# Saddle-nodes and period-doublings of Smale horseshoes: a case study near resonant homoclinic bellows

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**Abstract**

In unfoldings of resonant homoclinic bellows interesting bifurcation phenomena occur: two suspensed Smale horseshoes can collide and disappear in saddle-node bifurcations (all periodic orbits disappear through saddle-node bifurcations, there are no other bifurcations of periodic orbits), or a suspended horseshoe can go through saddle-node and period-doubling bifurcations of the periodic orbits in it to create an additional "doubled horseshoe".

## 1 Introduction

In these notes we discuss specific homoclinic bifurcations involving multiple homoclinic orbits to a hyperbolic equilibrium with a resonance condition among the eigenvalues of the linearized vector field about the equilibrium; the resonant homoclinic bellows. A homoclinic bellows consists of two homoclinic orbits $\gamma_1(t), \gamma_2(t)$ to a hyperbolic equilibrium with real leading eigenvalues, that are tangent to each other as $t \to \pm\infty$. If the homoclinic orbits are symmetry related through the action of a $\mathbb{Z}_2$ symmetry, the homoclinic bellows is a bifurcation of codimension one (we review the bifurcation theory in § 2); the additional resonance condition makes it a bifurcation of codimension two.

The resonant homoclinic bellows is an organizing center for an interesting bifurcation phenomenon involving suspended Smale horseshoes (this is our motivation for studying the bifurcation). In it, two suspended horseshoes collide through collisions of periodic orbits from one horseshoe with periodic orbits from the other horseshoe in saddle-node bifurcations. All periodic orbits disappear through saddle-node bifurcations. Or, starting from Morse-Smale dynamics one finds suspended Smale horseshoes being created, where the periodic orbits appear through saddle-node bifurcations alone. This creation of a horseshoe is much simpler than involved scenarios involving homoclinic tangencies (see e.g. [13]), but needs four dimensional state space to occur.

In the context of maps one can take the following simple geometric model illustrating the bifurcations: generic perturbations from a product map $(x, y) \mapsto (F(x), G_\mu(y))$ in $\mathbb{R}^2 \times \mathbb{R}$ of a map $x \mapsto F(x)$ on $\mathbb{R}^2$ possessing a two dimensional hyperbolic horseshoe with a family of maps $y \mapsto G_\mu(y)$ on $\mathbb{R}$ that unfolds a saddle-node bifurcation. Although it would be interesting to study this scenario directly, and discuss issues such as measure of the bifurcation set, in these notes we restrict to describing an organizing center that contains this scenario.

A related phenomenon lies in generic perturbations from a product map $(x, y) \mapsto (F(x), G_\mu(y))$ in $\mathbb{R}^2 \times \mathbb{R}$ with $F$ possessing a hyperbolic horseshoe and $G_\mu$ unfolding a period-doubling bifurcation. Resonant homoclinic bellows also serve as an organizing center for this bifurcation scenario, if an invariant plane bundle of center directions along the bellows is not orientable. Here we find that periodic orbits of odd period undergo period-doubling bifurcations, and periodic orbits of even period undergo saddle-node bifurcations. Because of the symmetry of the vector field, we will find pitchfork bifurcations instead of saddle-node bifurcations for some of the periodic orbits (namely the symmetric ones). These bifurcations lead to a 'doubled horseshoe', described by a symbolic dynamics on the doubled number of symbols (i.e. four symbols).

The bifurcation problem studied in this paper was first considered in the doctoral thesis [8] by A.C. Jukes. In this thesis resonance bifurcations of homoclinic loops in $\mathbb{D}_3$-symmetric systems are treated. Depending on the action of the symmetry, one of the occurring cases gives a bellows (with the minor difference that it contains three or six homoclinic orbits instead of two). The other cases, with multiple leading eigenvalues, are more complicated to study but seem to give rise to similar bifurcation phenomena (and more). It is straightforward to generalize the study of this paper to bellows in vector fields equivariant under the action of finite groups other than $\mathbb{Z}_2$. The description of the dynamics as used in this paper with symbolic dynamics on two symbols, for the two homoclinic orbits, is then replaced by symbolic dynamics on $k$ symbols if there are $k$ homoclinic orbits in a bellows configuration, compare [8].

Other scenarios along these lines, which we won't consider, could be Hopf-bifurcations of horseshoes, and further generalizations appear if horseshoes are replaced by other hyperbolic sets (attractors might also be possible). Moreover, other codimension two homoclinic bifurcations of bellows, such as inclination-flips and orbit-flips might lead to similar phenomena.

In the following section we start with the set-up of this paper, giving conditions that define a homoclinic bellows and recalling the bifurcation result for codimension-one homoclinic bellows in $\mathbb{Z}_2$-equivariant vector fields: a suspension of a hyperbolic horseshoe appears in an unfolding. After that we treat resonant homoclinic bellows.

Resonant bifurcations of single homoclinic orbits (with real leading eigenvalues) were considered by S.-N. Chow, B. Deng and B. Fiedler [1], in which the authors showed the appearance of saddle-node or period-doubling bifurcations depending on orientability of an invariant plane bundle. In resonant bellows the two phenomena, occurrence of horseshoes and saddle-node or period-doubling bifurcations, combine and lead to bifurcations of suspended horseshoes. Our bifurcation results are contained in § 3, the sections thereafter contain the bifurcation analysis. The techniques for the bifurcation analysis are borrowed from [6], to which we frequently refer for details. Resonant bifurcations of multiple homoclinic loops (without an assumption of $\mathbb{Z}_2$-equivariance) have been studied by R.W. Ghrist [3], who however did not consider homoclinic bellows.

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# 2 Nonresonant bellows

The purpose of this section is to introduce conditions that determine a homoclinic bellows and to present the result that an unfolding of a homoclinic bellows features the creation of suspended horseshoes. This result is due to D.V. Turaev [19] (that the horseshoes are hyperbolic can be concluded from the results in [6], see below). In the next section we continue the analysis with resonant bellows, focussing on the resulting bifurcations of the suspended horseshoes.

Starting point for all results is a $\mathbb{Z}_2$-equivariant vector field. Let

$$\dot{x} = f(x, \lambda) \tag{2.1}$$

be a smooth family of differential equations on $\mathbb{R}^n$, depending on a real parameter $\lambda \in \mathbb{R}$.

**Hypothesis 2.1 (Equivariance).** *The differential equation $\dot{x} = f(x, \lambda)$, $x \in \mathbb{R}^n$, is equivariant with respect to a linear involution $S$:*

$$Sf(Sx, \lambda) = f(x, \lambda). \tag{2.2}$$

Recall that the leading eigenvalues at a hyperbolic equilibrium are those closest to the imaginary axis.

