_1 and p_2 coexist; see §5.1.2.

We remark that T-points arise in the Lorenz equation [150, 151] as well as in systems that model the oxidation on platinum surfaces [193, 432, 433], the propagation of calcium waves [328, 379], Josephson junctions [408] and laser systems [420].

5.2.3 Heteroclinic cycles with nonhyperbolic equilibria

In generic two-parameter families of ODEs, heteroclinic cycles that involve a saddle-node equilibrium and a hyperbolic equilibrium may appear. We remark that heteroclinic cycles with two saddle-node equilibria may also occur: such a cycle is of codimension two if it forms a normally hyperbolic invariant circle that contains the two saddle-node equilibria.

Planar systems with heteroclinic cycles that consist of two heteroclinic orbits, which connect a hyperbolic saddle to a saddle-node equilibrium and back, have been studied by Grozovskiĭ [159]. Polycycles where both unstable separatrices of the hyperbolic saddle are heteroclinic orbits to the saddle-node equilibrium are also of codimension-two: they have also been treated in [159]. Homburg [184] studied general heteroclinic cycles between a hyperbolic equilibrium and a saddle-node equilibrium for ODEs in $\mathbb{R}^n$. We present here a single
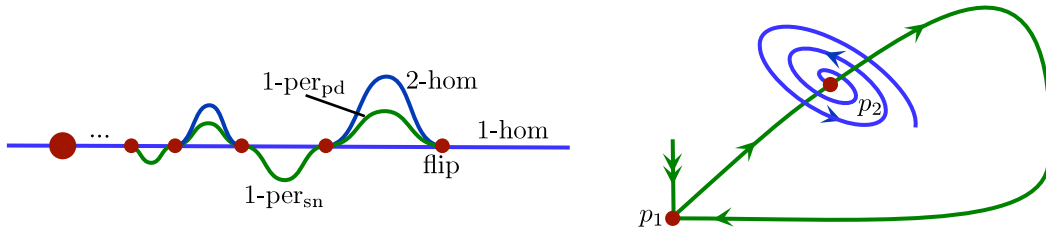
Figure 5.17: *Bifurcation set and phase-space dynamics at $\mu = 0$ for Theorem 5.34.*

theorem for the case where the hyperbolic equilibrium has a pair of complex conjugate leading eigenvalues: for simplicity, this result is stated for three-dimensional flows; generalizations to higher dimensions involve geometric conditions akin to no-flip conditions.

Theorem 5.34 ([184])**.** *Let $\dot{u} = f(u, \mu)$ be a two-parameter family of ODEs on $\mathbb{R}^3$ with $\mu \in \mathbb{R}^2$. Assume that a heteroclinic cycle exists for $\mu = 0$ which connects a hyperbolic equilibrium p_1 and a saddle-node equilibrium p_2 so that the following assumptions are met. The eigenvalues of $f_u(p_1, 0)$ at the hyperbolic equilibrium p_1 satisfy Hypothesis 2.3(ii) and are of Type B as explained in §5.1.6, while the saddle-node equilibrium p_2 satisfies Hypotheses 2.5 and 2.6(ii) with respect to μ_1, so that μ_1 unfolds the saddle node. Furthermore, we assume that $W^{\mathrm{u}}(p_1, 0) \not\subset W^{\mathrm{s}}(p_2, 0)$ and that the intersection of the stable manifold $W^{\mathrm{s}}(p_1, \mu)$ and the center manifold $W^{\mathrm{c}}(p_2, \mu)$ is unfolded generically with respect to the parameter μ_2 when $\mu_1 = 0$. The bifurcation set then contains a sequence of inclination-flip homoclinic bifurcations that converge to $\mu = 0$. Branching from the inclination-flip bifurcation points are curves of saddle-node and period-doubling bifurcations of periodic orbits and curves corresponding to 2-homoclinic orbits to p_1 as shown in Figure 5.17.*

The inclination-flip bifurcations occur as the stable manifold of p_1 undergoes an arbitrary number of rotations in the vicinity of p_2 for parameter values near $\mu_1 = 0$ (recall that μ_1 unfolds the saddle-node bifurcation). The above result assumed eigenvalues of type B that lead to the homoclinic-doubling bifurcation. For type-C eigenvalues, one likewise finds a converging sequence of inclination-flip homoclinic orbits with more complicated bifurcation diagrams as in §5.1.6.

We conclude this section with a few remarks on heteroclinic bifurcations for planar ODEs. Kotova and Stanzo [231] compiled a list of codimension one, two and three bifurcations of heteroclinic cycles (the 'Kotova zoo'). The interest lies in estimating the cyclicity, that is, the number of limit cycles that are born in an unfolding of these cycles. It turns out that there are heteroclinic cycles in three parameter families for which any number of limit cycles bifurcates.

Theorem 5.35 ([231])**.** *For each $N \in \mathbb{N}$, there is a planar ODE $\dot{u} = f(u)$ with a heteroclinic cycle that connects two saddle-node equilibria as shown in Figure 5.18 so that N limit cycles occur for an arbitrary small perturbation of $\dot{u} = f(u)$.*

Note that the heteroclinic cycle shown in Figure 5.18 contains a normally attracting saddle-node equilibrium p_- and a normally repelling saddle-node equilibrium p_+. To prove the preceding theorem, three-parameter families $\dot{u} = f(u, \mu)$ are considered. The normal form near the saddle-node equilibria $p_\pm$ are given by

$$
\begin{aligned}
\dot{x} &= \frac{x^2 + \delta_\pm(\mu)}{1 + a_\pm(\mu)} \\
\dot{y} &= \pm y,
\end{aligned}
$$

which can be integrated explicitly to construct local transition maps. Composing with global transition maps yields a return map that can be studied to prove the above result.

In contrast, generic k-parameter families that unfold elementary polycycles always lead to a bounded number of limit cycles: a polycycle (which, by definition, consists of finitely many heteroclinic orbits and equilibria) is called elementary if each of its equilibria has at least one nonzero eigenvalue.
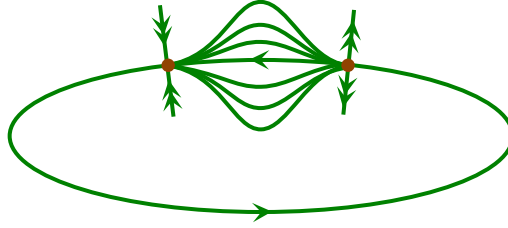
Figure 5.18: *An arbitrary number of limit cycles can occur in perturbations from a 'lips-shaped' heteroclinic cycle; see Theorem 5.35.*

Theorem 5.36 ([204, 210])**.** *There is a function $E : \mathbb{N} \to \mathbb{N}$ so that the following holds. Let $\dot{u} = f(u, \mu)$ be a generic family of planar ODEs that depends on a parameter $\mu \in \mathbb{R}^k$ and has an elementary polycycle at $\mu = 0$. The number of limit cycles that exist near the polycycle for small values of μ is then bounded by $E(k)$. Furthermore, the explicit estimate $E(k) \leq 2^{25k^2}$ holds.*

The existence of the function $E(k)$ is due to Il'yashenko and Yakovenko, while Kaloshin proved the explicit estimate.

5.2.4 Heteroclinic cycles containing periodic orbits

In this section, we consider heteroclinic cycles that involve heteroclinic connections between a hyperbolic equilibrium and a hyperbolic periodic orbit. We focus on three-dimensional flows and refer to [321] for results in higher dimensions for cycles of codimension one and two.

Let $\dot{u} = f(u, \mu)$ be a one-parameter family on $\mathbb{R}^3$. Suppose that the differential equation with $\mu = 0$ has a hyperbolic periodic orbit $q(x)$ of saddle type and a hyperbolic equilibrium p with eigenvalues $\nu^{ss} < \nu^s < 0 < \nu^u$. We assume that one separatrix in $W^u(p)$ is contained in the stable manifold $W^s(q)$ of q, and we denote this solution by h_1. The manifolds $W^u(q)$ and $W^s(p)$ are assumed to intersect transversally along a heteroclinic solution h_2 from $q(t)$ to p. Finally, the singular cycle $\overline{h_1 \cup h_2}$ is assumed to be the maximal invariant set in some neighborhood $\mathcal{U}$ of itself.

Hypothesis 5.22 (Genericity and unfolding conditions)**.** *Consider the following genericity conditions:*

(i) $W^u(q)$ and $W^s(p)$ intersect transversally along h_2.

(ii) $h_2 \notin W^{ss}(p)$.

(iii) The distance between $W^u(p)$ and $W^s(q)$, measured in a cross section, varies with nonzero speed in μ.

Hypothesis 5.23 (Expanding versus contracting cycles)**.** *We distinguish the following two cases:*

(i) Expanding singular cycle: $-\nu^s/\nu^u < 1$;

(ii) Contracting singular cycle: $-\nu^s/\nu^u > 1$.

Bifurcations from expanding and contracting singular cycles are quite different in nature, and we illustrate this with the following bifurcation result.

Theorem 5.37 ([28])**.** *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of ODEs that unfolds a singular cycle that satisfies Hypothesis 5.22.*

(i) If Hypothesis 5.23(i) is met, then the bifurcation set has zero Lebesgue measure.

(ii) If Hypothesis 5.23(ii) is met, then there are parameter values arbitrarily close to $\mu = 0$ for which an attracting periodic orbit exists near the singular cycle.

Further results on singular cycles for equilibria with real eigenvalues can be found in [245, 289, 301, 336], while results for singular cycles that contain an equilibrium with complex conjugate leading eigenvalues appear in [36, 62, 63]. Codimension-two singular cycles involving tangencies of stable and unstable manifolds are considered in [75, 279]. Papers in which singular cycles are treated using Lin's method include [222, 235, 322] and [29]; the latter paper uses Fenichel theory to describe trajectories near the periodic orbits.

5.3 Conservative and reversible systems

Systems that are reversible or conserve a certain energy-like quantity appear in many applications. In this section, we consider homoclinic bifurcations in such systems. We decided to review them together as they share several dynamical features and also since many models are both conservative and reversible: for instance, the energy $H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$ leads to a Hamiltonian system that is reversible and conservative.

In the introduction, we mentioned travelling waves in parabolic partial differential equations as an important source of ODE models where homoclinic orbits are of interest. One reason for the interest in reversible ODEs is the fact that standing waves $u(x)$ of reaction-diffusion systems $U_t = DU_{xx} + F(u)$ are captured by reversible systems. For surveys of reversible systems and their applications, we direct the reader to [69, 70, 248].

5.3.1 Introduction and hypotheses

In this section, we consider homoclinic orbits in systems

$$\dot{u} = f(u), \qquad u \in \mathbb{R}^{2n} \tag{5.6}$$

that are conservative or reversible. A differential system is conservative if it preserves an appropriate real-valued quantity. More precisely, we assume the following:

Hypothesis 5.24 (Conservative systems)**.** *There is a smooth function $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}$ with $\langle \nabla \mathcal{H}(u), f(u) \rangle = 0$ for all $u \in \mathbb{R}^{2n}$, and $\nabla \mathcal{H}(u) = 0$ only at a discrete set of points in $\mathbb{R}^{2n}$.*

Thus, if Hypothesis 5.24 is met, then $\mathcal{H}(u(t)) = \mathcal{H}(u(0))$ along any solution $u(t)$ of (5.6). A particular example of conservative systems are Hamiltonian systems given by

$$\dot{u} = J\nabla\mathcal{H}(u), \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad u \in \mathbb{R}^n \times \mathbb{R}^n.$$

If Hypothesis 5.24 is met and $h(t)$ is a homoclinic orbit, then

$$\psi(t) = \nabla\mathcal{H}(h(t)) \tag{5.7}$$

is a nontrivial bounded solution to the adjoint equation (2.3)

$$\dot{w} = -f_u(h(t))^* w. \tag{5.8}$$

Homoclinic orbits $h(t)$ to hyperbolic equilibria p are found as intersection of stable and unstable manifolds of p which all lie in the codimension-one surface $\mathcal{H}^{-1}(p)$:

Hypothesis 5.25 (Transversality)**.** *Assume that p is a hyperbolic equilibrium of (5.6) and that $h(t)$ is a homoclinic orbit with $h(0) \in W^{\mathrm{s}}(p) \pitchfork W^{\mathrm{u}}(p)$ in $\mathcal{H}^{-1}(p)$ (which is equivalent to assuming Hypothesis 2.2(i)).*

A differential system is reversible if it admits a reverser $\mathcal{R}$:

Hypothesis 5.26 (Reversible systems)**.** *There exists a linear operator $\mathcal{R} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\mathcal{R}^2 = 1$, $\dim \mathrm{Fix}(\mathcal{R}) = n$, and $\mathcal{R}f(u) = -f(\mathcal{R}u)$ for all $u \in \mathbb{R}^{2n}$.*

If Hypothesis 5.26 is met, then $v(t) := \mathcal{R}u(-t)$ is a solution of (5.6) whenever $u(t)$ is. In particular, we have $\mathcal{R}W^{\mathrm{s}}(p) = W^{\mathrm{u}}(p)$ for any equilibrium p. Furthermore, if $u(0) \in \mathrm{Fix}(\mathcal{R})$, then $u(t) = \mathcal{R}u(-t)$ for all $t \in \mathbb{R}$, and we call $u(t)$ reversible or symmetric:

Hypothesis 5.27 (Transversality). *We assume that*

  (i) $h(t)$ *is a symmetric homoclinic orbit of* (5.6) *to the hyperbolic equilibrium p.*

  (ii) $W^{\mathrm{s}}(p) \pitchfork \mathrm{Fix}(\mathcal{R})$ *in $\mathbb{R}^{2n}$ at $u = h(0)$.*

Homoclinic orbits in conservative or reversible systems are codimension-zero phenomena, that is, they persist under structure-preserving C^1-perturbations, and they are accompanied by periodic orbits.

Theorem 5.38 ([107, 414]). *A homoclinic orbit $h(t)$ to (5.6) that satisfies Hypotheses 5.24 and 5.25 (or Hypotheses 5.26 and 5.27) is accompanied by a family $q_T(t)$ of unique 1-periodic orbits which is parameterized by their period T for all T sufficiently large. For symmetric homoclinic orbits in reversible systems, the accompanying periodic orbits are also symmetric. Furthermore, such homoclinic orbits persist under C^1-perturbations of the vector field f that preserve the conservative (reversible) structure.*

We remark that the spectrum of the linearization of conservative or reversible systems about equilibria are symmetric:

Lemma 5.1. *We have*

$$\nu \in \mathrm{spec}(f_u(p)) \iff -\nu \in \mathrm{spec}(f_u(p)),$$

counted with multiplicity, if p is a hyperbolic equilibrium with $\mathcal{H}_{uu}(p)$ invertible in a conservative system or a symmetric equilibrium in a reversible system.

Proof. For reversible systems, we infer from Hypothesis 5.26 and $\mathcal{R}p = p$ that

$$\mathcal{R}f_u(p) = -f_u(\mathcal{R}p)\mathcal{R} = -f_u(p)\mathcal{R}$$

which shows that $f_u(p)$ and $-f_u(p)$ are similar. For conservative systems, Hypothesis 5.24 implies that $g(u) := \mathcal{H}_u(u)^* f(u) \equiv 0$. Thus, $0 = g_u(p) = \mathcal{H}_{uu}(p)f(p) + \mathcal{H}_u(p)^* f_u(p) = \mathcal{H}_u(p)^* f_u(p)$ which gives $\mathcal{H}_u(p) = 0$ since p is hyperbolic. Computing the second derivative of g, we get

$$
\begin{aligned}
0 = g_{uu}(p)[e_j, e_k] &= [\mathcal{H}_{uu}(p)e_j]^*[f_u(p)e_k] + [\mathcal{H}_{uu}(p)e_k]^*[f_u(p)e_j] \\
&= e_j^* \mathcal{H}_{uu}(p)f_u(p)e_k + e_k^* \mathcal{H}_{uu}(p)f_u(p)e_j \\
&= e_j^* \mathcal{H}_{uu}(p)f_u(p)e_k + e_j^* f_u(p)^* \mathcal{H}_{uu}(p)e_k \\
&= e_j^* [\mathcal{H}_{uu}(p)f_u(p) + f_u(p)^* \mathcal{H}_{uu}(p)]e_k
\end{aligned}
$$

for all j, k which shows that $\mathcal{H}_{uu}(p)f_u(p) = -f_u(p)^* \mathcal{H}_{uu}(p)$. Invertibility of $\mathcal{H}_{uu}(p)$ then implies that $f_u(p)$ and $-f_u(p)^*$ are similar. $\square$

5.3.2 Bi-foci homoclinic orbits

For conservative and reversible systems, homoclinic orbits to hyperbolic bi-foci are accompanied by infinitely many horseshoes. The result for conservative systems is as follows:

Theorem 5.39 ([106, 341]). *Assume that Hypotheses 5.24 and 5.25 are met. We also assume that there are precisely two leading stable eigenvalues $\nu^{\mathrm{s}} = -\alpha \pm \mathrm{i}\beta$ which are simple with $\beta > 0$, and that*

$$\lim_{t \to \infty} \mathrm{e}^{2\alpha t} \|h(t)\| \, \|h(-t)\| \neq 0$$

(which is equivalent to assuming Hypothesis 2.4(i) and (iii)). There are then infinitely many horseshoes close to the homoclinic orbit $h(t)$. Furthermore, for each $N \geq 2$ and each sequence $(k_1, \ldots, k_N) \in \mathbb{N}^N$, there are

numbers $k_*, T_* \geq 1$ and infinitely many N-homoclinic orbits near $h(t)$ which are parameterized by $k \geq k_*$ with return times given by

$$T_j = \frac{2\pi(k_j + k)}{\beta} + T_* + \mathrm{O}(\mathrm{e}^{-\delta k}), \qquad k \geq k_*, \qquad j = 1, \ldots, N.$$

The N-pulses with these return times are unique.

For Hamiltonian systems, the transversality condition can be relaxed significantly; see [55]. An application of the preceding theorem to fourth-order Hamiltonian systems can be found in [56]. Next, we state a similar result for reversible systems.

Theorem 5.40 ([68, 169, 341]). *Assume that Hypotheses 5.26, 5.27(i), and 2.2(i) are met. We also assume that Hypothesis 2.4(i)-(ii) hold and that there are precisely two leading stable eigenvalues $\nu^s = -\alpha \pm \mathrm{i}\beta$ which are simple with $\beta > 0$. There are then infinitely many horseshoes close to the homoclinic orbit $h(t)$, which consist of symmetric orbits. Furthermore, for each $N \geq 2$ and each sequence $(k_1, \ldots, k_{\lceil N/2 \rceil}) \in \mathbb{N}^{\lceil N/2 \rceil}$, there are numbers $k_*, T_* \geq 1$ and infinitely many symmetric N-homoclinic orbits near $h(t)$ which are parameterized by $k \geq k_*$ with return times given by*

$$T_j = \frac{2\pi(k_j + k)}{\beta} + T_* + \mathrm{O}(\mathrm{e}^{-\delta k}), \qquad k \geq k_*, \qquad j = 1, \ldots, \lceil N/2 \rceil.$$

The N-pulses with these return times are unique.

5.3.3 Belyakov–Devaney transition

For reversible and conservative systems that have a homoclinic orbit, we consider the codimension-one bifurcation where two real stable (and thus also the unstable) leading eigenvalues collide and become complex as the parameter is varied. This bifurcation was called Belyakov–Devaney in [69]; see also §5.1.4. For simplicity, we consider one-parameter families

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^4 \times \mathbb{R} \tag{5.9}$$

in $\mathbb{R}^4$ that are reversible or conservative. For $\mu = 0$, we assume that $h(t)$ is a homoclinic orbit to the hyperbolic equilibrium p. Recall that the imaginary parts of the two stable eigenvalues are nonzero if the discriminant $\Delta(\mu)$ of $f_u(p, \mu)$ restricted to the stable directions is negative.

Hypothesis 5.28 (Non-semisimple leading eigenvalues). *At $\mu = 0$, the real leading stable eigenvalue ν^s of $f_u(p, 0)$ has geometric multiplicity one and algebraic multiplicity two with $\partial_\mu \Delta(0) \neq 0$, and we have*

$$\lim_{t \to \infty} \frac{1}{|t|} \mathrm{e}^{|\nu^s t|} \|h(t)\| \neq 0, \qquad \lim_{t \to -\infty} \frac{1}{|t|} \mathrm{e}^{|\nu^s t|} \|\psi(t)\| \neq 0.$$

In the conservative case, we have the following result on the existence of N-homoclinic orbits.

Theorem 5.41. *Assume that (5.9) satisfies Hypothesis 5.24 for all μ and Hypotheses 5.25 and 5.28 at $\mu = 0$. Without loss of generality, assume that $\partial_\mu \Delta(0) = -1$ so that the eigenvalue at $p(\mu)$ are complex for $\mu > 0$. For each $N \geq 2$ and each sequence $(k_1, \ldots, k_N) \in \mathbb{N}^N$, there are numbers $k_*, T_* \geq 1$ and $0 < \mu_N \ll 1$ so that (5.9) has infinitely many N-homoclinic orbits near $h(t)$ for each $0 < \mu < \mu_N$ which are parameterized by $k \geq k_*$ with return times given by*

$$T_j = \frac{2\pi(k_j + k)}{\mu} + T_* + \mathrm{O}(\mathrm{e}^{-\delta k/\epsilon}), \qquad k \geq k_*, \qquad j = 1, \ldots, N.$$

The N-pulses with these return times are unique.

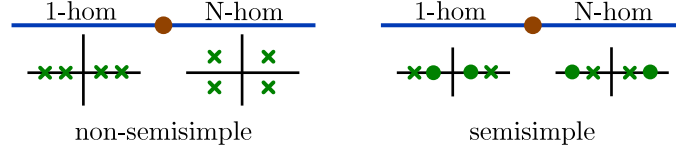An analogous theorem is true for reversible systems (see Figure 5.19).

Figure 5.19: *In reversible or conservative systems, eigenvalues that collide on the real axis will typically be non-semisimple and therefore become complex: the resulting Belyakov–Devaney transition, described in Theorems 5.41 and 5.42, leads to N-homoclinic orbits for all N [left]. If an additional $\mathbb{Z}_2$-symmetry is present, semisimple eigenvalues may be enforced that cannot leave the real axis: N-homoclinic orbits may still bifurcate as shown in Theorem 5.43 [right].*

Theorem 5.42. *Assume that (5.9) satisfies Hypothesis 5.26 for all μ and Hypotheses 5.27(i), 2.2(i) and 5.28 at $\mu = 0$. Without loss of generality, assume that $\partial_\mu \Delta(0) = -1$ so that the eigenvalues at $p(\mu)$ are complex for $\mu > 0$. For each $N \geq 2$ and each sequence $(k_1, \ldots, k_{\lceil N/2 \rceil}) \in \mathbb{N}^{\lceil N/2 \rceil}$, there are numbers $k_*, T_* \geq 1$ and $0 < \mu_N \ll 1$ so that (5.9) has infinitely many symmetric N-homoclinic orbits near $h(t)$ for each $0 < \mu < \mu_N$ which are parameterized by $k \geq k_*$ with return times given by*

$$T_j = \frac{2\pi(k_j + k)}{\beta} + T_* + \mathrm{O}(\mathrm{e}^{-\delta k}), \qquad k \geq k_*, \qquad j = 1, \ldots, \lceil N/2 \rceil.$$

The N-pulses with these return times are unique.

Theorem 5.41 was proved by Champneys and Toland [78] for a special class of Hamiltonian systems, while Theorem 5.42 was established by Peroueme (unpublished notes) for $N = 2$. Alternatively, these theorems follow as in [341] from Lin's method upon using [339, Lemma 1.5] or [429, Lemma 2.1] to write down expansions in t of the quantities that appear in (3.13).

5.3.4 Homoclinic orbits to equilibria with semisimple spectrum

We consider reversible Hamiltonian systems of second-order ODEs of the form

$$\begin{aligned} \ddot{u}_1 &= u_1 + g_{u_1}(u_1, u_2) \\ \ddot{u}_2 &= (1 + \mu)u_2 + g_{u_2}(u_1, u_2), \end{aligned} \tag{5.10}$$

where the Hamiltonian is given $H(u_1, u_2) = \dot{u}_1^2 + \dot{u}_2^2 + u_1^2 + u_2^2 + g(u_1, u_2)$. Systems of the above type arise in coupled nonlinear Schrödinger equations [429] and in the study of three-dimensional water waves [158]. The key feature in (5.10) is the semisimple eigenvalue of multiplicity two that is unfolded by the parameter μ: in contrast to the situation studied in §5.3.3, the unfolded eigenvalues are always real. We assume the following:

Hypothesis 5.29. *We have $g(0, 0) = g_u(0, 0) = 0$ and $g(-u_1, u_2) = g(u_1, u_2)$ for all $u = (u_1, u_2)$. Furthermore, we assume that there is a constant $\delta \notin \{0, 1\}$ so that $\delta g_{u_1}(u_1, \delta u_1) = g_{u_2}(u_1, \delta u_1)$ for all u_1.*

The preceding hypothesis implies that the first-order system associated with (5.10) is equivariant with respect to the $\mathbb{Z}_2$-action $\kappa : (u_1, u_2) \mapsto (-u_1, u_2)$. It also implies that the subspace $u_2 = \delta u_1$ is invariant under (5.10) when $\mu = 0$.

Hypothesis 5.30. *For $\mu = 0$, equation (5.10) has a pair $u = (\pm h_1(t), h_2(t))$ of homoclinic orbits that lie in the invariant subspace $u_2 = \delta u_1$ and satisfy Hypothesis 5.27.*

We refer to [158, 429] and [10] for conditions on $g(u_1, u_2)$ that guarantee the existence of transverse homoclinic orbits. The next theorem shows that (5.10) admits N-homoclinic orbits near $\mu = 0$.

Theorem 5.43 ([158, 429])**.** *Assume that Hypotheses 5.29 and 5.30 are met for (5.10). For each $N \geq 2$, there is then an $\mu_N > 0$ such that (5.10) has a unique pair of N-homoclinic orbits for each μ with $0 < |\mu| < \mu_N$ and $\mathrm{sign}\,\mu = \mathrm{sign}\,\ln|\delta|$: the N loops in each of the N-homoclinic orbits follow alternately (h_1, h_2) and $(-h_1, h_2)$. No other N-homoclinic orbits exist near $\mu = 0$.*

5.3.5 Reversible systems with SO(2)-symmetry

We consider the equation

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^4 \times \mathbb{R} \tag{5.11}$$

with $f \in C^2$ and assume that this equation is reversible for all μ; see Hypothesis 5.26. In addition, we assume (5.11) is equivariant with respect to an $\mathbb{S}^1$-action for all values of μ:

Hypothesis 5.31. *There is a one-parameter group of orthogonal matrices $T_\rho : \mathbb{R}^4 \to \mathbb{R}^4$, defined for $\rho \in \mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$, such that $T_{\rho_1} T_{\rho_2} = T_{\rho_1 + \rho_2}$ for all ρ_1, ρ_2 and $\dim \mathrm{Fix}(T_\rho) = \{0\}$ for $\rho \neq 0$. Furthermore, we assume that $f(T_\rho u, \mu) = T_\rho f(u, \mu)$ and $\mathcal{R} T_\rho = T_\rho \mathcal{R}$ for all (u, μ) and all ρ.*

In particular, $u = 0$ is an equilibrium for all μ which we assume to be hyperbolic. The presence of the $\mathbb{S}^1$-action precludes that homoclinic orbits are transversely constructed in the sense of Hypothesis 5.27(ii), since any homoclinic orbit $h(t)$ yields an $\mathbb{S}^1$-orbit $T_\rho h(t)$ of distinct symmetric homoclinic orbits for $\rho \in \mathbb{S}^1$. Thus, we assume that Hypothesis 5.27(i) is met with $p = 0$ when $\mu = 0$ and need then to assume that the parameter μ unfolds the $\mathbb{S}^1$-orbit of homoclinic orbits. To make this precise, note that we have $W^{\mathrm{s}}(0,0) = W^{\mathrm{u}}(0,0)$ due to the presence of the $\mathbb{S}^1$-action, which implies that the adjoint equation

$$w' = -f_u(h(t), 0, 0)^* w$$

has two bounded, linearly independent solutions $\psi_1(t)$ and $\psi_2(t)$, which can be chosen so that

$$\psi_1(0) \in \mathrm{Fix}(\mathcal{R}^*), \qquad \psi_2(0) \in \mathrm{Fix}(-\mathcal{R}^*).$$

Note that this implies $\mathcal{R}^* \psi_1(t) = \psi_1(-t)$ and $\mathcal{R}^* \psi_2(t) = -\psi_2(-t)$ for all t. We can now encode transversality with respect to μ using a Melnikov integral.

Hypothesis 5.32. *We assume that*

$$\int_{-\infty}^{\infty} \langle \psi_2(t), f_\mu(h(t), 0) \rangle \, \mathrm{d}t \neq 0.$$

We are interested in N-homoclinic orbits near the $\mathbb{S}^1$-orbit $\{T_\rho h(t) : \rho \in \mathbb{S}^1\}$: in particular, to each N-homoclinic orbit near this group orbit, we can associate a sequence $\{\rho_k\}_{k=1,\ldots,N}$ so that the kth loop of the N-homoclinic orbit follows $T_{\rho_k} h(t)$. We can now state the following result:

Theorem 5.44 ([2, 211, 267, 268]). *Assume that Hypotheses 5.26, 5.27(i), 5.31 and 5.32 are met. We also assume that the eigenvalues of $f_u(0,0)$ are $\pm\alpha \pm \mathrm{i}\beta$ with $\alpha, \beta > 0$ and that*

$$\lim_{t \to \infty} \frac{\langle \psi_1(t), \psi_2(t) \rangle}{\|\psi_1(t)\| \, \|\psi_2(t)\|} \neq 0.$$

For $N = 2, 3$, there are then sequences $\mu_N^{j,\pm}$ with $\mu_N^{j,\pm} \to 0$ as $j \to \infty$ and $\mu_N^{j,-} < 0 < \mu_N^{j,+}$ for all j so that (5.11) has an $\mathbb{S}^1$-orbit of symmetric N-homoclinic orbits for $\mu = \mu_N^{k,\pm}$ with $N = 2, 3$. The 2-homoclinic orbits associated with $\mu_2^{j,\pm}$ have $\rho_1 \rho_2 = (-1)^j$. Last, each of the N-homoclinic orbits constructed above for $N = 2, 3$ satisfies the assumptions of this theorem.

Of interest in applications to travelling waves, for instance in the complex Ginzburg–Landau equation, is the system

$$\dot{u} = f(u, \mu, c), \qquad (u, \mu, c) \in \mathbb{R}^4 \times \mathbb{R} \times \mathbb{R}$$

where we assume that the equation for $c = 0$ satisfies the hypotheses of Theorem 5.44, while the parameter c breaks the reversibility and the homoclinic orbit, while preserving the $\mathbb{S}^1$-action. In this case, asymmetric N-homoclinic orbits can also be found near $(\mu, c) = 0$, and we refer to [267, 268] for results.

5.3.6 Reversible and Hamiltonian flip bifurcations

The presence of reversers and conserved quantities has interesting implications for homoclinic flip bifurcations. Since the dynamics near reversible homoclinic orbits is symmetric with respect to time reflection, they undergo simultaneous orbit-flips in the stable and unstable direction, and the same is true for inclination-flips. For conservative systems, $\nabla\mathcal{H}(h(t))$ is the unique nontrivial solution of the adjoint variational equation; see (5.7). Since we have $\nabla\mathcal{H}(h(t)) \approx \mathcal{H}_{uu}(p)[h(t) - p]$ as $|t| \to \infty$, we find that if a homoclinic orbit in a conservative system undergoes an orbit-flip in the stable direction, say, it is at the same time in an inclination-flip configuration. Lastly, in reversible conservative systems, orbit-flips and inclination-flips of symmetric homoclinic orbits occur simultaneously and in both stable and unstable directions.

Throughout, we consider the equation

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^{2n} \times \mathbb{R} \tag{5.12}$$

with $f \in C^4$ and assume that this equation is conservative or reversible for all μ.

Hypothesis 5.33. *Equation (5.12) has $p = 0$ as equilibrium for $\mu = 0$. Furthermore, the linearization $f_u(0,0)$ has simple real eigenvalues $-\nu^{\mathrm{uu}} < -\nu^{\mathrm{u}} < 0 < \nu^{\mathrm{u}} < \nu^{\mathrm{uu}}$, and the real part of all other eigenvalues has modulus strictly larger than ν^{uu}.*

We begin with the reversible non-conservative orbit-flip.

Theorem 5.45 ([345]). *Assume that Hypothesis 5.26 is met for all μ, and that Hypotheses 2.2(i), 5.27(i) and 5.33 hold at $\mu = 0$. We assume that $h(t)$ is in an orbit-flip configuration so that*

$$v^{\mathrm{s}}(0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu}v^{\mathrm{s}}(0) \neq 0, \qquad v_*^{\mathrm{s}}(0) \neq 0, \qquad S := \lim_{t \to \infty} \mathrm{e}^{2\nu^{\mathrm{uu}}t}\langle \psi(-t), h(t)\rangle \neq 0, \tag{5.13}$$

where we used the notation from (2.9) and (2.11)[10]. We set $\delta := -\operatorname{sign} S\langle v_*^{\mathrm{s}}(0), \mathrm{d}v^{\mathrm{s}}/\mathrm{d}\mu(0)\rangle$, then, for each $N > 1$, there is a $\mu_N > 0$ so that (5.12) has a unique N-homoclinic orbit for each $|\mu| < \mu_N$ with $\operatorname{sign} \mu = \delta$.

Since conservative equations have (5.7) as the adjoint solution, they always violate assumption (5.13). To our knowledge, the reversible non-conservative inclination-flip has not been studied so far. Next, we state results on conservative non-reversible flip bifurcations.

Theorem 5.46 ([401]). *Assume that Hypothesis 5.24 is met for all μ, and that Hypotheses 5.9, 5.25 and 5.33 hold at $\mu = 0$. We assume that $h(t)$ is in a flip configuration in the stable direction so that*

$$v^{\mathrm{s}}(0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu}v^{\mathrm{s}}(0) \neq 0, \qquad v^{\mathrm{u}}(0) \neq 0, \tag{5.14}$$

see (2.9) and (2.11) for the notation[11]. Under these assumptions, there is then a $\delta \in \{\pm 1\}$ such that (5.12) has a unique N-homoclinic orbit for each $N > 1$ and each sufficiently small μ with $\operatorname{sign} \mu = \delta$, and no N-homoclinic orbits for $\operatorname{sign} \mu = -\delta$.

The bifurcation direction in the preceding theorem is made more explicit in [401], which contains also a complete characterization of the recurrent set and additional results about super-homoclinic orbits. Note again that the assumption (5.14) precludes reversibility of (5.12) since symmetric homoclinic orbits have $v^{\mathrm{u}}(0) = v^{\mathrm{s}}(0)$. It remains to consider the case where (5.12) is reversible and conservative.

Theorem 5.47 ([338]). *Assume that Hypotheses 5.24 and 5.26 are met for all μ, and that Hypotheses 5.25 and 5.33 hold at $\mu = 0$. We assume that $h(t)$ is in a flip configuration so that*

$$v^{\mathrm{s}}(0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\mu}v^{\mathrm{s}}(0) \neq 0, \tag{5.15}$$

see (2.9) and (2.11) for the notation[12]. Under these assumptions, (5.12) has a unique N-homoclinic orbit for each $N > 1$ and each sufficiently small nonzero μ.

---

[10]Reversibility and (5.13) imply that $v^{\mathrm{u}}(0) = 0$ and $v_*^{\mathrm{u}}(0) \neq 0$

[11]Equations (5.7) and (5.14) imply then that $v_*^{\mathrm{u}}(0) = 0$ and $v_*^{\mathrm{s}}(0) \neq 0$

[12]Reversibility, (5.7) and (5.14) together imply that $v^{\mathrm{u}}(0) = v_*^{\mathrm{u}}(0) = v_*^{\mathrm{s}}(0) = 0$

Thus, in contrast to the previous cases, N-homoclinic orbits bifurcate for either sign of the bifurcation parameter μ.

