ITERATED FUNCTION SYSTEMS OF LOGISTIC MAPS: SYNCHRONIZATION AND INTERMITTENCY

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Abstract. We discuss iterated function systems generated by finitely many logistic maps, with a focus on synchronization and intermittency. We provide sufficient conditions for synchronization, involving negative Lyapunov exponents and minimal dynamics. A number of results that clarify the scope of these conditions are included. We analyze a mechanism for intermittency that involves the full map $x \mapsto 4x(1-x)$ as one of the generators of the iterated function system. For iterated function systems generated by $x \mapsto 2x(1-x)$ and $x \mapsto 4x(1-x)$ we prove the existence of a σ-finite stationary measure.

1 Introduction

Iterated function systems are given by a (finite) collection of continuous maps on a metric space, that are composed for iterations. The maps are typically picked at random each iterate. They have been studied extensively because of their role in the study of fractals [10, 17]. The dynamics of iterated function systems can be studied using a description as a skew product system over a shift operator. As such they provide case studies for nonuniformly hyperbolic dynamics, which is another reason for their study. Indeed, phenomena that have been observed in skew product systems coming from iterated function systems frequently have analogues in more general skew product systems and more general dynamical systems (see [14] for an example where this line of thought is made explicit).

We take iterated function systems given by a collection of $k > 1$ logistic maps $f_i : I \to I$, $1 \leqslant i \leqslant k$, with $I = [0, 1]$ and

$$f_i(x) = \rho_i x(1-x),$$

$0 < \rho_i \leqslant 4$. The dynamics of logistic maps is a paradigm for chaotic dynamics, see [22], making it interesting to consider iterated function systems of logistic maps.

We focus on dynamics which can be viewed as simple dynamics in the skew product system setting, for instance characterized by negative Lyapunov exponents. We start with an investigation of

synchronization: orbits of different initial conditions converge to each other under identical compositions of logistic maps.

Figure 1 illustrates it by showing two orbit pieces of an iterated function system with a visible convergence of orbits. The figure contains also the histogram of a numerically computed orbit, illustrating that even though different orbits converge to each other, the orbits themselves occupy large parts of the interval. Steinsaltz [28] wrote a fundamental paper on contractive dynamics focusing on logistic maps with the parameter chosen from a continuous distribution. We continue this investigation in a context of iterated function systems generated by finitely many logistic maps. We use a dynamical systems theory approach,