**Hypothesis 2.2 (Real leading eigenvalues).** *At $\lambda = 0$, the equilibrium at the origin is hyperbolic and has real simple leading eigenvalues $\mu^s$, $\mu^u$.*

The leading stable and unstable directions, that is the eigendirections corresponding to $\mu^s$ and $\mu^u$, are thus one dimensional. At $\lambda = (0, 0)$, (2.1) possesses two homoclinic orbits $\gamma_1, \gamma_2 = S\gamma_1$ to the origin. That is, for $i = 1, 2$,

$$\lim_{t \to \pm\infty} \gamma_i(t) = 0.$$

A coexistence of homoclinic loops can lead to suspended subshifts of finite type in the unfolding. The next hypothesis poses the existence of several homoclinic orbits approaching the origin from the same direction for positive time, and similarly for negative time.

**Hypothesis 2.3 (Homoclinic bellows).** *The homoclinic orbits $\gamma_i$ approach the origin from the same direction for positive time and for negative time:*

$$\lim_{t\to\pm\infty} \gamma_1(t)/\|\gamma_1(t)\| = \lim_{t\to\pm\infty} \gamma_2(t)/\|\gamma_2(t)\|.$$

We will formulate conditions ensuring that the bifurcation is of codimension one. The resulting geometric configuration is called a homoclinic bellows, see [6]. All conditions apply to both homoclinic orbits by symmetry. The following hypothesis states that the homoclinic orbit is nondegenerate.

**Hypothesis 2.4 (Nondegenerate homoclinic orbits).** *At $\lambda = 0$, the tangent spaces of the unstable manifold $W^u(0)$ and of the stable manifold $W^s(0)$ intersect along $\gamma_i$ only along the vector field direction.*

As a consequence, the subspace $Z_i \subset T_{\gamma_i(0)}\mathbb{R}^n$ perpendicular to the tangent spaces of $W^s(0)$ and of $W^u(0)$ is one dimensional.

A local center unstable manifold $W^{cu}(0)$ is a locally invariant manifold with as tangent space at 0 the direct sum of the unstable and the leading stable directions. Likewise, a local center stable manifold $W^{cs}(0)$ is a locally invariant manifold with as tangent space at 0 the direct sum of the stable and the leading unstable directions. Local center (un)stable manifolds are not unique, but possess unique tangent spaces along the (un)stable manifold.

**Hypothesis 2.5 (Codimension one).** *For a nondegenerate homoclinic orbit with unique real leading eigenvalues, the following items, all at $\lambda = 0$, define a bifurcation of codimension one:*

*(i) Nonresonance condition: the leading eigenvalues satisfy $\beta = -\mu^s/\mu^u \neq 1$. By changing the direction of time, if necessary, this yields the condition*

$$\beta > 1. \tag{2.3}$$

*(ii) No orbit-flip condition: $\gamma_i$ is not within the strong stable manifold or strong unstable manifold of the origin:*

$$\gamma_i \not\subset W^{ss}(0), \quad \gamma_i \not\subset W^{uu}(0). \tag{2.4}$$

*(iii) No inclination-flip condition:*

$$W^s(0) \pitchfork_{\gamma_i} W^{cu}(0), \quad W^u(0) \pitchfork_{\gamma_i} W^{cs}(0). \tag{2.5}$$

See [1] for homoclinic bifurcation problems with resonant eigenvalues. Condition (2.4) excludes an orbit-flip condition, see [16], while (2.5) excludes the inclination-flip condition, see [7, 9]. By the no orbit-flip condition, the homoclinic orbits approach the equilibrium along the leading directions, both for positive and negative time.

**Hypothesis 2.6 (Generic unfolding).** *$W^u(0), W^s(0)$ split up with positive speed in $\lambda$.*

The following bifurcation theorem is a consequence of the geometric reductions presented in § 4.1.

**Theorem 2.1.** *Suppose that $\mathbb{Z}_2$ symmetric differential equations (2.1) unfold a homoclinic bellows: Hypotheses 2.1-2.6 are met. Then a hyperbolic set equivalent to a suspended full shift on 2 symbols exists for parameters on one side of $\lambda = 0$ and converges to the homoclinic orbits $\overline{\gamma_1 \cup \gamma_2}$ as $\lambda \to 0$.*

Under the assumption of Hypotheses 2.2, 2.4, 2.5(ii),(iii), there exists a continuous plane bundle $E^c$ along $\gamma_1 \cup \gamma_2 \cup \{0\}$, invariant under the first variation equation and equal to the sum of the leading direction at the origin (see e.g. [5, 14]). The plane bundle can be either orientable or nonorientable. Orientability of the bundle of center directions plays no role for the conclusion of the above result. It will be an essential factor in the next section, discussing homoclinic bellows near resonance.

# 3 Resonant homoclinic bellows

We state bifurcation results for two coexisting homoclinic orbits in a bellows configuration at resonant eigenvalues. The bifurcation analysis is in § 4 (treating reductions leading to bifurcation equations) and § 5 (analyzing the bifurcation equations).

For vector fields that are equivariant with respect to the linear action of an involution, this is a codimension two bifurcation problem in case the homoclinic orbits are related by symmetry. The parameter $\lambda = (\lambda_1, \lambda_2)$ will be from an open neighborhood of $(0, 0) \in \mathbb{R}^2$. At $\lambda = (0, 0)$, there are two homoclinic orbits $\gamma_1, \gamma_2 = S\gamma_1$ to the origin forming a homoclinic bellows. Specifically we assume that Hypotheses 2.2, 2.3, 2.4, 2.5(ii),(iii) are met. We assume however that Hypothesis 2.5(i) is violated:

**Hypothesis 3.1 (Resonance).** *At $\lambda = (0, 0)$, $\beta = -\mu^s/\mu^u = 1$.*

As before, the following genericity conditions apply to $\gamma_1$, $\gamma_2$ simultaneously.

**Hypothesis 3.2 (Generic unfolding).** *The bifurcation is generic and unfolds generically:*

*(i)* *At $\lambda = (0, 0)$, $\int_{-\infty}^{\infty} \mathrm{div}_2(\gamma_i(t))dt \neq 0$, where $\mathrm{div}_2$ denotes the rate of change of area within the plane field $E^c_{\gamma_i(t)}$. By changing the direction of time if necessary, we may assume*

$$B = e^{\int_{-\infty}^{\infty} \mathrm{div}_2(\gamma_i(t))dt} > 1, \tag{3.1}$$

*(ii)* *$W^u(0), W^s(0)$ split up with positive speed in $\lambda_1$,*

*(iii)* *$\frac{\partial}{\partial \lambda_2}\beta \neq 0$.*

The conditions formulated in Hypothesis 3.2 enable a reparameterization of the parameter plane, so that $\lambda_1$ unfolds the homoclinic bifurcation and $\lambda_2$ unfolds the resonance condition.