5.3.7 Coexisting homoclinic orbits

As symmetric homoclinic orbits in reversible systems are typically persistent, one can find multiple coexisting homoclinic orbits. On the other hand, the action of the reverser $\mathcal{R}$ forces nonsymmetric homoclinic orbits (which typically occur in one-parameter families of ODEs) to a symmetric hyperbolic equilibrium to come in pairs. The consequences of these two scenarios for the dynamics are reviewed in this section.

We consider homoclinic orbits to hyperbolic equilibria with unique real leading eigenvalues. Recall from Theorem 5.38 that nondegenerate symmetric homoclinic orbits in reversible systems, as well as nondegenerate homoclinic orbits in conservative systems, are accompanied by a sheet of periodic solutions. For two coexisting homoclinic orbits, one thus expects two sheets of periodic solutions. The sheets are normally hyperbolic and, depending on the geometry, transverse intersections of the stable and unstable manifolds of these sheets are feasible. This indeed occurs if both homoclinic orbits are in a bellows configuration, i.e. approach the equilibrium along the same direction for positive as well as negative time (see §5.1.8). For conservative systems, the transverse intersections of stable and unstable manifolds of single periodic orbits occur inside levels of the first integral and thus give a suspended horseshoe in each level. In other words, two homoclinic orbits in a bellows configuration for a conservative system give a one-parameter family of suspended horseshoes. The following theorem, contained in [188, 404], can be obtained from standard constructions of invariant laminations [177].

Theorem 5.48 ([188, 404])**.** *Let $\dot{u} = f(u)$ be a conservative, reversible ODE on $\mathbb{R}^{2n}$ with first integral $\mathcal{H}$ that satisfies Hypotheses 5.24 and 5.26. Suppose that it has a hyperbolic symmetric equilibrium p and two symmetric homoclinic orbits h_1, h_2 to p: we assume that both h_1 and h_2 satisfy Hypotheses 2.1, 2.2(i), and 2.4 and that they are in the bellows configuration so that Hypothesis 5.12(iv) is met. In a small neighborhood of $\overline{h_1 \cup h_2}$, there is then a one-parameter family of suspended horseshoes, parameterized by the energy $\mathcal{H}$, for values on one side of $\mathcal{H}(p)$.*

If the system is reversible but not conservative, then the dynamical picture is more complicated. The following two theorems summarize some of the features, and we refer to the cited papers for further properties such as the existence of sheets of almost periodic orbits and of heterodimensional cycles.

Theorem 5.49 ([188, 192])**.** *Let $\dot{u} = f(u)$ be a reversible ODE on $\mathbb{R}^{2n}$ that satisfies Hypothesis 5.26. Suppose that it has a hyperbolic symmetric equilibrium p and two symmetric homoclinic orbits h_1, h_2 to p: we assume that both h_1 and h_2 satisfy Hypotheses 2.1, 2.2(i), and 2.4 and that they are in the bellows configuration so that Hypothesis 5.12(iv) is met. There is then an invariant normally hyperbolic lamination that contains the nonwandering set near $\overline{h_1 \cup h_2}$. Furthermore, there are infinitely many sheets of symmetric periodic orbits, and arbitrarily small perturbations of f in the C^1 topology create saddle-node bifurcations of periodic orbits.*

The following theorem covers the case of nonsymmetric homoclinic orbits in a one-parameter family of reversible ODEs.

Theorem 5.50 ([188, 192])**.** *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of reversible ODEs on $\mathbb{R}^{2n}$ with reverser $\mathcal{R}$ as stated in Hypothesis 5.26. For $\mu = 0$, we assume that this system has a hyperbolic symmetric equilibrium p and two homoclinic orbits h_1 and h_2 with $h_1 = \mathcal{R}h_2$ to p that satisfy Hypotheses 2.1, 2.2, 2.4, and 5.12(iv). For each μ close to zero, there are then infinitely many sheets of symmetric periodic orbits. Furthermore, for μ on one side of $\mu = 0$, there are infinitely many hyperbolic periodic orbits. Arbitrarily small perturbations of f in the C^1 topology create saddle-node bifurcations of periodic orbits.*
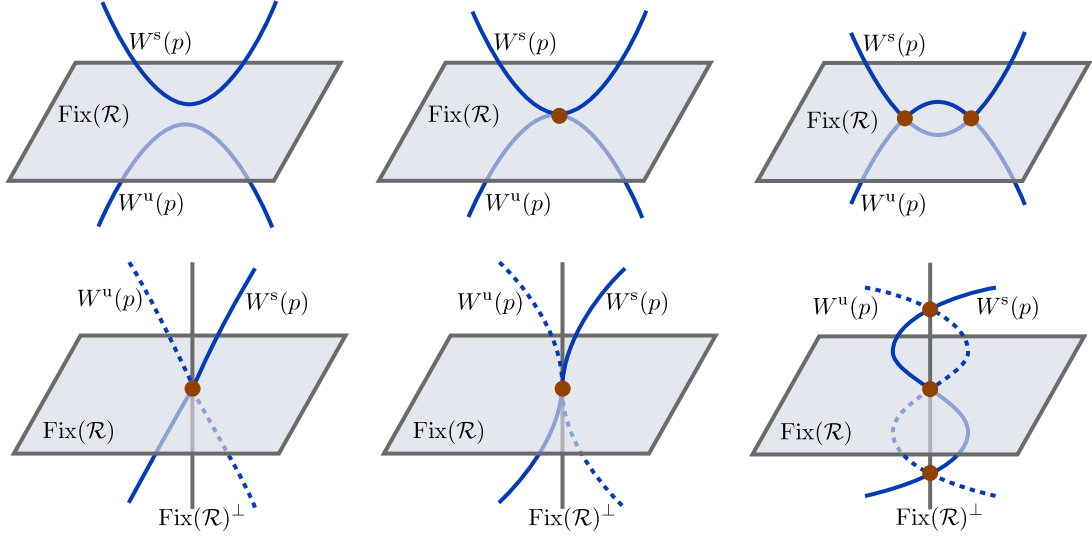
Figure 5.20: *The unfolding of degenerate reversible homoclinic orbits that satisfy Hypothesis 5.34(i) [top row] and 5.34(ii) [bottom row] is shown: the phase diagram in the middle column is for $\mu = 0$.*

5.3.8 Degenerate homoclinic orbits

Recall that a homoclinic orbit to a hyperbolic equilibrium is called degenerate if the tangent spaces of the stable and unstable manifolds along the homoclinic orbit have at least a plane in common. In generic systems, degenerate homoclinic orbits are of codimension three; see §5.1.9. In reversible systems, symmetric homoclinic orbits can be degenerate in two different ways which are of codimension one or codimension two. Such degenerate symmetric homoclinic orbits hence occur generically in one-parameter or two-parameter families. We review the situation near degenerate homoclinic orbits in reversible systems.

Thus, consider a family $\dot{u} = f(u, \mu)$, $\mu \in \mathbb{R}$ or $\mu \in \mathbb{R}^2$, of reversible ODEs on $\mathbb{R}^{2n}$ with reverser $\mathcal{R}$. We may assume that $\mathrm{Fix}(-\mathcal{R})$ is perpendicular to $\mathrm{Fix}(\mathcal{R})$. Assume that $h(t)$ is a symmetric homoclinic orbit for $\mu = 0$, then we can choose a cross section $\Sigma \subset \dot{h}(0)^{\perp}$ at $h(0)$ transverse to h, so that Σ is symmetric and thus contains part of $\mathrm{Fix}(\mathcal{R})$. A tangency of $W^{\mathrm{s}}(p, 0)$ and $W^{\mathrm{u}}(p, 0)$ can occur in two different ways, which we detail in the following hypothesis; see also Figure 5.20.

Hypothesis 5.34 (Generic Unfolding). *Consider the following nondegeneracy and unfolding conditions:*

(i) *The manifold $W^{\mathrm{u}}(p, 0) \cap \Sigma$ has a quadratic tangency with $\mathrm{Fix}(\mathcal{R})$ at $h(0)$, and $W^{\mathrm{u}}(p, \mu) \cap \Sigma \times \mathbb{R}$ intersects $\mathrm{Fix}(\mathcal{R}) \times \mathbb{R}$ transversally at $(h(0), 0)$ in the extended phase space $\Sigma \times \mathbb{R}$.*

(ii) *The manifold $W^{\mathrm{u}}(p, 0) \cap \Sigma$ has a cubic tangency with $[\mathrm{Fix}(\mathcal{R}) \cap \Sigma]^{\perp}$ at $h(0)$, and $W^{\mathrm{u}}(p, \mu) \cap \Sigma \times \mathbb{R}^2$ intersects $[\mathrm{Fix}(\mathcal{R}) \cap \Sigma]^{\perp} \times \mathbb{R}^2$ transversally at $(h(0), 0)$ in the extended phase space $\Sigma \times \mathbb{R}^2$.*

Homoclinic orbits near the degenerate homoclinic orbit can be found by studying the geometry of $W^{\mathrm{s}}(p, \mu)$ and $W^{\mathrm{u}}(p, \mu)$ in the vicinity of $\mathrm{Fix}(\mathcal{R}) \cap \Sigma$. Since $W^{\mathrm{s}}(p) = \mathcal{R}W^{\mathrm{u}}(p)$, any intersection of $W^{\mathrm{u}}(p)$ with $\mathrm{Fix}(\mathcal{R})$ in Σ yields a homoclinic orbit. We begin by stating a result in $\mathbb{R}^4$ due to Fiedler and Turaev that covers the case outlined in Hypothesis 5.34(i).

Hypothesis 5.35 (Transversality conditions). *Assume that $W^{\mathrm{s,lu}}(p, 0) \cap \Sigma$ is transverse to both $\mathrm{Fix}(\mathcal{R})$ and $[\mathrm{Fix}(\mathcal{R}) \cap \Sigma]^{\perp}$ in Σ at $h(0)$.*

Theorem 5.51 ([130]). *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of reversible ODEs on $\mathbb{R}^4$ with reverser $\mathcal{R}$ as in Hypothesis 5.26. Assume that $W^{\mathrm{s}}(p, 0)$ is tangent to $\mathrm{Fix}(\mathcal{R})$ at $h(0)$ and that Hypotheses 2.3(ii), 2.4(ii),(iv) and Hypotheses 5.34(i) and 5.35 are all met. At $\mu = 0$, two symmetric homoclinic orbits that exist on one side of $\mu = 0$ (say $\mu > 0$) collide and disappear when $\mu < 0$. Furthermore, one of the following two alternatives holds:*

(i) For $\mu > 0$, there is a single surface of symmetric periodic solutions joining the homoclinic orbits. No periodic solutions exist for $\mu < 0$.

(ii) For $\mu < 0$, there is a surface of symmetric periodic solutions that breaks into two components each bounded by a homoclinic orbit when $\mu > 0$.

Interestingly, the surfaces of periodic solutions in the above theorem contain both hyperbolic and elliptic periodic solutions with real and complex conjugate Floquet multipliers.

The next theorem is for degenerate homoclinic orbits that satisfy Hypothesis 5.34(ii): in this case, while the stable and unstable manifolds are tangent to each other, the tangent space of their intersection does not lie in $\mathrm{Fix}(\mathcal{R})$ but instead in $\mathrm{Fix}(-\mathcal{R})$. Since this bifurcation is of codimension two, we consider two-parameter families $\dot{u} = f(u, \mu)$ of reversible ODEs.

Theorem 5.52 ([221]). *Assume that the system $\dot{u} = f(u, \mu)$ with $(u, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}^2$ is reversible with reverser $\mathcal{R}$ as stated in Hypothesis 5.26. Suppose that $W^{\mathrm{u}}(p, 0)$ intersects $\mathrm{Fix}(\mathcal{R})$ transversally at $h(0)$ and that $T_{h(0)} W^{\mathrm{u}}(p, 0) \cap \mathrm{Fix}(\mathcal{R})^\perp$ is one-dimensional. If Hypothesis 5.34(ii) is met, then a symmetric homoclinic orbit to p exists for all small μ, and there is a one-sided curve in the parameter plane that terminates at $\mu = 0$ along which the family has two nonsymmetric homoclinic orbits.*

The coexisting homoclinic orbits that occur in the unfoldings described in the preceding two theorems come with complicated recurrent dynamics even when the leading eigenvalues are real: see Theorems 5.49 and 5.50 in the previous section.

5.3.9   Saddle-center homoclinic orbits

We begin by considering Hamiltonian systems

$$\dot{u} = J\nabla\mathcal{H}(u), \qquad u \in \mathbb{R}^4, \tag{5.16}$$

where J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which satisfy the following hypothesis:

Hypothesis 5.36. *We assume that $u = 0$ is a saddle-center equilibrium so that the linearization about 0 has two nonzero eigenvalues $\nu^{\mathrm{c}} = \pm\mathrm{i}\omega$ on the imaginary axis in addition to nonzero stable and unstable eigenvalues at $\pm\nu^{\mathrm{u}}$. We also assume that $h(t)$ is a homoclinic orbit to the origin.*

We record that homoclinic orbits to saddle-centers in Hamiltonian systems in $\mathbb{R}^4$ are a codimension-two phenomenon, which can be seen from the following observations. A cross section Σ is foliated by level sets of $\mathcal{H}$, and the center-stable manifold $W^{\mathrm{cs}}(0) \cap \Sigma$ is tangent, typically at a single point, to the level set of $\mathcal{H}(0)$. The same is true for $W^{\mathrm{cu}}(0) \cap \Sigma$, and a homoclinic orbit to the origin exists if these two tangencies occur at the same point. Since the level sets of $\mathcal{H}$ intersected with Σ are two dimensional, two parameters are needed to find homoclinic orbits to saddle-centers.

We first investigate the dynamics of (5.16) itself and discuss the unfolding under parameter variations afterwards. In appropriate coordinates (q_1, q_2, p_1, p_2), the Hamiltonian $\mathcal{H} : \mathbb{R}^4 \to \mathbb{R}$ has the Taylor expansion

$$\mathcal{H}(q_1, q_2, p_1, p_2) = -q_1 p_1 + \frac{\omega}{2}(q_2^2 + p_2^2) + \mathrm{O}(\|(q, p)\|^3) \tag{5.17}$$

where we may assume that $\omega > 0$ to make the energy increasing in the (q_2, p_2) coordinates. In particular, $\mathcal{H}(0) = 0$, and the origin has a two-dimensional center manifold which, by the Lyapunov-center theorem, is filled with periodic orbits $u_E(t)$, which are parameterized by their positive energy $E = \mathcal{H}(u_E(0))$ for $0 < E < E_0$ for some $E_0 > 0$.

To introduce a crucial genericity assumption needed below, we choose two-dimensional transverse sections $\Sigma_{\pm}$ to the homoclinic orbit in the energy level set $\mathcal{H}^{-1}(0)$ that contain the points $h(\pm\ell)$ for some sufficiently large times $\ell \geq 1$. Using the center coordinates (q_2, p_2) as coordinates in both sections $\Sigma_{\pm}$, it can be shown that the symplectic Poincaré map Π along the homoclinic orbit from Σ_- to Σ_+ is, after an appropriate adjustment of ℓ, of the form

$$\Pi(q_2, p_2) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \times \text{ rotation matrix } \times \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} + \mathrm{O}(\|(q_2, p_2)\|^2). \tag{5.18}$$

Thus, the linearization of the Poincaré map about the homoclinic orbit contains stretching and contracting directions when $a \neq 0$, while it is a pure rotation for $a = 0$. We shall also need a second geometric condition which we explain below after stating it:

Hypothesis 5.37. *Consider the following geometric conditions:*

- *(i) The quantity a in (5.18) is nonzero.*

- *(ii) The homoclinic orbit $h(t)$ approaches the origin along the positive p_1 and q_1 axes as $t \to \pm\infty$ or else along the negative p_1 and q_1 axes.*

The second assumption above implies that there are solutions in the level set $\mathcal{H}^{-1}(0)$ that pass from Σ_+ to Σ_- near the equilibrium p. If it is not met, then such solutions cannot exist.

Theorem 5.53 ([230, 255, 275]). *Assume that Hypotheses 5.36 and 5.37(i) are met, and that $\mathcal{H} \in C^3$, then there is an $E_0 > 0$ such that the stable and unstable manifolds of each periodic orbit $u_E(t)$ near the origin intersect transversally near $h(t)$ for $0 < E < E_0$. In particular, the level sets $\{u; \mathcal{H}(u) = E\}$ contain horseshoes for all $0 < E \ll 1$. This result is also valid in $\mathbb{R}^{2n}$ provided the leading stable and unstable eigenvalues are unique and simple (but possibly complex), $W^{\mathrm{cs}}(0) \pitchfork W^{\mathrm{u}}(0)$ and $W^{\mathrm{cu}}(0) \pitchfork W^{\mathrm{s}}(0)$ at $h(0)$ in $\mathcal{H}^{-1}(0)$, and the homoclinic orbit is not in an orbit-flip configuration (see Hypothesis 2.4(ii) and (iv)).*

Under additional assumptions, horseshoes can also be shown to exist in the zero-energy level set.

Theorem 5.54 ([156]). *Assume that Hypotheses 5.36 and 5.37 are met, that $\mathcal{H}$ is analytic, and that*

$$\frac{\omega|a - 1/a|}{\nu^{\mathrm{u}}} > \frac{3}{2},$$

then the energy level sets $\{u; \mathcal{H}(u) = E\}$ contain horseshoes for all E with $|E|$ sufficiently small.

See [157] for a study of Lyapunov stability of two saddle-center homoclinic orbits in Hamiltonian systems with an additional $\mathbb{Z}_2$-equivariance.

Next, we unfold the situation considered above for reversible Hamiltonian systems, and refer to [229, 230] for unfoldings in nonreversible Hamiltonian systems. In reversible Hamiltonian systems, symmetric homoclinic orbits to a saddle-center are of codimension one, so let

$$\dot{u} = J\nabla\mathcal{H}(u, \mu), \qquad (u, \mu) \in \mathbb{R}^4 \times \mathbb{R} \tag{5.19}$$

be a one-parameter family of Hamiltonian systems that satisfies Hypothesis 5.36 at $\mu = 0$. In this case, the Poincaré map (5.18) depends on μ and has the expansion

$$\Pi(q_2, p_2, \mu) = \mu\pi_0 + \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \times \text{ rotation matrix } \times \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} + \mathrm{O}((|q_2| + |p_2| + |\mu|)^2) \tag{5.20}$$

for some $\pi_0 \in \mathbb{R}^2$. We assume that $\pi_0 \neq 0$ and that (5.19) is reversible:

Hypothesis 5.38. *(i) The vector π_0 in (5.20) is not zero.*

(ii) *There is a linear map $\mathcal{R} : \mathbb{R}^4 \to \mathbb{R}^4$ with $\mathcal{R}^2 = 1$ and $\dim \text{Fix}(\mathcal{R}) = 2$ so that $\mathcal{R}J = -J\mathcal{R}$ and $\mathcal{H}$ is invariant under $\mathcal{R}$ (that is, $\mathcal{H}(\mathcal{R}u, \mu) = \mathcal{H}(u, \mu)$ for all (u, μ)). We assume that $h(0) \in \text{Fix}(\mathcal{R})$.*

We are interested in N-homoclinic orbits near $h(t)$ for parameter values μ near zero and therefore define

$$\Lambda_N := \{\mu;\ \mu \approx 0 \text{ and } (5.19) \text{ has an } N\text{-homoclinic orbit near } h(t)\}, \qquad \Lambda := \cup_{N \geq 1} \Lambda_N.$$

Theorem 5.55 ([156, 275])**.** *Assume that $\mathcal{H}$ is analytic in (u, μ) and that (5.19) satisfies Hypothesis 5.38 for all μ and Hypotheses 5.36 and 5.37 at $\mu = 0$.*

(i) *If μ is in Λ_k, then both $(\mu - \varepsilon, \mu)$ and $(\mu, \mu + \varepsilon)$ contain infinitely many points of Λ_{km} for any $m \geq 1$ and any $\epsilon > 0$.*

(ii) *The set Λ is countable, and each point in Λ is an accumulation point of Λ.*

In particular, for each $N \geq 2$, there are sequences $\mu_N^{\pm}(k) \to 0$ as $k \to \infty$ with $\mu_N^-(k) < 0 < \mu_N^+(k)$ so that (5.19) has an N-homoclinic orbit for $\mu = \mu_N^{\pm}(k)$.

Lastly, we consider reversible systems

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^4 \times \mathbb{R} \tag{5.21}$$

with homoclinic orbits to a saddle-center equilibrium that are not Hamiltonian.

Theorem 5.56 ([72])**.** *Assume that (5.21) satisfies Hypothesis 5.26 for all μ and Hypothesis 5.36 at $\mu = 0$. We also assume that $h(t)$ is symmetric and that the vector field $f(u, \mu)$ can be conjugated near the origin to a finite-order normal form by a C^1-diffeomorphism that commutes with $\mathcal{R}$. Lastly, we assume that*

$$\frac{\mathrm{d}}{\mathrm{d}\mu} h^{\mathrm{u}}(\mu)|_{\mu=0} \notin T_{h^{\mathrm{u}}(0)} W^{\mathrm{cs}}(0) \tag{5.22}$$

where $h^{\mathrm{u}}(\mu)$ is the unique intersection near $h(t)$ of the one-dimensional global unstable manifold $W^{\mathrm{u}}(0)$ with the section Σ_+. Under these assumptions, 2-homoclinic orbits can exist either for $\mu > 0$ or else for $\mu < 0$. Moreover, there is a sequence $\mu_k \to 0$ as $k \to \infty$ so that (5.21) has 2-homoclinic orbits near $h(t)$ for $\mu = \mu_k$ (and the μ_k have the same sign independently of k).

This theorem shows that reversible Hamiltonian and reversible non-Hamiltonian systems with homoclinic orbits to saddle-center equilibria behave in a fundamentally different way. It is worthwhile to remark that it is the assumption (5.22) that discriminates between the two cases: indeed, (5.17) shows that the level set $\mathcal{H}^{-1}(0)$ is tangent to $W^{\mathrm{cs}}(0)$ in Hamiltonian system; see [72, Lemma 2]. We refer to [72] for a comprehensive discussion and unfolding results in the situation where the Hamiltonian structure is broken while reversibility is retained; see also [219, 427] for results on homoclinic orbits to saddle-centers in reversible systems and to [361] for infinite-dimensional conservative systems.

5.3.10 Homoclinic orbits to nonhyperbolic equilibria

In §5.1.10, we discussed homoclinic orbits to equilibria which themselves undergo a local bifurcation. Obviously, analogous bifurcations are possible in ODEs that preserving a structure such as reversibility. Recall that homoclinic orbits to saddle-center equilibria were discussed in §5.3.9. Thus, we report here on reversible transcritical and pitchfork bifurcations.

We first discuss bifurcations from a symmetric homoclinic orbit to an equilibrium that undergoes a reversible transcritical bifurcation. Rather than stating detailed bifurcation results, for which we refer to [419], we focus on the geometric arguments that lead to these results. Thus, consider a two-parameter family of reversible ODEs on $\mathbb{R}^4$ that satisfies Hypothesis 5.26. Suppose that the origin has a double zero eigenvalue and two

real eigenvalues $\nu^{\mathrm{s}} < 0$ and $\nu^{\mathrm{u}} = -\nu^{\mathrm{s}} > 0$ when $\mu = 0$. Suppose that the unfolding is generic, so that the vector field on the two-dimensional center manifold $W^{\mathrm{c}}(0, \mu)$ is given by

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \mu_1 x + x^2.
\end{aligned}
$$

For $\mu_1 \neq 0$, the flow on the center manifold has a saddle and a focus equilibrium, and there exists a small symmetric homoclinic orbit to the saddle equilibrium that encloses the focus equilibrium. At $\mu = 0$, we now also assume the existence of a large symmetric homoclinic orbit h to the origin. Take a symmetric three-dimensional cross section Σ transverse to $h(0)$ for $\mu = 0$, then the fixed point space $\mathrm{Fix}(\mathcal{R})$ intersects Σ in a two-dimensional plane. The geometry of single-round homoclinic and heteroclinic orbits can now be understood from the following observations. First, we know that there are three-dimensional center-stable and center-unstable manifolds $W^{\mathrm{cs}}(0, \mu)$ and $W^{\mathrm{cu}}(0, \mu)$, which are fibered by the stable and unstable manifolds of orbits in the center manifolds. This yields copies of the dynamics on the center manifold in the two-dimensional intersections $W^{\mathrm{cs}}(0, \mu) \cap \Sigma$ and $W^{\mathrm{cu}}(0, \mu) \cap \Sigma$. Under the assumption that $W^{\mathrm{cs}}(0, 0) \cap \Sigma$ is transverse to $W^{\mathrm{cu}}(0, 0) \cap \Sigma$ at $h(0)$, these sets are also transverse to $\mathrm{Fix}(\mathcal{R})$ and intersect it along curves. The curves $W^{\mathrm{cs}}(0, \mu) \cap \Sigma \cap \mathrm{Fix}(\mathcal{R})$ provide symmetric homoclinic orbits to recurrent orbits in the center manifold.

Bifurcations in systems with an additional $\mathbb{Z}_2$ symmetry, where the equilibrium undergoes a reversible pitchfork bifurcation, are discussed in [417–419]. Depending on the group action and on normal-form coefficients, the bifurcating two equilibria may be connected by a heteroclinic cycle or be accompanied by symmetric or nonsymmetric figure-eight homoclinic orbits to the persisting equilibrium.

5.3.11 Heteroclinic cycles and snaking

We consider the equation

$$
\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^4 \times \mathbb{R} \tag{5.23}
$$

with $f \in C^2$ and assume that this equation is reversible for all μ, see Hypothesis 5.26, and that it has two symmetric hyperbolic equilibria, $u = 0$ and $u = p \neq 0$, for all μ near zero.

Hypothesis 5.39. *For $\mu = 0$, equation (5.23) has a heteroclinic orbit $h_1(t)$ that connects $u = 0$ to $u = p$ and is transversely constructed so that Hypothesis 2.2 is met.*

If Hypothesis 5.39 is met, then $h_2(t) := \mathcal{R} h_1(t)$ is a transversely constructed heteroclinic orbit between $u = p$ and $u = 0$. Our goal is to describe homoclinic orbits $h(t)$ that connect $u = 0$ to itself and are obtained by gluing the heteroclinic orbits $h_1(t)$ and $h_2(t)$ together for μ close to zero.

Theorem 5.57 ([223]). *Assume that Hypotheses 5.26 and 5.39 are met and that the eigenvalues of $f_u(p, 0)$ are $\pm \alpha \pm \mathrm{i}\beta$ with $\alpha, \beta > 0$. Then there are constants $a \neq 0$, $b \in \mathbb{R}$, and $L_* > 0$ such that (5.23) has a homoclinic orbit $h(t)$ to $u = 0$ for μ close to zero that spends time $L \geq L_*$ near $u = p$ if and only if*

$$
\mu = a \sin(\beta L + b) \mathrm{e}^{-\alpha L} + \mathrm{o}(\mathrm{e}^{-\alpha L}), \qquad L \geq L_*.
$$

In particular, there are infinitely many homoclinic orbits to $u = 0$ when $\mu = 0$; all but finitely many of them disappear for $\mu \neq 0$. In the same setting, the existence of multi-pulses was recently considered in [224] under the assumption that $u = 0$ is also a bi-focus.

We now discuss the situation where the equilibrium p is replaced by a periodic orbit $q(t)$. It it easier to formulate the hypotheses in the conservative context:

Hypothesis 5.40. *Equation (5.23) is conservative with an energy $\mathcal{H}(u, \mu)$ that is invariant under the reverser $\mathcal{R}$ for all μ. The origin $u = 0$ is a hyperbolic equilibrium of (5.23), and we may assume that $\mathcal{H}(0, \mu) = 0$ for all μ. Furthermore, for each μ, (5.23) has a symmetric periodic orbit $q(t, \mu)$ with $q(0, \mu) \in \mathrm{Fix}(\mathcal{R})$ and $\mathcal{H}(q(t, \mu), \mu) = 0$ that depends smoothly on μ and has two positive Floquet multipliers $\mathrm{e}^{\pm \alpha(\mu)}$ with $\alpha(\mu) > 0$ (the other two Floquet multipliers are necessarily equal to one).*

Next, we define
$$\Gamma := \{(\varphi, \mu) \in \mathbb{S}^1 \times \mathbb{R}; \; W^{\mathrm{u}}(0, \mu) \cap W^{\mathrm{ss}}(q(\varphi, \mu), \mu) \neq \emptyset\}$$
which encodes and captures all heteroclinic orbits that connect $u = 0$ to $q(t, \mu)$. We assume that Γ is a graph:

Hypothesis 5.41. *The set Γ is the graph of a smooth function $z : \mathbb{S}^1 \to \mathbb{R}$, and we assume that $z'(\varphi) = 0$ implies $z''(\varphi) \neq 0$.*

The next result shows that the heteroclinic orbits described in Hypothesis 5.41 and their symmetric counterparts can be glued together to construct homoclinic orbits that connect $u = 0$ to itself and spend a long time near the periodic orbits $q(t, \mu)$.

Theorem 5.58 ([29, 79, 95, 232, 424])**.** *Assume that Hypotheses 5.40-5.41 and an additional technical condition (which can be found in [29]) are met, then there are constants $L_* \gg 1$ and $\eta > 0$ so that the following is true: for each $L > L_*$, (5.23) has a symmetric homoclinic orbit $h(t)$ that spends time L near $q(t, \mu)$ if and only if $\mu = z(\varphi_0 + L) + \mathrm{O}\left(\mathrm{e}^{-\eta L}\right)$ for an appropriate $\varphi_0 \in \{0, \pi\}$, and $h(0)$ lies near $q(\varphi_0, \mu)$ in $\mathrm{Fix}(\mathcal{R})$.*

Geometric versions of the preceding theorem were first given in [95, 424]. The theorem as stated was proved in [29], and the results established there are, in fact, valid for higher-dimensional systems that are only reversible and not necessarily conservative. In [79, 232], the conclusions of Theorem 5.58 were shown to hold near degenerate Turing instabilities of the Swift–Hohenberg equation.

We remark that [58–60] gave numerical evidence for the existence of asymmetric homoclinic orbits of (5.23) whose existence was subsequently proved in [29] under assumptions similar to those stated above. Near degenerate Turing bifurcations, these results follow again from [79, 232]. We refer to the recent review [220] for a list of open problems.

5.4 Homoclinic orbits arising through local bifurcations

Homoclinic orbits can emerge in local bifurcations, and the study of local bifurcations therefore provides one way to obtain rigorous existence results for homoclinic orbits. We focus here on homoclinic bifurcations near nilpotent singularities, near Hopf/saddle-node bifurcations in generic and reversible systems, and at $0^{2+}\mathrm{i}\omega$ resonances in reversible systems. Other local bifurcations that lead to homoclinic orbits (in particular, 1:3 and 1:4 resonances, and codimension-three Hopf/Bogdanov–Takens singularities) will not be discussed here.

Recall that two differential equations on $\mathbb{R}^n$ are said to be topologically equivalent if there exists a homeomorphism h that maps orbits of the first system to orbits of the second equation, while preserving the direction of time. For two parameter-dependent families of ODEs on $\mathbb{R}^n$, one can seek homeomorphisms $\Phi(\cdot, \mu)$ of $\mathbb{R}^n$ and ϕ on the parameter space that provide an equivalence $v = \Phi(u, \mu)$ between $\dot{u} = f(u, \mu)$ and $\dot{v} = g(v, \phi(\mu))$. One speaks of a (fiber C^0, C^r)-equivalence if $\Phi(\cdot, \mu)$ is C^0 for each μ and ϕ is C^r. One speaks of a (C^0, C^r)-equivalence if $(u, \mu) \mapsto \Phi(u, \mu)$ is C^0 and ϕ is C^r. In local bifurcation theory, one uses, of course, local versions of these notions.

5.4.1 Nilpotent singularities

A nilpotent singularity is an equilibrium of an ODE $\dot{u} = f(u)$ for which the linearization about the equilibrium has multiple eigenvalues at zero and no other eigenvalues on the imaginary axis. The algebraic and geometric multiplicities of the zero eigenvalue distinguish different nilpotent singularities. By restricting to a center manifold, we may assume that the eigenvalues of the linearization about the equilibrium are all zero. The most elementary example of a nilpotent singularity is then the Bogdanov–Takens bifurcation in $\mathbb{R}^2$ where the linearization is a nontrivial 2×2 Jordan block. It is well known that small homoclinic orbits with real leading
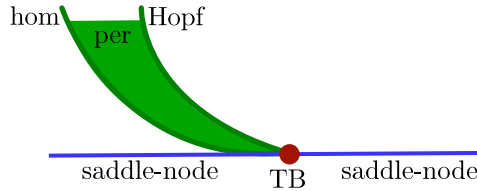
Figure 5.21: *The unfolding of a Bogdanov–Takens bifurcation (labelled TB) is illustrated.*

eigenvalues at the equilibrium occur in the unfolding of the Bogdanov–Takens singularity, and we will recall this statement and review homoclinic dynamics in unfoldings of higher-codimension nilpotent singularities that may lead to other related dynamics such as small Lorenz-like attractors.

First, we review the Bogdanov–Takens bifurcation where the eigenvalue at zero has algebraic multiplicity two and geometric multiplicity one. This bifurcation was studied by Bogdanov [42] and Takens [388] (reproduced in [393]); a streamlined analysis can be found in [332]. The result is that a generic two-parameter family that unfolds a geometrically simple but algebraically double eigenvalue at the origin is (fiber C^0, C^∞)-equivalent to

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= x^2 + \mu_1 + y(\mu_2 \pm x).
\end{aligned}$$

It was shown in [119] that this equivalence is, in fact, a (C^0, C^∞)-equivalence.

Theorem 5.59. *The two-parameter unfolding of a nilpotent Bogdanov–Takens singularity at $\mu = 0$ contains one-sided curves of Hopf bifurcations and of homoclinic loops that branch from a curve of saddle-node bifurcations at $\mu = 0$; see Figure 5.21.*

Codimension-three bifurcations of nilpotent equilibria in the plane have been studied by Dumortier, Sotomayor and Roussarie [121, 123]: intricate bifurcation diagrams, including homoclinic and heteroclinic bifurcations, arise, and we refer to these references for details.

Ibáñez and Rodríguez [199] considered nilpotent singularities with linearization

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.24}$$

in $\mathbb{R}^3$ and proved that saddle-focus homoclinic orbits occur in the three-parameter unfolding, confirming a conjecture in [18]. This singularity appears in differential equations for coupled Brusselators [115]. In passing, we note that saddle-focus homoclinic orbits occur also near nilpotent singularities of higher codimension; see [198, 386]. Returning to (5.24), a generic unfolding is given by the normal form

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= \mu_1 + \mu_2 y + \mu_3 z + x^2 + bxy + cxz + dy^2 + eyz + \mathrm{O}(\|(x, y, z, \mu)\|^3),
\end{aligned} \tag{5.25}$$

which depends on the parameters $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$.

Theorem 5.60 ([199])**.** *For any given neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^3$, there are parameter values μ arbitrary close to zero for which (5.25) has a wild saddle-focus homoclinic orbit in $\mathcal{U}$.*

The proof involves a singular rescaling that reduces the problem to a study of perturbations of

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= c^2 - \frac{1}{2}x^2 - y
\end{aligned}$$

for $c = 15\sqrt{22/19^3}$. This system has a T-point heteroclinic cycle between the equilibria $p_+ = (\sqrt{2}c, 0, 0)$ and $p_- = (-\sqrt{2}c, 0, 0)$: the two-dimensional stable manifold of p_- and the two-dimensional unstable manifold of p_+ intersect along an isolated heteroclinic orbit, while the codimension-two heteroclinic connection between p_- and p_+ lies in the x-axis.

Next, we consider nilpotent singularities with rank-one linearization

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The associated linear system is equivariant with respect to the $\mathbb{Z}_2$-action $(x, y, z) \mapsto (-x, -y, z)$. Within the class of $\mathbb{Z}_2$-equivariant perturbations, one finds miniature Lorenz-like attractors.

Theorem 5.61 ([118]). *Consider a family of ODEs on $\mathbb{R}^3$ with parameter $\mu \in \mathbb{R}^5$ whose third-order truncation is given by*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 x - x^3 + \mu_3 y + \mu_4 xz + \mu_5 yz, \\ \dot{z} &= \mu_2 z + x^2, \end{aligned}$$

then there exist arbitrarily small values of μ for which the differential equation has a small Lorenz-like attractor near the origin.