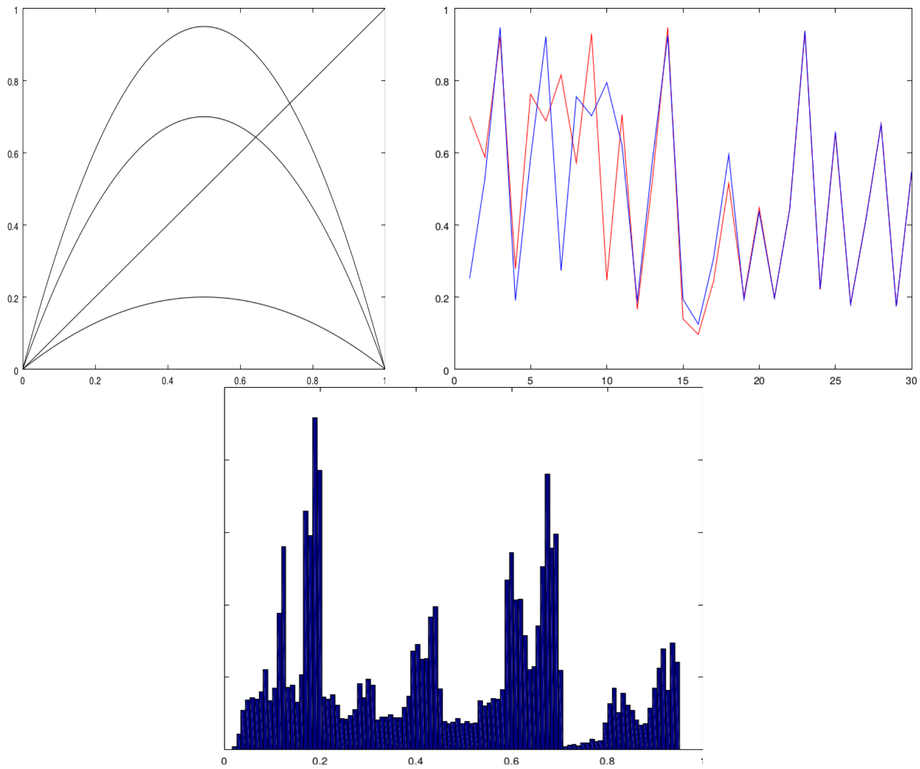

Figure 1. This figure illustrates the phenomenon of synchronization. Considered is the iterated function system generated by $f_1(x) = 0.8x(1-x)$ (chosen with probability $p_1 = 1/4$), $f_2(x) = 2.8x(1-x)$ (chosen with probability $p_2 = 1/2$), $f_2(x) = 3.8x(1-x)$ (chosen with probability $p_3 = 1/4$). Graphs are drawn in the first frame. The second frame shows numerically computed time series for two different initial conditions, where the points are connected by lines. The two orbits appear to converge to each other. The third frame shows the histogram of a numerically computed orbit, indicating that orbits occupy an entire interval.

adopting a skew product systems point of view. We prove a theorem on synchronization in iterated function systems of logistic maps under some assumptions, see Theorem 3.1 and Theorem 3.2. One of the assumptions is that the fixed point at 0 is repelling on average, so that typical orbits will not converge to 0. Other assumptions are on negative Lyapunov exponents and on minimality of the dynamics of the iterated function system. Precise statements and a detailed discussion are in Section 3.

If the fixed point at 0 is neutral on average, Athreya and Schuh [4] prove the occurrence of

intermittency: for typical orbits the set of iterates for which the orbit is near 0*, has full density, but orbits do not stay near* 0.

The left frame of Figure 2 illustrates a time series. This kind of intermittency has been called on-off intermittency [15, 25]. We discuss a different mechanism for intermittency, where the iterated function system contains both the map $x \mapsto 2x(1-x)$, for which the critical point is a superstable fixed point, and the map $x \mapsto 4x(1-x)$, for which the critical point is mapped onto 0 in two iterates. We assume the fixed point at 0 to be

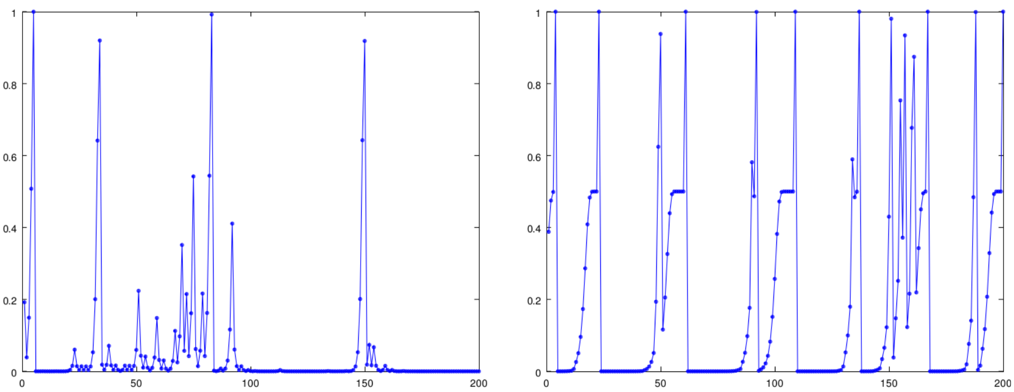

Figure 2. Two examples of intermittent time series. The left frame shows numerically computed time series for the iterated function system generated by $f_1(x) = \frac{1}{4}x(1-x)$ (chosen with probability $p_1 = 0.5$) and $f_2(x) = 4x(1-x)$ (chosen with probability $p_2 = 0.5$). This iterated function system has a vanishing Lyapunov exponent at 0. The right frame shows a numerically computed time series for the iterated function system generated by $f_1(x) = 4x(1-x)$ (chosen with probability $p_1 = 0.3$) and $f_2(x) = 2x(1-x)$ (chosen with probability $p_2 = 0.7$). Here 0 is repelling on average, but nonetheless typical orbits have full density of their iterates near 0.