**Lemma 3.1.** *Consider a two parameter family $\dot{x} = f(x, \lambda)$, $\lambda = (\lambda_1, \lambda_2)$, of ordinary differential equations unfolding a $\mathbb{Z}_2$-symmetric bellows at resonance: Hypotheses 2.1, 2.2, 2.4, 2.5(ii), (iii), 3.1 and 3.2 are met. By a reparameterization of the parameter plane,*

*(i)* *the primary homoclinic orbits $\gamma_1, \gamma_2$ exist along $\{\lambda_1 = 0\}$,*

*(ii)* $\beta = 1 + \lambda_2$.

We treat dynamics in a small tubular neighborhood $\mathcal{U}$ of the homoclinic orbits $\gamma_1$ and $\gamma_2$. It is convenient to assign a symbolic coding to orbits in the recurrent set in $\mathcal{U}$ associated to the two loops. This is introduced by considering a first return map on two cross sections, as follows. Let $\Sigma_1$ and $\Sigma_2$ be cross-sections of $\gamma_1$ and $\gamma_2$, respectively. Write $p_i = \gamma_i \cap \Sigma_i$. By

$$\Psi : \Sigma_1 \cup \Sigma_2 \to \Sigma_1 \cup \Sigma_2$$

we denote the first return map defined on a subset of $\Sigma_1 \cup \Sigma_2$. Associated to an orbit $x = \{x(i)\}$, $x(i+1) = \Psi(x(i))$ for $i \in \mathbb{Z}$, in the recurrent set of $\Psi$, there is an itinerary $\Upsilon(x) : \mathbb{Z} \to \{1, 2\}$ defined by

$$\Upsilon(x)(i) = j, \text{ if } x(i) \in \Sigma_j.$$

Obviously one can associate this itinerary to the corresponding orbit of the vector field. Let $\mathcal{B}_2$ be the set of itineraries $\mathbb{Z} \mapsto \{1, 2\}$, endowed with the product topology. The shift operator $\sigma : \mathcal{B}_2 \to \mathcal{B}_2$ is, as usual, given by $\sigma y(k) = y(k+1)$. Orbits in the recurrent set in $\mathcal{U}$ that do not lie in the stable or unstable manifold of the origin, give rise to two sided infinite itinerary. When we speak of the period of a periodic itinerary, we will always mean the minimal period. The other orbits have a one-sided infinite itinerary or, for a homoclinic orbit, a finite itinerary. For instance the code (12) stands for a homoclinic orbit that follows closely $\gamma_1$ and then $\gamma_2$ before converging to the origin. We call a homoclinic orbit with an itinerary of $k$ symbols a $k$-homoclinic orbit (apart from the primary homoclinic orbits $\gamma_1$ and $\gamma_2$, only 2-homoclinic orbits occur in our bifurcation study).

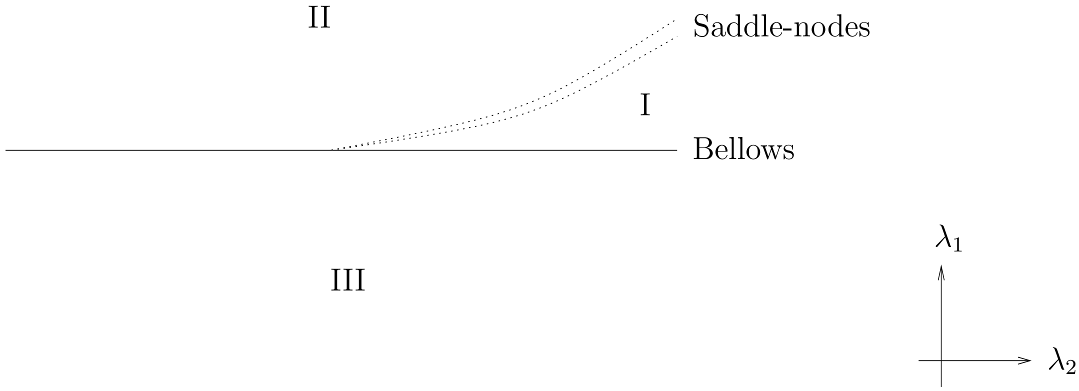
The symmetry $S$ induces an action $\mathcal{S}$ on $\mathcal{B}_2$ by interchanging symbols 1 and 2. A periodic itinerary $y$ of period $k$ is symmetric if $\mathcal{S}y = \sigma^s y$ for some $s$. It is not hard to see that necessarily $k$ is even and $s = k/2$.

Recall from the previous section that the bundle of center directions $E^c$ can be either orientable or nonorientable. We formulate two bifurcation theorems for these two possible cases. We start with bifurcations from a nontwisted resonant bellows, where $E^c$ is an orientable bundle.

**Theorem 3.1.** *Consider a two parameter family $\dot{x} = f(x, \lambda)$, $\lambda = (\lambda_1, \lambda_2)$, of ordinary differential equations unfolding a $\mathbb{Z}_2$-symmetric bellows at resonance: Hypotheses 2.1-2.4, 2.5(ii), (iii), 3.1 and 3.2 are met. Suppose the bundle of center directions $E^c$ along $\gamma_1 \cup \gamma_2 \cup \{0\}$ is orientable. Up to a reparameterization of the parameters given by Lemma 3.1 the bifurcation diagram is as depicted.*

*Homoclinic bellows occur along $\{\lambda_1 = 0\}$. The recurrent set consists in the different regions of the following:*

- *I: the equilibrium and two hyperbolic sets each equivalent to a suspended full shift on the two symbols 1,2,*
- *II: the equilibrium,*
- *III: the equilibrium and a single hyperbolic set equivalent to a suspended full shift on the two symbols 1,2.*