To prove this result, Dumortier, Kokubu, and Oka showed that the unfolding of the singularity contains inclination-flip homoclinic orbits and that the eigenvalues at the equilibrium are such that Rychlik's theorem in [334] (see Theorem 5.81) applies which gives the existence of Lorenz-like attractors. In [364], further information on the connection between $\mathbb{Z}_2$-equivariant unfoldings of the nilpotent singularity with a triple zero eigenvalue of rank one and the Shimizu–Morioka system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - \bar{\mu}_2 y - xz, \\ \dot{z} &= -\bar{\mu}_3 z + x^2 \end{aligned}$$

is obtained: Consider $\mathbb{Z}_2$-equivariant systems with third-order truncation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 x - \mu_2 y + axz - a_1 x(x^2 + y^2) - a_2 yz + a_3 y(x^2 + y^2), \\ \dot{z} &= -\mu_3 + z^2 + b(x^2 + y^2) \end{aligned}$$

for $ab > 0$ and $\mu_3 > 0$, and let $\tau^2 = \mu + a\sqrt{\mu_3}$. The time scaling $t \to s/\tau$, the phase-space scaling $x \to x\sqrt{\tau^3/(ab)}$, $y \to y\tau\sqrt{\tau^3/(ab)}$, $z \to \sqrt{\mu_3} + z\tau^2/a$, and the parameter scaling $\mu_2 = \bar{\mu}_2\tau$, $\mu_3 = (\bar{\mu}_3\tau/2)^2$ then produces the Shimizu–Morioka system in the limit $\tau \to 0$. We refer to [362] for a bifurcation study of the Shimizu–Morioka system.

5.4.2 Hopf/saddle-node bifurcations in generic and reversible systems

In the previous section, we reviewed local bifurcations that give rise to small homoclinic orbits; in particular, wild saddle-focus homoclinic orbits appear in the unfolding of a codimension-three nilpotent singularity. We emphasize that the unfoldings considered so far were determined by finite jets.

Wild saddle-focus homoclinic orbits can also arise in Hopf/saddle-node bifurcations, which can occur in two-parameter families of ODEs. In this case, the normal form is invariant under rotations, and the perturbations that create homoclinic orbits are flat and therefore not captured by finite jets. It is an open problem to

determine whether saddle-focus homoclinic orbits bifurcate for analytic two-parameter families that do not possess rotational symmetry.

To set the scene, let $\dot{u} = f(u, \mu)$ be a two-parameter family of ODEs on $\mathbb{R}^3$. For $\mu = 0$, we assume that p is an equilibrium whose linearization has a simple eigenvalue at zero and a pair of purely imaginary eigenvalues. Under certain nondegeneracy conditions, the normal form near p is given by

$$\begin{aligned}
\dot{x} &= \nu_1 + x^2 + s|z|^2 + \mathrm{O}(\|(x, z, \bar{z})\|^4), \\
\dot{z} &= (\nu_2 + \mathrm{i}\omega)z + (a + \mathrm{i}b)xz + x^2 z + \mathrm{O}(\|(x, z, \bar{z})\|^4),
\end{aligned}$$

where $(x, z) \in \mathbb{R} \times \mathbb{C}$, and $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ is the unfolding parameter; see, for instance, [242]. In the above expression, we have $s = \pm 1$, while ω, a, and b are smooth functions of ν with $\omega(0) \neq 0$ and $a(0) \neq 0$. The truncated normal form, in which only terms of order at most three are retained, is equivariant under rotations of z. The following proposition, proved in [52, 389], shows that a normal form exists that is rotationally symmetric except possibly for flat terms.

Proposition 5.2. *There exists a smooth coordinate change that transforms the above system into a differential equation of the form*

$$\begin{aligned}
\dot{x} &= \nu_1 + x^2 + s|z|^2 + R_1(x, |z|, \nu) + S_1(x, z, \bar{z}, \nu) \\
\dot{z} &= (\nu_2 + \mathrm{i}\omega)z + (a + \mathrm{i}b)xz + x^2 z + z R_2(x, |z|, \nu) + S_2(x, z, \bar{z}, \nu),
\end{aligned}$$

where $R_1(x, |z|, \nu)$ and $z R_2(x, |z|, \nu)$ are $\mathrm{O}(\|(x, z)\|^4)$, and S_j are smooth functions that are flat in $(x, z, \bar{z}, \nu)$ at the origin in $\mathbb{R} \times \mathbb{C} \times \mathbb{R}^2$.

Writing the z-equation of the truncated system in polar coordinates (ρ, ϕ), we can ignore the equation for the angle ϕ due to rotational symmetry and obtain the equation

$$\begin{cases}
\dot{x} &= \nu_1 + x^2 + s\rho^2, \\
\dot{\rho} &= \nu_2 \rho + ax\rho + \rho x^2
\end{cases} \tag{5.26}$$

for the amplitudes (x, ρ). Depending on the signs of s and a, different unfoldings need to be considered. Here, we shall discuss only the case $s = 1$, $a < 0$, where a heteroclinic cycle occurs in the unfolding of the amplitude system. Thus, we impose the condition:

Hypothesis 5.42 (Coefficient condition). *Assume that $s = 1$ and $a < 0$ in (5.26).*

An analysis of the amplitude equation gives the following picture; see also Figure 5.22. For $\nu_1 < 0$, there are two hyperbolic equilibria $p_+ = (\sqrt{-\nu_1}, 0, 0)$ and $p_- = (-\sqrt{-\nu_1}, 0, 0)$. Inside the region bounded by the Hopf bifurcation curve, the equilibrium p_+ has two complex conjugate stable eigenvalues, while p_- has two complex conjugate unstable eigenvalues. Along a curve Γ that emerges from the origin in the parameter plane, we find an invariant sphere of heteroclinic connections from p_- to p_+, while the x-axis contains a heteroclinic orbit from p_+ to p_-. Generic perturbations from the truncated normal form will create transverse intersections of $W^{\mathrm{u}}(p_-)$ and $W^{\mathrm{s}}(p_+)$ [160]. Moreover, the x-axis may no longer be invariant, so that homoclinic orbits to p_- and to p_+ could exist. A detailed analysis yields the existence of parameter values for which saddle-focus homoclinic orbits exists in generic families.

Theorem 5.62 ([53]). *A generic two-parameter family on $\mathbb{R}^3$ that unfolds a Hopf/saddle-node equilibrium at $\nu = 0$ and satisfies Hypothesis 5.42 has saddle-focus homoclinic orbits for parameter values arbitrarily close to $\nu = 0$. These parameter values are contained in a wedge-shaped region whose width is flat in ν as $\nu \to 0$ and that is tangent at $\nu = 0$ to the curve Γ for which a sphere of heteroclinic connections from p_+ to p_- exists in the truncated normal form. Furthermore, the saddle-focus homoclinic orbits are wild for $-2 < a < 0$.*
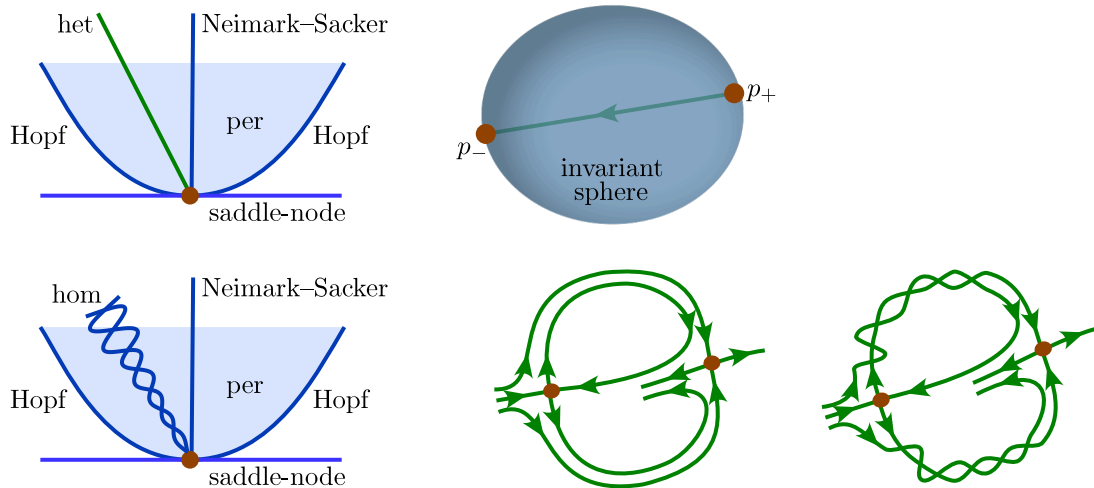
Figure 5.22: *The upper-left panel contains the bifurcation diagram of the truncated amplitude system in the (ν_1, ν_2)-parameter plane: the curve labelled 'het' corresponds to the existence of an invariant sphere in phase space, shown in the upper-right panel, whose surface is filled with heteroclinic connections from p_- to p_+, while the axis connecting these equilibria contains the indicated heteroclinic orbit from p_+ back to $p+-$. Numerical computations for the non-truncated equation show entwined wiggling curves of saddle-focus homoclinic orbits of which the lower-left panel gives an impression. The center and right panels in the bottom row indicate possible geometries of the invariant manifolds of p_- and p_+ at parameter values where saddle-focus homoclinic orbits appear: homoclinic orbits to p_- can appear for parameter values where transverse intersections of $W^{\mathrm{u}}(p_-)$ and $W^{\mathrm{s}}(p_+)$ exist [right] but also for parameter values for which an invariant region exists [center].*

Gaspard [141] noted the possible existence of a flow-invariant region for parameter values for which saddle-focus homoclinic orbits to p_- occur (the invariant region is bounded by the stable manifold of p_+), which implies the existence of an attractor in this region; see Figure 5.22.

Much subsequent work has focused on the question whether specific families, which consist of the truncated normal form with small analytic terms added to it that break the rotational symmetry, would unavoidably contain saddle-focus homoclinic bifurcations. When adding small third-order symmetry-breaking terms to the normal form, one finds two curves of saddle-focus homoclinic orbits (to p_+ and to p_-) that cross infinitely often on parameter space; see [74, 141, 214, 215] for earlier results, [27] for recent progress on this topic, and Figure 5.22 for an illustration of the bifurcation diagram.

Next, we discuss Hopf/saddle-node bifurcations in reversible systems in $\mathbb{R}^3$, where the fixed-point space of the reverser is a line of equilibria. In contrast to the situation for generic systems, where this bifurcation has codimension two (see above), Hopf/saddle-node bifurcation have codimension one in reversible systems. We remark that the conservative case is also of codimension one and refer to [53, 98] for results on small homoclinic orbits that emerge in this setting.

The reversible Hopf/saddle-node bifurcation occurs in the Michelson system [274] given by

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= c^2 - \frac{1}{2}x^2 - y,
\end{aligned}
$$

which is reversible with respect to the involution $\mathcal{R}(x, y, z) = (-x, y, -z)$. Michelson's system is equivalent to the third-order system $\dddot{x} + \dot{x} + \frac{1}{2}x^2 = c^2$, which is the integrated travelling-wave equation of the Kuramoto–Sivashinsky equation $u_t + uu_x + u_{xx} + u_{xxxx} = 0$. Analogous to the preceding analysis of generic Hopf/saddle-node bifurcation, the truncated normal form for reversible systems is invariant under rotations and reduces
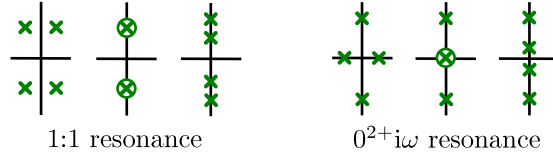
Figure 5.23: *The eigenvalues of an equilibrium in the unfolding of a 1:1 resonance [left] and a $0^{2+}\mathrm{i}\omega$ resonance are shown; see §5.4.3 and §5.4.4.*

therefore to a planar amplitude system of the form

$$
\begin{aligned}
\dot{x} &= \nu + x^2 + \rho^2, \\
\dot{\rho} &= a\rho x
\end{aligned}
$$

with unfolding parameter ν, which replaces the amplitude equation (5.26) for the generic Hopf/saddle-node bifurcation. Lamb, Teixeira and Webster showed in [249] that reversible perturbations of this normal form lead to the existence of infinitely many N-homoclinic orbits, where the number number of rounds is defined with respect to a rescaled normal form.

Theorem 5.63 ([249])**.** *A generic one-parameter family on $\mathbb{R}^3$ that is reversible with respect to a reverser $\mathcal{R}$ with one-dimensional fixed-point space and that unfolds a Hopf/saddle-node equilibrium admits infinitely many hyperbolic basic sets, infinitely many parameter values that correspond to saddle-focus N-homoclinic orbits, and infinitely many parameter values that correspond to symmetric N-heteroclinic cycles.*

In contrast to the situation for dissipative and conservative Hopf/saddle-node bifurcations, where heteroclinic cycles are of codimension two, symmetric heteroclinic cycles are codimension one in the reversible context. It was further established in [249] that the Michelson system has infinitely many bifurcations to N-homoclinic orbits and to symmetric N-heteroclinic cycles when varying the parameter c. For additional information on global bifurcations in the Michelson system, we refer to [117, 250, 421, 422].

5.4.3   1:1 resonances in reversible systems

We now turn to local bifurcations in reversible systems and begin with the 1:1 resonance: this bifurcation occurs at equilibria where two purely imaginary pairs of eigenvalues collide on the imaginary axis and turn in to a quadruplet of complex conjugate eigenvalues as a one-dimensional parameter is varied; see Figure 5.23. At the bifurcation point, the equilibrium therefore has a pair of two non-semisimple eigenvalues at $\pm\mathrm{i}\omega$ for some $\omega > 0$. Restricting to a center manifold, we may therefore consider a one-parameter family $\dot{u} = f(u, \mu)$ on $\mathbb{R}^4$ that is reversible for all μ. The generic unfolding of the 1:1 resonance was studied by Iooss and Peroueme [207], and we report first on their results.

Thus, suppose that we are given a reversible family of ODEs on $\mathbb{R}^4$ so that the origin is an equilibrium for all μ. For $\mu = 0$, we assume that $\mathrm{i}\omega$ is a non-semisimple eigenvalue of multiplicity two for some $\omega > 0$ (and so is then $-\mathrm{i}\omega$). The normal form of a reversible family near the origin is, to any finite order, given by

$$
\dot{A} = \mathrm{i}\omega A + B + \mathrm{i}AP\left(|A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B); \mu\right) \tag{5.27}
$$

$$
\dot{B} = \mathrm{i}\omega B + + AQ\left(|A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B); \mu\right) + \mathrm{i}AP\left(|A|^2, \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B); \mu\right),
$$

where $(A, B) \in \mathbb{C}^2$ and the reverser acts as $(A, B) \mapsto (\bar{A}, -\bar{B})$. The functions $P(u, v; \mu)$ and $Q(u, v; \mu)$ are real polynomials in (u, v) that vanish at $(u, v, \mu) = 0$; we write

$$
Q(u, v; \mu) = q_1\mu + q_2 u + q_3 v + \mathrm{O}((|u| + |v| + |\mu|)^2). \tag{5.28}
$$

Note that the eigenvalues at the origin have nonzero real part for all μ with $q_1\mu > 0$.

Theorem 5.64 ([207])**.** *If the normal-form coefficients of a one-parameter family of reversible ODEs with a 1:1 resonance at $\mu = 0$ satisfy $q_1 \neq 0$ and $q_2 < 0$, then the system has a pair of small symmetric homoclinic orbits to the origin for each μ close to zero for which $q_1\mu > 0$, and these homoclinic orbits satisfy Hypothesis 5.27.*

If $q_2 > 0$, then a pair of homoclinic orbits to periodic orbits bifurcates, and we refer to [207] for the precise results.

To prove the preceding theorem, Iooss and Peroueme first observed that the normal form (5.27) is integrable: two conserved quantities are given by $v = \frac{\mathrm{i}}{2}(A\bar{B} - \bar{A}B)$ and $|B|^2 - \int_0^{|A|^2} Q(u, v; \mu)\,\mathrm{d}u$. The normal form is furthermore equivariant under the $\mathbb{S}^1$-action $(A, B) \mapsto \mathrm{e}^{\mathrm{i}\phi}(A, B)$. If $q_2 < 0$, an $\mathbb{S}^1$-orbit of small homoclinic orbits to the origin bifurcates into $q_1\mu > 0$, and Iooss and Peroueme showed that the symmetric homoclinic orbits survive the perturbations by nonflat terms that are neglected in the normal form.

We remark that Bolle and Buffoni [43] constructed algebraically decaying homoclinic orbits for Hamiltonian systems that are exactly at a 1:1 resonance, provided a certain normal-form coefficient has the correct sign.

Next, we consider the situation where q_2 changes sign as a second parameter is varied. This case is important in applications to snaking that we discussed in §5.3.11. When $q_2 = 0$ and the normal-form coefficient q_4 corresponding to the term q_4u^2 in the expansion of Q in (5.28) satisfies $q_4 \neq 0$, then the normal form can exhibit homoclinic orbits to the origin and heteroclinic cycles that connect the origin to small periodic orbits and back: this scenario was first studied by Woods and Champneys [424]. The persistence of these solutions for the full system involves asymptotics beyond all orders and has not yet been proved rigorously: we refer to recent work by Chapman and Kozyreff [79, 232], who provided a detailed analysis of this bifurcation using formal methods.

5.4.4 $0^{2+}\mathrm{i}\omega$ resonances in reversible systems

In this section, we discuss $0^{2+}\mathrm{i}\omega$ resonances in reversible systems: these bifurcations occur at equilibria that have eigenvalues 0, 0, and $\pm\mathrm{i}\omega$ with $\omega > 0$; see Figure 5.23. We outline the results contained in the monograph [259] by Lombardi who studied homoclinic loops near $0^{2+}\mathrm{i}\omega$ resonances in both four- and infinite-dimensional state space. Consider a one-parameter family $\dot{u} = f(u, \mu)$ of reversible ODEs on $\mathbb{R}^4$ with reverser $\mathcal{R}$ that satisfies the following hypothesis.

Hypothesis 5.43 ($0^{2+}\mathrm{i}\omega$ resonance)**.** *At $\mu = 0$, the origin is an equilibrium so that $f_u(0, 0)$ has eigenvalues $0, 0, \pm\mathrm{i}\omega$ for some $\omega > 0$ with (generalized) eigenvectors $v_0, v_1, v_\pm$ that satisfy*

$$f_u(0, 0)v_0 = 0, \quad f_u(0, 0)v_1 = v_0, \quad f_u(0, 0)v_\pm = \pm\mathrm{i}\omega v_\pm,$$

and the reverser $\mathcal{R}$ maps v_0 to v_0.

We remark that the case where $\mathcal{R}v_0 = -v_0$ is referred to as the $0^{2-}\mathrm{i}\omega$ resonance. In the following, we write $v_0^*, v_1^*, v_\pm^*$ for a basis that is dual to $v_0, v_1, v_\pm$.

Hypothesis 5.44 (Generic unfolding of $0^{2+}\mathrm{i}\omega$ resonances)**.** *Consider the following conditions on the quadratic terms and the dependence on the parameter:*

(i) $c_1 = \langle v_1^*, f_{u\mu}(0, 0)v_0 \rangle \neq 0$;

(ii) $c_2 = \langle v_1^*, f_{uu}(0, 0)[v_0, v_0] \rangle \neq 0$.

Without loss of generality, we may assume that $c_1 > 0$ is positive. Under these assumptions, the equilibrium persists for μ near zero, and the double zero eigenvalues of the linearization about it move from the imaginary axis for $\mu < 0$ onto the real axis for $\mu > 0$. Thus, for $\mu > 0$, the linearization about the origin has a fast oscillatory part corresponding to the purely imaginary eigenvalues at $\pm\mathrm{i}\omega$ and a slow hyperbolic part with real

eigenvalues $\pm\sqrt{\mu[c_1 + O(\mu)]}$. The question then is whether small-amplitude homoclinic orbits can bifurcate for $\mu > 0$, and the following result shows that this will not happen for generic families due to exponentially small terms beyond all orders.

Theorem 5.65 ([259]). *Assume that the reversible one-parameter family $\dot{u} = f(u, \mu)$ is analytic in $(u, \mu) \in \mathbb{R}^4 \times \mathbb{R}$ and satisfies the Hypotheses 5.43 and 5.44. For all $c_1\mu > 0$ close to zero, the system then admits a sheet of small periodic solutions $q_{\kappa,\mu}$ that are parameterized by their amplitude κ, and there are positive constants $\kappa_1, \kappa_2, \sigma > 0$ so that the ODE has a pair of symmetric homoclinic orbits to the periodic solution $q_{\kappa,\mu}$ for each κ with*

$$\kappa_1 c_1 \mu e^{-\omega[\pi - \sigma(c_1\mu)^{3/10}]/\sqrt{c_1\mu}} < \kappa < \kappa_2 c_1 \mu.$$

Furthermore, there is a constant $\kappa_0 \geq 0$, which generically is positive, so that the ODE has no single-round symmetric homoclinic orbits to $q_{\kappa,\mu}$ for

$$0 \leq \kappa < \kappa_0 c_1 \mu e^{-\pi/[\omega\sqrt{c_1\mu}]}.$$

The qualifier 'single-round' in the preceding theorem is with respect to a homoclinic orbit to the origin in a rescaled normal-form family of ODEs that we now describe. Using a singular rescaling and the rescaled parameter $\nu = \sqrt{c_1\mu}$, the one-parameter family can be transformed into the normal form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} y \\ x - \frac{3}{2}x^2 - c(v^2 + w^2) \\ -w(\omega/\nu + x\nu + b\nu x) \\ v(\omega/\nu + x\nu + b\nu x) \end{pmatrix} + R(x, y, v, w, \nu) \tag{5.29}$$

with higher-order terms $R(x, y, v, w, \nu)$; see [259]. The truncated normal form is integrable and has a sheet of small symmetric periodic orbits $q_{\kappa,\nu}$ for $\nu > 0$ that is parameterized by the amplitude κ. Moreover, the truncated normal form admits a symmetric homoclinic orbit $h(t)$ to the origin, and the stable and unstable manifolds of each periodic orbit $q_{\kappa,\nu}$ coincide to form a two-parameter family of homoclinic orbits between symmetric periodic orbits. Among the circle of homoclinic orbits to a periodic orbit, two intersect the fixed point space of $\mathcal{R}$ and therefore correspond to symmetric homoclinic orbits. Integrability holds not only for the normal form truncated at second-order terms but, in fact, for the normal forms truncated at any order.

The geometry becomes clearer in a symmetric three-dimensional cross section Σ placed at $h(0)$. The two-dimensional stable and unstable manifolds $W^s(q_{\kappa,\nu})$ and $W^u(q_{\kappa,\nu})$ of each individual periodic orbit $q_{\kappa,\nu}$ are identical and, for $\kappa > 0$, their intersection with Σ are circles that intersect the plane $\text{Fix}(\mathcal{R})$ transversally. The one-dimensional stable and unstable manifolds of the origin, on the other hand, each intersect Σ in a point, and these intersection points coincide and lie in $\text{Fix}(\mathcal{R})$ for all truncated normal forms. We remark that the truncated system also admits elliptic periodic orbits and invariant tori.

The central issue is to study the effect of perturbations of the truncated normal form which will perturb the intersections $W^s(q_{\kappa,\nu}) \cap \Sigma$ and $W^u(q_{\kappa,\nu}) \cap \Sigma$. In particular, we expect that the homoclinic orbit to the origin breaks as there is no reason why the zero-dimensional intersections of the stable and unstable manifolds of the origin with Σ should coincide. The precise statement for analytic families that unfold the local bifurcation forms the content of the preceding theorem; we remark that this result does not address the possible existence of multi-round homoclinic orbits. The proof of the theorem involves the complexification of the system and the time variable which leads to a complex differential equation in $\mathbb{C}^4$ in a complex time variable.

We remark that the geometry sketched above suggests that an additional parameter could be used to control the existence of symmetric homoclinic orbits to the origin: these homoclinic orbits should then occur along curves in the parameter plane. This idea is expounded in [71], which also contains careful numerical experiments.

The occurrence of exponentially small phenomena is illustrated in the following toy model which is taken from [259]. Consider the differential equation

$$
\begin{aligned}
\dot{x} &= 1 - x^2, \\
\dot{z} &= \frac{\mathrm{i}\omega z}{\varepsilon} + \mathrm{i}\varepsilon\rho(1 - x^2)
\end{aligned}
$$

on $\mathbb{R} \times \mathbb{C}$, which respects the reverser $\mathcal{R}(x, z) = (-x, \bar{z})$; note that $\mathrm{Fix}(\mathcal{R})$ is the real axis in $\{0\} \times \mathbb{C}$. This system has two families of periodic solutions given by $q_k^{\pm}(t) = (\pm 1, k\mathrm{e}^{\mathrm{i}\omega t/\varepsilon})$. For $\rho = 0$, there is a sheet of heteroclinic solutions, given by $h_{k,\phi}(t) = (\tanh t, k\mathrm{e}^{\mathrm{i}\omega t/\varepsilon + \mathrm{i}\phi})$, which connect q_k^- to q_k^+. The connections corresponding to $\phi = 0, 1$ are symmetric. Thus, this toy problem has features reminiscent of those of the normal form of the $0^{2+}\mathrm{i}\omega$ resonance, where the parameter ρ can now be thought of as breaking the truncated normal form. To determine the fate of the stable manifold $W^{\mathrm{s}}(1, 0)$ of the equilibrium $(x, z) = (1, 0)$ upon varying ρ, we note that it is given explicitly by

$$
u(t) = \left(\tanh t, -\mathrm{i}\rho\varepsilon \int_t^\infty \frac{\mathrm{e}^{\mathrm{i}\omega(t-s)/\varepsilon}}{\cosh^2 s} \, \mathrm{d}s \right).
$$

For $t \to -\infty$, $u(t)$ converges to the periodic solution $q_{K(\varepsilon)}^-$ with

$$
K(\varepsilon) = \rho\varepsilon \int_{-\infty}^\infty \frac{\mathrm{e}^{-\mathrm{i}\omega s/\varepsilon}}{\cosh^2 s} \, \mathrm{d}s = \frac{\pi\omega\rho}{\sinh(\omega\pi/2\varepsilon)},
$$

which is asymptotic to $2\pi\omega\rho\mathrm{e}^{-\omega\pi/2\varepsilon}$ as $\varepsilon \to 0$. Thus, the splitting distance between stable and unstable manifolds is exponentially small in ε. For more general problems such as the $0^{2+}\mathrm{i}\omega$ resonance, oscillatory integrals of the form $\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\omega s/\varepsilon} g(u(s)) \, \mathrm{d}s$ with a small parameter ε for a solution $u(t)$ and an analytic function g need to be studied. We refer to [205, 206, 258] for further details and results in this direction.

5.5 Equivariant systems

In flows that are equivariant under the action of a symmetry group, global bifurcations can differ substantially from those in generic flows. The restriction to equivariant perturbations often reduces the codimension of bifurcations and may also constrain the dynamics found in an unfolding. One striking example are heteroclinic cycles which can appear robustly in equivariant systems.

Homoclinic and heteroclinic bifurcation theory for equivariant systems is a broad area in itself. To stay within the central theme of this survey, we limit ourselves to flows that are equivariant under the linear action of a finite group and discuss existence, stability, and bifurcations of heteroclinic cycles within this framework. Thus, continuous group actions will not be considered, and we will also not discuss how homoclinic and heteroclinic dynamics can emerge in local bifurcations. For general background on equivariant flows, we refer the reader to the books [153] and [87, 132, 133]; the latter references also contain sections on homoclinic and heteroclinic bifurcation theory.

To set the scene, let Γ be a finite group with a linear action $x \mapsto \gamma x$ on $\mathbb{R}^n$. We may assume that $\Gamma \subset O(n)$, so that its action leaves the inner product $\langle \cdot, \cdot \rangle$ invariant. We may also assume that the action of Γ is faithful[13]. By definition, a differential equation

$$
\dot{u} = f(u), \qquad u \in \mathbb{R}^n \tag{5.30}
$$

is Γ-equivariant if the following statement holds:

$u(t)$ is a solution to (5.30) if, and only if, $\gamma u(t)$ is a solution to (5.30) for each $\gamma \in \Gamma$.

Equivalently, Γ-equivariance means that

$$
f(\gamma u) = \gamma f(u), \qquad \forall \gamma \in \Gamma, \quad \forall u \in \mathbb{R}^n. \tag{5.31}
$$

---

[13]That is, for each $\gamma \in \Gamma$, there is an $x \in \mathbb{R}^n$ with $\gamma x \neq x$

For each $u \in \mathbb{R}^n$, we write $\Gamma u = \{\gamma u \in \mathbb{R}^n; \ \gamma \in \Gamma\}$ for its group orbit and $\Gamma_u = \{\gamma \in \Gamma; \ \gamma u = u\}$ for its isotropy group. If $\Sigma \subset \Gamma$ is a subgroup of Γ, we write $\mathrm{Fix}(\Sigma) = \{u \in \mathbb{R}^n; \ \gamma u = u \ \forall \gamma \in \Sigma\}$ for the fixed-point space of Σ: note that (5.31) implies that $\mathrm{Fix}(\Sigma)$ is flow invariant. For any set $\{\gamma_i\}$ of elements in Γ, we denote by $\langle \{\gamma_i\} \rangle$ the group generated by $\{\gamma_i\}$: this is the smallest subgroup of Γ that contains the set $\{\gamma_i\}$. Finally, we recall that an isotypic component is a subspace of $\mathbb{R}^n$ which is given as the sum of isomorphic irreducible subspaces of the action of Γ. The associated isotypic decomposition of $\mathbb{R}^n$ is the decomposition of $\mathbb{R}^n$ into isotypic components; see, for instance, [153].

To avoid any possible confusion between group and flow orbits, we shall use the terms 'trajectory' and 'solution' to refer to flow orbits and reserve, in this section, the term 'orbit' for group orbits.

5.5.1 Robust heteroclinic cycles

Equivariant flows may admit heteroclinic cycles between equilibria that persist under equivariant perturbations: this is because symmetry may force subspaces to be invariant, and a heteroclinic solution can now connect an equilibrium within an invariant subspace to a second equilibrium that is stable within this invariant subspace. We note that robust heteroclinic cycles can also occur in flows that are not equivariant but have invariant subspaces for all parameter values: this is a common feature of models in population dynamics, where the coordinate subspaces $\{u \in \mathbb{R}^n; \ u_i = 0\}$ are typically invariant (if species i is not present at $t = 0$, its population $u_i(t)$ vanishes for all t); see [179]. An extensive review of robust heteroclinic cycles is [236], which also includes a historical overview and descriptions of relevant experiments.

We ought to mention that the terminology of homoclinic and heteroclinic dynamics in equivariant systems varies widely in the literature: for consistency, we shall adhere to the notation introduced earlier in this paper, though this may not always agree with the prevalent terminology used in the literature.

Let $\dot{u} = f(u)$ be a Γ-equivariant differential equation. A collection of different hyperbolic equilibria $p_1, \ldots, p_\ell$ and heteroclinic solutions $h_j(t)$ from p_j to p_{j+1} is called a heteroclinic cycle[14]. Since the pointwise isotropy groups $\Sigma_j = \Gamma_{h_j(t)}$ along each heteroclinic solution do not depend on t, we refer to it as the isotropy group of h_j. Define the fixed point spaces $S_j = \mathrm{Fix}(\Sigma_j)$ and recall that these spaces are flow invariant.

Hypothesis 5.45 (Robustness). *We distinguish the following properties:*

(i) $W^{\mathrm{u}}(p_j) \cap S_j$ and $W^{\mathrm{s}}(p_{j+1}) \cap S_j$ intersect transversally in S_j.

(ii) $\dim W^{\mathrm{u}}(p_j) = 1$, and p_{j+1} is a sink in S_j.

(iii) Each fixed point space S_j is two-dimensional.

If Hypothesis 5.45(i) is met, then the manifold of heteroclinic connections from p_j to p_{j+1} is robust under Γ-equivariant perturbations, since the subspace S_j will continue to be invariant. Let $\mathrm{ind}_{S_j}(p_j)$ denote the Morse index $\dim[W^{\mathrm{u}}(p_j) \cap S_j]$ of p_j inside S_j, then the dimension of the manifold of heteroclinic connections in S_j is equal to $\mathrm{ind}_{S_j}(p_j) - \mathrm{ind}_{S_j}(p_{j+1})$. If this dimension is one, then the space S_j contains a robust isolated heteroclinic trajectory that connects p_j to p_{j+1}. We call the resulting heteroclinic cycle a *robust heteroclinic cycle*. Note that Hypothesis 5.45(ii), which is often found in the literature, is stronger: heteroclinic cycles that satisfy this assumption are often attracting, and we refer to §5.5.2 for stability results in this direction.

If Hypothesis 5.45(ii)-(iii) is met, we call the heteroclinic cycle a *simple heteroclinic cycle*. Any connected component in the image of a heteroclinic cycle under the group Γ is called a *heteroclinic network*. A *homoclinic cycle* is a polycycle[15] that is equal to the group orbit $\langle \gamma \rangle \overline{h}$ of a heteroclinic trajectory $h(t)$ that connects the equilibrium p to γp for some $\gamma \in \Gamma$. The element $\gamma \in \Gamma$ is called the twist of the homoclinic cycle: the twist is well defined modulo the isotropy group of p. A homoclinic network is a connected component of the group orbit ΓH of a homoclinic cycle $H = \langle \gamma \rangle \overline{h}$. We may now also define robust and simple homoclinic cycles in an analogous fashion. The following lemma characterizes homoclinic cycles:

---

[14]Throughout this section, all indices are taken modulo ℓ

[15]Recall that any connected invariant set that is the union of finitely many heteroclinic cycles is called a polycycle
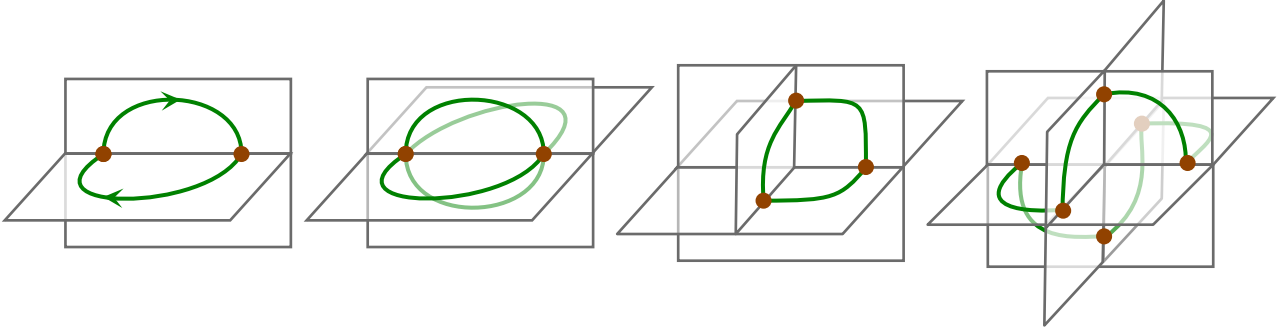
Figure 5.24: *Shown are the four different robust homoclinic cycles in $\mathbb{R}^3$: the symmetry groups are $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$ in the left two panels pictures and $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ in the right two panels. For each Γ, the two different homoclinic cycles correspond to the same homoclinic network.*

Lemma 5.2 ([187])**.** *Let h be a heteroclinic trajectory that connects the equilibria p and γp for some $\gamma \in \Gamma$, then $\Gamma \overline{h}$ is connected, and thus a homoclinic cycle, if and only if $\Gamma = \langle \gamma, \Gamma_p \rangle$.*

Proof. Note that $H_1 = \langle \gamma \rangle \overline{h}$ is trivially connected. Define inductively $H_{i+1} = \cup_{q \in H_i} \Gamma_q H_i$, where the union is over equilibria in H_i, and note that each H_i is connected. Since Γ is finite, this process terminates and yields the homoclinic cycle H. For Abelian groups Γ, the isotropy groups Γ_p are identical for all equilibria p in H, and thus $H = H_2$. For general Γ, the preceding construction shows that the isotropy groups of equilibria in H are conjugate to Γ_p via elements of $\langle \gamma, \Gamma_p \rangle$. Therefore, $\Gamma = \langle \gamma, \Gamma_p \rangle$. $\qquad\square$

We will now discuss the classification of simple homoclinic cycles in $\mathbb{R}^3$ and $\mathbb{R}^4$, the only dimensions for which a complete classification is known, and refer the reader to [382] for details and an historical overview. Homoclinic cycles for which $\mathrm{ind}_{S_j}(p_j) - \mathrm{ind}_{S_j}(p_{j+1}) = 0$ will be discussed in §5.5.3: such cycles are typically of codimension one and will break under equivariant perturbations.