repelling on average. In our discussion of synchronization the possibility of iterating the map $x \mapsto 4x(1-x)$ is ruled out. That inclusion of this map can give rise to new phenomena was earlier observed by Högnäs and Carlsson [8]. An intermittent time series under this mechanism is illustrated in the right frame of Figure 2. Precise statements and proofs of intermittency are in Section 4, see in particular Theorem 4.1 and Theorem 4.2. The first of these two theorems establishes intermittency by studying time series. The second theorem gives more precise information through σ-finite stationary measures. Section 4.3 includes a brief discussion of other examples of intermittency, such as involving superstable periodic orbits. Figure 3 pictures two examples.

For a more precise discussion of our results, we start with some generalities to introduce concepts and notation. Then we present our main results on synchronization in Section 3 and intermittency in Section 4.

2 Generalities

Throughout the paper, $f_1, \ldots, f_k$ will stand for logistic maps

$$f_i(x) = \rho_i x(1-x),\ 0 < \rho_i \leqslant 4,$$

on $I = [0,1]$. The study in this paper makes use of descriptions with skew product systems and properties of their invariant measures. The necessary material is introduced in this section. This fixes also notation for the rest of the paper.

2.1 Iterated function systems

Denoting $\mathbb{F} = \{f_1, \ldots, f_k\}$, the iterated function system IFS $(\mathbb{F})$ is the action on I of the semi-group generated by $f_1, \ldots, f_k$. A set $A \subset I$ is called invariant for the iterated function

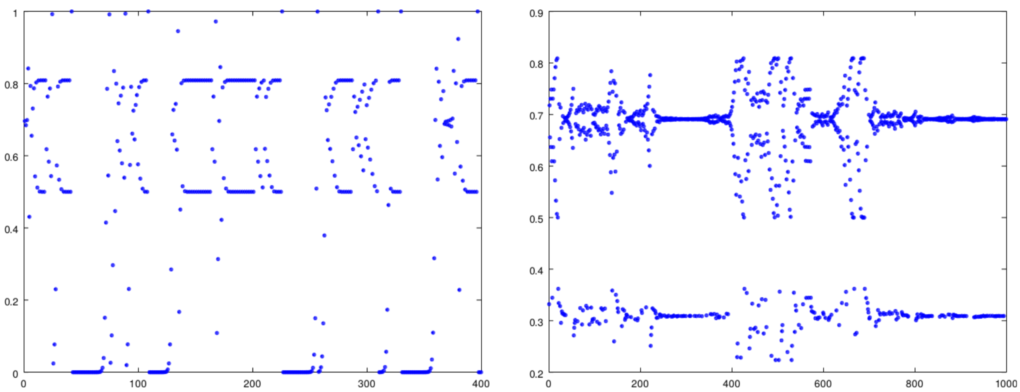

Figure 3. The left frame illustrates Theorem 4.3: numerically computed time series for the iterated function system generated by $f_1(x) = (1+\sqrt{5})x(1-x)$ (chosen with probability $p_1 = 0.85$) and $f_2(x) = 4x(1-x)$ (chosen with probability $p_2 = 0.15$). The map f_1 possesses a superstable period two orbit. The right frame illustrates Theorem 4.4: numerically computed time series for the iterated function system generated by $f_1(x) = (1+\sqrt{5})x(1-x)$ and $f_2(x) = (1+\frac{\sqrt{5}}{5})x(1-x)$ chosen with probabilities that yield zero Lyapunov exponents on the invariant set consisting of the two positive fixed points of f_1 and f_2.

system IFS $(\mathbb{F})$ if $\mathbb{F}(A) = A$, where $\mathbb{F}(A) = \cup_{i=1}^k f_i(A)$. A sequence $\{x_n : n \in \mathbb{N}\}$ is called a branch of an orbit of IFS $(\mathbb{F})$ if for each $n \in \mathbb{N}$ there is $f_n \in \mathbb{F}$ such that $x_{n+1} = f_n(x_n)$. The iterated function system IFS $(\mathbb{F})$ is minimal on an invariant set A if any orbit of $x \in A$ under IFS $(\mathbb{F})$ has a branch which is dense in A.