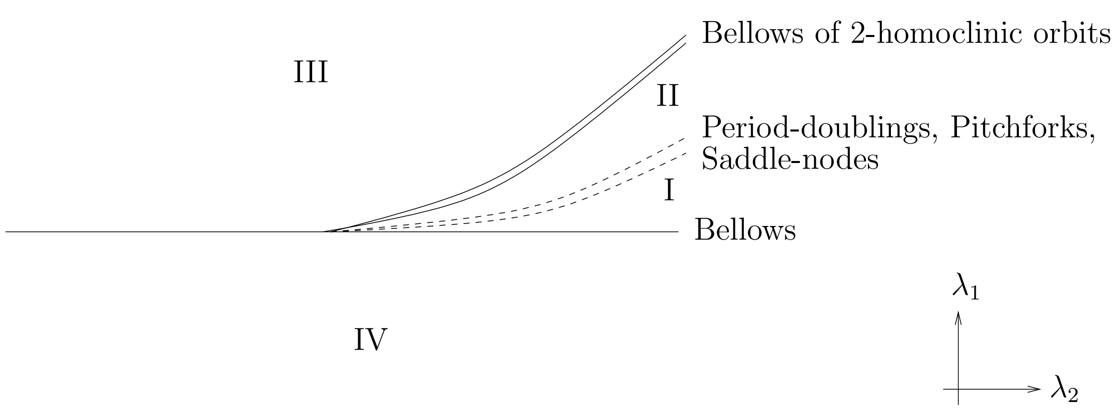

*The suspended horseshoes in region I disappear through bifurcations that take place in the wedge between regions I and II.*

*There is $\lambda_2^0 > 0$ so that for each periodic itinerary $\eta$ there is a smooth curve $\{(s^\eta(\lambda_2), \lambda_2)\}$, $0 < \lambda_2 < \lambda_2^0$, inside the wedge and with a flat tangency at $\lambda_2 = 0$, of generically unfolding saddle-node bifurcations of periodic orbits with itinerary $\eta$. The curves have the same asymptotics*

$$\lim_{\lambda_2 \to 0} \frac{s^\eta(\lambda_2)}{\lambda_2 (1/B)^{1/\lambda_2}} = \frac{1}{e}.$$

Bifurcations from a twisted resonant bellows, with nonorientable bundle $E^c$, are treated in the following theorem. The bifurcation statements are weaker then in the previous theorem on nontwisted resonant bellows. We do not provide proofs that the period-doubling and pitchfork bifurcations are generically unfolding and occurring along smooth curves. We note though that the arguments below establish that for each periodic orbit with a period-doubling or pitchfork bifurcation, the bifurcation occurs along a smooth curve. We have however no uniform bound on the lenght of these curves, so that this information does not help to explain the bifurcations of entire horseshoes. Moreover, the saddle-node bifurcations are close to degenerate saddle-node bifurcations, we expect them to be either of codimension-one or codimension-two. They may therefore not occur along smooth curves.

**Theorem 3.2.** *Consider a two parameter family $\dot{x} = f(x, \lambda)$, $\lambda = (\lambda_1, \lambda_2)$, of ordinary differential equations unfolding a $\mathbb{Z}_2$-symmetric bellows at resonance: Hypotheses 2.1-2.4, 2.5(ii), (iii), 3.1 and 3.2 are met. Suppose the bundle of center directions $E^c$ along $\gamma_1 \cup \gamma_2 \cup \{0\}$ is nonorientable. Up to a reparameterization of the parameters given by Lemma 3.1 the bifurcation diagram is as depicted.*




*The recurrent set consists in the different regions of the following:*

*I: the equilibrium and a hyperbolic set equivalent to a suspended full shift on the two symbols 1,2,*

*II: the equilibrium and two hyperbolic sets, one equivalent to a suspended full shift on the two symbols 1,2 the other equivalent to a suspended full shift on four symbols corresponding to the four 2-homoclinic orbits (12), (21), (11), (22).*

*III: the equilibrium and a single hyperbolic set equivalent to a suspended full shift on the two symbols 1,2.*

*IV: the equilibrium.*

*The suspended horseshoe in region I doubles through bifurcations that take place in the wedge between regions I and II. The doubled horseshoe disappears through homoclinic bifurcations in the wedge between regions II and III.*

*Homoclinic bellows occur along* $\{\lambda_1 = 0\}$*. Homoclinic bifurcations of* $n$*-homoclinic orbits,* $n \geq 2$*, can take place only in an exponentially flat wedge of the following asymptotics: a curve* $(s(\lambda_2), \lambda_2)$ *inside the wedge satisfies*

$$\lim_{\lambda_2 \to 0} \frac{s(\lambda_2)}{\lambda_2 (1/B)^{1/\lambda_2}} = 1.$$

*Bifurcations of 2-homoclinic bellows occur along two smooth curves in this wedge branching from* $\lambda = 0$*; a curve* $(s^{11}(\lambda_2), \lambda_2)$ *of homoclinic orbits with itineraries (11) and (22), and a curve* $(s^{12}(\lambda_2), \lambda_2)$ *of homoclinic orbits with itineraries (12) and (21).*

*Bifurcations involving nonhyperbolic periodic orbits take place in an exponentially flat wedge of the following asymptotics: a curve* $(s(\lambda_2), \lambda_2)$ *inside the wedge satisfies*

$$\lim_{\lambda_2 \to 0} \frac{s(\lambda_2)}{\lambda_2 (1/B)^{1/\lambda_2}} = \frac{1}{e}.$$

*There is* $\lambda_2^0 > 0$ *so that the following bifurcation statements hold.*

*For each periodic itinerary with odd period, there is for each* $\lambda_2$ *with* $0 < \lambda_2 < \lambda_2^0$ *a value* $d^\eta(\lambda_2)$*, inside the wedge, of possibly degenerate period-doubling bifurcations of periodic orbits with itinerary* $\eta$*.*

*For each symmetric periodic itinerary* $\eta$ *of period* $2k$ *with* $k$ *odd, there is for each* $\lambda_2$ *with* $0 < \lambda_2 < \lambda_2^0$ *a value* $d^\eta(\lambda_2)$*, inside the wedge, of possibly degenerate pitchfork bifurcations of periodic orbits with itinerary* $\eta$*.*

*For all other periodic itineraries of period* $2k$ *(nonsymmetric or symmetric with* $k$ *even), there is for each* $\lambda_2$ *with* $0 < \lambda_2 < \lambda_2^0$ *a value* $d^\eta(\lambda_2)$*, inside the wedge, of a possibly degenerate saddle-node bifurcation of a periodic orbit with itinerary* $\eta$*.*

We do not know the mutual position of the curves of 2-homoclinic bellows; they have the same asymptotics and we cannot exclude that they intersect each other. We also do not have precise statements on possible bifurcations of $n$-homoclinic bellows, $n > 2$. Such bifurcations may for instance be absent for values of $\lambda_2$ where the curves of 2-homoclinic orbits coincide.

# 4 Geometric and analytic reductions

Combining geometric techniques (constructions of invariant manifolds) and analytic techniques (derivation of reduced bifurcation equations applying a Lyapunov-Schmidt argument) proves the bifurcation theorems. In this section we discuss both approaches.