In $\mathbb{R}^3$, two different homoclinic networks exist that arise from four different homoclinic cycles:

Theorem 5.66. *Figure 5.24 and the table*

ℓ	Γ
2	$\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$
4	$\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$
3	$\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$
6	$\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$

contain the classification of simple homoclinic cycles in $\mathbb{R}^3$: listed are the number ℓ of heteroclinic trajectories and the group Γ for which the cycle exists.

Figure 5.24 illustrates the four different homoclinic cycles in $\mathbb{R}^3$. Consider first the left two panels: the associated symmetry group is $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$, where $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$ are generated by $(x, y, z) \mapsto (-x, z, y)$ and $(x, y, z) \mapsto (x, \pm y, \pm z)$, respectively. Two equilibria lie on the x-axis, and the homoclinic cycle consists of two heteroclinic trajectories when $\gamma(x, y, z) = (-x, z, y)$ and of four trajectories when $\gamma(x, y, z) = (-x, z, -y)$. Both cycles generate the same homoclinic network. Such homoclinic networks were studied, for instance, in [16, 316]. An explicit system which contains these homoclinic networks is given by [346]

$$
\begin{aligned}
\dot{x} &= \nu x + z^2 - y^2 - x^3 + \beta x(y^2 + z^2), \\
\dot{y} &= y(\lambda + a y^2 + b z^2 + c x^2) + y x, \\
\dot{z} &= z(\lambda + a z^2 + b y^2 + c x^2) - z x
\end{aligned}
$$

where $\nu > 0$ is small and $\lambda \in (\lambda_H(\nu), \sqrt{\nu} + c\nu)$ for some $\lambda_H(\nu) = -\frac{1}{2}\nu + \mathrm{O}(\nu)$; the homoclinic cycle connects the equilibria $(\pm\sqrt{\nu}, 0, 0)$.

Next, consider the homoclinic cycles in the two rightmost panels in Figure 5.24, which again generate the same homoclinic network. The associated symmetry group is $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ where $\mathbb{Z}_3$ and $\mathbb{Z}_2^3$ are generated by $(x, y, z) \mapsto (y, z, x)$ and $(x, y, z) \mapsto (\pm x, \pm y, \pm z)$, respectively. An example of a $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$-equivariant system that contains this network is given by

$$
\begin{aligned}
\dot{x} &= x(\lambda + ax^2 + by^2 + cz^2), \\
\dot{y} &= y(\lambda + ay^2 + bz^2 + cx^2), \\
\dot{z} &= z(\lambda + az^2 + bx^2 + cy^2)
\end{aligned}
$$

with $a < 0$ and $\lambda > 0$: this ODE has a robust homoclinic cycle if, and only if, $b < a < c$ or $c < a < b$ [161]. To see how the robust homoclinic cycle arises, assume that $b < a < c$. The equilibria in the cycle are given by the group orbit of $p_1 = (\sqrt{-\lambda/a}, 0, 0)$: it is easy to check that $(\pm\sqrt{-\lambda/a}, 0, 0)$ are sinks and $(0, 0, \pm\sqrt{-\lambda/a})$ are saddles in the (x, z)-plane. Since $b < 0$, the region $\{0 < z < \sqrt{-\lambda/a}\}$ is forward invariant, and it is also easy to verify that the unstable manifold of $(0, 0, \sqrt{-\lambda/a})$ is bounded. This implies that $W^{\mathrm{u}}(p_1)$ goes to $(\pm\sqrt{-\lambda/a}, 0, 0)$. It follows from the results presented in §5.5.2 that the corresponding homoclinic cycle is asymptotically stable when $2a > b + c$.

We will now sketch the arguments that lead to Theorem 5.66 and refer to [382] for further details. Let $\gamma \in \Gamma$ be the twist so that $p_{j+1} = \gamma p_j$. Choose a basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ so that[16] $S_1 = \langle e_1, e_2 \rangle$, $S_2 = \gamma S_1 = \langle e_2, e_3 \rangle$ and $S_3 = \gamma S_2 = \langle \cos(t)e_2 + \sin(t)e_3, e_1 \rangle$. The angle t between consecutive equilibria p_j and p_{j+1} is called the *connecting angle*. The matrix A that represents γ in the basis $\{e_j\}$ is

$$
A = \begin{pmatrix}
0 & 0 & 1 \\
\alpha \sin(t) & \cos(t) & 0 \\
-\alpha \cos(t) & \sin(t) & 0
\end{pmatrix}
$$

with $\det A = \alpha$ and $\alpha = 1$ or $\alpha = -1$. We claim that the connecting angle is either $t = \pi/2$ or $t = \pi$. Indeed, let $R = \mathrm{diag}(1, 1, -1)$ be the matrix that fixes S_1, then $ARA^{-1} = \mathrm{diag}(-1, 1, 1)$ is a matrix that represents an element of Γ, and we conclude that the matrices

$$
\begin{pmatrix}
0 & 0 & 1 \\
\sin(t) & \cos(t) & 0 \\
-\cos(t) & \sin(t) & 0
\end{pmatrix}, \qquad
\begin{pmatrix}
0 & 0 & 1 \\
-\sin(t) & \cos(t) & 0 \\
\cos(t) & \sin(t) & 0
\end{pmatrix}
$$

both represent elements of Γ. The real parts of the complex eigenvalues of these matrices satisfy

$$
\frac{1}{2}\cos(t) - \frac{1}{2} = \cos(\pi a), \qquad \frac{1}{2}\cos(t) + \frac{1}{2} = \cos(\pi b)
$$

for some rational numbers a, b. Hence $\sin((a + b)\pi/2)\sin((a - b)\pi/2) = 1/2$, which is only possible when $a + b = a - b = 1/2$, so that $t = \pi/2$ or $t = \pi$. Finally, if $t = \pi$, the homoclinic cycle contains two equilibria. On the other hand, for $t = \pi/2$, it contains either three or six equilibria depending on the sign of $\det A$. This concludes the sketch of the arguments for Theorem 5.66.

Breaking the $\mathbb{Z}_2$-symmetry in $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$ creates a heteroclinic cycle with two equilibria from the homoclinic cycle. Likewise, breaking the $\mathbb{Z}_3$-symmetry in $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ creates a heteroclinic cycle with three or six equilibria near the homoclinic cycle. Hawker and Ashwin [172], see also [173], classify heteroclinic cycles in $\mathbb{Z}_3$-equivariant ODEs in $\mathbb{R}^3$; see Figure 5.25 for simple heteroclinic cycles in $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$- and $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$-equivariant ODEs in $\mathbb{R}^3$ (clearly, many more heteroclinic cycles exist).

Simple homoclinic cycles in $\mathbb{R}^4$ come in three types. Type B homoclinic cycles are contained in a three-dimensional invariant subspace.

Hypothesis 5.46 (Type A,B,C homoclinic cycles). *We distinguish the following configurations for simple heteroclinic trajectories in $\mathbb{R}^4$:*

---

[16]We use the notation $\langle\{u_j\}\rangle$ for the vector space spanned by the vectors u_j in $\mathbb{R}^3$
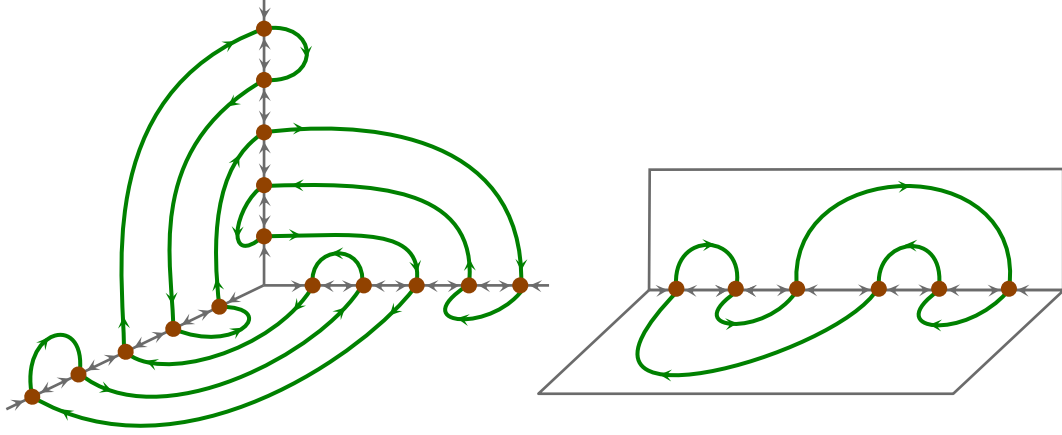
Figure 5.25: *Left panel: a heteroclinic cycle in a $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$-equivariant ODE in $\mathbb{R}^3$, where the action of the symmetry is defined by reflections in the coordinate planes and a permutation of the coordinates. Right panel: a heteroclinic cycle in a $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$-equivariant ODE in $\mathbb{R}^3$, where the action of the symmetry is defined by reflection in two coordinate planes and a permutation of two coordinates.*

   (i) *Type A: $S_j + S_{j+1}$ is not a fixed point space.*

  (ii) *Type B: $S_j + S_{j+1}$ is a fixed point space that contains the homoclinic cycle.*

 (iii) *Type C: $S_j + S_{j+1}$ is a fixed point space that does not contains the homoclinic cycle.*

Simple homoclinic cycles of type B and C in $\mathbb{R}^4$ were investigated in [239], while Sottocornola [380–382] studied type A simple homoclinic cycles in $\mathbb{R}^4$. To state the classification result, we need the definition of structure angles that was introduced in [380–382]. Recall that the twist is the group element $\gamma \in \Gamma$ for which $p_{j+1} = \gamma p_j$. As for homoclinic cycles in $\mathbb{R}^3$, we choose a basis $\{e_1, e_2, e_3, e_4\}$ in which S_1, $S_2 = \gamma S_1$, and $S_3 = \gamma S_2$ are given by

$$S_1 = \langle e_1, e_2 \rangle, \qquad S_2 = \langle e_2, e_3 \rangle, \qquad S_3 = \langle \cos(t)e_2 + \sin(t)e_3, \cos(s)e_1 + \sin(s)e_4 \rangle$$

for *structure angles* s, t, which can be thought of as the tilt and connecting angles of the hyperplane. The matrix that represents the twist γ is given by

$$A_{t,s}^\alpha = \begin{pmatrix} 0 & 0 & \cos(s) & -\sin(s) \\ \alpha\sin(t) & \cos(t) & 0 & 0 \\ -\alpha\cos(t) & \sin(t) & 0 & 0 \\ 0 & 0 & \sin(s) & \cos(s) \end{pmatrix}, \tag{5.32}$$

where $\det(A) = \alpha$ with $\alpha \in \{\pm\}$. The symmetry group Γ contains the matrix $R_1 = \mathrm{diag}(1, 1, -1, -1)$ since the elements of Σ_1 are the identity on S_1 and -1 on $\langle e_3 \rangle$. For type B and C homoclinic cycles, Γ also contains $R_2 = \mathrm{diag}(1, 1, 1, -1)$.

Theorem 5.67 ([239, 380–382]). *The classification of simple homoclinic cycles in $\mathbb{R}^4$ is in Table 1 where we list the homoclinic network, the generators of Γ, and the number of heteroclinic trajectories of the homoclinic cycle $\langle \gamma \rangle \overline{h}$ for possible twists γ, where h is a heteroclinic trajectory in $S = \mathrm{Fix}(R_1)$ that connects equilibria in $\gamma^{-1}S \cap S$ to $S \cap \gamma S$.*

We briefly outline the strategy for proving the preceding theorem. First, the simple homoclinic cycles we found in $\mathbb{R}^3$ occur in $\mathbb{R}^4$ as robust type A or type B homoclinic cycles, depending on whether $\mathbb{R}^3$ is a fixed point space or not. We note that they can also occur as homoclinic cycles that are not simple. To see this, consider the action of $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$ on $\mathbb{R}^4$ given by the linear maps $(x, y, z, u) \mapsto (-x, z, y, -u)$, which generates $\mathbb{Z}_2$, and $(x, y, z, u) \mapsto (x, \pm y, \pm z, u)$, which generate $\mathbb{Z}_2^2$. The subspaces $S_1 = \{y = 0\}$ and $S_2 = \{z = 0\}$ are three-dimensional fixed-point spaces. Suppose that a differential equation with two hyperbolic

type	homoclinic network	generators	twist	cycle length
A with $\Gamma \subset SO(4)$	$\mathcal{H}_2^{A,+}$	$A^1_{\pi,0}, R_1$	$A^1_{\pi,0}$	2
			$R_1 A^1_{\pi,0}$	4
	$\mathcal{H}_6^{A,+}$	$A^1_{\frac{\pi}{2},0}, R_1$	$A^1_{\frac{\pi}{2},0}$	3
			$R_1 A^1_{\frac{\pi}{2},0}$	6
	$\mathcal{H}_8^{A,+}$	$A^1_{\frac{\pi}{2},\frac{\pi}{2}}, R_1$	$A^1_{\frac{\pi}{2},\frac{\pi}{2}}, R_1 A^1_{\frac{\pi}{2},\frac{\pi}{2}}$	8
	$\mathcal{H}_{48}^{A,+}$	$A^1_{\frac{\pi}{4},\frac{\pi}{4}}, R_1$	$A^1_{\frac{\pi}{4},\frac{\pi}{4}}$	12
			$R_1 A^1_{\frac{\pi}{4},\frac{\pi}{4}}$	24
A with $\Gamma \not\subset SO(4)$	$\mathcal{H}_{2k^2}^{A,-}, k \geq 1$	$A^{-1}_{\frac{\pi}{k},\frac{\pi}{k}}, R_1$	$A^{-1}_{\frac{\pi}{k},\frac{\pi}{k}}$	$2k$
			$R_1 A^{-1}_{\frac{\pi}{k},\frac{\pi}{k}}$	4
B	$\mathcal{H}_2^{B}$	$A^1_{\pi,0}, R_1, R_2$	$A^1_{\pi,0}, R_2 A^1_{\pi,0}$	2
			$R_1 A^1_{\pi,0}, R_1 R_2 A^1_{\pi,0}$	4
	$\mathcal{H}_6^{B}$	$A^1_{\frac{\pi}{2},0}, R_1, R_2$	$A^1_{\frac{\pi}{2},0}, R_2 A^1_{\frac{\pi}{2},0}$	3
			$R_1 A^1_{\frac{\pi}{2},0}, R_1 R_2 A^1_{\frac{\pi}{2},0}$	6
C	$\mathcal{H}_8^{C}$	$A^1_{\frac{\pi}{2},\frac{\pi}{2}}, R_1, R_2$	$A^1_{\frac{\pi}{2},\frac{\pi}{2}}, R_1 A^1_{\frac{\pi}{2},\frac{\pi}{2}}$	8
			$R_2 A^1_{\frac{\pi}{2},\frac{\pi}{2}}, R_1 R_2 A^1_{\frac{\pi}{2},\frac{\pi}{2}}$	4

Table 1: *The classification of simple homoclinic cycles in $\mathbb{R}^4$ is shown: the columns contain the type (including, for type A cycles, a sign that indicates whether or not $\Gamma \subset SO(4)$ and the number of equilibria it contains), the generators of Γ, and the number of heteroclinic trajectories of the homoclinic cycle $\langle \gamma \rangle \overline{h}$.*

equilibria $p_1 = (x_*, 0, 0, u_*)$ and $p_2 = (-x_*, 0, 0, -u_*)$ is given and assume further that the equilibria have different indices when restricting the flow to the fixed-point space S_1 so that $\dim[W^u(p_1) \cap S_1] = 2$ and $\dim[W^u(p_2) \cap S_1] = 1$. If the manifolds $W^u(p_1) \cap S_1$ and $W^u(p_2) \cap S_1$ have a transverse intersection, its Γ image is a robust homoclinic network. The other three-dimensional simple homoclinic cycles can similarly provide models of robust homoclinic cycles in $\mathbb{R}^4$ that are not simple. Next, consider any other simple homoclinic cycle of Type A in $\mathbb{R}^4$. If $\Gamma \subset SO(4)$, Sottocornola proved that the structure angles are multiples of $\pi/4$, and a direct analysis leads to the different simple homoclinic cycles listed in Table 1, where the homoclinic cycles of length 12 and 24 give the same homoclinic network. The structure angles for the remaining type A cycles that have $\Gamma \not\subset SO(4)$ turn out to satisfy either $t = s$ or $t + s = \pi$. In the second case, there is an infinite family of simple homoclinic cycles, all with 4 equilibria, that are parameterized by $t = \pi/k$ with $k > 1$. In the first case, there is likewise an infinite family of simple homoclinic cycles with $l = 2k$ equilibria for $t = \pi/k$ with $k > 1$: each cycle gives the same network. For the existence of ODEs that admit these homoclinic cycles, Sottocornola relies on a result by Ashwin and Montaldi [26].

In passing, we note that the classification of simple homoclinic cycles of type B and C in $\mathbb{R}^4$ extends to a classification of those simple heteroclinic cycles that intersect each connected component of $[S_{j-1} \cap S_j] \setminus \{0\}$ in at most one point [239]. It turns out that there are four simple heteroclinic networks of type B and three of type C which satisfy the preceding condition: in [239], these networks are denoted by $B_1^+, B_2^+, B_1^-, B_3^-$ and C_1^-, C_2^-, C_4^-, where the subscript m is the number of different group orbits of equilibria ($m = 1$ corresponds to homoclinic networks) and the superscript $\pm$ indicates whether or not $-\mathrm{id} \in \Gamma$ (the minus sign means that $-\mathrm{id} \in \Gamma$).

Finally, we briefly discuss the construction of homoclinic and heteroclinic cycles in $\mathbb{R}^n$ with symmetry groups $\Gamma = \mathbb{Z}_n \ltimes \mathbb{Z}_2^n$: details can be found in [132], and the reference [108] contains further information on their appearance in local bifurcations. A very useful tool is the invariant-sphere theorem which we explain first. Consider differential equations on $\mathbb{R}^n$ of the form

$$\dot{u} = \lambda u + Q(u), \tag{5.33}$$

where $\lambda > 0$ and Q is a homogeneous polynomial of degree d that satisfies $\langle Q(x), x \rangle < 0$ for all $x \in \mathbb{S}^{n-1}$: we refer to such maps Q as *contracting*. We can now split the vector field into its spherical and radial vector components so that

$$
\begin{aligned}
\dot{x} &= r^{d-1} \left(Q(x) - \langle Q(x), x \rangle \right), \\
\dot{r} &= \lambda r + r^d \langle Q(x), x \rangle.
\end{aligned}
$$

The following invariant-sphere theorem due to Field shows that the radial part is, in a certain sense, irrelevant.

Theorem 5.68 ([131]). *The differential equation (5.33) admits a unique invariant topological manifold S^{n-1} which is homeomorphic to the $(n-1)$-sphere and attracts every point except the origin. The flow of (5.33) restricted to S^{n-1} is topologically equivalent to the phase differential equation $x' = Q(x) - \langle Q(x), x \rangle$ on $\mathbb{S}^{n-1}$.*

To illustrate its use, consider the family

$$
\begin{cases}
\dot{x}_1 &= x_1 + ax_1 r^2 + bx_1 x_2^2 + cx_1 x_3^2 + dx_1 x_4^2, \\
\dot{x}_2 &= x_2 + ax_2 r^2 + bx_2 x_3^2 + cx_2 x_4^2 + dx_2 x_1^2, \\
\dot{x}_3 &= x_3 + ax_3 r^2 + bx_3 x_4^2 + cx_3 x_1^2 + dx_3 x_2^2, \\
\dot{x}_4 &= x_4 + ax_4 r^2 + bx_4 x_1^2 + cx_4 x_2^2 + dx_4 x_3^2,
\end{cases}
\tag{5.34}
$$

of $\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4$-equivariant ODEs, where $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. The action of the symmetry group $\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4$ is generated by $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$ and $(x_1, x_2, x_3, x_4) \mapsto (x_4, x_1, x_2, x_3)$. Theorem 5.68 can now be applied to yield an attracting invariant topological sphere provided $a < 0$, $2a + c < 0$, $4a + b + d < 0$, and $4a + b + c + d < 0$ [425]. An analysis of the flow inside the invariant planes shows that a homoclinic cycle of type C exists when $a < 0$ and $bd < 0$.

More generally, if the differential equation (5.33) is $\mathbb{Z}_n \ltimes \mathbb{Z}_2^n$-equivariant for an appropriate contracting homogeneous polynomial Q, then the invariant-sphere theorem shows that it suffices to consider the associated phase differential equation for $x \in \mathbb{S}^{n-1}$. Define $\Lambda_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{S}^{n-1}; x_1, \ldots, x_n \geq 0\}$, then Λ_{n-1} is left invariant by the action of $\mathbb{Z}_n \subset \Gamma$, we have $\mathbb{S}^{n-1} = \mathbb{Z}_2^n \Lambda_{n-1}$, and $\gamma \operatorname{int}(\Lambda_{n-1}) \cap \Lambda_{n-1} = \emptyset$ for each $\gamma \in Z_2^n$ with $\gamma \neq \operatorname{id}$. A homoclinic network is thus given by a heteroclinic cycle for the phase differential equation on Λ_{n-1}. A given homoclinic cycle is called a k-face homoclinic cycle if the heteroclinic trajectories for the phase differential equation are on the k-faces of the simplex Λ_{n-1}.

We end this section with a brief discussion of more general heteroclinic cycles that contain infinitely many connecting trajectories. For instance, the definition of heteroclinic cycle given in [237], in which unstable manifolds of groups orbits of equilibria connect to stable manifolds of group orbits of equilibria, allows for manifolds of heteroclinic trajectories. Ashwin and Field [25] gave a definition of a heteroclinic network which, in the context of trajectories that connect hyperbolic equilibria, consists of the following ingredients. A compact invariant set Σ which consists of finitely many hyperbolic equilibria $p_1, \ldots, p_k$ and trajectories connecting them is a heteroclinic network if

(i) Σ is indecomposable: for every $\varepsilon > 0$ and $T > 0$, any two points in Σ can be connected by an (ε, T)-pseudo trajectory[17];

(ii) Σ has finite depth: if Λ_{i+1} denotes the recurrent set of the flow restricted to Λ_i, then there is a finite sequence $\Lambda_0 = \Sigma, \ldots, \Lambda_N = \{p_1, \ldots, p_k\}$ (the minimal such N is called the depth of Σ; note the similarity with the Birkhoff center, which is defined analogously for nonwandering sets).

Ashwin and Field further generalize the notion of heteroclinic network to sets that contain periodic solutions or other recurrent invariant sets instead of equilibria. The sometimes intricate dynamics near invariant polycycles and heteroclinic networks is discussed in various papers: see, or instance, [7, 133] for heteroclinic networks in $\mathbb{R}^4$ that contain equilibria with complex conjugate eigenvalues and produce suspended horseshoes. Some other key references are [8, 97, 314].

---

[17]An (ε, T)-pseudo trajectory from x to y is a finite set of points $x_0 = x, \ldots, x_n = y$ and times $t_i > T$ with $\|x_{i+1} - \varphi_{t_i}(x_i)\| < \varepsilon$

5.5.2 Asymptotic stability of heteroclinic networks

Consider a robust heteroclinic cycle for which Hypothesis 5.45(ii) is met: the ω-limit point p_{j+1} of the heteroclinic trajectory h_j is an attracting equilibrium inside the fixed-point space $S_j = \text{Fix}(\Sigma_j)$ of the isotropy group Σ_j of the heteroclinic orbit $h_j(t)$. The geometry given by the subspaces S_j allows us to divide the spectrum of $f_u(p_j)$ into four disjoint sets:

  (i) Radial eigenvalues, whose generalized eigenspaces lie in $V_j^r = S_{j-1} \cap S_j$;

 (ii) Contracting eigenvalues, whose generalized eigenspaces lie in $V_j^c = S_{j-1} \ominus V_j^r$;

(iii) Transverse eigenvalues, whose generalized eigenspaces lie in $V_j^t = (S_{j-1} + S_j)^\perp$;

 (iv) Expanding eigenvalues, whose generalized eigenspaces lie in $V_j^e = S_j \ominus V_j^r$.

We remark that not all eigenvalues in V_j^e need to have positive real part. Define

$$
\begin{aligned}
r_j &= \min\{\text{Re}\,\lambda;\ \lambda \text{ is an eigenvalue of } f_u(p_j)|_{V_j^r}\}, \\
c_j &= \min\{\text{Re}\,\lambda;\ \lambda \text{ is an eigenvalue of } f_u(p_j)|_{V_j^c}\}, \\
t_j &= \max\{\text{Re}\,\lambda;\ \lambda \text{ is an eigenvalue of } f_u(p_j)|_{V_j^t}\}, \\
e_j &= \max\{\text{Re}\,\lambda;\ \lambda \text{ is an eigenvalue of } f_u(p_j)|_{V_j^e}\}.
\end{aligned}
$$

If $S_{j-1} + S_j = \mathbb{R}^n$, that is, when there are no transverse directions, we set $t_j = -\infty$. Note that the eigenspaces corresponding to c_j, t_j, e_{j+1}, and t_{j+1} all lie in $S_j^\perp$. The following condition is the analogue of type A, which we defined in Hypothesis 5.46(i) for systems in $\mathbb{R}^4$, in higher-dimensional systems.

Hypothesis 5.47. *The eigenspaces corresponding to $c_j, t_j, e_{j+1}, t_{j+1}$ lie in the same Σ_j isotypic component.*

Theorem 5.69 ([237])**.** *Suppose that Γ is a finite group that acts linearly, and assume that $\dot{u} = f(u)$ is a Γ-equivariant equation on $\mathbb{R}^n$ that admits a heteroclinic network H which satisfies Hypothesis 5.45(ii). Assume, furthermore, that there are C^1 linearizing coordinates near the equilibria in H. Write $C_j = |c_j/e_j|$ and $T_j = |t_j/e_j|$, then the network H is asymptotically stable if $\prod_{i=1}^k \min\{C_j, 1-T_j\} > 1$. If Hypothesis 5.47 is met, this spectral condition is generically necessary and sufficient for asymptotic stability.*

Thus, radial eigenvalues play no role in the spectral conditions for asymptotic stability. Note that for homoclinic networks, where the spectral bounds are independent of j, the stability condition becomes $-c > e$. We shall now work out the asymptotic stability conditions for some simple heteroclinic networks of type B and C in $\mathbb{R}^4$.

Theorem 5.70 ([239])**.** *Let $\dot{u} = f(u)$ be a Γ-equivariant equation on $\mathbb{R}^4$ that admits a simple heteroclinic network of type B or C. Assume that the heteroclinic network intersects each connected component of $V_j^r \setminus \{0\}$ in at most one point. Generically, the conditions listed in Table 2 are then necessary and sufficient for asymptotic stability.*

We now attempt to make the conditions in Table 2 more transparent. The proof of Theorem 5.70 makes use of transition matrices[134, 179], which we now introduce. Near each equilibrium p_j, take coordinates $x = (x^r, x^c, x^t, x^e)$ corresponding to the splitting $\mathbb{R}^4 = V_j^r \oplus V_j^c \oplus V_j^t \oplus V_j^e$. Pick the incoming and outgoing cross sections[18] $\Sigma_j^{\text{in}} = \{x;\ |x^t|, |x^e| \leq 1, |x^r|^2 + |x^c|^2 = 1\}$ and $\Sigma_j^{\text{out}} = \{x;\ |x^r|, |x^c|, |x^t| \leq 1, x^e = 1\}$, define transition maps $\Pi_j^{\text{loc}} : \Sigma_j^{\text{in}} \to \Sigma_j^{\text{out}}$ and $\Pi_j^{\text{far}} : \Sigma_j^{\text{out}} \to \Sigma_j^{\text{in}}$, and write $\Pi_j = \Pi_j^{\text{far}} \circ \Pi_j^{\text{loc}}$. The first-return map is therefore given by the composition $\Pi_k \circ \cdots \circ \Pi_1$. In linearizing coordinates near p_j, we have

$$
\Pi_j^{\text{loc}}(x^r, x^c, x^t, x^e) = (x^r(x^e)^{-r_j/e_j}, x^c(x^e)^{-c_j/e_j}, x^t(x^e)^{-t_j/e_j}, 1).
$$

---

[18]We abuse notation by denoting these section by $\Sigma_j^{\text{in,out}}$ which are *not* the isotropy groups Σ_j used earlier

type	Γ	stability condition
B_1^+	$\mathbb{Z}_2 \ltimes \mathbb{Z}_2^3$	$C_1 > 1$
B_2^+	$\mathbb{Z}_2^3$	$C_1 C_2 > 1$
B_1^-	$\mathbb{Z}_3 \ltimes \mathbb{Z}_2^4$	$C_1 > 1$
B_3^-	$\mathbb{Z}_2^4$	$C_1 C_2 C_3 > 1$
C_1^-	$\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4$	$C_1 - T_1 > 1$
C_2^-	$\mathbb{Z}_2 \ltimes \mathbb{Z}_2^4$	$C_1 + C_2 + T_1 T_2 > \min\{2, 1 + C_1 C_2\}$
C_4^-	$\mathbb{Z}_2^4$	$C_1 C_3 + C_2 C_4 + T_1 T_2 C_3 + T_2 T_3 C_4 + T_3 T_4 C_1 + T_4 T_1 C_2 +$ $T_1 T_2 T_3 T_4 + T_1 T_2 \min\{2, 1 + C_1 C_2 C_3 C_4\}$

Table 2: *Necessary and sufficient conditions for asymptotic stability of simple heteroclinic networks of type B and C in $\mathbb{R}^4$, where $C_j = |c_j/e_j|$ and $T_j = |t_j/e_j|$.*

For x near $h_{j-1} \cap \Sigma_j^{\text{in}}$, the radii $|x^r|^2$ and $|x^c|^2$ will generically be nonzero, which suggests that the (x^t, x^e) components are more important. Expanding the transition maps Π_j^{far} in a Taylor series, the (x^t, x^e) components Π_j^t, Π_j^e of Π_j are, at lowest order, given by $\Pi_j^t(x) = (a_j(x^e)^{-c_j/e_j} + b_j x^t (x^e)^{-t_j/e_j})$ and $\Pi_j^e(x) = (c_j(x^e)^{-c_j/e_j} + d_j x^t (x^e)^{-t_j/e_j})$, respectively. As the (x^r, x^c) coordinates are absent, we can consider the lowest-order terms of (Π_j^t, Π_j^e) as a map $\pi_j : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\pi_j(x^t, x^e) = \left(a_j(x^e)^{-c_j/e_j} + b_j x^t (x^e)^{-t_j/e_j}, c_j(x^e)^{-c_j/e_j} + d_j x^t (x^e)^{-t_j/e_j} \right).$$

For type B cycles, we have $a = d = 0$, while we have $b = c = 0$ for type C cycles. Hence, in both cases, we can write $\pi_j(x^t, x^e) = (E(x^t)^{\alpha_j}(x^e)^{\beta_j}, F(x^t)^{\gamma_j}(x^e)^{\delta_j})$ for suitable coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j$. The transition matrix is now defined as the matrix $M_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}$, and asymptotic stability is deduced from the matrix $M = M_k \cdots M_2 M_1$: the network is asymptotically stable if the row sums of iterates M^l diverge to infinity as $l \to \infty$. For type C cycles, this conditions becomes trace $M > \min\{2, 1 + \det M\}$, which translates into the specific conditions stated in Theorem 5.70.

An interesting phenomenon occurs for robust heteroclinic networks that contain equilibria of different indices: although the basin of attraction of such networks may not necessarily contain an open neighborhood of the network, it may have full Lebesgue measure at points of the heteroclinic network. Such a heteroclinic network is called *essentially asymptotically stable* [51, 273].

An example that exhibits essentially asymptotically stable networks can be constructed as follows. Consider a differential equation in $\mathbb{R}^4$ that is $\mathbb{Z}_2^2$-equivariant under the action generated by $(u_1, u_2, u_3, u_4) \mapsto (u_1, -u_2, -u_3, u_4)$ and $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, -u_3, -u_4)$. Consider a heteroclinic cycle with equilibria $p_1 = (-1, 0, 0, 0)$ and $p_2 = (1, 0, 0, 0)$, and assume that both $f_u(p_1)$ and $f_u(p_2)$ have four distinct real eigenvalues with $\text{ind}(p_1) = 2$ and $\text{ind}(p_2) = 1$. Suppose that, inside the fixed-point plane $\{(u_3, u_4) = 0\}$, there are two heteroclinic trajectories that are related by symmetry and connect p_1 to p_2: we assume that these trajectories form an attracting normally hyperbolic circle inside $\{(u_3, u_4) = 0\}$. Similarly, we assume that there are two symmetry-related heteroclinic trajectories inside $\{(u_2, u_3) = 0\}$ that connect p_2 to p_1 and again form an attracting normally hyperbolic circle. If the eigenvalues are such that the strong unstable direction at p_1 and the leading stable direction at p_2 both lie in $\{(u_3, u_4) = 0\}$, the $\mathbb{Z}_2^2$-invariant heteroclinic network can be essentially asymptotically stable. Recall the definitions of radial, transversal, contracting and expanding eigenvalues from the beginning of this section.

Theorem 5.71 ([273]). *Consider a $\mathbb{Z}_2^2$-equivariant equation on $\mathbb{R}^4$ that admits a robust heteroclinic network as described above. If the open conditions*

$$c_1 c_2 > e_1 e_2, \qquad -c_2(e_1 - t_1) > e_1 e_2, \qquad t_2 < c_2$$

on the transversal, contracting and expanding eigenvalues are satisfied, then the heteroclinic network is, generically, essentially asymptotically stable.
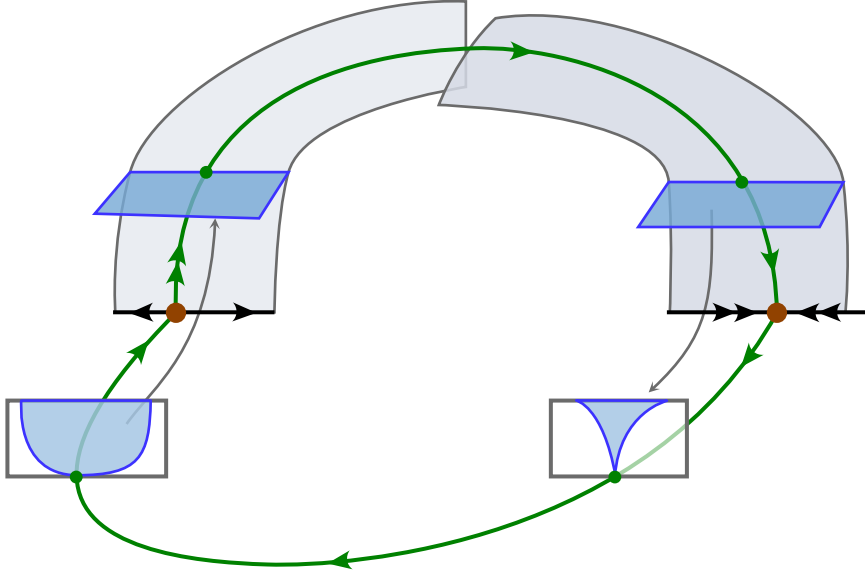
Figure 5.26: *The geometry behind a non-asymptotically stable heteroclinic network is illustrated. Only two heteroclinic trajectories are drawn. Starting near the upper heteroclinic trajectory, all points converge to the heteroclinic network, whereas a small part of points near the lower heteroclinic trajectory escape.*

Note that the heteroclinic network can be constructed so that is is contained in an attracting normally hyperbolic three-sphere $\{\|u\| = 1\}$. For this to hold, the additional eigenvalue conditions $r_1 < c_1$ and $r_2 < \min\{t_2, c_2\}$ must be satisfied. The flow near the heteroclinic network restricted to the three-sphere is illustrated in Figure 5.26. Further information on the stability properties of heteroclinic networks can be found in [238].