The logistic maps $f_i(x) = \rho_i x(1-x)$ are chosen independently from a fixed distribution; f_i is picked with probability p_i with $0 < p_i < 1$ and $\sum_{i=1}^k p_i = 1$. As this defines a Markov process, a central role is played by stationary measures: a stationary measure m is a probability measure on I that satisfies

$$m = \sum_{i=1}^k p_i f_i m.$$

That is, a stationary measure is equal to its averaged push forward under the maps f_i, where the pushforward $f_i m$ is defined by $f_i m(A) = m(f_i^{-1}(A))$ for Borel sets $A \subset I$. The support supp (m) of a stationary measure m is the complement of the largest open set A for which $m(A) = 0$.

Let $\mathcal{M}_I$ be the space of all Borel probability measures on I endowed with the weak-star topology. The topological space $\mathcal{M}_I$ is metrizable, we will take a metric $d_{\mathcal{M}_I}$ on $\mathcal{M}_I$ that generates the weak star topology, see e.g. [21].

Define the transformation $\mathcal{T} : \mathcal{M}_I \to \mathcal{M}_I$ by

$$\mathcal{T} m = \sum_{i=1}^k p_i f_i m. \qquad (1)$$

Note that the fixed points of $\mathcal{T}$ are precisely the stationary measures of the iterated function system.

Lemma 2.1. *The map $\mathcal{T}$ is continuous. It also depends continuously on the parameters $p_1, \ldots, p_k$ and $\rho_1, \ldots, \rho_k$.*

Proof. The proof of [13, Lemma A.1] applies. □

2.2 Skew product systems

Associated to an iterated function system is a skew product system over the shift on a symbol space. Let $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$ and $F^+ : \Sigma_k^+ \times I \to \Sigma_k^+ \times I$ be given by

$$F^+(\omega, x) = (\sigma^n \omega, f_{\omega_0}(x)).$$

Here σ is the left shift operator acting on a sequence $\omega = (\omega_i)_{i \in \mathbb{N}}$ by $(\sigma\omega)_i = \omega_{i+1}$. For notational convenience we write

$$\left(F^+\right)^n(\omega, x) = (\sigma^n \omega, f_\omega^n(x))$$

for $n \geqslant 1$, so

$$f_\omega^n = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}.$$

We will adopt the notation f_ω^n also if only $\omega_0, \ldots, \omega_{n-1}$ are given.

Given probabilities p_i, one has the product measure, or Bernoulli measure, ν^+ on Σ_k^+. If $C_{0,\ldots,n-1}^{i_0,\ldots,i_{n-1}} \subset \Sigma_k^+$ is the cylinder

$$C_{0,\ldots,n-1}^{i_0,\ldots,i_{n-1}} = \{\omega \in \Sigma_k^+ \; ; \; \omega_j = i_j, 0 \leqslant j \leqslant n-1\},$$

then

$$\nu^+(C_{0,\ldots,n-1}^{i_0,\ldots,i_{n-1}}) = p_{i_0}^{\ell_0} \cdots p_{i_{n-1}}^{\ell_{n-1}}$$

if ℓ_i is the number of symbols i in $i_0, \ldots, i_{n-1}$.

A direct computation gives the following well known correspondence between stationary measures and invariant measures for F^+ with marginal ν^+.

Lemma 2.2. *A probability measure m is a stationary measure if and only if $\mu^+ = \nu^+ \times m$ is an invariant measure of F^+ with marginal ν^+ on Σ_k^+.*

Proof. See e.g. [13, Lemma A.2]. □

A stationary measure m is called ergodic if $\nu^+ \times m$ is ergodic for F^+. As a consequence of Birkhoff's ergodic theorem, see e.g. [11, Corollary 4.20], for an ergodic stationary measure m we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_\omega^i(x)} = m, \tag{2}$$

with convergence in the weak star topology, for $\nu^+ \times m$-almost all (ω, x). So we recover m from the distribution of points in typical orbits. A point (ω, x) is said to be a generic point for an ergodic measure $\nu^+ \times m$, if the orbit is distributed according to the measure.