The context of this section is, as in the bifurcation theorems above, of a two parameter family of differential equations unfolding a $\mathbb{Z}_2$-symmetric homoclinic bellows at resonance (defined by Hypotheses 2.1-2.4, 2.5(ii), (iii), 3.1 and 3.2).

## 4.1 Cantor books of center manifolds

The clearest insight in the geometry of the flow is obtained from the construction of invariant center manifolds. We recall a result from [6] (also applicable in the context here) in which a collection of center manifolds, indexed by itineraries on two symbols, is constructed. The recurrent set near the homoclinic bellows is contained in the collection of center manifolds. For $\lambda = (0, 0)$, when the homoclinic bellows exist, this leads to a Cantor book of two dimensional center manifolds all containing the homoclinic bellows (as spine of the book). The center manifolds persist for small $\lambda$. They are normally hyperbolic, so that hyperbolicity of a periodic orbit is deduced from hyperbolicity within a center manifold. They are however in general only continuously differentiable (the precise smoothness depends on spectral gap conditions on the spectrum of the linearized vector field about the origin). This renders them useless for a detailed study of bifurcations of nonhyperbolic periodic orbits. In the next section we show how smooth bifurcation formulas are obtained from Lin's method. The geometric information from the center manifolds and the analytic information from Lin's method combined provide the information needed to prove the bifurcation results.

The following theorem shows that the recurrent set of $\Psi$ is contained in a Cantor book of center manifolds.

**Theorem 4.1 ([6]).** *For each small $\lambda$ and each $\eta \in \mathcal{B}_2$, there is a one-dimensional normally hyperbolic center manifold $W^c_\eta$ for $\Psi_\lambda$, so that any orbit $x$ with itinerary $\Upsilon(x) = \eta$, satisfies $x(0) \in W^c_\eta$. The manifold $W^c_\eta$ is continuously differentiable, depends differentiable on $\lambda$, and depends continuously on $\eta$. It satisfies $W^c_{\sigma(\eta)} = \Psi_\lambda(W^c_\eta)$.*

Fix a sequence $\eta : \mathbb{Z} \to \{1, 2\}$. Let $x(j + 1) = \Psi(x(j))$, $j \in \mathbb{Z}$, be an orbit of $\Psi$ contained in the recurrent set of $\Psi$, with

$$x(j) \in \Sigma_{\eta(j)} \tag{4.1}$$

(we suppress the dependence of $x$ on $\eta$ from the notation). For the following analysis we use coordinates as in [6]. In particular we parameterize a center manifold by a coordinate $x_u$ corresponding to the real leading unstable direction. It follows from [6], see also [5], that points $x(j)$ on $W^c_{\sigma^j\eta}$, parameterized by $x_u(j)$, satisfy

$$x_u(j + 1) \;=\; \tilde{G}^{\sigma^j\eta}(x_u(j), \lambda)$$

where $\tilde{G}^{\sigma^j\eta}$ is a continuously differentiable function with asymptotics

$$\tilde{G}^{\sigma^j\eta}(x_u(j), \lambda) = a + b\,(x_u(j))^\beta + o(x_u(j)^\beta),$$

where $b \neq 0$. In fact, $|b|$ equals the integral in (3.1) and the sign of $b$ corresponds to the orientation of the homoclinic center manifold [14, 15]. Further,

$$\frac{d}{dx_u(j)}\tilde{G}^{\sigma^j\eta}(x_u(j),\lambda) = b\,(x_u(j))^{\beta-1} + o(x_u(j)^{\beta-1}),$$

where these asymptotics hold uniformly in $\eta$.

By a smooth reparameterization we may assume $a = \lambda_1$ and $\beta = 1+\lambda_2$, so that

$$\tilde{G}^{\sigma^j\eta}(x_u(j),\lambda) \quad = \quad \lambda_1 + bx_u(j)^{1+\lambda_2} + o(x_u(j)^{1+\lambda_2}). \tag{4.2}$$

## 4.2 Reduced bifurcation equations from Lin's method

We continue the bifurcation study with the description of reduced bifurcation equations (for homoclinic and periodic orbits) obtained from Lin's method [10, 12, 16, 21] (the general theory of this method simplifies under the conditions of this paper, see [6]). In lowest order terms the formulas are the same as those given by the center manifold reduction in the previous section. However, reminiscent of reduced bifurcation equations in local bifurcation theory obtained from a Lyapunov-Schmidt reduction (see e.g. [2]) the reduced bifurcation equations will be smooth.

Recall that $\Psi$ is the first return map on $\Sigma_1 \cup \Sigma_2$. Fix a sequence $\eta : \mathbb{Z} \to \{1,2\}$. Let $x(j+1) = \Psi(x(j))$, $j \in \mathbb{Z}$, be an orbit of $\Psi$ contained in the recurrent set of $\Psi$, with $x(j) \in \Sigma_{\eta(j)}$ (again supressing the dependence of $x$ on $\eta$ from the notation). Treat the equality $x(j+1) = \Psi(x(j))$, $j \in \mathbb{Z}$, as an equation on $l^\infty(\mathbb{R}^n)$ of bounded sequences $x : \mathbb{Z} \to \mathbb{R}^n$. We quote from [6] that reduced bifurcation equations are given by

$$x_u(j+1) \quad = \quad G^{\sigma^j\eta}(x_u(j), x_u, \lambda) \tag{4.3}$$

where $G$ is a smooth function with asymptotics

$$G^{\sigma^j\eta}(x_u(j), x_u, \lambda) = a + bx_u(j)^\beta + R_j(x_u,\lambda) \tag{4.4}$$

where the remainder terms $R_j(x_u,\lambda)$ are smooth in $x_u > 0$ and $\lambda$ and satisfy estimates

$$|D^k_{x_u} D^m_\lambda R_j(x_u,\lambda)| \leq C_{j,m}|x_u(j)|^{\beta+\omega-k}, \tag{4.5}$$

for some $\omega > 0$, uniformly in $\eta$. Note that the higher order terms depend on the entire sequence $x_u(i)$, $i \in \mathbb{Z}$.

As in the previous section, after a reparameterization of the parameter plane,

$$G^{\sigma^j\eta}(x_u(j), x_u, \lambda) \quad = \quad \lambda_1 + bx_u(j)^{1+\lambda_2} + \mathcal{O}(|x_u(j)|^{1+\omega}) \tag{4.6}$$

(where the higher order terms are understood in the sense of (4.5)). Note that a periodic orbit of periodic $k$, is determined by $k$ equations of the form

$$x_u(j+1) = G^{\sigma^j\eta}(x_u(j), x_u, \lambda) \tag{4.7}$$

with indices taken modulo $k$.