Lauterbach and Roberts [252] have shown how such essentially asymptotic robust heteroclinic networks appear in symmetry-breaking bifurcations of systems with spherical symmetry. Starting point in their work is a normally hyperbolic invariant three-sphere for an $SO(3)$-equivariant system for which the effects of forced symmetry-breaking are considered; see also §5.5.4.

Kirk and Silber gave another example of an essentially asymptotically stable heteroclinic network. It involves an equation on $\mathbb{R}^4$ that is equivariant under the action of $\mathbb{Z}_2^4$ generated by reflections of the coordinate axes. Assume that the ith coordinate axis contains the hyperbolic equilibrium p_i for each $i = 1, \ldots, 4$ and that there is a robust heteroclinic cycle inside the (u_1, u_2, u_3)-space that connects the equilibria p_1, p_2 and p_3. We write H_{123} for the heteroclinic network in (u_1, u_2, u_3)-space that is the image under the action of the group. Likewise, assume the existence of a robust heteroclinic cycle that connects the equilibria p_1, p_2 and p_4 and of a corresponding heteroclinic network H_{124} in (u_1, u_2, u_4)-space. The connecting trajectories from p_i to p_j lie in the (u_i, u_j)-plane, and the equilibria p_1, p_3, p_4 have one-dimensional unstable manifolds, while the equilibrium p_2 has a two-dimensional unstable manifold. The following hypothesis captures the asymptotic stability of H_{123} and H_{124} restricted to the (u_1, u_2, u_3)-space and (u_1, u_2, u_4)-space respectively; see Theorem 5.70. Denote by c_{ij} the contracting eigenvalue of $f_u(p_j)$ that corresponds to the connecting trajectory from p_i to p_j and by e_{jk} the expanding eigenvalue of $f_u(p_j)$ restricted to the (u_j, u_k)-plane, which corresponds to the trajectory from p_j to p_k.

Hypothesis 5.48 (Asymptotic stability of three-dimensional heteroclinic networks). *The heteroclinic networks H_{123} and H_{124} are asymptotically stable within the enclosing three-dimensional spaces: we have*

$$-\frac{c_{12}c_{23}c_{31}}{e_{23}e_{31}e_{12}} > 1, \qquad -\frac{c_{12}c_{24}c_{41}}{e_{24}e_{41}e_{12}} > 1.$$

Theorem 5.72 ([216]). *Let $\dot{u} = f(u)$ be a Z_2^4-equivariant ODE with a robust heteroclinic network as above, and assume Hypothesis 5.48 is met, then the robust heteroclinic network is, generically, essentially asymptotically stable.*

Postlethwaite and Dawes [314] considered heteroclinic networks in $\mathbb{Z}_6 \ltimes \mathbb{Z}_2^6$-equivariant differential equations in $\mathbb{R}^6$ for which the equilibria lie on a single group orbit. They established the existence of trajectories that follow different heteroclinic trajectories in an irregular order, while converging to the heteroclinic network. Finally, we remark that homoclinic networks can also be essentially asymptotically stable [86, 113].

5.5.3 Bifurcations from heteroclinic cycles

In this section, we consider bifurcations from both robust and non-robust heteroclinic cycles.

First, we consider bifurcations from robust cycles: more specifically, we consider so-called *resonant bifurcations* where, by definition, the eigenvalue conditions for asymptotic stability become violated; see also §5.1.5. Certain resonant bifurcations from homoclinic cycles with real leading eigenvalues have been considered in [87, 351] and, in more generality, in [114]; a specific bifurcation that involves complex leading eigenvalues can be found in [315]. Here, we focus instead on *transverse bifurcations* where a transverse eigenvalue crosses the imaginary axis. Obviously, other types of bifurcations exist as well, but we are not aware of any systematic studies of bifurcations from robust heteroclinic cycles.

To outline the available results for transverse bifurcations of simple homoclinic cycles, we consider a Γ-equivariant system in $\mathbb{R}^4$ and assume that H is a Γ-invariant simple homoclinic cycle consisting of equilibria $p_1, \ldots, p_\ell$. Recall from §5.5.2 that the linearization about each of the equilibria in a simple homoclinic cycle has four real eigenvalues r, c, e, t, and we denote the corresponding eigenspaces at p_j by $V_j^r, \ldots, V_j^t$. A transverse bifurcation occurs when the transverse eigenvalue t crosses through zero in V_j^t: the presence of symmetry implies that this is a pitchfork bifurcation.

Hypothesis 5.49 (Transverse bifurcation). *Assume that an asymptotically stable homoclinic cycle loses its stability through a supercritical pitchfork bifurcation in the transverse directions:*

(i) *The pitchfork bifurcation is supercritical: on a one-dimensional center manifold, the differential equation has the normal form $\dot{z} = \mu z + h(z, \mu)$ with $h(0, \mu) = h_z(0, \mu) = 0$ and $h_{zzz}(0, \mu) < 0$.*

(ii) *At $\mu = 0$, the contracting and expanding eigenvalue satisfy $-c > e$.*

We then have the following bifurcation result.

Theorem 5.73 ([85]). *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of ODEs on $\mathbb{R}^4$, which is equivariant under the linear action of a finite group Γ. We assume that the equation for $\mu = 0$ admits a simple Γ-invariant homoclinic cycle H whose equilibria undergo a pitchfork bifurcation so that Hypothesis 5.49 is met. Depending on the type of cycle (as defined in Hypothesis 5.46), we then have the following cases:*

(i) *If H is of type A, then, generically, there is a unique branch of limit cycles that bifurcate from the homoclinic cycle. Furthermore, there is a constant $d > 0$, which depends only on the flow at $\mu = 0$, so that each periodic cycle is asymptotically stable for $d < 1$ and unstable for $d > 1$.*

(ii) *If H is of type B, then there is a supercritical pitchfork bifurcation to two asymptotically stable homoclinic cycles.*

(iii) *If H is of type C, then there is a supercritical pitchfork bifurcation to four asymptotically stable homoclinic cycles.*

We will now briefly discuss the geometry for homoclinic cycles H of type B. In this case, H lies in a three-dimensional fixed-point space $\mathrm{Fix}(\tau)$ for some $\tau \in \Gamma$. If h_j is the heteroclinic trajectory of the homoclinic cycle inside the fixed-point space $S_j = \mathrm{Fix}(\xi_j)$ that connects the equilibrium p_j to p_{j+1}, then $\mathrm{Fix}(\xi_j \tau)$ is a three-dimensional fixed-point space that contains S_j and the transverse directions to S_j. Furthermore, the equilibria p_j', $\tau p_j'$, p_{j+1}', and $\tau p_{j+1}'$ that are created in the pitchfork bifurcation are contained in $\mathrm{Fix}(\xi_j \tau)$: in fact, p_j' and $\tau p_j'$ are saddles, while p_{j+1}' and $\tau p_{j+1}'$ are sinks in $\mathrm{Fix}(\xi_j \tau)$. A continuity argument establishes

the existence of heteroclinic trajectories from p_j' to p_{j+1}' and, through the action of τ, from $\tau p_j'$ to $\tau p_{j+1}'$, which are all robust homoclinic cycles. Checking the eigenvalue conditions at the equilibrium p_j', where the eigenvalues are close to those about p_j, proves asymptotic stability.

Transverse bifurcations are among the bifurcation scenarios that occur in models of magnetic dynamos in rotating Bénard convection; see [86]. In [113], transverse bifurcations from an asymptotically stable homoclinic network in five-dimensional differential equations were considered, which lead to an essentially asymptotically stable homoclinic network (see §5.5.2 for this notion) that are contained in an asymptotically stable heteroclinic network.

Our next example of bifurcations from robust cycles is an inclination-flip bifurcation of simple type-A homoclinic cycle in $\mathbb{R}^4$ that has been studied by Worfolk [425]. Consider the action of $\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4$ generated by $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$ and $(x_1, x_2, x_3, x_4) \mapsto (x_4, x_1, x_2, x_3)$. Let Γ be the subgroup of index two consisting of those elements that preserve orientation.

Theorem 5.74 ([425]). *Let $\dot{u} = f(u, \mu)$ with $u = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ be a one-parameter family on $\mathbb{R}^4$ which is equivariant under the action of Γ as stated above. Suppose that the family has a robust homoclinic cycle and that, at $\mu = 0$, the stable manifold $W^{\mathrm{s}}(p_{i+1}, 0)$ is in an inclination-flip configuration so that Hypothesis 2.12(i) is violated. For a generic family of equivariant ODEs for which Hypotheses 2.9, 2.10, 2.11(ii), and 2.12(ii),(ii),(iv) are met, there exist values of μ arbitrarily close to zero for which there exist hyperbolic invariant chaotic sets.*

The genericity conditions referred to in the preceding theorem involve smooth linearizability assumptions. We remark that local bifurcations in Γ-equivariant ODEs have been studied in [163]: starting point is the system

$$\begin{cases} \dot{x}_1 &= x_1 + ax_1r^2 + bx_1x_2^2 + cx_1x_3^2 + dx_1x_4^2 + ex_2x_3x_4, \\ \dot{x}_2 &= x_2 + ax_2r^2 + bx_2x_3^2 + cx_2x_4^2 + dx_2x_1^2 - ex_1x_3x_4, \\ \dot{x}_3 &= x_3 + ax_3r^2 + bx_3x_4^2 + cx_3x_1^2 + dx_3x_2^2 + ex_1x_2x_4, \\ \dot{x}_4 &= x_4 + ax_4r^2 + bx_4x_1^2 + cx_4x_2^2 + dx_4x_3^2 - ex_1x_2x_3, \end{cases} \tag{5.35}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, see also (5.34), and it is illustrated how complicated dynamics, including Shil'nikov homoclinic loops, arise in unfoldings. The inclination-flip homoclinic cycle from Theorem 5.74 may possibly occur in the preceding cubic normal form.

Having discussed bifurcations from robust homoclinic cycles, we focus now on bifurcations from codimension-one homoclinic cycles. An example of such a bifurcation is the Takens–Bogdanov bifurcation in $\mathbb{R}^4$ with $\mathbb{D}_3$-symmetry that has been studied by Matthies [269]. The action of $\mathbb{D}_3$ on $\mathbb{R}^4 \sim \mathbb{C}^2$ is generated by $(u_1, u_2) \mapsto (e^{2\pi i/3}u_1, e^{2\pi i/3}u_2)$ and $(u_1, u_2) \mapsto (\overline{u_1}, \overline{u_2})$, and we assume that the $\mathbb{D}_3$ invariant equilibrium at the origin undergoes the equivariant analogue of the Takens–Bogdanov bifurcation, where the linearization at the origin is given by $\dot{u}_1 = u_2$, $\dot{u}_2 = 0$. The unfolding of this bifurcation contains a $\mathbb{D}_3$-symmetric configuration of three homoclinic loops to the origin (resembling a clover of homoclinic loops; see Figure 5.27) that occurs a one-sided branch. The linearization of the ODE on the two-dimensional stable and unstable directions is semi-simple, and the unfolding of the homoclinic cycle near the homoclinic branch contains a suspended topological Markov chain.

Below, we discuss generalizations of this example in more detail. An important difference compared with homoclinic loops in generic flows is the possibility that multiple semisimple eigenvalues can be enforced by the symmetry, as is the case in the situation studied in [269]. Symmetry can also force heteroclinic trajectories to approach an equilibrium along non-leading directions, thus enforcing orbit-flip configurations (or inclination flips, if this occurs in the adjoint system). The bifurcation result we present below allows for multiple semisimple eigenvalues, but assumes that heteroclinic trajectories approach equilibria along the leading directions.

Thus, consider a Γ-equivariant system that admits a homoclinic cycle which consists of the heteroclinic trajectories $h_1, \ldots, h_\ell$, where h_j connects hyperbolic equilibria p_j to p_{j+1} (taking indices modulo ℓ). We
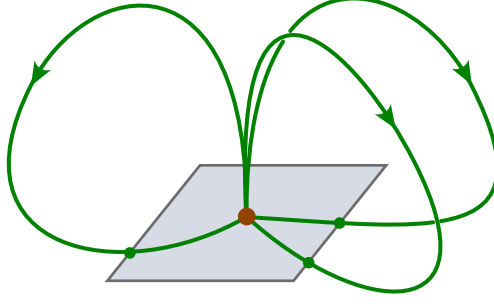
Figure 5.27: *Illustration of three homoclinic trajectories approaching p when* $\dim E_p^{\mathrm{s}} = 2$. *Note that the state space has to be at least four dimensional.*

will now assume that this connection is of codimension one inside the fixed-point space S_j of Γ_{h_j}. We write $\mathrm{ind}_{S_j}(p) := \dim[W^{\mathrm{u}}(p) \cap S_j]$ for the Morse index of an equilibrium p inside S_j.

Hypothesis 5.50. *We have* $\mathrm{ind}_{S_j}(p_j) = \mathrm{ind}_{S_j}(p_{j+1})$ *for all j.*

Recall that E_j^{ls} and E_j^{lu} denote the leading stable and unstable eigenspaces of the Jacobian $f_u(p_j, 0)$.

Hypothesis 5.51. *The isotropy groups Γ_{p_j} act real absolutely irreducibly on E_j^{ls} (so that ν_j^{s} is real), and* $0 < -\nu_j^{\mathrm{s}} < \mathrm{Re}\,\nu_j^{\mathrm{u}}$.

Denoting by $E_{j,*}^{\mathrm{ls}}$ and $E_{j,*}^{\mathrm{lu}}$ the leading stable and unstable eigenspaces of the adjoint Jacobian $f_u(p_j, 0)^*$. Assuming the nondegeneracy condition Hypothesis 2.10(i), there is a unique bounded solution ψ_j of the adjoint variational equation $\dot{w} = -f_u(h_j, 0)^* w$ along the solution h_j, and we record that ψ_j is contained in S_j. The following conditions, which require that h_j and ψ_j decay with rates coming from the leading directions, are analogues of the orbit-flip and inclination-flip conditions we introduced for generic systems. For equivariant systems, these conditions are open, but not necessarily generic.

Hypothesis 5.52. *Consider the following conditions on the geometry of h_j:*

(i) *The limit $v_{j+1}^{\mathrm{s}} = \lim_{t\to\infty} h_j(t)/\|h_j(t)\|$ lies in the leading stable eigenspace so that $v_{j+1}^{\mathrm{s}} \in E_{j+1}^{\mathrm{ls}}$;*

(ii) *$S_j \cap E_{j,*}^{\mathrm{ls}} \neq \{0\}$, and the limit $v_{j,*}^{\mathrm{s}} = \lim_{t\to-\infty} \psi_j(t)/\|\psi_j(t)\|$ satisfies $v_{j,*}^{\mathrm{s}} \in E_{j,*}^{\mathrm{ls}}$.*

Take cross sections Σ_j, transverse to the heteroclinic trajectory h_j, related by symmetry. For a given $\ell \times \ell$ matrix M with entries $M_{ij} \in \{0, 1\}$, let $\mathcal{B}_M$ be the topological Markov chain defined by M. For $\kappa \in \mathcal{B}_M$, we denote a solution by $u(\mu, \kappa)(\cdot)$ if there is a monotonically increasing sequence $(\tau_i)_{i\in\mathbb{Z}}$ such that

$$u(\mu, \kappa)(\tau_i) \in \Sigma_{\kappa_i}, \text{ but } u(\mu, \kappa)(t) \notin \bigcup_{j=1}^{\ell} \Sigma_j \text{ for } t \notin \{\tau_i, i \in \mathbb{Z}\}.$$

We call κ the itinerary of $u(\mu, \kappa)(\cdot)$. Let $\Pi(\cdot, \mu)$ be the first-return map defined on $\cup_{j=1}^{\ell} \Sigma_j$ (more precisely, the domain of $\Pi(\cdot, \mu)$ will be a subset of $\cup_{j=1}^{\ell} \Sigma_j$):

$$\Pi(\cdot, \mu) : \bigcup_{j=1}^{\ell} \Sigma_j \to \bigcup_{j=1}^{\ell} \Sigma_j, \quad \Pi(u(\mu, \kappa)(\tau_i), \mu) = u(\mu, \kappa)(\tau_{i+1}). \tag{5.36}$$

The next theorem describes how recurrent sets change through the bifurcation. For a given matrix A, we write $|A| = (|a_{ij}|)_{ij}$.

Theorem 5.75 ([187])**.** *Let Γ be a finite group and $\dot{u} = f(u, \mu)$ be a one-parameter Γ-equivariant family of differential equations on $\mathbb{R}^n$ that admits a homoclinic network for $\mu = 0$. Suppose that Hypotheses 5.50, 5.51, 5.52, and 2.10 are met, then the system contains a recurrent set near the homoclinic network that is given as follows. Define the matrix $A = (a_{ij})_{i,j\in\{1,\dots,m\}}$ by*

$$a_{ij} = \begin{cases} 0 & \text{if } \omega(h_i) \neq \alpha(h_j), \\ \mathrm{sign}\langle v_i^{\mathrm{s}}, v_{j,*}^{\mathrm{s}} \rangle, & \text{if } \omega(h_i) = \alpha(h_j). \end{cases}$$
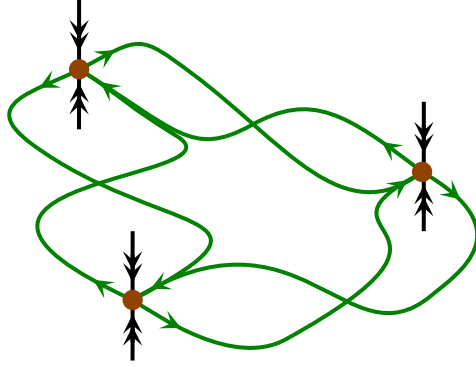
Figure 5.28: *An impression of a codimension-one $\mathbb{D}_3$-invariant homoclinic cycle for a $\mathbb{D}_3$-equivariant flow in $\mathbb{R}^3$ with three $\mathbb{Z}_2$-symmetric equilibria. Note that a codimension-two bifurcation that involves resonance conditions, an inclination flip, or an orbit flip condition would lead to $\mathbb{D}_3$-symmetric singular hyperbolic attractors in an unfolding (akin to Lorenz-like attractors); see also §5.5.5.*

Write $M_+ = \frac{1}{2}(A + |A|)$ and $M_- = -\frac{1}{2}(A - |A|)$. For $\mu > 0$ small enough, there is an invariant set $\mathcal{D}_\mu \subset \cup_{j=1}^\ell \Sigma_j$ of $\Pi(\cdot, \mu)$ such that, for each $\kappa \in \mathcal{B}_{M_+}$, there exists a unique solution $u(\mu, \kappa)(t)$ with $u(\mu, \kappa)(0) \in \mathcal{D}_\mu$. Moreover, $\Pi(\cdot, \mu)|_{\mathcal{D}_\mu}$ is topologically conjugated to the left shift on $\mathcal{B}_{M_+}$. An analogous statement holds for $\mu < 0$ with $\mathcal{B}_{M_+}$ replaced by $\mathcal{B}_{M_-}$.

The recurrent trajectories described in the preceding theorem generate the entire recurrent set for $\mu \neq 0$ only when the inner products $\langle e_i^s, e_j^- \rangle$ are nonzero for all i, j with $\omega(h_i) = \alpha(h_j)$.

5.5.4 Forced symmetry breaking

In this section, we consider equivariant systems in which the some of the symmetries are broken upon changing parameters: in other words, we consider systems $\dot{u} = f(u, \mu)$ that are equivariant under Γ for $\mu = 0$ but respect only a proper subgroup $\tilde{\Gamma}$ of Γ for $\mu \neq 0$. This situation is commonly referred to as *forced symmetry breaking*[19].

Forced symmetry breaking of relative equilibria in systems with continuous symmetries provides a mechanism for creating robust heteroclinic cycles. This research was initiated by Lauterbach and Roberts, who used forced symmetry breaking from $SO(3)$ to the tetrahedral group $\mathbb{T}$ as an example.

Theorem 5.76 ([252]). *Let $\dot{u} = f(u)$ be an $SO(3)$-equivariant system with an $SO(3)$-orbit $\mathcal{E}$ of equilibria, whose isotropy subgroups are conjugate to $O(2)$, so that $\mathcal{E}$ is a normally hyperbolic invariant manifold. Let $\dot{u} = \tilde{f}(u)$ be a $\mathbb{T}$-equivariant system that is close to $\dot{u} = f(u)$ in the C^1 topology. Restricted to the perturbed invariant manifold $\tilde{\mathcal{E}}$ near $\mathcal{E}$, the system $\dot{u} = \tilde{f}(u)$ then admits equilibria with isotropy subgroups conjugate to $\mathbb{D}_2$ and $\mathbb{Z}_3$; in addition, it admits either equilibria with isotropy conjugate to $\mathbb{Z}_2$ or robust heteroclinic cycles that connect the equilibria with $\mathbb{D}_2$ symmetry. Moreover, there exist $\mathbb{T}$-equivariant equations $\dot{u} = \tilde{f}(u)$ arbitrarily close to $\dot{u} = f(u)$ in the C^1 topology for which $\tilde{\mathcal{E}}$ contains one of the following:*

(i) stable equilibria with $\mathbb{Z}_3$ symmetry;

(ii) stable equilibria with $\mathbb{D}_2$ symmetry and equilibria with $\mathbb{Z}_2$ symmetry;

(iii) stable relative homoclinic cycles that connect the equilibria with $\mathbb{D}_2$ symmetry.

There is a dual statement on forced symmetry breaking from $SO(3)$ to $O(2)$-symmetry for equilibria with isotropy $\mathbb{T}$, in which a circle of heteroclinic cycles occurs that connect equilibria with $\mathbb{D}_2$ symmetry. If we break symmetry from $SO(3)$ to $\mathbb{D}_n$, then non-asymptotically stable heteroclinic cycles, see Theorem 5.71, can bifurcate:

---

[19]This contrasts so-called *spontaneous symmetry breaking* which involves changes in the isotropy group of a critical element (or a attractor or any transitive invariant set) for systems that are equivariant under a fixed symmetry group

Theorem 5.77 ([252]). *Let $\dot{u} = f(u)$ be an $SO(3)$-equivariant system with an $SO(3)$-orbit $\mathcal{E}$ of equilibrium points with isotropy subgroups conjugate to $\mathbb{T}$ and assume that $\mathcal{E}$ is a normally hyperbolic invariant manifold. Let $\dot{u} = \tilde{f}(u)$ be a $\mathbb{D}_n$-equivariant system close to $\dot{u} = f(u)$ in the C^1 topology, then the perturbed invariant manifold $\tilde{\mathcal{E}}$ near $\mathcal{E}$ has the following properties:*

(i) If $3|n$, then $\tilde{\mathcal{E}}$ contains equilibria or periodic solutions with $\mathbb{Z}_3$ symmetry;

(ii) If n is odd, then $\tilde{\mathcal{E}}$ contains equilibria or periodic solutions with $\mathbb{Z}_2$ symmetry;

(iii) If n is even, then equilibria with $\mathbb{D}_2$-symmetry and heteroclinic cycles and/or equilibria with $\mathbb{Z}_2$-symmetry exist in $\tilde{\mathcal{E}}$.

We refer the reader to [82, 195, 251, 310] for further results of a similar flavor.

Next, we consider a few specific examples of the effect of forced symmetry breaking on robust relative homoclinic cycles. Recall the classification of robust relative homoclinic cycles in $\mathbb{R}^3$ that we outlined in Theorem 5.66. Let $\Gamma = \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$ and suppose that $\dot{u} = f_0(u)$ is a Γ-equivariant system on $\mathbb{R}^3$ that admits a robust relative homoclinic cycle. We will consider perturbations $\dot{u} = f(u, \mu)$ with $f(u, 0) = f_0(u)$ that are only $\mathbb{Z}_3$-equivariant for all μ. Generically, the two-dimensional fixed-point spaces that contain the homoclinic cycle for $\mu = 0$ will no longer be invariant for $\mu \neq 0$, and we expect that the homoclinic cycle ceases to exist. The following result, adapted from [346], shows that the original symmetry Γ may force the homoclinic cycle to be in an inclination-flip configuration, which is then unfolded by small perturbations that retain $\mathbb{Z}_3$ symmetry. We remark that it is not hard to think of other examples where the symmetry enforces orbit flips that can be unfolded by forced symmetry breaking. Let $\mu \in \mathbb{R}^2$ and consider the following condition:

Hypothesis 5.53 (Inclination flip). *We assume that $v_j^{\mathrm{s}}, v_j^{\mathrm{u}}, v_{j,*}^{\mathrm{u}} \neq 0$ at $\mu = 0$, while $v_{j,*}^{\mathrm{s}}(0) = 0$ with $\partial_{\mu_2} v_{j,*}^{\mathrm{s}}(0) \neq 0$.*

Theorem 5.78 ([346]). *Let $\dot{u} = f(u, \mu)$ be a $\mathbb{Z}_3$-equivariant two-parameter family on $\mathbb{R}^3$ which, for $\mu = 0$, is $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$-equivariant and has a simple relative homoclinic cycle. Suppose that Hypothesis 2.10(i) with $d_i = 0$ and Hypothesis 2.12(ii) are met. If $c < r$, where c, r are the contracting and radial eigenvalues at the equilibria, then the relative homoclinic cycle is of inclination-flip type. Furthermore, if the unfolding condition Hypothesis 5.53 is met, then the bifurcation diagram is as in Theorems 5.1, 5.16, and 5.17, depending on the eigenvalue $\nu^{\mathrm{s}} = r$ and $\nu^{\mathrm{u}} = e$ (the expanding eigenvalue).*

Proof. Symmetry forces the bounded solution ψ_j to the adjoint variational equation along the heteroclinic trajectory h_j to be perpendicular to the fixed point space of Γ_{h_j}. For $c < r$, the relative homoclinic cycle is therefore of inclination-flip type. An orbit space reduction [83] reduces the problem to a generic inclination-flip bifurcation of a homoclinic orbit. $\square$

An analogous result holds for forced symmetry breaking of robust homoclinic cycles from $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$ to $\mathbb{Z}_2$ symmetry in systems in $\mathbb{R}^3$. A more detailed classification can be found in [346], where also four-dimensional representations are considered. The additional transverse eigenvalue of the robust homoclinic cycle could be a leading eigenvalue and must therefore be taken into account.

5.5.5 Homoclinic orbits in systems with $\mathbb{Z}_2$-symmetry

The unfolding of multiple homoclinic orbits to a hyperbolic equilibrium can lead to nontrivial dynamics, including suspended horseshoes. One reason to devote an individual section to homoclinic orbits in differential equations with reflection symmetries is that additional degeneracies may create Lorenz-like strange attractors. To see this, consider a $\mathbb{Z}_2$-equivariant differential equation with homoclinic orbits to a symmetric equilibrium p. We write ρ for the linear map that generates the action of $\mathbb{Z}_2$. If h is a homoclinic solution to p, then ρh is a second homoclinic solution to p. Analogous to Hypothesis 5.12, we distinguish different geometries of the closure of $h \cup \rho h$.

Hypothesis 5.54 (Geometric configurations). *Suppose that p has unique real leading eigenvalues ν^{s} and ν^{u} with eigenvectors v^{s} and v^{u}. We distinguish:*

(i) *Figure eight:* $v^{\mathrm{s}} = -\rho v^{\mathrm{s}}$ *and* $v^{\mathrm{u}} = -\rho v^{\mathrm{u}}$;

(ii) *Butterfly (expanding):* $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} < 1$ *and* $v^{\mathrm{s}} = \rho v^{\mathrm{s}}$, $v^{\mathrm{u}} = -\rho v^{\mathrm{u}}$;

(iii) *Butterfly (contracting):* $-\nu^{\mathrm{s}}/\nu^{\mathrm{u}} > 1$ *and* $v^{\mathrm{s}} = \rho v^{\mathrm{s}}$, $v^{\mathrm{u}} = -\rho v^{\mathrm{u}}$;

(iv) *Bellows:* $v^{\mathrm{s}} = \rho v^{\mathrm{s}}$ *and* $v^{\mathrm{u}} = \rho v^{\mathrm{u}}$.

Theorem 5.79. *Let $\dot{u} = f(u, \mu)$ be a one-parameter family of $\mathbb{Z}_2$-equivariant ODEs with admits two different homoclinic loops h and ρh to the hyperbolic equilibrium p when $\mu = 0$. Suppose that Hypotheses 2.1, 2.2, 2.3(ii), and 2.4 are met.*

(i) *If Hypothesis 5.54(i) or 5.54(iii) is met, then there are two nonsymmetric periodic solutions for μ on one side of $\mu = 0$ and a single symmetric periodic solution for μ on the other side of $\mu = 0$.*

(ii) *If Hypothesis 5.54(ii) or Hypothesis 5.54(iv) holds, then there is no recurrent dynamics (apart from the equilibrium) for μ on one side of $\mu = 0$ and a suspended horseshoe for μ on the other side of $\mu = 0$.*

Shil'nikov [372, 373] observed that Lorenz-like attractors could be created via three different codimension-two bifurcations in $\mathbb{Z}_2$-equivariant systems with two homoclinic loops: these are resonant leading eigenvalues, inclination flips, and orbit flips. The resonant bifurcation has subsequently been studied by Robinson, the inclination flip by Rychlik, and the orbit flip by Golmakani and Homburg. We now present the results of these analyses in the following three theorems.

First, we discuss $\mathbb{Z}_2$ equivariant systems with homoclinic loops at resonance, which have been studied by Robinson; see also [283, 284]. Although we only state a result on the existence of Lorenz-like attractors, unfoldings of homoclinic loops at resonance can also give rise to contracting Lorenz models [284, 327]; see also §4.2.

Theorem 5.80 ([325–327]). *There exists an open set of two-parameter families of $\mathbb{Z}_2$-equivariant systems with the following properties. Each such family has two homoclinic loops for $\mu = 0$ that satisfy Hypotheses 2.1, 2.2, 2.3(ii), and 2.4, and there is an open set in the parameter plane for which a Lorenz-like attractor exists.*

In fact, Robinson showed that Lorenz-like attractors occur in the cubic system

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y \pm yz, \\
\dot{z} &= -\gamma z + \delta x^2.
\end{aligned}
$$

Recall that the Lorenz equations are quadratic. Rychlik proved a similar phenomenon in the unfolding of $\mathbb{Z}_2$-equivariant ODEs with inclination-flip homoclinic loops.

Theorem 5.81 ([334]). *Let $\dot{u} = f(u, \mu)$ be a two-parameter family of $\mathbb{Z}_2$-equivariant ODEs in $\mathbb{R}^3$ that admit a hyperbolic equilibrium p with one-dimensional unstable and two-dimensional stable manifolds and has two symmetry-related homoclinic orbits to p when $\mu = 0$. Suppose furthermore that Hypotheses 2.1, 2.2, and 5.8 are met, and that this bifurcation is of Type B as explained in §5.1.6. There are then parameter values arbitrarily close to $\mu = 0$ for which the system has a Lorenz-like attractor.*

In fact, Rychlik showed that Lorenz attractors occur in the $\mathbb{Z}_2$-equivariant cubic system

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y + \delta xz, \\
\dot{z} &= -\gamma z + x^2,
\end{aligned}
$$

which, up to a change of coordinates, coincides with the original Lorenz equations when $\beta = 0$. Finally, orbit-flip homoclinic bifurcations can give rise to Lorenz-like attractors in much the same way.

Theorem 5.82 ([152]). *Let $\dot{u} = f(u, \mu)$ be a two-parameter family of $\mathbb{Z}_2$-equivariant ODEs in $\mathbb{R}^3$ that admit a hyperbolic equilibrium p with one-dimensional unstable and two-dimensional stable manifolds and has two symmetry-related homoclinic orbits to p when $\mu = 0$. Assume that the action of the symmetry has a one-dimensional fixed-point space, that Hypotheses 2.1, 2.2, and 5.10 are met, and that this bifurcation is of Type B as explained in §5.1.7. There are then parameter values arbitrarily close to $\mu = 0$ for which the system has a Lorenz-like attractor.*

Degenerate homoclinic loops of certain $\mathbb{Z}_2$-equivariant ODEs in $\mathbb{R}^4$ have been investigated in [23, 410]. These bifurcation studies were motivated by the equation

$$
\begin{aligned}
\dot{s}_1 &= s_2, \\
\dot{s}_2 &= \frac{I - s_2 - \sin s_1 \cos r_1}{3 + \beta}, \\
\dot{r}_1 &= r_2, \\
\dot{r}_2 &= -\frac{r_2 + \sin r_1 \cos r_1}{\beta}
\end{aligned}
$$

which models two coupled Josephson junctions with a capacitive load [22]. This system is equivariant under the action of $\mathbb{Z}_2$ given by $(s_1, s_2, r_1, r_2) \mapsto (s_1, s_2, -r_1, -r_2)$, which fixes elements of the plane $\{(s_1, s_2, r_1, r_2) \in \mathbb{R}^4;\ r_1 = r_2\}$. There is a value of β and I for which this system has a symmetric homoclinic loop in the fixed-point space along which the stable and unstable manifolds of the origin have a two-dimensional common tangent space. Among the phenomena found for nearby parameter is the existence of a one-sided curve in the parameter plane along which a pair of nonsymmetric homoclinic loops in a bellows configuration exists. Further analytic work revealed the existence of curves of pitchfork and periodic-doubling bifurcations of periodic solutions. Numerical work suggests a complicated bifurcation diagram including chaotic dynamics.

6 Related topics

In this section, a number of topics are reviewed that do not fit into the previous sections but are of interest to homoclinic bifurcation theory. Specifically, we discuss topological index theory for homoclinic orbits and moduli of stability associated to heteroclinic and homoclinic orbits in some detail. We also give an overview of existence theorems for homoclinic orbits, numerical techniques for continuing homoclinic orbits and detecting their bifurcations, variational methods for constructing homoclinic orbits in Hamiltonian systems, techniques for studying homoclinic orbits in singularly perturbed systems, and, finally, extensions to infinite-dimensional systems. However, for the latter topics, we will be brief and focus primarily on giving pointers to existing literature.

6.1 Topological indices

Homoclinic orbits occur along curves in two-dimensional parameter spaces. Of interest is then the fate of these curves for generic systems: the issue is whether we can list generic possibilities for start and end points of such curves and identify bifurcations along them that do not rely on unfolding and nondegeneracy conditions.

First, we discuss topological bifurcations that do not rely on unfolding conditions and present a prototypical result due to Nii. Recall that two-dimensional homoclinic center manifolds are orientable annuli or nonorientable Möbius strips. Nii proved that, if we are given a path of homoclinic orbits in the parameter plane so that the orientations of the homoclinic center manifold are different at the end points, then a 2-homoclinic orbit has branched somewhere along the path. We stress that there is no need to check unfolding conditions at the bifurcation, though we need nondegeneracy conditions at the end points of the path that guarantee the existence of two-dimensional homoclinic center manifolds. The proof given by Nii uses Conley index
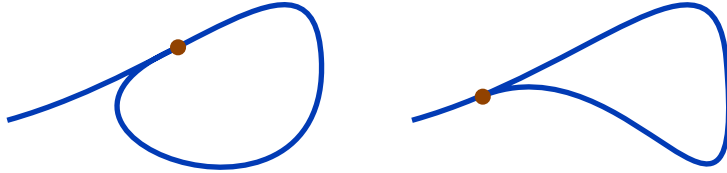
Figure 6.1: *Bifurcation curves of homoclinic orbits that contain a homoclinic noose.*

theory, see [136, 276] for an overview, to find the 2-homoclinic solutions; we remark that Lin's method can also be utilized to prove the following theorem.