Similar to F^+, an associated skew product system over the two sided shift is given: with $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$, $F : \Sigma_k \times I \to \Sigma_k \times I$ is defined by

$$F(\omega, x) = (\sigma\omega, f_\omega(x)).$$

The map F on $\Sigma_k \times I$ is an extension of F^+ on $\Sigma_k^+ \times I$. In the set-up of invertible fiber maps f_ω, F is invertible (in such a set-up F is the natural extension). In our case F is not invertible, because the fiber maps are not invertible. Write ν for the product measure on Σ_k corresponding to the probabilities p_i.

A key role in our study is played by ergodic invariant measures for the skew product systems. The relation between invariant measures with marginal ν^+ for F^+ and invariant measures with marginal ν for F is discussed in [13]. The material comes from standard sources such as [1]. A difference with the material in [1, 13] is that here the fiber maps f_ω are not invertible and so F is an extension of F^+ but not the natural extension. The following material is usually developed for natural extensions, as in [13], but applies to the current setting as well.

Denote by $\mathbf{B}$ the Borel sigma-algebra on I. Write $\pi : \Sigma_k \to \Sigma_k^+$ for the natural projection

$$\pi(\omega_n)_{-\infty}^{\infty} = (\omega_n)_0^{\infty}.$$

The Borel sigma-algebra $\mathbf{F}^+$ on Σ_k^+ yields a sigma-algebra $\mathbf{F}_0 = \pi^{-1}\mathbf{F}^+$ on Σ_k. A measure μ on $\Sigma_k \times I$ with marginal ν has conditional measures μ_ω on the fibers $\{\omega\} \times I$, such that

$$\mu(A) = \int_{\Sigma_k} \mu_\omega(A_\omega)\, d\nu(\omega)$$

for measurable sets A, where we have written

$$A_\omega = A \cap (\{\omega\} \times I).$$

A measure μ^+ on $\Sigma_k^+ \times I$ with marginal ν^+ likewise has conditional measures μ_ω^+. It is convenient to consider ν^+ as a measure on Σ_k with sigma-algebra $\mathbf{F}_0$ and μ^+ as a measure on $\Sigma_k \times I$ with sigma-algebra $\mathbf{F}_0 \otimes \mathbf{B}$. When $\omega \in \Sigma_k$ we will write μ_ω^+ for the conditional measures $\mu_{\pi\omega}^+$. The spaces of measures are equipped with the weak star topology.

Invariant measures for F^+ with marginal ν^+ correspond to invariant measures for F with marginal ν in a one-to-one relationship, as detailed in Proposition 2.1 below. This is a special case of [1, Theorem 1.7.2], see [13, Proposition A.1]. The result implies that stationary measures correspond one-to-one to specific invariant measures for F with marginal ν.

Write $\Sigma_k = \Sigma_k^- \times \Sigma_k^+$, where Σ_k^- consists of the past parts $(\omega_i)_{-\infty}^{-1}$ of sequences ω. We have a projection

$$\Pi : \Sigma_k^- \times \Sigma_k^+ \times I \to \Sigma_k^+ \times I.$$

Proposition 2.1. *Let μ^+ be an F^+-invariant probability measure with marginal ν^+. Then there exists an F-invariant probability measure μ with marginal ν and conditional measures*

$$\mu_\omega = \lim_{n\to\infty} f^n_{\sigma^{-n}\omega} \mu^+_{\sigma^{-n}\omega}, \tag{3}$$

ν-almost surely.

Let μ be an F-invariant probability measure with marginal ν. Then,

$$\mu^+ = \Pi\mu \tag{4}$$

is an F^+-invariant probability measure with marginal ν^+.

The correspondence $\mu \leftrightarrow \mu^+$ given by (3), (4) is one-to-one. Furthermore, through these relations, F^+-invariant product measures $\mu^+ = \nu^+ \times m$ correspond one-to-one with F-invariant product measures $\mu = \nu^+ \times \vartheta$ on $\Sigma_k^+ \times (\Sigma_k^- \times I)$. The measure μ is ergodic if and only if μ^+ is ergodic.