For later use we add a few words on the derivation of the reduced bifurcation equations. A Shil'nikov variable approach leads to bifurcation equations for orbits $x = (x_{ss}, x_u, x_{uu})$, of the form

$$(x_{ss}(j+1), x_u(j+1), x_{uu}(j)) - T(x_{ss}(j), x_u(j), x_{uu}(j+1)) = 0, \quad j \in \mathbb{Z}, \tag{4.8}$$

for a map $T$, smooth when $x_u > 0$, with asymptotic expansions

$$\begin{aligned} x_{ss}(j+1) - \mathcal{O}(x_u(j)^{\beta+\omega}) &= 0, \\ x_u(j+1) - a - bx_u(j)^\beta + \mathcal{O}(x_u(j)^{\beta+\omega}) &= 0, \\ x_{uu}(j) - \mathcal{O}(x_u(j+1)^{1+\omega}) &= 0. \end{aligned} \tag{4.9}$$

The asymptotics apply in suitable smooth coordinates on the cross sections on which the return map $\Psi$ acts. Let $l^\infty_{\mathbb{R}^k}$ be the space of bi-infinite sequences of elements of $\mathbb{R}^k$ equipped with the supremum norm. Then equation (4.8) can be considered as an equation in $l^\infty_{\mathbb{R}^{q_{ss}}} \times l^\infty_{\mathbb{R}} \times l^\infty_{\mathbb{R}^{q_{uu}}}$. In the same way the first and third equation in (4.9) can be seen. The first and third equation of this system can be solved for

$$\Big(x_{ss}(j), x_{uu}(j)\Big)_{j\in\mathbb{Z}}\Big((x_u(j))_{j\in\mathbb{Z}}\Big).$$

Plugging this into the second equation of (4.9) we arrive at (4.3).

# 5 Bifurcations of horsehoes

We combine the different reduction results to prove the bifurcation theorems. We focus on the bifurcations of the horseshoes, it is easy to complete the bifurcation picture. Compare also the analysis of resonant homoclinic orbits in [1].

## 5.1 Nontwisted resonant bellows

Here we treat the saddle-node bifurcation of horseshoes, as presented in Theorem 3.1.

Fix a periodic itinerary $\eta$ of periodic $k$. First we consider the center manifold reduction. A periodic point $x_u$ with itinerary $\eta$ satisfies the $k$ equations

$$x_u(i+1) = \tilde{G}^{\sigma^i\eta}(x_u(i), \lambda),$$

with indices taken modulo $k$. For a saddle-node bifurcation, we moreover have

$$\prod_{i=0}^{k-1}\left(\tilde{G}^{\sigma^i\eta}\right)'(x_u(i), \lambda) = 1.$$

Straightforward continuity arguments, making use of the asymptotic expansions (4.2), give the following: for each small positive $\lambda_2$ and each periodic itinerary $\eta$ there exists $\lambda_1$ for which a saddle-node bifurcation of a periodic orbit with itinerary $\eta$ occurs. Note that

$$x_u(j) \sim \frac{1}{e}\left(\frac{1}{b}\right)^{1/\lambda_2} \tag{5.1}$$

in the sense that all $x_u(j)$ are contained in a small interval $[1-\epsilon, 1+\epsilon]\frac{1}{e}\left(\frac{1}{b}\right)^{1/\lambda_2}$. With no more at our disposal then continuous differentiability of $\tilde{G}$, more can not be deduced at this point.

If the center manifolds $W^c_\eta$ are $C^2$ (and depending continuously on $\eta$ in the $C^2$ topology) one can use the center manifold reduction to prove uniqueness by estimating second order derivatives. Indeed, one has

$$\left(\frac{d}{dx_u(0)}\right)^2 \tilde{G}^{\sigma^{k-1}\eta}\circ\cdots\circ\tilde{G}^\eta(x_u(0)) = \sum_{i=0}^{k-1}\left(\tilde{G}^{\sigma^i\eta}\right)''(x_u(i))\prod_{j=0}^{i-1}\left(\tilde{G}^{\sigma^j\eta}\right)'(x_u(j)), \tag{5.2}$$

which is positive as all terms are. For $C^2$ regularity however gap conditions on the spectrum of $Df(0,0)$ (derivative with respect to the state variable) are needed. Such higher regularity of center manifolds, depending on spectral conditions, is not considered in [6] (for results on higher regularity of center manifolds, see [17, 20]). For the general case we rely on the reduced bifurcations from Lin's method.

Recall the reduced bifurcation equations (4.7) obtained from Lin's method. Take a periodic orbit $x(i)$ of periodic $k$, satisfying the reduced bifurcation equations

$$\begin{aligned} x_u(1) &= G^{\eta}(x_u(0), x_u, \lambda), \\ x_u(2) &= G^{\sigma\eta}(x_u(1), x_u, \lambda), \\ \vdots &= \vdots \\ x_u(0) &= G^{\sigma^{k-1}\eta}(x_u(k-1), x_u, \lambda). \end{aligned} \tag{5.3}$$

We remark that system (5.3) can be solved successively, to end up with a single equation for $x_u(0)$, as follows. The implicit function theorem allows to solve the first equation for

$$x_u(1) = x_u^*(1)(x_u(0), x_u(2), \ldots, x_u(k-1), \lambda).$$

Plugging this into the second equation allows one to solve for

$$x_u(2) = x_u^*(2)(x_u(0), x_u(3), \ldots, x_u(k-1), \lambda).$$

The final equation yields a fixed point equation

$$x_u(0) = G^{\eta,k}(x_u(0), \lambda). \tag{5.4}$$

Add a defining equation for the saddle-node bifurcation,

$$1 = \frac{d}{dx_u(0)} G^{\eta,k}(x_u(0), \lambda). \tag{5.5}$$

We already know that (5.4), (5.5) can be solved for some $x_u(0)$, $\lambda_1$ (for each given $\lambda_2$ small and positive) and we claim this solution to be unique. A uniform treatment in $\eta$ is required for the conclusion that the curves of saddle-node bifurcations branch a distance uniformly away from 0 in $\eta$.