Theorem 6.1 ([295]). *Let $\dot{u} = f(u, \mu)$ be a two-parameter family of ODEs on $\mathbb{R}^n$ so that a homoclinic orbit to a hyperbolic equilibrium p exists along a path $s : [0, 1] \to \mathbb{R}^2$ in the parameter plane. Suppose that the equilibria along the path have one-dimensional unstable manifolds and that Hypotheses 2.1, 2.2, and 2.3(ii) are met for all $\mu \in s([0, 1])$. Furthermore, we assume that the homoclinic orbit is not in a flip configuration so that Hypotheses 2.4(i)-(ii) are met for $\mu = s(0), s(1)$. Finally, suppose that at $\mu = s(0)$ the homoclinic center manifold is orientable, while at $\mu = s(1)$ the homoclinic center manifold is nonorientable, then there exists a $t \in (0, 1)$ so that a 2-homoclinic orbit is created in the homoclinic bifurcation at $\mu = s(t)$. The statement remains true if the homoclinic center manifolds are nonorientable for $\mu = s(0), s(1)$ but $-\nu^s/\nu^u > 1$ at $\mu = s(0)$, and $-\nu^s/\nu^u < 1$ at $\mu = s(1)$.*

Next, we discuss the fate of curves of homoclinic orbits in the bifurcation plane. In the spirit of the continuation theory for periodic orbits in [9], Fiedler [129] introduced an index for homoclinic orbits in the interior of the closure of the class of Morse–Smale flows (that is, for tame homoclinic orbits) and used this index to derive a pathfollowing result for homoclinic orbits in generic families. The idea is to follow curves of homoclinic orbits up to the boundary (of the interior of the closure) of the class of Morse–Smale flows or to a collection of bifurcations of higher codimension.

Let us illustrate the outcomes of the resulting homoclinic continuation theory by showing that certain bifurcation diagrams cannot occur in generic two-parameter families of three-dimensional flows. Here, we use the term 'generic' to refer to two-parameter systems that contain homoclinic bifurcations of codimension one and two only that, furthermore, unfold generically, that is, occur only along curves and at isolated points, respectively. Adopting the term 'noose' from the continuation theory of periodic orbits, we say that a bifurcation diagram of homoclinic orbits has a homoclinic noose if it contains the structures shown in Figure 6.1. The following result can be deduced from the general homoclinic continuation theory that we describe further below.

Proposition 6.1. *The bifurcation diagram of a generic two-parameter family of ODEs $\dot{u} = f(u, \mu)$ in $\mathbb{R}^3$ cannot contain homoclinic nooses.*

Proof. Suppose that the bifurcation diagram of $\dot{u} = f(u, \mu)$ contains a homoclinic noose. The branching point is then necessarily a homoclinic-doubling bifurcation, that is, either a resonant homoclinic bifurcation, an inclination flip or an orbit flip (see §5), and the noose consists of the primary orbit that returns to itself as a 2-homoclinic orbit. Draw a smooth curve in the parameter plane close to the homoclinic noose and consider the bifurcations of periodic orbits along this curve. The bifurcation diagram for the periodic orbits in the parameter that parameterizes the chosen curve then contains a noose, which is impossible by [147, 213]. $\square$

Other applications of a topological continuation theory for homoclinic orbits include the existence of homoclinic-doubling cascades, see §4.5, and of cascades of T-points in [193]. We remark that these results are similar in spirit and detail to the existence proofs of cascades of period-doubling bifurcations that use continuation theory for periodic orbits [430].

We now outline homoclinic continuation theory itself and follow the account given in [190] where a continuation result without genericity conditions is proved. Let $\dot{u} = f(u, \mu)$ be a two-parameter family of ODEs in

$\mathbb{R}^3$. We denote by $\mathcal{P}$ the set of compact subsets of $\mathbb{R}^3$, equipped with the Hausdorff metric, and define

$$G \quad = \quad \left\{ (\mu, h) \in \mathbb{R}^2 \times \mathcal{P}; \begin{array}{l} h \text{ is the union of an equilibrium and a} \\ \text{homoclinic orbit of } \dot{u} = f(u, \mu) \end{array} \right\}. \tag{6.1}$$

For $(\mu, h) \in G$, let $l(\mu, h)$ denote the arclength of h. For simplicity, we assume $l(\mu, h)$ is finite, which is guaranteed, for instance, if the equilibrium in h is hyperbolic. We say that k is a virtual length of (μ, h) if there exists a sequence of smooth perturbations $f_i(\cdot, \nu)$ of the family $f(\cdot, \nu)$ with $f_i(\cdot, \nu) \to f(\cdot, \nu)$ as $i \to \infty$ so that $\dot{u} = f_i(u, \nu)$ has a homoclinic orbit h_i at parameter values μ_i with $\mu_i \to \mu$, $h_i \to h$ in the Hausdorff topology, and $l(\mu_i, h_i) \to k$ as $i \to \infty$. We write $\tau(\mu, h)$ for the set of virtual lengths of $(\mu, h) \in G$.

To decide whether a homoclinic orbit can be continued globally, we associate an index to each such orbit. First, pick an $(\mu, h) \in G$ and assume that it is a codimension-one homoclinic orbit that is unfolded generically upon varying μ and satisfies $\tau(\mu, h) = \{l(\mu, h)\}$. There are then a sequence μ_i of parameter values that converges to μ and a sequence of periodic orbits q_i for these parameter values that converges to h in the Hausdorff topology as $i \to \infty$. Our assumptions imply furthermore that q_i is the unique periodic orbit for parameter value equal to μ_i for all sufficiently large i and that its unstable manifold $W^{\mathrm{u}}(q_i)$ is either orientable or nonorientable. Define the index

$$\phi(\mu, h) \quad = \quad \left\{ \begin{array}{ll} 0 & \text{if } W^{\mathrm{u}}(q_i) \text{ is nonorientable for large } i; \\ 1 & \text{if } W^{\mathrm{u}}(q_i) \text{ is orientable for large } i. \end{array} \right. \tag{6.2}$$

Note that one- and three-dimensional unstable manifolds are always orientable. If $W^{\mathrm{u}}(q_i)$ is two-dimensional, then there exists a two-dimensional homoclinic center manifold $W^{\mathrm{c}}(h)$ of h; see §3.4: this manifold is orientable if $\phi(\mu, h) = 1$ and nonorientable otherwise. In particular, it is possible to define $\phi(\mu, h)$ using only the equation at the parameter value μ.

Next, we extend this definition of $\phi(\mu, h)$ as follows to the entire set G. For each $(\mu, h) \in G$, we set $\phi(\mu, h) = 1$ if the virtual length of (μ, h) is bounded and if there exists a sequence of families $f_i(\cdot, \nu)$ with $f_i(\cdot, \nu) \to f(\cdot, \nu)$ as $i \to \infty$ and $f_i(\cdot, \nu)$ has a generically unfolded homoclinic orbit h_i of codimension one at parameter values μ_i so that $\mu_i \to \mu$, $h_i \to h$ in the Hausdorff topology, and $\phi(\mu_i, \tau_i) = 1$ as $i \to \infty$. For all other $(\mu, h) \in G$, we set $\phi(\mu, h) = 0$. Let

$$G_1 \quad = \quad \{ (\mu, h) \in G; \; \phi(\mu, h) = 1 \} \tag{6.3}$$

be the set of (μ, h) of index one.

Finally, we can state precisely what global continuation of homoclinic orbits (μ, h) in G_1 refers to. Let $(\mu, h) \in G_1$ so that h is the union of a homoclinic orbit and a hyperbolic equilibrium, and write Γ_1 for the connected component of G_1 that contains (μ, h). We call (μ, h) globally continuable if either

- $\Gamma_1 \backslash \{(\mu, h)\}$ is connected

or else each component C_1 of $\Gamma_1 \backslash \{(\mu, h)\}$ satisfies at least one of the following conditions:

- C_1 is unbounded;

- there exists a sequence $(\mu_i, h_i) \in C_1$ so that $\sup_i \tau(\mu_i, h_i) = \infty$ or so that $(\mu_i, h_i) \to (\tilde{\mu}, \tilde{h}) \in G$ as $i \to \infty$ and $(\tilde{\mu}, \tilde{h})$ has unbounded virtual length;

- there exists a sequence $(\mu_i, h_i) \in C_1$ so that μ_i has a limit as $i \to \infty$, and h_i converges, in the Hausdorff topology, to a closed invariant set that contains either a nonhyperbolic equilibrium or more than two orbits.

Note that the closure of a homoclinic orbit always consists of two orbits: thus, if the closed invariant set consists of more than two orbits, it may contain two homoclinic orbits or a heteroclinic cycle.

Theorem 6.2 ([190]). *A generically unfolded codimension-one homoclinic orbit in G_1 is globally continuable.*

	$\alpha(h)$	equilibrium	equilibrium	periodic orbit	periodic orbit
$\omega(h)$	leading	real	complex	real	complex
equilibrium	real				
equilibrium	complex		$\frac{\operatorname{Re}\nu^{\mathrm{s}}\,\operatorname{Im}\nu^{\mathrm{u}}}{\operatorname{Im}\nu^{\mathrm{s}}\,\operatorname{Re}\nu^{\mathrm{u}}}$	$\frac{\operatorname{Re}\nu^{\mathrm{s}}}{\operatorname{Im}\nu^{\mathrm{s}}}\frac{1}{\ln|\nu^{\mathrm{u}}|}$	$\frac{\operatorname{Re}\nu^{\mathrm{s}}}{\operatorname{Im}\nu^{\mathrm{s}}}\frac{1}{\ln|\nu^{\mathrm{u}}|},\arg\nu^{\mathrm{u}}$
periodic	real			$\frac{\ln|\nu^{\mathrm{s}}|}{\ln|\nu^{\mathrm{u}}|}$	$\frac{\ln|\nu^{\mathrm{s}}|}{\ln|\nu^{\mathrm{u}}|},\arg\nu^{\mathrm{u}}$
periodic	complex				$\frac{\ln|\nu^{\mathrm{s}}|}{\ln|\nu^{\mathrm{u}}|},\arg\nu^{\mathrm{s}},\arg\nu^{\mathrm{u}}$

Table 3: *Moduli for heteroclinic orbits are listed, where ν^{s} is the leading stable eigenvalue (or Floquet multiplier) at $\alpha(h)$ and ν^{u} is the leading unstable eigenvalue (or Floquet multiplier) at $\omega(h)$.*

6.2 Moduli

Recall that two differential equations $\dot{u}=f(u)$ and $\dot{u}=g(u)$ on $\mathbb{R}^n$ are topologically equivalent if there is a homeomorphism that maps orbits of $\dot{u}=f(u)$ to orbits of $\dot{u}=g(u)$, while preserving the direction of time. Any invariant of topological equivalence is called a modulus.

Palis [304] proved that heteroclinic orbits that involve a tangency between the stable and unstable manifolds of two periodic orbits give rise to a modulus that can be expressed as the quotient of the leading Floquet multipliers of the periodic orbits. Homoclinic tangencies of stable and unstable manifolds of periodic orbits can give rise to infinitely many moduli: see [154] for a survey of results. Homoclinic and heteroclinic orbits to equilibria can also give rise to moduli, and we remark that moduli near saddle-focus homoclinic orbits were treated in Theorem 5.5 in §5.1.2.

In this section, we review some other results in this direction. Moduli of stability that occur for systems with heteroclinic connections between critical elements (and for families that unfold these heteroclinic connections) have been studied by van Strien, extending and generalizing work by Beloqui [32] and Newhouse, Takens and Palis [291, 292, 390, 391]. The following result lists necessary and sufficient conditions for the existence of a topological equivalence near a heteroclinic orbit for two nearby vector fields.

Theorem 6.3 ([411]). *Let $\dot{u}=f(u)$ be an ODE on $\mathbb{R}^n$ that has a heteroclinic orbit $h(t)$ connecting hyperbolic critical elements (equilibria or periodic orbits) $\alpha(h)$ to $\omega(h)$ and suppose that additional generic conditions are met (for equilibria, these are Hypotheses 2.9, 2.10(i) with $d=1$, 2.11(iii), and 2.12). Table 3 contains the complete list of moduli of topological equivalence in this situation: if the moduli of two nearby vector fields with such heteroclinic orbits are equal, then there exists a topological equivalence by a near-identity homeomorphism in a neighborhood of the closure of the heteroclinic orbit.*

Heteroclinic connections of codimension two are considered by Bonatti and Dufraine [45]. In [392], a complete set of three invariants of conjugacy are constructed for attracting planar heteroclinic cycles with two hyperbolic equilibria: the moduli arise because the time averages of continuous functions along orbits that converge to the heteroclinic cycle typically do not converge [143], and we refer to [392] for the relation between the moduli and these time averages.

Moduli are also relevant for the comparison of families of vector fields. Consider two families $\dot{u}=f(u,\mu)$ and $\dot{v}=g(v,\nu)$ on $\mathbb{R}^n$. A topological equivalence between these families is given by a family $\Phi(\cdot,\mu)$ of homeomorphisms of $\mathbb{R}^n$ and a homeomorphism ϕ on the parameter space so that $(v,\nu)=(\Phi(u,\mu),\phi(\mu))$ relates their orbits. As in §5.4, one distinguishes different regularity properties of $\Phi(\cdot,\mu)$ and ϕ: the properties most relevant here are (fiber C^0,C^0)-equivalence (the above definition) and (C^0,C^0)-equivalence (where $\Phi(u,\mu)$ is continuous in (u,μ)). We consider one-parameter families and assume the following:

Hypothesis 6.1 (Generic unfolding). *The unions of $W^{\mathrm{s}}(\omega(h))$ and $W^{\mathrm{u}}(\alpha(h))$ in the product $\mathbb{R}^n\times\mathbb{R}$ of state and parameter space intersect transversally.*

The next theorem is the analogue of Theorem 6.3 for families that unfold a heteroclinic bifurcation of codimension one.

	$\alpha(h)$	equilibrium	equilibrium	periodic orbit	periodic orbit
$\omega(h)$	leading	real	complex	real	complex
equilibrium	real				
equilibrium	complex		$\frac{\text{Re}\,\nu^{\text{s}}}{\text{Im}\,\nu^{\text{s}}}, \frac{\text{Re}\,\nu^{\text{u}}}{\text{Im}\,\nu^{\text{u}}}$	$\frac{\text{Re}\,\nu^{\text{s}}}{\text{Im}\,\nu^{\text{s}}}, \ln|\nu^{\text{u}}|$	$\frac{\text{Re}\,\nu^{\text{s}}}{\text{Im}\,\nu^{\text{s}}}, \ln|\nu^{\text{u}}|, \arg\nu^{\text{u}}$
periodic	real			$\ln|\nu^{\text{s}}|, \ln|\nu^{\text{u}}|$	$\ln|\nu^{\text{s}}|, \ln|\nu^{\text{u}}|, \arg\nu^{\text{u}}$
periodic	complex				$\ln|\nu^{\text{s}}|, \ln|\nu^{\text{u}}|, \arg\nu^{\text{s}}, \arg\nu^{\text{u}}$

Table 4: *Moduli for unfoldings of heteroclinic orbits. In the table, ν^{s} is the leading stable eigenvalue (or Floquet multiplier) at $\alpha(h)$ and ν^{u} is the leading unstable eigenvalue (or Floquet multiplier) at $\omega(h)$.*

Theorem 6.4 ([411]). *Let $\dot{u} = f(u,\mu)$ be a one-parameter family of ODEs on $\mathbb{R}^n$ that has a heteroclinic orbit $h(t)$ for $\mu = 0$ which connects hyperbolic critical elements (equilibria or periodic orbits) $\alpha(h)$ to $\omega(h)$. Suppose that certain generic conditions are met (for equilibria , these are Hypotheses 2.9, 2.10(i) with $d = 1$, 2.11(iii), and 2.12) and that the generic unfolding condition Hypothesis 6.1 holds. Table 4 then lists moduli of topological equivalence: If the moduli of two nearby families with such a heteroclinic orbit are equal, then there exists a (C^0, C^0)-equivalence by near-identity homeomorphisms in a neighborhood of the closure of the heteroclinic orbit.*

A topological equivalence between two nearby families in $\mathbb{R}^2$ that unfold a homoclinic bifurcation gives rise to a modulus if one requires that the parameter change is a diffeomorphism.

Theorem 6.5 ([119]). *For $j = 1, 2$, let $\dot{u} = f_j(u,\mu)$ be one-parameter families in $\mathbb{R}^2$ that have a homoclinic orbit to hyperbolic equilibrium p_i at $\mu = 0$. Suppose that Hypotheses 2.2 and 2.3(ii) are met for both vector fields and denote the stable and unstable leading eigenvalues at p_j by ν_j^{s} and ν_j^{u}. Suppose that there exists a topological equivalence (Φ, ϕ) between these two systems by a homeomorphism $\Phi(\cdot, \mu)$ that depends continuously on μ and a diffeomorphism ϕ, then necessarily $-\nu_1^{\text{s}}/\nu_1^{\text{u}} = -\nu_2^{\text{s}}/\nu_2^{\text{u}}$ at $\mu = 0$.*

The proof uses C^1 linearizations near the equilibria to obtain the expressions $x \mapsto \alpha_j(\mu) + x^{-\nu_j^{\text{s}}/\nu_j^{\text{u}}}(1 + \vartheta_j(x, \mu))$ for the first-return maps on curves that are transverse to the homoclinic orbits, where α_j is differentiable, and ϑ_j is continuous and vanishes along $\{x = 0\}$ and $\{\mu = 0\}$. A conjugacy of the vector fields then implies a conjugacy of these maps that can be further analyzed to find moduli. Theorem 6.5 can be applied to prove (C^0, C^∞)-equivalence of generic families with Bogdanov–Takens bifurcations and their normal forms; see §5.4.1.

We briefly comment on extensions to global stability where the conjugacy is not restricted to a neighborhood of the heteroclinic connection. A family of vector fields is called structurally stable if every nearby family is (C^0, C^0)-equivalent. The following result by Labarca and Plaza characterizes structurally stable families that unfold heteroclinic bifurcations.

Theorem 6.6 ([244, 247]). *A generic one-parameter family $\dot{u} = f(u,\mu)$ on a compact three-dimensional manifold whose nonwandering set consists of finitely many hyperbolic critical elements and that has no heteroclinic cycles is structurally stable provided the following holds.*

(i) Stable and unstable manifolds of periodic orbits intersect transversally;

(ii) If p is an equilibrium with one-dimensional unstable (stable) manifold and complex conjugate stable (unstable) eigenvalues, then $W^{\text{u}}(p)$ is contained in the stable (unstable) manifold of an attracting (repelling) critical element.

We refer to [416] for the global stability of families with non-trivial recurrent set and unfoldings of heteroclinic bifurcations. An investigation of how the geometry of stable and unstable manifolds induces moduli can also be found in [183]. An extension of Theorem 6.6 by Plaza and Vera, incorporating local bifurcations, is

contained in [312]. One-parameter families of gradient vector fields on compact manifolds of any dimension turn out to be generically structurally stable [308], and the same is true for two-parameter families of gradient vector fields [66, 415].

Finally, we consider Lorenz-like attractors. Guckenheimer and Williams established that geometric Lorenz models have two moduli.

Theorem 6.7 ([162]). *There is an open set $\mathcal{U}$ in the space of smooth ODEs on $\mathbb{R}^3$ and a continuous mapping $k : \mathcal{U} \to \mathbb{R}^2$ so that the following holds. Each $f \in \mathcal{U}$ has a Lorenz-like attractor, and $f, \tilde{f} \in \mathcal{U}$ are topologically equivalent by a homeomorphism close to identity precisely if $k(f) = k(\tilde{f})$.*

The natural mapping k is obtained by considering the kneading sequences of the unstable manifolds [196, 323]. The above theorem becomes more precise in the language of interval maps; the statement below on expanding Lorenz maps applies to Lorenz-like vector fields by identifying points on leaves of the stable foliation on a cross section. We refer to [162, 323, 423] for the construction; see also §3.5.

Hypothesis 6.2 (Expanding Lorenz maps). *Consider $f : [-1, 1] \to [-1, 1]$ that satisfy:*

(i) *f is continuous and strictly increasing away from zero;*

(ii) *$\lim_{x \uparrow 0} f(x) = 1$ and $\lim_{x \downarrow 0} f(x) = -1$;*

(iii) *f is topologically expanding: there exists an $\varepsilon > 0$ so that, for all x_0, y_0 whose orbits do not contain zero, $|f^i(x_0) - f^i(y_0)| > \varepsilon$ for some $i \in \mathbb{N}$.*

Given an expanding Lorenz map and a point x that is not a preimage of 0, define its kneading sequence $k(x) \in \{-1, 1\}^{\mathbb{N}}$ by

$$k(x)(i) = \begin{cases} -1, & f^i(x) < 0, \\ 1, & f^i(x) > 0. \end{cases}$$

For general $x \in [-1, 1]$, we define its upper and lower kneading sequences by

$$k^+(x) = \lim_{y \downarrow x} k(x), \qquad k^-(x) = \lim_{y \uparrow x} k(x),$$

where the limits run over all points y that are not preimages of 0, and define the *kneading invariant*

$$K(f) = (k^+(-1), k^-(1)).$$

We write $\sigma : \{-1, 1\}^{\mathbb{N}} \to \{-1, 1\}^{\mathbb{N}}$ for the left shift operator defined by $[\sigma\alpha](i) = \alpha(i + 1)$ and take the lexicographical ordering on $\{-1, 1\}^{\mathbb{N}}$.

Theorem 6.8 ([196]). *If f is an expanding Lorenz map, then the kneading invariant $K(f) = (\alpha, \beta)$ satisfies*

$$\alpha \leq \sigma^n \alpha < \beta, \qquad \alpha < \sigma^n \beta \leq \beta \tag{6.4}$$

for all $n \in \mathbb{N}$. Conversely, given a pair of sequences $\alpha, \beta \in \{-1, 1\}^{\mathbb{N}}$ satisfying (6.4), there exists an expanding Lorenz map f with $K(f) = (\alpha, \beta)$, and f is unique up to conjugacy.

The combinatorial structure encoded by the kneading invariant is also apparent in the organization of heteroclinic bifurcation curves as presented in the numerical study of the Lorenz equations in [112].

6.3   Existence results

The existence of homoclinic orbits has been proved in many concrete models and applications. We give a few examples here and refer otherwise to some of the references listed in §5 for many other examples: in particular, we mention the review of local bifurcations in §5.4 that lead to small homoclinic orbits, and the

cubic equivariant vector fields given in §5.5.5 that admit two homoclinic loops to the origin at a resonance or inclination-flip bifurcation, which lead to geometric Lorenz attractors in appropriate unfoldings.

Sandstede provided in [340] a general construction of vector fields which have homoclinic orbits that undergo various codimension-two bifurcations. The idea is to begin with a two-dimensional vector field that leaves a planar algebraic curve invariant and has a homoclinic solution which lies on this curve. A third coordinate is then added in such a way that the geometry near the algebraic curve can be changed, while the curve itself remains invariant. Further perturbations can now be added to break the homoclinic orbit. This construction results in the system

$$\begin{aligned}
\dot{x} &= ax + by - ax^2 + (\mu_2 - \alpha z)x(2 - 3x) + \delta z, \\
\dot{y} &= bx + ay - \frac{3}{2}bx^2 - \frac{3}{2}axy - (\mu_2 - \alpha z)2y - \delta z, \\
\dot{z} &= cz + \mu_1 x + \gamma xz + \alpha\beta(x^2(1-x) - y^2).
\end{aligned}$$

that involves the real parameters $a, b, c, \alpha, \beta, \gamma, \mu_1, \mu_2$ and $\delta \in \{0, 1\}$.

Theorem 6.9 ([340])**.** *For $\mu = 0$, the above system has a homoclinic orbit which is contained in the Cartesian leaf $\Gamma = \{(x, y, z) \in \mathbb{R}^3; \ x^2(1-x) - y^2 = 0, z = 0\}$. First, suppose that $\delta = 0$, then the eigenvalues of the linearization at the origin are real, and the following codimension-two homoclinic bifurcations occur:*

- *A resonant bifurcation occurs for $a = 0$ if $c < -\sqrt{b^2 - 4\mu_2^2}$ and for $c = a + \sqrt{b^2 - 4\mu_2^2}$ otherwise. This bifurcation is unfolded by μ_1 and a.*

- *An inclination-flip occurs for $c < a - b$ and $\beta = 1$. This bifurcation is unfolded by μ_1 and $\alpha - \alpha_0$ for a certain α_0 that depends on a, b, c and γ.*

- *An orbit-flip occurs for $c > a - b$, $\beta = 0$ and sufficiently small $\alpha > 0$. The unfolding parameters are μ_1, μ_2.*

Next, suppose that $\delta = 1$, then the eigenvalues of the linearization at the origin consists of a complex conjugate pair and a real eigenvalue, and a saddle-focus homoclinic orbit occurs for $c = a - b$, $\gamma = 0$, $\alpha = 0$.

Finally, we comment on the Lorenz system

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= \rho x - y - xz, \\
\dot{z} &= -\beta z + xy
\end{aligned}$$

for which a number of results have been obtained that prove rigorously the existence of homoclinic orbits for some parameter values and which do not rely on numerical computations (rigorous or otherwise). In particular, a theoretical existence proof for homoclinic orbits has been given in [171], using an analytic implementation of the shooting method: the authors prove that, for σ near 10 and β near 1, there exists a $\rho \in (1, 1000)$ for which the Lorenz equations have a homoclinic orbit. Leonov [253] states that, for $\sigma > (2\beta + 1)/3$, there is a $\rho > 1$ for which the Lorenz equations have a homoclinic orbit; this refines earlier results from [35].

6.4 Numerical techniques

Since homoclinic and heteroclinic orbits are genuinely global dynamical objects, it is typically very hard to prove their existence and nondegeneracy in a given explicit system of differential equation. In these situations, numerical computations are often the only way to get insight into the existence and bifurcation structure of connecting orbits.

The most efficient and accurate algorithms seek homoclinic and heteroclinic orbits as solutions to appropriate boundary-value problems on bounded time intervals $(-T, T)$. Path-following in systems parameters can then be used to continue connecting orbits and to locate parameter values where they undergo bifurcations. We refer to the recent surveys [40, §6 and §8] and [344, §6.1] for details and references.

Algorithms of this kind can also be used for large systems, and specifically for discretizations of partial differential equations, and we refer to [77] for a recent overview. They have also been applied to delay differential equations [335] and even to equations with advanced and retarded terms [1].

These methods have been implemented via the driver HOMCONT [76] in the software package AUTO [111] by Doedel. AUTO also allows users to switch to N-homoclinic orbits at bifurcation points using a numerical implementation of Lin's method developed in [297]. DDE-BIFTOOL is a MATLAB package that implements a similar functionality for delay differential equations [125].

6.5 Variational methods

For Hamiltonian systems of the form

$$
\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla H(p, q), \qquad (p, q) \in \mathbb{R}^{2n},
\tag{6.5}
$$

global methods from the calculus of variations can often be used to prove the existence of homoclinic, heteroclinic and periodic orbits. To find connecting orbits in this manner, an appropriate variational formulation needs to be set up whose critical points are the desired heteroclinic orbits. The difficulty lies in finding variational formulations to which minimization techniques, mountain-pass theorems or other global methods can be applied by verifying the necessary hypotheses.

This approach has been used successfully to construct N-pulses (multibump orbits) near a given primary homoclinic orbits without having to impose any nondegeneracy conditions. In particular, multibump orbits have been constructed near homoclinic orbits to bi-foci in [55, 57, 209]. Such orbits have also been found in four-dimensional systems where the primary homoclinic orbit converges to a center with non-semisimple eigenvalues on the imaginary axis [43] and in systems that have two primary homoclinic orbits to the same saddle equilibrium [39]. In [378, 407, 409], variational methods were used to find multibump orbits in the Swift–Hohenberg equation.

We refer to the recent survey [320] for more details and a comprehensive list of references.

6.6 Singularly perturbed systems

Many physical systems admit two or more natural time scales. In ODE models, multiple time scales typically manifest themselves via the presence of small parameters that multiply the time derivatives of some of the variables. We shall give a very brief outlook of such singularly perturbed problem. Specifically, we consider systems of the form

$$
\begin{aligned}
\dot{u} &= f_1(u, v) \\
\dot{v} &= \epsilon f_2(u, v),
\end{aligned}
\tag{6.6}
$$

where t is the time variable and $(u, v) \in \mathbb{R}^{n_1 \times n_2}$. The singular character of these equations is reflected in the assumption that $0 < \epsilon \ll 1$. That the perturbation caused by ϵ is singular becomes more visible if we use the slow time $s = \epsilon t$ which yields the slow system

$$
\begin{aligned}
\epsilon u' &= f_1(u, v) \\
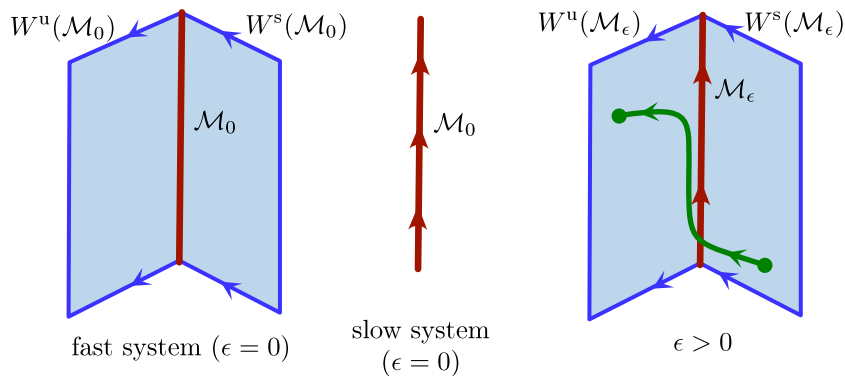v' &= f_2(u, v)
\end{aligned}
\tag{6.7}
$$

Figure 6.2: *Shown are the dynamics of the fast system [left] and the slow system [center] for $\epsilon = 0$, and the anticipated dynamics for $\epsilon > 0$ [right].*

in the slow time variable s. For $\epsilon > 0$, equations (6.6) and (6.7) are equivalent. However, for $\epsilon = 0$, we obtain the fast system

$$\dot{u} = f_1(u, v), \qquad \dot{v} = 0, \tag{6.8}$$

where v plays the role of a parameter, and the slow system

$$v' = f_2(u, v), \qquad f_1(u, v) = 0, \tag{6.9}$$

which is a differential-algebraic system, where v is constrained to the surface $\mathcal{M}_0 = \{(u, v); \ f_1(u, v) = 0\}$, which we refer to as the slow manifold. Note that the elements of $\mathcal{M}_0$ correspond to the equilibria of (6.8). Since the systems (6.8) and (6.9) are equations with fewer dependent variables, we can exploit this reduction in dimension to understand the occurrence and bifurcations of homoclinic and heteroclinic orbits for $\epsilon > 0$ by investigating the two systems (6.8) and (6.9) for $\epsilon = 0$ separately and gluing their solutions together to get solutions that persist for $\epsilon > 0$.

Rigorous matched asymptotic expansion provides one possible avenue for gluing slow and fast solutions together, and we refer to [257] for further details. Geometric singular perturbation theory, originating in work by Fenichel, offers an alternative approach: if the slow manifold $\mathcal{M}_0$ is normally hyperbolic for (6.8), then it persists as an invariant manifold $\mathcal{M}_\epsilon$ of (6.6) for $\epsilon > 0$ with dynamics that is close to the dynamics of (6.9) on $\mathcal{M}_0$. The so-called Exchange Lemma [54, 208, 240, 350] due to Jones and Kopell then describes the dynamics near the manifold $\mathcal{M}_\epsilon$ and allows one to carry out the matching of slow and fast solutions, and we refer to [208] for a review of this approach; see Figure 6.2 for an illustration. If $\mathcal{M}_0$ is not normally hyperbolic, then geometric blowup can often be used to analyze the dynamics, and [120, 241] contain recent results in this direction.

Last, we mention that homoclinic orbits in near-integrable Hamiltonian systems are often the building blocks of complicated chaotic behavior, and we refer to [168] for a comprehensive book on this topic.

6.7   Infinite-dimensional systems

Many of the results reviewed and summarized in this survey can be generalized to infinite-dimensional dynamical systems. We give a brief list of such systems and a few pointers to the relevant literature.

Delay differential equations are systems where the evolution of the solution $u(t)$ depends not only on its state at time t but also on its history: they occur often as models in population dynamics, in laser systems, and in system with time-delayed feedback. A typical delay differential equation is of the form

$$\dot{u}(t) = f(u(t), u(t - \tau)), \qquad u \in \mathbb{R}^n,$$

where $\tau > 0$ is the temporal delay. Such equations generate dynamical systems on the function space $C^0([-\tau, 0], \mathbb{R}^n)$. Both geometric and analytical approaches can be used to study homoclinic and heteroclinic

103

bifurcations in delay differential equations, and we refer to [166, 167] and [426] for results and further references.

Functional differential equations of mixed type (FDEs) are of the form

$$\dot{u}(t) = \sum_{j=-m}^{m} \alpha_j u(t+j) + f(u(t)), \qquad u \in \mathbb{R}^n, \tag{6.10}$$

for constants $\alpha_j \in \mathbb{R}$ and $m \geq 1$. Thus, in contrast to delay differential equation, the equation for the rate of change of u at time t depends here not only on the past but also on the future. FDEs are ill-posed in the sense that, given an initial condition $u(t)$ defined on the interval $[-m, m]$, a solution to (6.10) may not exist. In particular, FDEs do not generate a flow on an appropriate function space, which prevents us from studying homoclinic bifurcations using geometric Poincare-map based approaches. Recently, however, the existence of exponential dichotomies was established for FDEs; see [170, 266]. This opens up the possibility of using Lin's method for studying homoclinic bifurcations for FDEs and recent work in this direction can be found in [145, 197].

Partial differential equations (PDEs) provide another important class of infinite-dimensional dynamical systems. Consider, for instance, systems of parabolic partial differential equations

$$u_t = D\Delta u + f(u), \qquad x \in \Omega \subset \mathbb{R}^d, \ u \in \mathbb{R}^n$$

with Dirichlet or Neumann conditions, where $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ denotes the Laplace operator on a domain Ω with smooth boundary, D is a diagonal positive matrix, and f is a smooth nonlinearity. More generally, we may consider an abstract system of the form

$$\dot{u} = Au + f(u), \tag{6.11}$$

where A is a sectorial operator with dense domain defined on some Banach space X. These equations generate semiflows: solutions with prescribed initial data at $t = 0$ exist for $t > 0$ but not necessarily for $t < 0$ in backward time. Having a semiflow available is sufficient to use many of the analytical and geometric techniques we discussed in §3, and we refer to [176] for results in this direction. In particular, the bifurcation of periodic orbits with large period from a homoclinic orbit considered in §3.6 was investigated in [88]. We also mention that the homoclinic center-manifold theory developed in [343] is applicable to equations (6.11) with A sectorial.

Homoclinic bifurcation theory for elliptic partial differential equations

$$\Delta u + f(u) = 0, \qquad x \in \Omega \times \mathbb{R}, \ u \in \mathbb{R}^n$$

on cylindrical domains is often of interest as elliptic PDEs on cylinders arise when studying travelling waves of parabolic PDEs on such domains. Similar to FDEs, elliptic PDEs are ill-posed as initial-value problems. Exponential dichotomy theory for such systems was developed in [311], and we refer to [311, 347, 348] for results on homoclinic and heteroclinic bifurcations for elliptic and pseudo-elliptic PDEs.

References

[1] K. A. Abell, C. E. Elmer, A. R. Humphries and E. S. Van Vleck. Computation of mixed type functional differential boundary value problems. *SIAM J. Appl. Dyn. Syst.* **4** (2005) 755–781.

[2] A. Afendikov and A. Mielke. Bifurcation of homoclinic orbits to a saddle-focus in reversible systems with SO(2)-symmetry. *J. Differential Equations* **159** (1999) 370–402.

[3] V. S. Afraĭmovich, V. V. Bykov and L. P. Shil′nikov. On structurally unstable attracting limit sets of Lorenz attractor type. *Trans. Mosc. Math. Soc.* **1983** (1983) 153–216.

[4] V. S. Afraĭmovich, S.-N. Chow and W. Liu. Lorenz-type attractors from codimension one bifurcation. *J. Dynam. Differential Equations* **7** (1995) 375–407.

[5] V. S. Afraĭmovich and S.-B. Hsu. *Lectures on chaotic dynamical systems*. American Mathematical Society, Providence, RI, 2003.

[6] V. S. Afraĭmovich and Y. B. Pesin. Dimension of Lorenz type attractors. In: *Mathematical physics reviews 6*. Harwood Academic Publishers, Chur, 1987, 169–241.

[7] M. A. D. Aguiar, S. B. S. D. Castro and I. S. Labouriau. Dynamics near a heteroclinic network. *Nonlinearity* **18** (2005) 391–414.