2.3 Lyapunov exponents

Our discussion of contractive dynamics in the next chapter requires the notion of Lyapunov exponents. The Lyapunov exponent of F^+ at a point $(\omega, x) \in \Sigma_k^+ \times I$ is

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| f'_{\omega_{n-1}}(f^{n-1}_\omega(x)) \cdots f'_{\omega_0}(x) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'_{\sigma^i\omega}(f^i_\omega(x)) \right|,$$

in case the limit exists. Given an ergodic stationary measure m, by Birkhoff's ergodic theorem

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln \left| f'_{\omega_{n-1}}(f^{n-1}_\omega(x)) \cdots f'_{\omega_0}(x) \right| &= \int_{\Sigma_k^+} \int_0^1 \ln \left| f'_\omega(x) \right| \, dm(x) d\nu^+(\omega) \\ &= \int_0^1 \sum_{i=1}^k p_i \ln \left| f'_i(x) \right| \, dm(x), \end{aligned}$$

for $\nu^+ \times m$-almost all $(\omega, x) \in \Sigma_k^+ \times I$ (the value can be $-\infty$). The number on the right hand side is referred to as the Lyapunov exponent with respect to the ergodic stationary measure m. Since 0 is a fixed point of f_i for every i, the delta measure at 0 is a stationary measure. As $f'_i(0) = \rho_i$, we obtain for $x = 0$ that

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| f'_{\omega_{n-1}}(f^{n-1}_\omega(0)) \cdots f'_{\omega_0}(0) \right| = \sum_{i=1}^k p_i \ln(\rho_i),$$

for ν^+-almost all $\omega \in \Sigma_k^+$. We write

$$L(0) = \sum_{i=1}^k p_i \ln(\rho_i). \tag{5}$$

For $L(0) < 0$, the boundary point 0 is attracting on average [2]. The case $L(0) = 0$ is considered in [4]. We will be interested in the case $L(0) > 0$.

3 Synchronization

We formulate a general result on synchronization in iterated function systems of logistic maps. The result comes with assumptions on invariant measures and Lyapunov exponents; in addition to proving a general result we will provide checkable conditions for its assumptions. Figure 1 in the introduction provides an illustration of synchronized time series.

Theorem 3.1. *Consider an iterated function system IFS*$(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* $I = [0,1]$ *with* $0 < \rho_i < 4$. *Suppose*

- *(i) For some* $1 \leqslant i_1 \leqslant k$, $\rho_{i_1} \in (1,3)$: *the map* f_{i_1} *possesses an attracting fixed point in* $(0,1)$ *with basin of attraction equal to* $(0,1)$;
- *(ii)* $L(0) > 0$: *the fixed point at* $x = 0$ *is repelling on average;*
- *(iii) There is an ergodic stationary probability measure* m *such that*
  - *(a) with respect to* m, *the iterated function system has negative Lyapunov exponents;*
  - *(b) the iterated function system is minimal on supp* $(m) \setminus \{0\}$.

For ν^+-almost all $\omega \in \Sigma_k^+$ there is an open set $W^s(\omega) \subset I$ with $m(W^s(\omega)) = 1$, so that

$$\lim_{n\to\infty} |f_\omega^n(x) - f_\omega^n(y)| = 0 \tag{6}$$

for $x, y \in W^s(\omega)$.

We prove Theorem 3.1 in Section 3.2, using Pesin theory for local stable sets in fibers $\{\omega\} \times I$ as developed in Section 3.1. Section 3.5 contains a more elementary proof of a somewhat stronger conclusion valid under stronger assumptions. In Sections 3.3 and 3.4 we present results on the existence of iterated function systems for which the conditions of Theorem 3.1, in particular the two parts of condition (iii), hold.

3.1 Local stable sets

Given an ergodic stationary measure with negative Lyapunov exponents, Pesin theory gives stable manifolds inside fibers $\{\omega\}\times I$. The following proposition extracts the statement for our setting. We provide a direct argument along the lines of [19, Lemma 2.2] or [7, Lemma A.1]. General statements for skew product systems with diffeomorphisms as fiber maps are in e.g. [9] or [23, Section 10]. Extensions to endomorphisms are treated in [26, Section 5].