One strategy for showing uniqueness would be to mimic the computation (5.2). A slightly different approach, which we will adopt, is to trace the graph of the map defined by the system of equations (5.3). Define

$$\mathcal{G}_u^{\eta}(x_u(0), \ldots, x_u(k-1)) = \begin{pmatrix} G^{\eta}(x_u(0), x_u, \lambda) - x_u(1) \\ \vdots \\ G^{\sigma^{k-1}\eta}(x_u(k-1), x_u, \lambda) - x_u(0) \end{pmatrix}. \tag{5.6}$$

Fix $\lambda$ for which $x$ is a periodic orbit at a saddle-node bifurcation. Local uniqueness of $x$ follows from

$$\left\langle M_u, D^2\mathcal{G}_u^{\eta}(x_u)[N_u, N_u] \right\rangle > 0, \tag{5.7}$$

where $N_u, M_u$ are the right and left eigenvectors respectively for the single zero eigenvalue of $D\mathcal{G}_u^{\eta}(x_u)$ (see [4, 18]). As $M_u$ is perpendicular to the image of $D\mathcal{G}_u^{\eta}(x_u)$, (5.7) expresses transversality of $D^2\mathcal{G}_u^{\eta}(x_u)[N_u, N_u]$ to the image of $D\mathcal{G}_u^{\eta}(x_u)$. A generic unfolding is guaranteed by

$$\left\langle M_u, \frac{\partial}{\partial \lambda_1} \mathcal{G}_u^{\eta}(x_u) \right\rangle > 0. \tag{5.8}$$

Let us indicate how the computation goes when $\mathcal{G}_u^\eta$ is replaced by the map obtained when ignoring the higher order terms:

$$\mathcal{H}_u(\tilde{x}_u(0), \ldots, \tilde{x}_u(k-1)) = \begin{pmatrix} \lambda_1 + b\tilde{x}_u(0)^{1+\lambda_2} - \tilde{x}_u(1) \\ \vdots \\ \lambda_1 + b\tilde{x}_u(k-1)^{1+\lambda_2} - \tilde{x}_u(0) \end{pmatrix}. \tag{5.9}$$

Note that $\mathcal{H}_u$ does not depend on $\eta$. Compute its derivative

$$D\mathcal{H}_u(\tilde{x}_u(0), \ldots, \tilde{x}_u(k-1)) = \begin{pmatrix} b(1+\lambda_2)\tilde{x}_u(0)^{\lambda_2} & -1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & -1 \\ -1 & 0 & 0 & b(1+\lambda_2)\tilde{x}_u(k-1)^{\lambda_2} \end{pmatrix}.$$

A simple calculation shows that $\mathcal{H}_u(\tilde{x}_u) = \tilde{x}_u$, $\det D\mathcal{H}_u(\tilde{x}_u) = 0$ has the solution $\tilde{x}_u(i) = (\frac{1}{b(1+\lambda_2)})^{1/\lambda_2}$, $0 \le i < k$, at the following value for $\lambda_1$: $\lambda_1 = \left(\frac{1}{b(1+\lambda_2)}\right)^{1/\lambda_2} \frac{\lambda_2}{1+\lambda_2}$. Further, $D\mathcal{H}_u(\tilde{x}_u)$ has one dimensional kernel spanned by $N_u = (1, \ldots, 1)$. Let $M_u$ be the left eigenvector of $D\mathcal{H}_u(x_u)$; $M_u = (1, \ldots, 1)$. Computing second order derivatives gives

$$\left\langle M_u, D^2\tilde{\mathcal{H}}_u(\tilde{x}_u)[N_u, N_u] \right\rangle = k\lambda_2(1/b)^{-1/\lambda_2} > 0 \tag{5.10}$$

(it is in fact much larger than 0). As a consequence, the graph of $\mathcal{H}_u$ has a quadratic tangency to 0 at $\tilde{x}_u$.

Now, fix $\lambda$ for which $x$ is a periodic orbit at a saddle-node bifurcation. Recall that $\lambda$ is fixed such that $x$ is a periodic orbit at a saddle-node bifurcation and $x_u(i) \sim \frac{1}{e}(1/b)^{1/\lambda_2}$. Note that $D\mathcal{G}_u^\eta(x)$ is a $k \times k$-matrix near $D\mathcal{H}_u(x)$. The asymptotic expansion identity (4.6) allows for many (if $k$ is large) small nonzero entries in $D\mathcal{G}_u^\eta(x)$, which might complicate the analysis. For better control of the higher order terms we work with the full bifurcation equations (4.8), (4.9) instead of directly with the reduced bifurcation equations. With reference to the bifurcation equations (4.8), define

$$\mathcal{G}^\eta(x(0), \ldots, x(k-1)) = \begin{pmatrix} T(x_{ss}(0), x_u(0), x_{uu}(1)) - (x_{ss}(1), x_u(1), x_{uu}(0)) \\ \vdots \\ T(x_{ss}(k-1), x_u(k-1), x_{uu}(0)) - (x_{ss}(0), x_u(0), x_{uu}(k-1)) \end{pmatrix}.$$

The lowest order terms yield the map

$$\mathcal{H}(\tilde{x}(0), \ldots, \tilde{x}(k-1)) = \begin{pmatrix} (-\tilde{x}_{ss}(1), \lambda_1 + b\tilde{x}_u(0)^{1+\lambda_2} - \tilde{x}_u(1), \tilde{x}_{uu}(0)) \\ \vdots \\ (-\tilde{x}_{ss}(0), \lambda_1 + b\tilde{x}_u(k-1)^{1+\lambda_2} - \tilde{x}_u(0), \tilde{x}_{uu}(k-1)) \end{pmatrix}$$

for which $D\mathcal{H}(\tilde{x})$ (with $\tilde{x}(i) = (0, \tilde{x}_u(i), 0)$) has left and right eigenvectors for the nullspace $N, M = ((0,1,0), \ldots, (0,1,0))$.

By the center manifold theorem, the right eigenvector $N = (N_i)$ of $D\mathcal{G}^\eta(x)$ is such that

$$N_i \in T_{x(i)} W^c_{\sigma^i \eta}.$$

Thus, writing $N_i = (v_i^{ss}, v_i^u, v_i^{uu})$, we obtain $v_i^{ss}, v_i^{uu}$ as function of $v_i^u$ alone with moreover $|v_i^{ss}|, |v_i^{uu}| = \mathcal{O}(|x_u(i)|^\omega)|v_i^u|$. Plugging this into the equations for the kernel of $D\mathcal{G}^\eta(x)$ leads to equations for $N_u$ in the kernel of $D\mathcal{G}_u^\eta(x)$. These equations are of the form $\alpha_i v_i^u - \alpha_{i+1} v_{i+1}^u = 0$ with $\alpha_i$ close to 1 (recall that for the kernel of $D\mathcal{H}_u(\tilde{x}_u)$, $N_u$ is given by equations $v_i^u - v_{i+1}^u = 0$). It follows that each entry in $N_u$ is positive, the other entries of $N$ are small in comparison.