[8] M. A. D. Aguiar, S. B. S. D. Castro and I. S. Labouriau. Simple vector fields with complex behavior. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **16** (2006) 369–381.

[9] K. T. Alligood and J. A. Yorke. Families of periodic orbits: virtual periods and global continuability. *J. Differential Equations* **55** (1984) 59–71.

[10] C. J. Amick and J. F. Toland. Global uniqueness of homoclinic orbits for a class of fourth order equations. *Z. Angew. Math. Phys.* **43** (1992) 591–597.

[11] A. A. Andronov, E. A. Leontovich, I. I. Gordon and A. G. Maĭer. *Theory of bifurcations of dynamic systems on a plane*. Halsted Press, 1973.

[12] V. Araújo. Large deviations bound for semiflows over a non-uniformly expanding base. *Bull. Braz. Math. Soc. (N.S.)* **38** (2007) 335–376.

[13] V. Araújo and M. J. Pacífico. *Three dimensional flows*. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2007.

[14] V. Araújo and M. J. Pacífico. Physical measures for infinite-modal maps. *Fund. Math.* **203** (2009) 211–262.

[15] V. Araújo, M. J. Pacífico, E. R. Pujals and M. Viana. Singular-hyperbolic attractors are chaotic. *Trans. Amer. Math. Soc.* **361** (2009) 2431–2485.

[16] D. Armbruster, J. Guckenheimer and P. J. Holmes. Heteroclinic cycles and modulated travelling waves in systems with O(2) symmetry. *Phys. D* **29** (1988) 257–282.

[17] D. Armbruster, E. Stone and V. Kirk. Noisy heteroclinic networks. *Chaos* **13** (2003) 71–86.

[18] A. Arneodo, P. Coullet, E. A. Spiegel and C. Tresser. Asymptotic chaos. *Phys. D* **14** (1985) 327–347.

[19] A. Arneodo, P. Coullet and C. Tresser. A possible new mechanism for the onset of turbulence. *Phys. Lett. A* **81** (1981) 197–201.

[20] A. Arneodo, P. Coullet and C. Tresser. Possible new strange attractors with spiral structure. *Comm. Math. Phys.* **79** (1981) 573–579.

[21] V. I. Arnol′d, V. S. Afraĭmovich, Y. S. Il′yashenko and L. P. Shil′nikov. *Bifurcation theory and catastrophe theory*. Springer-Verlag, Berlin, 1999.

[22] D. G. Aronson, M. Golubitsky and M. Krupa. Coupled arrays of Josephson junctions and bifurcation of maps with S_N symmetry. *Nonlinearity* **4** (1991) 861–902.

[23] D. G. Aronson, S. A. van Gils and M. Krupa. Homoclinic twist bifurcations with Z_2 symmetry. *J. Nonlinear Sci.* **4** (1994) 195–219.

[24] A. Arroyo and F. Rodriguez Hertz. Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003) 805–841.

[25] P. Ashwin and M. J. Field. Heteroclinic networks in coupled cell systems. *Arch. Ration. Mech. Anal.* **148** (1999) 107–143.

[26] P. Ashwin and J. Montaldi. Group theoretic conditions for existence of robust relative homoclinic trajectories. *Math. Proc. Cambridge Philos. Soc.* **133** (2002) 125–141.

[27] I. Baldomá and T. M. Seara. Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity. *J. Nonlinear Sci.* **16** (2006) 543–582.

[28] R. Bamón, R. Labarca, R. Mañé and M. J. Pacífico. The explosion of singular cycles. *Inst. Hautes Études Sci. Publ. Math.* **78** (1993) 207–232 (1994).

[29] M. Beck, J. Knobloch, D. J. B. Lloyd, B. Sandstede and T. Wagenknecht. Snakes, ladders, and isolas of localised patterns. *SIAM J. Math. Anal.* **41** (2009) 936–972.

[30] G. R. Belitskiĭ. Functional equations, and conjugacy of local diffeomorphisms of finite smoothness class. *Funkcional. Anal. i Priložen.* **7** (1973) 17–28.

[31] G. R. Belitskiĭ and A. Y. Kopanskii. Equivariant Sternberg-Chen theorem. *J. Dynam. Differential Equations* **14** (2002) 349–367.

[32] J. A. Beloqui. Modulus of stability for vector fields on 3-manifolds. *J. Differential Equations* **65** (1986) 374–395.

[33] L. A. Belyakov. Bifurcation set in a system with homoclinic saddle curve. *Math. Notes* **28** (1981) 910–916.

[34] L. A. Belyakov. Bifurcations of systems with a homoclinic curve of the saddle-focus with a zero saddle value. *Mat. Zametki* **36** (1984) 681–689, 798.

[35] V. N. Belykh. Bifurcation of separatrices of a saddle point of the Lorenz system. *Differ. Equations* **20** (1984) 1184–1191.

[36] V. N. Belykh and V. V. Bykov. Bifurcations for heteroclinic orbits of a periodic motion and a saddle-focus and dynamical chaos. *Chaos Solitons Fractals* **9** (1998) 1–18.

[37] P. Bergé, Y. Pomeau and C. Vidal. *Order within chaos: Towards a deterministic approach to turbulence*. John Wiley & Sons, 1984.

[38] M. S. Berger. *Nonlinearity and functional analysis*. Academic Press, New York, 1977.

[39] M. Berti and P. Bolle. Variational construction of homoclinics and chaos in presence of a saddle-saddle equilibrium. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **27** (1998) 331–377.

[40] W.-J. Beyn, A. R. Champneys, E. Doedel, W. Govaerts, Y. A. Kuznetsov and B. Sandstede. Numerical continuation, and computation of normal forms. In: *Handbook of dynamical systems 2*. North-Holland, Amsterdam, 2002, 149–219.

[41] V. S. Biragov and L. P. Shil′nikov. On the bifurcation of a saddle-focus separatrix loop in a three-dimensional conservative dynamical system. *Selecta Math. Soviet* **11** (1992) 333–340.

[42] R. Bogdanov. Versal deformation of a singularity of a vector field on the plane in the case of zero eigenvalues. *Sel. Math. Sov.* **1** (1976) 389–421.

[43] P. Bolle and B. Buffoni. Multibump homoclinic solutions to a centre equilibrium in a class of autonomous Hamiltonian systems. *Nonlinearity* **12** (1999) 1699–1716.

[44] C. Bonatti, L. J. Díaz and M. Viana. *Dynamics beyond uniform hyperbolicity*. Springer-Verlag, Berlin, 2005.

[45] C. Bonatti and E. Dufraine. Équivalence topologique de connexions de selles en dimension 3. *Ergodic Theory Dynam. Systems* **23** (2003) 1347–1381.

[46] C. Bonatti, A. Pumariño and M. Viana. Lorenz attractors with arbitrary expanding dimension. *C. R. Acad. Sci. Paris Sér. I Math.* **325** (1997) 883–888.

[47] P. Bonckaert. Conjugacy of vector fields respecting additional properties. *J. Dynam. Control Systems* **3** (1997) 419–432.

[48] P. Bonckaert and P. De Maesschalck. Gevrey and analytic local models for families of vector fields. *Discrete Contin. Dyn. Syst. Ser. B* **10** (2008) 377–400.

[49] A. R. Borisyuk. Global bifurcations on the Klein bottle. The unimodal case. *Math. Notes* **71** (2002) 316–329.

[50] A. R. Borisyuk. Global bifurcations on a Klein bottle. The general case. *Sb. Math.* **196** (2005) 465–483.

[51] W. Brannath. Heteroclinic networks on the tetrahedron. *Nonlinearity* **7** (1994) 1367–1384.

[52] H. W. Broer. Formal normal form theorems for vector fields and some consequences for bifurcations in the volume preserving case. In: *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, volume 898 of *Lecture Notes in Math.* Springer, Berlin, 1981, 54–74.

[53] H. W. Broer and G. Vegter. Subordinate shil'nikov bifurcations near some singularities of vector fields having low codimension. *Ergodic Theory Dynam. Systems* **4** (1984) 509–525.

[54] P. Brunovský. Tracking invariant manifolds without differential forms. *Acta Math. Univ. Comenian. (N.S.)* **65** (1996) 23–32.

[55] B. Buffoni. Infinitely many large amplitude homoclinic orbits for a class of autonomous Hamiltonian systems. *J. Differential Equations* **121** (1995) 109–120.

[56] B. Buffoni, A. R. Champneys and J. F. Toland. Bifurcation and coalescence of a plethora of homoclinic orbits for a Hamiltonian system. *J. Dynam. Differential Equations* **8** (1996) 221–279.

[57] B. Buffoni and E. Séré. A global condition for quasi-random behavior in a class of conservative systems. *Comm. Pure Appl. Math.* **49** (1996) 285–305.

[58] J. Burke and E. Knobloch. Localized states in the generalized Swift–Hohenberg equation. *Phys. Rev. E* **73** (2006) 056211.

[59] J. Burke and E. Knobloch. Homoclinic snaking: structure and stability. *Chaos* **17** (2007) 037102.

[60] J. Burke and E. Knobloch. Snakes and ladders: localized states in the Swift–Hohenberg equation. *Phys. Lett. A* **360** (2007) 681–688.

[61] V. V. Bykov. The bifurcations of separatrix contours and chaos. *Phys. D* **62** (1993) 290–299.

[62] V. V. Bykov. One mechanism of appearance of a spiral quasiattractor involving heteroclinic contours. *Comput. Math. Appl.* **34** (1997) 229–244.

[63] V. V. Bykov. On bifurcations leading to chaos in Chua's circuit. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8** (1998) 685–699.

[64] V. V. Bykov. On systems with separatrix contour containing two saddle-foci. *J. Math. Sci. (New York)* **95** (1999) 2513–2522.

[65] V. V. Bykov. Orbit structure in a neighborhood of a separatrix cycle containing two saddle-foci. In: *Methods of qualitative theory of differential equations and related topics*. Amer. Math. Soc., Providence, RI, 2000, 87–97.

[66] M. J. D. Carneiro and J. Palis. Bifurcations and global stability of families of gradients. *Inst. Hautes Études Sci. Publ. Math.* **70** (1989) 103–168 (1990).

[67] J. C. Ceballos and R. Labarca. A note on modulus of stability for cycles of the complex type. *Phys. D* **55** (1992) 37–44.

[68] A. R. Champneys. Subsidiary homoclinic orbits to a saddle-focus for reversible systems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **4** (1994) 1447–1482.

[69] A. R. Champneys. Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics. *Phys. D* **112** (1998) 158–186.

[70] A. R. Champneys. Homoclinic orbits in reversible systems ii: Multi-bumps and saddle-centres. *CWI Quarterly* **12** (1999) 185–212.

[71] A. R. Champneys. Codimension-one persistence beyond all orders of homoclinic orbits to singular saddle centres in reversible systems. *Nonlinearity* **14** (2001) 87–112.

[72] A. R. Champneys and J. Härterich. Cascades of homoclinic orbits to a saddle-centre for reversible and perturbed Hamiltonian systems. *Dyn. Stab. Syst.* **15** (2000) 231–252.

[73] A. R. Champneys, J. Härterich and B. Sandstede. A non-transverse homoclinic orbit to a saddle-node equilibrium. *Ergodic Theory Dynam. Systems* **16** (1996) 431–450.

[74] A. R. Champneys and V. Kirk. The entwined wiggling of homoclinic curves emerging from saddle-node/Hopf instabilities. *Phys. D* **195** (2004) 77–105.

[75] A. R. Champneys, V. Kirk, E. Knobloch, B. E. Oldeman and J. D. M. Rademacher. Unfolding a tangent equilibrium-to-periodic heteroclinic cycle. *SIAM J. Appl. Dyn. Syst.* **8** (2009) 1261–1304.

[76] A. R. Champneys, Y. A. Kuznetsov and B. Sandstede. A numerical toolbox for homoclinic bifurcation analysis. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 867–887.

[77] A. R. Champneys and B. Sandstede. Numerical computation of coherent structures. In: *Numerical Continuation Methods for Dynamical Systems* (B. Krauskopf, H. M. Osinga and J. Galan-Vioque, eds.). Springer, 2007, 331–358.

[78] A. R. Champneys and J. F. Toland. Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems. *Nonlinearity* **6** (1993) 665–721.

[79] S. J. Chapman and G. Kozyreff. Exponential asymptotics of localized patterns and snaking bifurcation diagrams. *Phys. D* **238** (2008) 319–354.

[80] K.-T. Chen. Equivalence and decomposition of vector fields about an elementary critical point. *Amer. J. Math.* **85** (1963) 693–722.

[81] L. Cherkas. Structure of the sequence function in the neighborhood of a separatrix cycle during perturbation of an analytic autonomous system on the plane. *Differ. Equations* **17** (1981) 321–328.

[82] D. Chillingworth and R. Lauterbach. Dynamics and geometry in forced symmetry breaking: a tetrahedral example. *Math. Proc. Cambridge Philos. Soc.* **137** (2004) 411–432.

[83] P. Chossat. The reduction of equivariant dynamics to the orbit space for compact group actions. *Acta Appl. Math.* **70** (2002) 71–94.

[84] P. Chossat and D. Armbruster. Dynamics of polar reversals in spherical dynamos. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003) 577–596.

[85] P. Chossat, M. Krupa, I. Melbourne and A. Scheel. Transverse bifurcations of homoclinic cycles. *Phys. D* **100** (1997) 85–100.

[86] P. Chossat, M. Krupa, I. Melbourne and A. Scheel. Magnetic dynamos in rotating convection—a dynamical systems approach. *Dynam. Contin. Discrete Impuls. Systems* **5** (1999) 327–340.

[87] P. Chossat and R. Lauterbach. *Methods in equivariant bifurcations and dynamical systems.* World Scientific Publishing, River Edge, NJ, 2000.

[88] S.-N. Chow and B. Deng. Bifurcation of a unique stable periodic orbit from a homoclinic orbit in infinite-dimensional systems. *Trans. Amer. Math. Soc.* **312** (1989) 539–587.

[89] S.-N. Chow, B. Deng and B. Fiedler. Homoclinic bifurcation at resonant eigenvalues. *J. Dynam. Differential Equations* **2** (1990) 177–244.

[90] S.-N. Chow, B. Deng and D. Terman. The bifurcation of homoclinic and periodic orbits from two heteroclinic orbits. *SIAM J. Math. Anal.* **21** (1990) 179–204.

[91] S.-N. Chow, B. Deng and D. Terman. The bifurcation of homoclinic orbits from two heteroclinic orbits - a topological approach. *Appl. Anal.* **42** (1991) 275–299.

[92] S.-N. Chow and X.-B. Lin. Bifurcation of a homoclinic orbit with a saddle-node equilibrium. *Differential Integral Equations* **3** (1990) 435–466.

[93] E. Colli. Infinitely many coexisting strange attractors. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998) 539–579.

[94] M. J. Costa. Chaotic behaviour of one-dimensional saddle-node horseshoes. *Discrete Contin. Dyn. Syst.* **9** (2003) 505–548.

[95] P. Coullet, C. Riera and C. Tresser. Stable static localized structures in one dimension. *Phys. Rev. Lett.* **84** (2000) 3069–3072.

[96] S. Crovisier. Saddle-node bifurcations for hyperbolic sets. *Ergodic Theory Dynam. Systems* **22** (2002) 1079–1115.

[97] M. Dellnitz, M. J. Field, M. Golubitsky, J. Ma and A. Hohmann. Cycling chaos. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **5** (1995) 1243–1247.

[98] A. Delshams and T. M. Seara. On the existence of homoclinic orbits in a family of vector fields near an equilibrium point. In: *European Conference on Iteration Theory (Caldes de Malavella, 1987)*. World Sci. Publishing, Teaneck, NJ, 1989, 167–177.

[99] B. Deng. The Shil′nikov problem, exponential expansion, strong λ-lemma, C^1-linearization, and homoclinic bifurcation. *J. Differential Equations* **79** (1989) 189–231.

[100] B. Deng. Homoclinic bifurcations with nonhyperbolic equilibria. *SIAM J. Math. Anal.* **21** (1990) 693–720.

[101] B. Deng. The bifurcations of countable connections from a twisted heteroclinic loop. *SIAM J. Math. Anal.* **22** (1991) 653–679.

[102] B. Deng. Homoclinic twisting bifurcations and cusp horseshoe maps. *J. Dynam. Differential Equations* **5** (1993) 417–467.

[103] B. Deng. On Shil′nikov's homoclinic-saddle-focus theorem. *J. Differential Equations* **102** (1993) 305–329.

[104] B. Deng. Exponential expansion with principal eigenvalues. Nonlinear dynamics, bifurcations and chaotic behavior. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 1161–1167.

[105] B. Deng and K. Sakamoto. Shil′nikov-Hopf bifurcations. *J. Differential Equations* **119** (1995) 1–23.

[106] R. L. Devaney. Homoclinic orbits in Hamiltonian systems. *J. Differential Equations* **21** (1976) 431–438.

[107] R. L. Devaney. Blue sky catastrophes in reversible and Hamiltonian systems. *Indiana Univ. Math. Journal* **26** (1977) 247–263.

[108] A. P. S. Dias, B. Dionne and I. Stewart. Heteroclinic cycles and wreath product symmetries. *Dyn. Stab. Syst.* **15** (2000) 353–385.

[109] L. J. Díaz and R. Ures. Critical saddle-node cycles: Hausdorff dimension and persistence of tangencies. *Ergodic Theory Dynam. Systems* **22** (2002) 1117–1140.

[110] K. Díaz-Ordaz, M. P. Holland and S. Luzzatto. Statistical properties of one-dimensional maps with critical points and singularities. *Stoch. Dyn.* **6** (2006) 423–458.

[111] E. Doedel, R. C. Paffenroth, A. R. Champneys, T. F. Fairgrieve, Y. A. Kuznetsov, B. E. Oldeman, B. Sandstede and X. Wang. AUTO2000: Continuation and bifurcation software for ordinary differential equations (with HOMCONT). Technical report, Concordia University, 2002.

[112] E. J. Doedel, B. Krauskopf and H. M. Osinga. Global bifurcations of the Lorenz manifold. *Nonlinearity* **19** (2006) 2947–2972.

[113] R. Driesse and A. J. Homburg. Essentially asymptotically stable homoclinic networks. *Dyn. Syst.* **24** (2009) 459–471.

[114] R. Driesse and A. J. Homburg. Resonance bifurcation from homoclinic cycles. *Journal of Differential Equations* **246** (2009) 2681–2705.

[115] F. Drubi, S. Ibáñez and J. A. Rodríguez. Coupling leads to chaos. *J. Differential Equations* **239** (2007) 371–385.

[116] E. Dufraine. Some topological invariants for three-dimensional flows. *Chaos* **11** (2001) 443–448.

[117] F. Dumortier, S. Ibáñez and H. Kokubu. Cocoon bifurcation in three-dimensional reversible vector fields. *Nonlinearity* **19** (2006) 305–328.

[118] F. Dumortier, H. Kokubu and H. Oka. A degenerate singularity generating geometric Lorenz attractors. *Ergodic Theory Dynam. Systems* **15** (1995) 833–856.

[119] F. Dumortier and R. Roussarie. On the saddle-loop bifurcation. In: *Bifurcations of planar vector fields*, volume 1455 of *LNM*, Springer-Verlag, 1990.

[120] F. Dumortier and R. Roussarie. Geometric singular perturbation theory beyond normal hyperbolicity. In: *Multiple-time-scale dynamical systems (Minneapolis, MN, 1997)*. Springer, New York, 2001, 29–63.

[121] F. Dumortier, R. Roussarie and J. Sotomayor. Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case of codimension 3. *Ergodic Theory Dynam. Systems* **7** (1987) 375–413.

[122] F. Dumortier, R. Roussarie and J. Sotomayor. Bifurcations of cuspidal loops. *Nonlinearity* **10** (1997) 1369–1408.

[123] F. Dumortier, R. Roussarie, J. Sotomayor and H. Żołpolhkadek. *Bifurcations of planar vector fields*, volume 1480 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[124] M. El Morsalani and A. Mourtada. Degenerate and non-trivial hyperbolic 2-polycycles: appearance of two independant écalle-Roussarie compensators and Khovanskiï's theory. *Nonlinearity* **7** (1994) 1593–1604.

[125] K. Engelborghs, T. Luzyanina and G. Samaey. DDE-BIFTOOL: A matlab package for bifurcation analysis of delay differential equations. Technical report, University of Leuven, 2001.

[126] J. W. Evans, N. Fenichel and J. A. Feroe. Double impulse solutions in nerve axon equations. *SIAM J. Appl. Math.* **42** (1982) 219–234.

[127] J. A. Feroe. Existence of traveling wave trains in nerve axon equations. *SIAM J. Appl. Math.* **46** (1986) 1079–1097.

[128] J. A. Feroe. Homoclinic orbits in a parametrized saddle-focus system. *Phys. D* **62** (1993) 254–262. Homoclinic chaos (Brussels, 1991).

[129] B. Fiedler. Global pathfollowing of homoclinic orbits in two-parameter flows. *Pitman Res. Notes Math.* **352** (1996) 79–146.

[130] B. Fiedler and D. V. Turaev. Coalescence of reversible homoclinic orbits causes elliptic resonance. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 1007–1027.

[131] M. J. Field. Equivariant bifurcation theory and symmetry breaking. *J. Dynam. Differential Equations* **1** (1989) 369–421.

[132] M. J. Field. *Lectures on bifurcations, dynamics and symmetry*, volume 356 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1996.

[133] M. J. Field. *Dynamics and symmetry*, volume 3 of *ICP Advanced Texts in Mathematics*. Imperial College Press, London, 2007.

[134] M. J. Field and J. W. Swift. Stationary bifurcation to limit cycles and heteroclinic cycles. *Nonlinearity* **4** (1991) 1001–1043.

[135] A. C. Fowler and C. Sparrow. Bifocal homoclinic orbits in four dimensions. *Nonlinearity* **4** (1991) 1159–1182.

[136] J. Franks and M. Misiurewicz. Topological methods in dynamics. In: *Handbook of dynamical systems 1A*. North-Holland, Amsterdam, 2002, 547–598.

[137] J.-M. Gambaudo, P. Glendinning and C. Tresser. The gluing bifurcation. I. Symbolic dynamics of the closed curves. *Nonlinearity* **1** (1988) 203–214.

[138] J.-M. Gambaudo and C. Tresser. On the dynamics of quasi-contractions. *Bol. Soc. Brasil. Mat.* **19** (1988) 61–114.

[139] P. Gaspard. Generation of a countable set of homoclinic flows through bifurcation. *Phys. Lett. A* **97** (1983) 1–4.

[140] P. Gaspard. Generation of a countable set of homoclinic flow through bifurcation in multidimensional systems. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **70** (1984) 61–83.

[141] P. Gaspard. Local birth of homoclinic chaos. *Phys. D* **62** (1993) 94–122.

[142] P. Gaspard, R. Kapral and G. Nicolis. Bifurcation phenomena near homoclinic systems: a two-parameter analysis. *J. Statist. Phys.* **35** (1984) 697–727.

[143] A. Gaunersdorfer. Time averages for heteroclinic attractors. *SIAM J. Appl. Math.* **52** (1992) 1476–1489.

[144] N. K. Gavrilov and A. L. Shil′nikov. Example of a blue sky catastrophe. In: *Methods of qualitative theory of differential equations and related topics*, volume 200 of *Amer. Math. Soc. Transl. Ser. 2*. Amer. Math. Soc., Providence, RI, 2000, 99–105.

[145] M. Georgi, A. Jukes, J. S. W. Lamb and K. Webster. Homoclinic period blow-up and the detection of multipulse solutions in lattice differential equations. Preprint (2008).

[146] R. W. Ghrist. Resonant gluing bifurcations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **10** (2000) 2141–2160.

[147] R. W. Ghrist, P. J. Holmes and M. C. Sullivan. *Knots and links in three-dimensional flows*, volume 1654 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[148] R. W. Ghrist and T. Young. From Morse-Smale to all knots and links. *Nonlinearity* **11** (1998) 1111–1125.

[149] P. Glendinning. Subsidiary bifurcations near bifocal homoclinic orbits. *Math. Proc. Cambridge Philos. Soc.* **105** (1989) 597–605.

[150] P. Glendinning and C. Sparrow. Local and global behavior near homoclinic orbits. *J. Statist. Phys.* **35** (1984) 645–696.

[151] P. Glendinning and C. Sparrow. T points: a codimension two heteroclinic bifurcation. *J. Statist. Phys.* **43** (1986) 479–488.

[152] A. Golmakani and A. J. Homburg. Lorenz attractors in unfoldings of homoclinic flip bifurcations. *Dyn. Syst.* (2010). (to appear).

[153] M. Golubitsky, I. Stewart and D. G. Schaeffer. *Singularities and groups in bifurcation theory II*. Springer-Verlag, New York, 1988.

[154] S. V. Gonchenko, L. P. Shil′nikov, O. V. Sten′kin and D. V. Turaev. Bifurcations of systems with structurally unstable homoclinic orbits and moduli of Ω-equivalence. *Comput. Math. Appl.* **34** (1997) 111–142.

[155] S. V. Gonchenko, D. V. Turaev, P. Gaspard and G. Nicolis. Complexity in the bifurcation structure of homoclinic loops to a saddle-focus. *Nonlinearity* **10** (1997) 409–423.

[156] C. Grotta Ragazzo. Irregular dynamics and homoclinic orbits to Hamiltonian saddle centers. *Comm. Pure Appl. Math.* **50** (1997) 105–147.

[157] C. Grotta Ragazzo. On the stability of double homoclinic loops. *Comm. Math. Phys.* **184** (1997) 251–272.

[158] M. D. Groves and B. Sandstede. A plethora of three-dimensional periodic travelling gravity-capillary water waves with multipulse transverse profiles. *J. Nonlinear Sci.* **14** (2004) 297–340.

[159] T. M. Grozovskiĭ. Bifurcations of polycycles "apple" and "half-apple" in typical two-parameter families. *Differ. Equations* **32** (1996) 459–470.

[160] J. Guckenheimer. On a codimension two bifurcation. In: *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, volume 898 of *Lecture Notes in Math.* Springer, Berlin, 1981, 99–142.

[161] J. Guckenheimer and P. J. Holmes. Structurally stable heteroclinic cycles. *Math. Proc. Cambridge Philos. Soc.* **103** (1988) 189–192.

[162] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.* **50** (1979) 59–72.

[163] J. Guckenheimer and P. A. Worfolk. Instant chaos. *Nonlinearity* **5** (1992) 1211–1222.

[164] L.-S. Guimond. Homoclinic loop bifurcations on a Möbius band. *Nonlinearity* **12** (1999) 59–78.

[165] L.-S. Guimond and C. Rousseau. Finite cyclicity of finite codimension nondegenerate homoclinic loops with real eigenvalues in $\mathbb{R}^3$. *Qual. Theory Dyn. Syst.* **2** (2001) 151–204.

[166] J. K. Hale and X.-B. Lin. Examples of transverse homoclinic orbits in delay equations. *Nonlinear Anal.* **10** (1986) 693–709.

[167] J. K. Hale and X.-B. Lin. Heteroclinic orbits for retarded functional-differential equations. *J. Differential Equations* **65** (1986) 175–202.

[168] G. Haller. *Chaos near resonance*, volume 138 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1999.

[169] J. Härterich. Cascades of reversible homoclinic orbits to a saddle-focus equilibrium. *Physica D* **112** (1998) 187–200.

[170] J. Härterich, B. Sandstede and A. Scheel. Exponential dichotomies for linear non-autonomous functional differential equations of mixed type. *Indiana Univ. Math. J.* **51** (2002) 1081–1109.

[171] S. P. Hastings and W. C. Troy. A proof that the Lorenz equations have a homoclinic orbit. *J. Differential Equations* **113** (1994) 166–188.

[172] D. Hawker and P. Ashwin. Classification of robust heteroclinic cycles for vector fields in $\mathbb{R}^3$ with symmetry. *J. Phys. A* **38** (2005) 8319–8335.

[173] D. Hawker and P. Ashwin. Robust bursting to the origin: heteroclinic cycles with maximal symmetry equilibria. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **15** (2005) 2819–2832.

[174] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. of Math. (2)* **145** (1997) 81–137.

[175] S. Hayashi. Correction to: "Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows" [Ann. of Math. (2) **145** (1997), no. 1, 81–137]. *Ann. of Math. (2)* **150** (1999) 353–356.

[176] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1981.

[177] M. W. Hirsch, C. C. Pugh and M. Shub. *Invariant manifolds.* Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 583.

[178] P. Hirschberg and E. Knobloch. Shil'nikov-Hopf bifurcation. *Physica D* **62** (1993) 202–216.

[179] J. Hofbauer and K. Sigmund. *Evolutionary games and population dynamics.* Cambridge University Press, Cambridge, 1998.

[180] P. J. Holmes. A strange family of three-dimensional vector fields near a degenerate singularity. *J. Differential Equations* **37** (1980) 382–403.

[181] P. J. Holmes. Symmetries, heteroclinic cycles and intermittency in fluid flow. In: *Turbulence in fluid flows.* Springer, New York, 1993, 49–58.

[182] A. J. Homburg. Global aspects of homoclinic bifurcations of vector fields. *Mem. Amer. Math. Soc.* **121** (1996) viii+128.

[183] A. J. Homburg. Heteroclinic bifurcations of Ω-stable vector fields on 3-manifolds. *Discrete Contin. Dynam. Systems* **4** (1998) 559–580.

[184] A. J. Homburg. Singular heteroclinic cycles. *J. Differential Equations* **161** (2000) 358–402.

[185] A. J. Homburg. Periodic attractors, strange attractors and hyperbolic dynamics near homoclinic orbits to saddle-focus equilibria. *Nonlinearity* **15** (2002) 1029–1050.

[186] A. J. Homburg, A. Jukes, J. Knobloch and J. S. W. Lamb. Saddle-nodes and period-doublings of smale horseshoes: a case study near resonant homoclinic bellows. *Bull. Belg. Math. Soc. Simon Stevin* **15** (2008) 833–850.

[187] A. J. Homburg, A. Jukes, J. Knobloch and J. S. W. Lamb. Bifurcations from codimension one homoclinic cycles in equivariant flows. *Trans. Amer. Math. Soc.* (2010). (to appear).

[188] A. J. Homburg and J. Knobloch. Multiple homoclinic orbits in conservative and reversible systems. *Trans. Amer. Math. Soc.* **358** (2006) 1715–1740.

[189] A. J. Homburg, H. Kokubu and M. Krupa. The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic orbit. *Ergodic Theory Dynam. Systems* **14** (1994) 667–693.

[190] A. J. Homburg, H. Kokubu and V. Naudot. Homoclinic-doubling cascades. *Arch. Rational Mech. Anal.* **160** (2001) 195–243.

[191] A. J. Homburg and B. Krauskopf. Resonant homoclinic flip bifurcations. *J. Dynam. Differential Equations* **12** (2000) 807–850.

[192] A. J. Homburg and J. S. W. Lamb. Symmetric homoclinic tangles in reversible systems. *Ergodic Theory Dynam. Systems* **26** (2006) 1769–1789.

[193] A. J. Homburg and M. A. Natiello. Accumulations of T-points in a model for solitary pulses in an excitable reaction-diffusion medium. *Phys. D* **201** (2005) 212–229.

[194] A. J. Homburg and T. Young. Universal scalings in homoclinic doubling cascades. *Comm. Math. Phys.* **222** (2001) 269–292.

[195] C. Hou and M. Golubitsky. An example of symmetry breaking to heteroclinic cycles. *J. Differ. Equations* **133** (1997) 30–48.

[196] J. H. Hubbard and C. Sparrow. The classification of topologically expansive Lorenz maps. *Comm. Pure Appl. Math.* **43** (1990) 431–443.

[197] H. J. Hupkes and S. M. Verduyn-Lunel. Lin's method and homoclinic bifurcations for functional differential equations of mixed type. *Indiana Univ. Math. J.* **58** (2009) 2433–2487.

[198] S. Ibáñez and J. A. Rodríguez. Shil′nikov bifurcations in generic 4-unfoldings of a codimension-4 singularity. *J. Differential Equations* **120** (1995) 411–428.

[199] S. Ibáñez and J. A. Rodríguez. Shil′nikov configurations in any generic unfolding of the nilpotent singularity of codimension three on $\mathbb{R}^3$. *J. Differential Equations* **208** (2005) 147–175.

[200] Y. S. Il′yashenko. Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane. *Funct. Anal. Appl.* **18** (1984) 199–209.

[201] Y. S. Il′yashenko. Centennial history of Hilbert's 16th problem. *Bull. Amer. Math. Soc. (N.S.)* **39** (2002) 301–354.

[202] Y. S. Il′yashenko and W. Li. *Nonlocal bifurcations*, volume 66 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[203] Y. S. Il′yashenko and S. Y. Yakovenko. Finitely-smooth normal forms of local families of diffeomorphisms and vector fields. *Russ. Math. Surv.* **46** (1991) 1–43.

[204] Y. S. Il'yashenko and S. Y. Yakovenko. Finite cyclicity of elementary polycycles in generic families. In: *Concerning the Hilbert 16th problem*. Amer. Math. Soc., Providence, RI, 1995, 21–95.

[205] G. Iooss and E. Lombardi. Normal forms with exponentially small remainder: application to homoclinic connections for the reversible $0^{2+}i\omega$ resonance. *C. R. Math. Acad. Sci. Paris* **339** (2004) 831–838.

[206] G. Iooss and E. Lombardi. Polynomial normal forms with exponentially small remainder for analytic vector fields. *J. Differential Equations* **212** (2005) 1–61.

[207] G. Iooss and M.-C. Pérouème. Perturbed homoclinic solutions in reversible 1 : 1 resonance vector fields. *J. Differential Equations* **102** (1993) 62–88.

[208] C. K. R. T. Jones, T. J. Kaper and N. Kopell. Tracking invariant manifolds up to exponentially small errors. *SIAM J. Math. Anal.* **27** (1996) 558–577.

[209] W. D. Kalies, J. Kwapisz and R. C. A. M. Vandervorst. Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria. *Comm. Math. Phys.* **193** (1998) 337–371.

[210] V. Y. Kaloshin. The existential Hilbert 16-th problem and an estimate for cyclicity of elementary polycycles. *Invent. Math.* **151** (2003) 451–512.

[211] T. Kapitula and S. Maier-Paape. Spatial dynamics of time periodic solutions for the Ginzburg-Landau equation. *Z. Angew. Math. Phys.* **47** (1996) 265–305.

[212] J. P. Keener. Chaotic behavior in piecewise continuous difference equations. *Trans. Amer. Math. Soc.* **261** (1980) 589–604.

[213] P. Kent and J. Elgin. Noose bifurcation of periodic orbits. *Nonlinearity* **4** (1991) 1045–1061.

[214] V. Kirk. Breaking of symmetry in the saddle-node Hopf bifurcation. *Phys. Lett. A* **154** (1991) 243–248.

[215] V. Kirk. Merging of resonance tongues. *Phys. D* **66** (1993) 267–281.

[216] V. Kirk and M. Silber. A competition between heteroclinic cycles. *Nonlinearity* **7** (1994) 1605–1621.

[217] M. Kisaka, H. Kokubu and H. Oka. Bifurcations to N-homoclinic orbits and N-periodic orbits in vector fields. *J. Dynam. Differential Equations* **5** (1993) 305–357.

[218] M. Kisaka, H. Kokubu and H. Oka. Supplement to homoclinic doubling bifurcation in vector fields. In: *Dynamical systems (Santiago, 1990)*. Longman Sci. Tech., Harlow, 1993, 92–116.

[219] J. Klaus and J. Knobloch. Bifurcation of homoclinic orbits to a saddle-center in reversible systems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003) 2603–2622.

[220] E. Knobloch. Spatially localized structures in dissipative systems: open problems. *Nonlinearity* **21** (2008) T45–T60.

[221] J. Knobloch. Bifurcation of degenerate homoclinics in reversible and conservative systems. *J. Dynam. Differential Equations* **9** (1997) 427–444.

[222] J. Knobloch and T. Rieß. Lin's method for heteroclinic chains involving periodic orbits. *Nonlinearity* **23** (2010) 23–54.

[223] J. Knobloch and T. Wagenknecht. Homoclinic snaking near a heteroclinic cycle in reversible systems. *Phys. D* **206** (2005) 82–93.