Now we consider the left eigenvector $M = (M_i)$ of $D\mathcal{G}^\eta(x)$. First note that $T_x W^c_\eta$ is a center manifold, indexed by $\eta$, for $v \mapsto D\Psi(x)v$ and can be constructed with the procedure in [6]. Similarly one constructs center manifolds, indexed by $\eta$, for $v \mapsto D\Psi^*(x)v$. These center manifolds contain the left eigenvector $M$, which is thus obtained similarly as $N$ and satisfies analogous properties (in particular $M_u$ has all positive entries and the other entries are small in comparison).

Having computed $N_u$ and $M_u$, one checks that $D\mathcal{G}^\eta(x)$ has one dimensional kernel and

$$\left\langle M, D^2\tilde{\mathcal{G}}^\eta(x)[N,N] \right\rangle > 0. \tag{5.11}$$

The graph of $\mathcal{G}^\eta$ therefore has a quadratic tangency to 0 and the solution to $\mathcal{G}^\eta(x) = 0$ is locally unique. Since the partial derivative of $G^{\sigma^{i-1}\eta}(x_u(i), x_u, \lambda)$ with respect to $\lambda_1$ is positive,

$$\left\langle M, \frac{\partial}{\partial \lambda_1} \mathcal{G}^\eta(x) \right\rangle > 0 \tag{5.12}$$

and there is a unique value of $\lambda_1$ for which a saddle-node bifurcation occurs. This proves that the saddle-node bifurcations of all periodic orbits are codimension one and unfold generically. Note also that the saddle-node bifurcations occur along smooth curves that extend a uniform distance (uniform in the period) in the $\lambda_2$ coordinate.

## 5.2 Twisted resonant bellows

The analysis of period-doubling and pitchfork bifurcations of periodic orbits in the suspended horseshoe follows the arguments of the saddle-node bifurcations in the previous section, with obvious changes to cope with the different defining and genericity conditions. We will indicate but not pursue the analysis, as the necessary computations become tedious.

We start with period-doubling bifurcations of periodic orbits with odd period. Take a periodic orbit $x(i)$ of period $k$ (that is, $x(j+k) = x(j)$), satisfying the reduced bifurcation equations

$$\begin{aligned} x_u(0) &= G^{\sigma^{k-1}\eta}(x_u(k-1), x_u, \lambda), \\ x_u(1) &= G^{\eta}(x_u(0), x_u, \lambda), \\ \vdots &= \vdots \\ x_u(k-1) &= G^{\sigma^{k-2}\eta}(x_u(k-2), x_u, \lambda). \end{aligned} \tag{5.13}$$

The center manifold reduction yields the existence of a parameter value $\lambda_1$, for each small positive $\lambda_2$, for which (5.13) has a solution $x$ with itinerary $\eta$ at a period-doubling bifurcation. Note that the period of $x(i)$ has to be odd for this to be true (the product of the derivatives at the points of the periodic orbit has to be -1).

As in the previous section, one can trace the graph of the map defined by the system of equations (5.13). Define

$$\mathcal{G}_u^\eta(x_u(0),\ldots,x_u(k-1)) = \begin{pmatrix} G^\eta(x_u(0),x_u,\lambda) - x_u(1) \\ \vdots \\ G^{\sigma^{k-1}\eta}(x_u(k-1),x_u,\lambda) - x_u(0) \end{pmatrix}. \tag{5.14}$$

Fix $\lambda$ for which $x$ is a periodic orbit with itinerary $\eta$ at a period-doubling bifurcation. Let $N_u, M_u$ are the right and left eigenvectors respectively for the single zero eigenvalue of $D\mathcal{G}_u^\eta(x_u)$.

Consider the equation $\mathcal{G}_u^\eta(x_u) = x_u$ defining $x_u$ as function of $\lambda_1$ for a fixed small positive value of $\lambda_2$. A generic unfolding of the period-doubling bifurcation is implied by

$$\frac{\partial}{\partial \lambda_1} D\mathcal{G}_u^\eta(x_u) N_u \neq 0 \tag{5.15}$$

at the bifurcating orbit $x_u$ (note that both $x_u$ and $N_u$ depend on $\lambda_1$). For a proof that the period-doubling bifurcation is of codimension-one, and not degenerate, one has to check the condition (see [11, Chapter 5])

$$\begin{aligned} &\left\langle M_u, D^3\mathcal{G}_u^\eta(x_u)[N_u,N_u,N_u]\right\rangle - \\ &\qquad \left\langle M_u, D^2\mathcal{G}_u^\eta(x_u)[N_u, \left(D\mathcal{G}_u^\eta(x_u)\right)^{-1} D^2\mathcal{G}_u^\eta(x_u)[N_u,N_u]\right\rangle \neq 0. \end{aligned} \tag{5.16}$$

We will not go into the required tedious estimates. Note though that for fixed $k$ the equations are more tractable when one considers sufficiently small parameter values. As in the previous section, estimates that hold uniformly in $k$ are harder to obtain.

An analogous analysis can be performed for a symmetric periodic orbit $x(i)$ with itinerary $\eta$ of period $2k$ with $k$ odd. The center manifold reduction explains the occurrence of a pitchfork bifurcation: the composition $S \circ \Psi^k$ maps one half of the center manifold $W_\eta^c$ bounded by $x(0)$ to the other half. Note that this symmetry argument breaks down if the periodic orbit is symmetric but of period $2k$ with $k$ even. There is hence a (possibly degenerate) saddle-node bifurcation for such periodic orbits.

The statements on the 2-homoclinic orbits follow just as in [1] for bifurcations near a single resonant homoclinic orbit. The equations are of the form

$$\begin{aligned} x_u(1) &= \lambda_1 + R_1(x_u(1),\lambda), \\ 0 &= \lambda_1 + x_u(1)^{1+\lambda_2} + R_2(x_u(1),\lambda), \end{aligned}$$

where the higher order terms $R_1, R_2$ contain a dependence on the itinerary. Clearly the itineraries (12), (21) are symmetry related, as are (11) and (22). All periodic orbits from the basic set in region II that is equivalent to a suspended full shift on four symbols must likewise disappear through homoclinic bifurcations. We do not have precise knowledge on this scenario.

The suspended basic set that exists in parameter region II, can be described using symbolic dynamics on four symbols, corresponding to each of the four homoclinic orbits that exists along the curves of doubled bellows. A straightforward and convenient way to code orbits is by distinguishing two symbols $1^+, 1^-$ that replace the symbol 1 and two symbols $2^+, 2^-$ that replace the symbol 2. Here one makes use of the observation that the $x_u$ coordinates of points in a recurrent orbit of $\Psi$ are close to two different numbers: the symbols $1^+, 1^-$, and likewise $2^+, 2^-$, correspond to the larger and smaller of these numbers. This gives a full shift on four symbols.

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