[224] J. Knobloch and T. Wagenknecht. Snaking of multiple homoclinic orbits in reversible systems. *SIAM J. Appl. Dyn. Syst.* **7** (2008) 1397–1420.

[225] H. Kokubu. Homoclinic and heteroclinic bifurcations of vector fields. *Japan J. Appl. Math.* **5** (1988) 455–501.

[226] H. Kokubu, M. Komuro and H. Oka. Multiple homoclinic bifurcations from orbit flip. I. Successive homoclinic-doublings. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 833–850.

[227] H. Kokubu and V. Naudot. Existence of infinitely many homoclinic-doubling bifurcations from some codimension- three homoclinic orbits. *J. Dynam. Differential Equations* **9** (1997) 445–462.

[228] H. Kokubu and R. Roussarie. Existence of a singularly degenerate heteroclinic cycle in the Lorenz system and its dynamical consequences. I. *J. Dynam. Differential Equations* **16** (2004) 513–557.

[229] O. Y. Koltsova. Families of multi-round homoclinic and periodic orbits near a saddle-center equilibrium. *Regul. Chaotic Dyn.* **8** (2003) 191–200.

[230] O. Y. Koltsova and L. M. Lerman. Families of transverse Poincaré homoclinic orbits in 2N-dimensional Hamiltonian systems close the system with a loop to a saddle -center. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 991–1006.

[231] A. Kotova and V. Stanzo. On few-parameter generic families of vector fields on the two-dimensional sphere. In: *Concerning the Hilbert 16th problem.* Amer. Math. Soc., Providence, RI, 1995, 155–201.

[232] G. Kozyreff and S. J. Chapman. Asymptotics of large bound states of localized structures. *Phys. Rev. Lett.* **97** (2006) 044502.

[233] B. Krauskopf and B. E. Oldeman. A planar model system for the saddle-node Hopf bifurcation with global reinjection. *Nonlinearity* **17** (2004) 1119–1151.

[234] B. Krauskopf and B. E. Oldeman. Bifurcations of global reinjection orbits near a saddle-node hopf bifurcation. *Nonlinearity* **19** (2006) 2149–2167.

[235] B. Krauskopf and T. Rieß. A Lin's method approach to finding and continuing heteroclinic connections involving periodic orbits. *Nonlinearity* **21** (2008) 1655–1690.

[236] M. Krupa. Robust heteroclinic cycles. *J. Nonlinear Sci.* **7** (1997) 129–176.

[237] M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry. *Ergodic Theory Dynam. Systems* **15** (1995) 121–147.

[238] M. Krupa and I. Melbourne. Nonasymptotically stable attractors in O(2) mode interactions. In: *Normal forms and homoclinic chaos (Waterloo, ON, 1992).* Amer. Math. Soc., Providence, RI, 1995, 219–232.

[239] M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry. II. *Proc. Roy. Soc. Edinburgh Sect. A* **134** (2004) 1177–1197.

[240] M. Krupa, B. Sandstede and P. Szmolyan. Fast and slow waves in the FitzHugh-Nagumo equation. *J. Differential Equations* **133** (1997) 49–97.

[241] M. Krupa and P. Szmolyan. Extending slow manifolds near transcritical and pitchfork singularities. *Nonlinearity* **14** (2001) 1473–1491.

[242] Y. A. Kuznetsov. *Elements of applied bifurcation theory.* Springer-Verlag, New York, 2004.

[243] Y. A. Kuznetsov, O. De Feo and S. Rinaldi. Belyakov homoclinic bifurcations in a tritrophic food chain model. *SIAM J. Appl. Math.* **62** (2001) 462–487.

[244] R. Labarca. Stability of parametrized families of vector fields. In: *Dynamical systems and bifurcation theory (Rio de Janeiro, 1985).* Longman Sci. Tech., Harlow, 1987, 121–213.

[245] R. Labarca. Bifurcation of contracting singular cycles. *Ann. Sci. École Norm. Sup. (4)* **28** (1995) 705–745.

[246] R. Labarca and M. J. Pacífico. Stability of singularity horseshoes. *Topology* **25** (1986) 337–352.

[247] R. Labarca and S. Plaza. Global stability of families of vector fields. *Ergodic Theory Dynam. Systems* **13** (1993) 737–766.

[248] J. S. W. Lamb and J. A. G. Roberts. Time-reversal symmetry in dynamical systems: a survey. *Phys. D* **112** (1998) 1–39.

[249] J. S. W. Lamb, M. A. Teixeira and K. N. Webster. Heteroclinic bifurcations near Hopf-zero bifurcation in reversible vector fields in $\mathbb{R}^3$. *J. Differential Equations* **219** (2005) 78–115.

[250] Y.-T. Lau. The "cocoon" bifurcations in three-dimensional systems with two fixed points. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **2** (1992) 543–558.

[251] R. Lauterbach, S. Maier-Paape and E. Reissner. A systematic study of heteroclinic cycles in dynamical systems with broken symmetries. *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996) 885–909.

[252] R. Lauterbach and M. Roberts. Heteroclinic cycles in dynamical systems with broken spherical symmetry. *J. Differential Equations* **100** (1992) 22–48.

[253] G. A. Leonov. On estimates of the bifurcation values of the parameters of a Lorenz system. *Russ. Math. Surv.* **43** (1988) 216–217.

[254] E. A. Leontovich. On the generation of limit cycles from separatrices. *Doklady Akad. Nauk SSSR (N.S.)* **78** (1951) 641–644.

[255] L. M. Lerman. Hamiltonian systems with a separatrix loop of a saddle-center. In: *Methods of the qualitative theory of differential equations (Russian)*. Gor′kov. Gos. Univ., Gorki, 1987, 89–103.

[256] X.-B. Lin. Using Melnikovs method to solve Shil′nikovs problems. *Proc. Roy. Soc. Edinburgh* **116A** (1990) 295–325.

[257] X.-B. Lin. Construction and asymptotic stability of structurally stable internal layer solutions. *Trans. Amer. Math. Soc.* **353** (2001) 2983–3043.

[258] E. Lombardi. Non-persistence of homoclinic connections for perturbed integrable reversible systems. *J. Dynam. Differential Equations* **11** (1999) 129–208.

[259] E. Lombardi. *Oscillatory integrals and phenomena beyond all algebraic orders*, volume 1741 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[260] E. N. Lorenz. Deterministic non-periodic flow. *J. Atmos. Sci.* **20** (1963) 130–141.

[261] V. I. Lukyanov. Bifurcations of dynamical systems with a saddle-point-separatrix loop. *Differ. Equations* **18** (1983) 1049–1059.

[262] S. Luzzatto, I. Melbourne and F. Paccaut. The Lorenz attractor is mixing. *Comm. Math. Phys.* **260** (2005) 393–401.

[263] S. Luzzatto and W. Tucker. Non-uniformly expanding dynamics in maps with singularities and criticalities. *Inst. Hautes Études Sci. Publ. Math.* **89** (1999) 179–226.

[264] S. Luzzatto and M. Viana. Positive Lyapunov exponents for Lorenz-like families with criticalities. *Astérisque* **261** (2000) xiii, 201–237.

[265] D. V. Lyubimov, A. S. Pikovsky and M. A. Zaks. *Universal scenarios of transitions to chaos via homoclinic bifurcations*, volume 8 of *Soviet Scientific Reviews, Section C: Mathematical Physics Reviews*. Harwood Academic Publishers, Chur, 1989.

[266] J. Mallet-Paret and S. M. Verduyn-Lunel. Exponential dichotomies and Wiener-Hopf factorizations for mixed-type functional differential equations. *J. Diff. Eqns.* (to appear).

[267] V. Manukian. *Existence and stability of multi-pulses with applications to nonlinear optics*. Ph.D. thesis, Ohio State University, 2005.

[268] V. Manukian and B. Sandstede. Multi-hump pulses in systems with reflection and phase invariance. *J. Differential Equations* **247** (2009) 1866–1898.

[269] K. Matthies. A subshift of finite type in the Takens-Bogdanov bifurcation with D_3 symmetry. *Doc. Math.* **4** (1999) 463–485.

[270] V. S. Medvedev. On a new type of bifurcations on manifolds. *Math. USSR, Sb.* **41** (1982) 403–407.

[271] V. S. Medvedev. On the "blue sky catastrophe" bifurcation on two-dimensional manifolds. *Mat. Zametki* **51** (1992) 118–125.

[272] I. Melbourne. Intermittency as a codimension-three phenomenon. *J. Dynam. Differential Equations* **1** (1989) 347–367.

[273] I. Melbourne. An example of a nonasymptotically stable attractor. *Nonlinearity* **4** (1991) 835–844.

[274] D. Michelson. Steady solutions of the Kuramoto-Sivashinsky equation. *Phys. D* **19** (1986) 89–111.

[275] A. Mielke, P. J. Holmes and O. O'Reilly. Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle-center. *J. Dynam. Differential Equations* **4** (1992) 95–126.

[276] K. Mischaikow and M. Mrozek. Conley index. In: *Handbook of dynamical systems 2.* North-Holland, Amsterdam, 2002, 393–460.

[277] C. A. Morales. Lorenz attractor through saddle-node bifurcations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** (1996) 589–617.

[278] C. A. Morales. On inclination-flip homoclinic orbit associated to a saddle-node singularity. *Bol. Soc. Brasil. Mat. (N. S.)* **27** (1996) 145–160.

[279] C. A. Morales and M. J. Pacífico. Degenerated singular cycles of inclination-flip type. *Ann. Sci. École Norm. Sup. (4)* **31** (1998) 1–16.

[280] C. A. Morales and M. J. Pacífico. Inclination-flip homoclinic orbits arising from orbit-flip. *Nonlinearity* **14** (2001) 379–393.

[281] C. A. Morales, M. J. Pacífico and E. R. Pujals. Strange attractors across the boundary of hyperbolic systems. *Comm. Math. Phys.* **211** (2000) 527–558.

[282] C. A. Morales, M. J. Pacífico and E. R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math. (2)* **160** (2004) 375–432.

[283] C. A. Morales, M. J. Pacífico and B. San Martín. Expanding Lorenz attractors through resonant double homoclinic loops. *SIAM J. Math. Anal.* **36** (2005) 1836–1861.

[284] C. A. Morales, M. J. Pacífico and B. San Martín. Contracting Lorenz attractors through resonant double homoclinic loops. *SIAM J. Math. Anal.* **38** (2006) 309–332.

[285] C. A. Morales and E. R. Pujals. Singular strange attractors on the boundary of Morse-Smale systems. *Ann. Sci. École Norm. Sup. (4)* **30** (1997) 693–717.

[286] A. Mourtada. Cyclicité finie des polycycles hyperboliques de champs de vecteurs du plan: mise sous forme normale. In: *Bifurcations of planar vector fields (Luminy, 1989)*, volume 1455 of *Lecture Notes in Math.* Springer, Berlin, 1990, 272–314.

[287] A. Mourtada. Cyclicité finie des polycycles hyperboliques de champs de vecteurs du plan. Algorithme de finitude. *Ann. Inst. Fourier (Grenoble)* **41** (1991) 719–753.

[288] A. Mourtada. Degenerate and non-trivial hyperbolic polycycles with two vertices. *J. Differential Equations* **113** (1994) 68–83.

[289] E. M. Muñoz Morales, B. San Martín and J. A. Vera Valenzuela. Nonhyperbolic persistent attractors near the Morse-Smale boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003) 867–888.

[290] V. Naudot. Strange attractor in the unfolding of an inclination flip homoclinic orbit. *Ergodic Theory Dynam. Systems* **16** (1996) 1071–1086.

[291] S. Newhouse, J. Palis and F. Takens. Stable arcs of diffeomorphisms. *Bull. Amer. Math. Soc.* **82** (1976) 499–502.

[292] S. Newhouse, J. Palis and F. Takens. Bifurcations and stability of families of diffeomorphisms. *Inst. Hautes Études Sci. Publ. Math.* **57** (1983) 5–71.

[293] H. K. Nguyen and A. J. Homburg. Global bifurcations to strange attractors in a model for skewed varicose instability in thermal convection. *Phys. D* **211** (2005) 235–262.

[294] H. K. Nguyen and A. J. Homburg. Resonant heteroclinic cycles and Lorenz type attractors in models for skewed varicose instability. *Nonlinearity* **18** (2005) 155–173.

[295] S. Nii. *N*-homoclinic bifurcations for homoclinic orbits changing their twisting. *J. Dynam. Differential Equations* **8** (1996) 549–572.

[296] V. P. Nozdracheva. Bifurcations of a structurally unstable separatrix loop. *Differentsial'nye Uravneniya* **18** (1982) 1551–1558, 1654.

[297] B. E. Oldeman, A. R. Champneys and B. Krauskopf. Homoclinic branch switching: a numerical implementation of Lin's method. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003) 2977–2999.

[298] B. E. Oldeman, B. Krauskopf and A. R. Champneys. Death of period-doublings: locating the homoclinic-doubling cascade. *Phys. D* **146** (2000) 100–120.

[299] I. M. Ovsyannikov and L. P. Shil'nikov. On systems with saddle-focus homoclinic curve. *Math. USSR Sbornik* **58** (1987) 557–574.

[300] I. M. Ovsyannikov and L. P. Shil'nikov. Systems with a homoclinic curve of multidimensional saddle-focus type, and spiral chaos. *Math. USSR Sbornik* **73** (1992) 415–443.

[301] M. J. Pacífico and A. Rovella. Unfolding contracting singular cycles. *Ann. Sci. École Norm. Sup. (4)* **26** (1993) 691–700.

[302] M. J. Pacífico, A. Rovella and M. Viana. Infinite-modal maps with global chaotic behavior. *Ann. of Math. (2)* **148** (1998) 1–44.

[303] M. J. Pacífico, A. Rovella and M. Viana. Corrigendum: "Infinite-modal maps with global chaotic behavior" [Ann. of Math. (2) **148** (1998), no. 2, 441–484]. *Ann. of Math. (2)* **149** (1999) 705.

[304] J. Palis. A differentiable invariant of topological conjugacies and moduli of stability. In: *Dynamical systems, Vol. III—Warsaw*. Soc. Math. France, Paris, 1978, 335–346. Astérisque, No. 51.

[305] J. Palis. A global perspective for non-conservative dynamics. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005) 485–507.

[306] J. Palis. Open questions leading to a global perspective in dynamics. *Nonlinearity* **21** (2008) T37–T43.

[307] J. Palis and C. C. Pugh. Fifty problems in dynamical systems. In: *Dynamical systems—Warwick 1974 (Proc. Sympos. Appl. Topology and Dynamical Systems, Univ. Warwick, Coventry, 1973/1974; presented to E. C. Zeeman on his fiftieth birthday)*. Springer, Berlin, 1975, 345–353. Lecture Notes in Math., Vol. 468.

[308] J. Palis and F. Takens. Stability of parametrized families of gradient vector fields. *Ann. of Math. (2)* **118** (1983) 383–421.

[309] J. Palis and F. Takens. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*. Cambridge University Press, 1993.

[310] M. J. Parker, M. G. M. Gomes and I. Stewart. Forced symmetry-breaking of square lattice planforms. *J. Dynam. Differential Equations* **18** (2006) 223–255.

[311] D. Peterhof, B. Sandstede and A. Scheel. Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders. *J. Differential Equations* **140** (1997) 266–308.

[312] S. Plaza and J. Vera. Bifurcation and stability of families of hyperbolic vector fields in dimension three. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997) 119–142.

[313] J. Porter and E. Knobloch. Dynamics in the 1 : 2 spatial resonance with broken reflection symmetry. *Phys. D* **201** (2005) 318–344.

[314] C. M. Postlethwaite and J. H. P. Dawes. Regular and irregular cycling near a heteroclinic network. *Nonlinearity* **18** (2005) 1477–1509.

[315] C. M. Postlethwaite and J. H. P. Dawes. A codimension-two resonant bifurcation from a heteroclinic cycle with complex eigenvalues. *Dyn. Syst.* **21** (2006) 313–336.

[316] M. R. E. Proctor and C. A. Jones. The interaction of two spatially resonant patterns in thermal convection. I. Exact 1 : 2 resonance. *J. Fluid Mech.* **188** (1988) 301–335.

[317] E. R. Pujals and M. Sambarino. Homoclinic bifurcations, dominated splitting, and robust transitivity. In: *Handbook of dynamical systems. Vol. 1B.* Elsevier B. V., Amsterdam, 2006, 327–378.

[318] A. Pumariño and J. A. Rodríguez. *Coexistence and persistence of strange attractors*, volume 1658 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1997.

[319] A. Pumariño and J. A. Rodríguez. Coexistence and persistence of infinitely many strange attractors. *Ergodic Theory Dynam. Systems* **21** (2001) 1511–1523.

[320] P. H. Rabinowitz. Variational methods for Hamiltonian systems. In: *Handbook of dynamical systems, Vol. 1A.* North-Holland, Amsterdam, 2002, 1091–1127.

[321] J. D. M. Rademacher. Homoclinic orbits near heteroclinic cycles with one equilibrium and one periodic orbit. *J. Differential Equations* **218** (2005) 390–443.

[322] J. D. M. Rademacher. Lyapunov-scmidt reduction for unfolding heteroclinic networks of equilibria and periodic orbits with tangencies. Preprint (2008).

[323] D. Rand. The topological classification of Lorenz attractors. *Math. Proc. Cambridge Philos. Soc.* **83** (1978) 451–460.

[324] R. C. Robinson. Differentiability of the stable foliation for the model Lorenz equations. In: *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, volume 898 of *Lecture Notes in Math.* Springer, Berlin, 1981, 302–315.

[325] R. C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. *Nonlinearity* **2** (1989) 495–518.

[326] R. C. Robinson. Homoclinic bifurcation to a transitive attractor of Lorenz type. II. *SIAM J. Math. Anal.* **23** (1992) 1255–1268.

[327] R. C. Robinson. Nonsymmetric Lorenz attractors from a homoclinic bifurcation. *SIAM J. Math. Anal.* **32** (2000) 119–141.

[328] M. M. Romeo and C. K. R. T. Jones. The stability of traveling calcium pulses in a pancreatic acinar cell. *Phys. D* **177** (2003) 242–258.

[329] R. Roussarie. On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields. *Bol. Soc. Brasil. Mat.* **17** (1986) 67–101.

[330] R. Roussarie. *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, volume 164 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 1998.

[331] R. Roussarie and C. Rousseau. Almost planar homoclinic loops in $\mathbb{R}^3$. *Journ. Diff. Eq.* **126** (1996) 1–47.

[332] R. Roussarie and F. Wagener. A study of the Bogdanov-Takens bifurcation. *Resenhas* **2** (1995) 1–25.

[333] A. Rovella. The dynamics of perturbations of contracting Lorenz maps. *Bol. Soc. Brasil. Mat. (N. S.)* **24** (1993) 233–259.

[334] M. R. Rychlik. Lorenz attractors through Shil'nikov-type bifurcation. Part I. *Ergod. Th. & Dynam. Syst.* **10** (1990) 793–821.

[335] G. Samaey, K. Engelborghs and D. Roose. Numerical computation of connecting orbits in delay differential equations. *Numer. Algorithms* **30** (2002) 335–352.

[336] B. San Martín. Contracting singular cycles. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998) 651–659.

[337] B. San Martín. Saddle-focus singular cycles and prevalence of hyperbolicity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998) 623–649.

[338] B. Sandstede. Multi-pulses in reversible conservative vector fields. In preparation .

[339] B. Sandstede. *Verzweigungstheorie homokliner Verdopplungen.* Ph.D. thesis, University of Stuttgart, 1993.

[340] B. Sandstede. Constructing dynamical systems having homoclinic bifurcation points of codimension two. *J. Dynam. Differential Equations* **9** (1997) 269–288.

[341] B. Sandstede. Instability of localized buckling modes in a one-dimensional strut model. *Philos. Trans. Roy. Soc. London Ser. A* **355** (1997) 2083–2097.

[342] B. Sandstede. Stability of *N*-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh-Nagumo equation. *SIAM J. Math. Anal.* **29** (1998) 183–207.

[343] B. Sandstede. Center manifolds for homoclinic solutions. *J. Dynam. Differential Equations* **12** (2000) 449–510.

[344] B. Sandstede. Stability of travelling waves. In: *Handbook of dynamical systems 2.* North-Holland, Amsterdam, 2002, 983–1055.

[345] B. Sandstede, C. K. R. T. Jones and J. C. Alexander. Existence and stability of N-pulses on optical fibres with phase-sensitive amplifiers. *Phys. D* **106** (1997) 167–206.

[346] B. Sandstede and A. Scheel. Forced symmetry breaking of homoclinic cycles. *Nonlinearity* **8** (1995) 333–365.

[347] B. Sandstede and A. Scheel. Essential instabilities of fronts: bifurcation, and bifurcation failure. *Dyn. Syst.* **16** (2001) 1–28.

[348] B. Sandstede and A. Scheel. Defects in oscillatory media: toward a classification. *SIAM J. Appl. Dynam. Syst.* **3** (2004) 1–68.

[349] S. Schecter. The saddle-node separatrix-loop bifurcation. *SIAM J. Math. Anal.* **18** (1987) 1142–1156.

[350] S. Schecter. Exchange lemmas. II. General exchange lemma. *J. Differential Equations* **245** (2008) 411–441.

[351] A. Scheel and P. Chossat. Bifurcation d'orbites périodiques à partir d'un cycle homocline symétrique. *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992) 49–54.

[352] G. R. Sell. Smooth linearization near a fixed point. *Amer. J. Math.* **107** (1985) 1035–1091.

[353] M. V. Shashkov. On bifurcations of separatrix contours with two saddles. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **2** (1992) 911–915.

[354] M. V. Shashkov. On the bifurcations of separatrix contours on two-dimensional surfaces. I [translation of *methods in qualitative theory and bifurcation theory (russian)*, 116–129, Gor′kov. Gos. Univ., Gorki, 1989; MR1102920 (92d:34086)]. *Selecta Math. Soviet.* **11** (1992) 341–353. Selected translations.

[355] M. V. Shashkov. On bifurcations of separatrix contours on two-dimensional surfaces. II [translation of *methods in qualitative theory and bifurcation theory (russian)*, 41–49, Nizhegorod. Gos. Univ., Nizhniĭ Novgorod, 1990; MR1224008 (94f:34081)]. *Selecta Math.* **13** (1994) 175–182. Selected translations.

[356] M. V. Shashkov. Impossibility of complete bifurcations description for some classes of systems with simple dynamics. In: *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999).* World Sci. Publ., River Edge, NJ, 2000, 68–70.

[357] M. V. Shashkov and L. P. Shil′nikov. The existence of a smooth invariant foliation for Lorentz-type maps. *Differ. Equations* **30** (1994) 536–544.

[358] M. V. Shashkov and D. V. Turaev. On the complex bifurcation set for a system with simple dynamics. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 949–968.

[359] M. V. Shashkov and D. V. Turaev. An existence theorem of smooth nonlocal center manifolds for systems close to a system with a homoclinic loop. *J. Nonlinear Sci.* **9** (1999) 525–573.

[360] M. V. Shashkov and D. V. Turaev. A proof of Shilnikov's theorem for C^1-smooth dynamical systems. In: *Methods of qualitative theory of differential equations and related topics*. Amer. Math. Soc., Providence, RI, 2000, 149–163.

[361] J. Shatah and C. Zeng. Orbits homoclinic to centre manifolds of conservative PDEs. *Nonlinearity* **16** (2003) 591–614.

[362] A. L. Shil'nikov. On bifurcations of the Lorenz attractor in the Shimizu-Morioka model. *Phys. D* **62** (1993) 338–346.

[363] A. L. Shil'nikov and G. Cymbalyuk. Homoclinic bifurcations of periodic orbits en a route from tonic-spiking to bursting in neuron models. *Regul. Chaotic Dyn.* **9** (2004) 281–297.

[364] A. L. Shil'nikov, L. P. Shil'nikov and D. V. Turaev. Normal forms and Lorenz attractors. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **3** (1993) 1123–1139.

[365] A. L. Shil'nikov, L. P. Shil'nikov and D. V. Turaev. Blue-sky catastrophe in singularly perturbed systems. *Mosc. Math. J.* **5** (2005) 269–282.

[366] L. P. Shil'nikov. Some cases of generation of period motions from singular trajectories. *Mat. Sb. (N.S.)* **61(103)** (1963) 443–466.

[367] L. P. Shil'nikov. A case of the existence of a countable number of periodic motions. *Soviet Math. Dokl.* **6** (1965) 163–166.

[368] L. P. Shil'nikov. On the generation of a periodic motion from a trajectory which leaves and re-enters a saddle-saddle state of equilibrium. *Sov. Math., Dokl.* **7** (1966) 1155–1158.

[369] L. P. Shil'nikov. On the generation of a periodic motion from trajectories doubly asymptotic to an equilibrium state of saddle type. *Math. USSR Sbornik* **6** (1968) 427–438.

[370] L. P. Shil'nikov. A certain new type of bifurcation of multidimensional dynamic systems. *Dokl. Akad. Nauk SSSR* **189** (1969) 59–62.

[371] L. P. Shil'nikov. A contribution to the problem of the structure of an extended neighborhood of a rough state of saddle-focus type. *Math. USSR Sbornik* **10** (1970) 91–102.

[372] L. P. Shil'nikov. The bifurcation theory and quasi-hyperbolic attractors. *Uspekhi Mat. Nauk* **36** (1981) 240–241.

[373] L. P. Shil'nikov. Bifurcations and strange attractors. In: *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*. Higher Education, 2002, 349–372.

[374] L. P. Shil'nikov, A. L. Shil'nikov, D. V. Turaev and L. O. Chua. *Methods of qualitative theory in nonlinear dynamics. Part I.* World Scientific Publishing Co., 1998.

[375] L. P. Shil'nikov, A. L. Shil'nikov, D. V. Turaev and L. O. Chua. *Methods of qualitative theory in nonlinear dynamics. Part II.* World Scientific Publishing Co., 2001.

[376] L. P. Shil'nikov and D. V. Turaev. Super-homoclinic orbits and multi-pulse homoclinic loops in Hamiltonian systems with discrete symmetries. *Regul. Khaoticheskaya Din.* **2** (1997) 126–138. V. I. Arnol' d (on the occasion of his 60th birthday) (Russian).

[377] L. P. Shil'nikov and D. V. Turaev. A new simple bifurcation of a periodic orbit of "blue sky catastrophe" type. In: *Methods of qualitative theory of differential equations and related topics*. Amer. Math. Soc., Providence, RI, 2000, 165–188.

[378] D. Smets and J. B. van den Berg. Homoclinic solutions for Swift-Hohenberg and suspension bridge type equations. *J. Differential Equations* **184** (2002) 78–96.

[379] J. Sneyd, A. LeBeau and D. Yule. Traveling waves of calcium in pancreatic acinar cells: model construction and bifurcation analysis. *Phys. D* **145** (2000) 158–179.

[380] N. Sottocornola. Complete classification of homoclinic cycles in $\mathbb{R}^4$ in the case of a symmetry group $G \subset \mathrm{SO}(4)$. *C. R. Math. Acad. Sci. Paris* **334** (2002) 859–864.

[381] N. Sottocornola. Robust homoclinic cycles in $\mathbb{R}^4$. *Nonlinearity* **16** (2003) 1–24.

[382] N. Sottocornola. Simple homoclinic cycles in low-dimensional spaces. *J. Differential Equations* **210** (2005) 135–154.

[383] C. Sparrow. *The Lorenz equations: bifurcation, chaos and strange attractors*. Springer-Verlag, 1982.

[384] S. Sternberg. On the structure of local homeomorphisms of euclidean n-space. II. *Amer. J. Math.* **80** (1958) 623–631.

[385] E. Stone and P. J. Holmes. Random perturbations of heteroclinic attractors. *SIAM J. Appl. Math.* **50** (1990) 726–743.

[386] J. H. Sun. Homoclinic bifurcations in symmetric unfoldings of a singularity with three-fold zero eigenvalue. *Acta Math. Sin. (Engl. Ser.)* **21** (2005) 65–80.

[387] F. Takens. Partially hyperbolic fixed points. *Topology* **10** (1971) 133–147.

[388] F. Takens. Forced oscillations and bifurcations. In: *Applications of global analysis, I (Sympos., Utrecht State Univ., Utrecht, 1973)*. Math. Inst. Rijksuniv. Utrecht, Utrecht, 1974, 1–59. Comm. Math. Inst. Rijksuniv. Utrecht, No. 3–1974.

[389] F. Takens. Singularities of vector fields. *Inst. Hautes Études Sci. Publ. Math.* **43** (1974) 47–100.

[390] F. Takens. Global phenomena in bifurcations of dynamical systems with simple recurrence. *Jahresber. Deutsch. Math.-Verein.* **81** (1978/79) 87–96.

[391] F. Takens. Moduli and bifurcations; nontransversal intersections of invariant manifolds of vectorfields. In: *Functional differential equations and bifurcation (Proc. Conf., Inst. Ciênc. Mat. São Carlos, Univ. São Paulo, São Carlos, 1979)*, volume 799 of *Lecture Notes in Math.* Springer, Berlin, 1980, 368–384.

[392] F. Takens. Heteroclinic attractors: time averages and moduli of topological conjugacy. *Bol. Soc. Brasil. Mat. (N.S.)* **25** (1994) 107–120.

[393] F. Takens. Forced oscillations and bifurcations. In: *Global analysis of dynamical systems*. Inst. Phys., Bristol, 2001, 1–61.

[394] F. Takens and A. Vanderbauwhede. Local invariant manifolds and normal forms. In: *Handbook of Dynamical Systems 3*. North-Holland, 2010.

[395] Y. Togawa. A modulus of 3-dimensional vector fields. *Ergod. Th. & Dynam. Syst.* **7** (1987) 295–301.

[396] C. Tresser. About some theorems by L. P. Shil'nikov. *Ann. Inst. Henri Poincaré Phys. Théor.* **40** (1984) 441–461.

[397] W. Tucker. The Lorenz attractor exists. *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999) 1197–1202.

[398] W. Tucker. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.* **2** (2002) 53–117.

[399] D. V. Turaev. Bifurcations of a homoclinic "figure eight" of a multidimensional saddle. *Russian Math. Surveys* **43** (1988) 264–265.

[400] D. V. Turaev. On dimension of non-local bifurcational problems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6** (1996) 919–948.

[401] D. V. Turaev. Multi-pulse homoclinic loops in systems with a smooth first integral. In: *Ergodic theory, analysis, and efficient simulation of dynamical systems*. Springer, Berlin, 2001, 691–716.

[402] D. V. Turaev. An example of a resonant homoclinic loop of infinite cyclicity. *Mosc. Math. J.* **5** (2005) 283–293.

[403] D. V. Turaev and L. P. Shil'nikov. On bifurcations of a homoclinic "figure eight" for a saddle with negative saddle value. *Soviet. Math. Dokl.* **34** (1987) 397–401.

[404] D. V. Turaev and L. P. Shil'nikov. On Hamiltonian systems with homoclinic saddle curves. *Soviet Math. Dokl.* **39** (1989) 165–168.

[405] D. V. Turaev and L. P. Shil'nikov. Blue sky catastrophes. *Dokl. Akad. Nauk* **342** (1995) 596–599.

[406] D. V. Turaev and L. P. Shil'nikov. An example of a wild strange attractor. *Mat. Sb.* **189** (1998) 137–160.

[407] J. B. van den Berg, J. Hulshof and R. C. A. M. Vandervorst. Traveling waves for fourth order parabolic equations. *SIAM J. Math. Anal.* **32** (2001) 1342–1374.

[408] J. B. van den Berg, S. A. van Gils and T. P. P. Visser. Parameter dependence of homoclinic solutions in a single long Josephson junction. *Nonlinearity* **16** (2003) 707–717.

[409] J. B. van den Berg and R. C. A. M. Vandervorst. Stable patterns for fourth-order parabolic equations. *Duke Math. J.* **115** (2002) 513–558.

[410] S. A. van Gils, M. Krupa and V. Tchistiakov. Homoclinic twist bifurcation in a system of two coupled oscillators. *J. Dynam. Differential Equations* **12** (2000) 733–806.

[411] S. J. van Strien. *One parameter families of vector fields, bifurcations near saddle-connections*. Ph.D. thesis, Utrecht University, 1982.

[412] A. Vanderbauwhede. Centre manifolds, normal forms and elementary bifurcations. In: *Dynamics reported 2*. Wiley, Chichester, 1989, 89–169.

[413] A. Vanderbauwhede. Bifurcation of degenerate homoclinics. *Results in Mathematics* **21** (1992) 211–223.

[414] A. Vanderbauwhede and B. Fiedler. Homoclinic period blow-up in reversible and conservative systems. *Z. angew. Math. Phys.* **43** (1992) 292–318.

[415] G. Vegter. Global stability of generic two-parameter families of gradients on three-manifolds. In: *Dynamical systems and bifurcations (Groningen, 1984)*, volume 1125 of *Lecture Notes in Math.* Springer, Berlin, 1985, 107–129.

[416] J. Vera. Stability of quasi-transversal bifurcation of vector fields on 3-manifolds. *Nonlinearity* **9** (1996) 943–972.

[417] T. Wagenknecht. About a homoclinic pitchfork bifurcation in reversible systems with additional $\mathbb{Z}_2$-symmetry. *Nonlinearity* **15** (2002) 2097–2119.

[418] T. Wagenknecht. Two-heteroclinic orbits emerging in the reversible homoclinic pitchfork bifurcation. *Nonlinearity* **18** (2005) 527–542.

[419] T. Wagenknecht and A. R. Champneys. When gap solitons become embedded solitons: a generic unfolding. *Phys. D* **177** (2003) 50–70.

[420] S. Wieczorek and B. Krauskopf. Bifurcations of n-homoclinic orbits in optically injected lasers. *Nonlinearity* **18** (2005) 1095–1120.

[421] D. Wilczak. Chaos in the Kuramoto-Sivashinsky equations—a computer-assisted proof. *J. Differential Equations* **194** (2003) 433–459.

[422] D. Wilczak. Symmetric homoclinic solutions to the periodic orbits in the Michelson system. *Topol. Methods Nonlinear Anal.* **28** (2006) 155–170.

[423] R. F. Williams. The structure of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.* **50** (1979) 73–99.

[424] P. D. Woods and A. R. Champneys. Heteroclinic tangles and homoclinic snaking in the unfolding of a degenerate reversible Hamiltonian-Hopf bifurcation. *Phys. D* **129** (1999) 147–170.

[425] P. A. Worfolk. An equivariant, inclination-flip, heteroclinic bifurcation. *Nonlinearity* **9** (1996) 631–647.

[426] J. Wu. *Theory and applications of partial functional-differential equations*, volume 119 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996.

[427] K. Yagasaki and T. Wagenknecht. Detection of symmetric homoclinic orbits to saddle-centres in reversible systems. *Phys. D* **214** (2006) 169–181.

[428] E. Yanagida. Branching of double pulse solutions from single pulse solutions in nerve axon equations. *Journ. Diff. Eq.* **66** (1987) 243–262.

[429] A. C. Yew. Multipulses of nonlinearly coupled Schrödinger equations. *J. Differential Equations* **173** (2001) 92–137.

[430] J. A. Yorke and K. T. Alligood. Period doubling cascades of attractors: a prerequisite for horseshoes. *Comm. Math. Phys.* **101** (1985) 305–321.

[431] E. C. Zeeman. Bifurcation, catastrophe, and turbulence. In: *New directions in applied mathematics (Cleveland, Ohio, 1980).* Springer, New York, 1982, 109–153.

[432] M. G. Zimmermann, S. O. Firle, M. A. Natiello, M. Hildebrand, M. Eiswirth, M. Bär, A. K. Bangia and I. G. Kevrekidis. Pulse bifurcation and transition to spatiotemporal chaos in an excitable reaction-diffusion model. *Phys. D* **110** (1997) 92–104.

[433] M. G. Zimmermann and M. A. Natiello. Homoclinic and heteroclinic bifurcations close to a twisted heteroclinic cycle. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8** (1998) 359–375.

[434] M. G. Zimmermann, M. A. Natiello and H. G. Solari. Shil′nikov-saddle–node interaction near a codimension-2 bifurcation: laser with injected signal. *Phys. D* **109** (1997) 293–314.