Proposition 3.1. *Consider an iterated function system IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, of logistic maps $f_i(x) = \rho_i x(1-x)$ on I with $0 < \rho_i < 4$. Suppose there is an ergodic stationary measure m so that the iterated function system has negative Lyapunov exponents with respect to m. For each $\zeta > 0$, there are $\eta > 0$, a set $\mathcal{A}^+ \subset \Sigma_k^+$ of positive measure $\nu^+(\mathcal{A}) > 1 - \zeta$ and constants $C > 0, 0 < \lambda < 1$, so that for $\omega \in \mathcal{A}^+$, there is an interval $B_\omega^s \subset I$ of length at least η with*

$$|f_\omega^n(x) - f_\omega^n(y)| \leqslant C\lambda^n \, |x - y|$$

for $x, y \in B_\omega^s$.

Proof. By Birkhoff's ergodic theorem, there is a set of full $\nu^+ \times m$ measure so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'_{\sigma^i\omega}(f_\omega^i(x))| < 0$$

for (ω, x) in this set. By Fubini's theorem there is a subset $B \subset I$ of full m-measure, so that for any $x \in B$ there is a set of full ν^+-measure $\Theta_x \subset \Sigma_k^+$ with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'_{\sigma^i\omega}(f_\omega^i(x))| < 0, \ \forall \omega \in \Theta_x.$$

Take $x_0 \in B$. For $\varepsilon > 0$, $\omega \in \Theta_{x_0}$ and $i = 0, 1, \dots$ write

$$a(\omega, i) = \ln\left(|f'_{\sigma^i\omega}(f_\omega^i(x_0))| + \varepsilon\right).$$

For every $\omega \in \Theta_{x_0}$ there is a small $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$

$$L_\varepsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a(\omega, i) = \int_0^1 \sum_{i=1}^{k} p_i \ln\left(|f_i'(x)| + \varepsilon\right) \, dm(x) \tag{7}$$

and is negative (by monotone convergence). So, for ν^+-almost all $\omega \in \Sigma_k^+$ and $0 < \varepsilon \leqslant \varepsilon_0$, $\sum_{i=0}^{n-1} a(\omega, i)$ converges to $-\infty$ as $n \to \infty$ and

$$A(\omega) = \max\{0, \max_{n\geqslant 1} \sum_{i=0}^{n-1} a(\omega, i)\}$$

exists.

Take $0 < \varepsilon \leqslant \varepsilon_0$. There exists $\delta > 0$ so that if $|x - y| < \delta$, then

$$|f_j'(x)| \leqslant |f_j'(y)| + \varepsilon, \ \forall 1 \leqslant j \leqslant k.$$

Let B_ω^s be a neighborhood of x_0 with radius $\delta e^{-A(\omega)}$. For every x in B_ω^s, there exists $c_0 \in (x_0, x)$ such that

$$\big|f_\omega(x) - f_\omega(x_0)\big| = |f_\omega'(c_0)||x - x_0|.$$

Since $|c_0 - x_0| < \delta$ we find $|f_\omega'(c_0)| \leqslant |f_\omega'(x_0)| + \varepsilon$ and

$$|f_\omega(x) - f_\omega(x_0)| \leqslant e^{a(\omega,0)}|x - x_0|.$$

Note that $|f_\omega(x) - f_\omega(x_0)| < \delta e^{a(\omega,0)} e^{-A(\omega)} \leqslant \delta$. So with a similar reasoning as above,

$$|f_\omega^n(x) - f_\omega^n(x_0)| < e^{\sum_{i=0}^{n-1} a(\omega,i)}|x - x_0|,$$

for every $n \in \mathbb{N}$. Write $\Lambda_n = \frac{1}{n}\sum_{i=0}^{n-1} a(\omega, i)$. By (7), for every $\varepsilon' > 0$ there exists $N_{\varepsilon'} > 0$ (depending on ω and ε) such that for every $n > N_{\varepsilon'}$,

$$|\Lambda_n - L_\varepsilon| < \varepsilon'.$$

Since $L_\varepsilon < 0$ we can take $\varepsilon' > 0$ such that $L_\varepsilon + \varepsilon' < 0$. Now for $\omega \in \Theta_{x_0}$, $\lambda = e^{L_\varepsilon + \varepsilon'} < 1$ and $n > N_{\varepsilon'}$,

$$|f_\omega^n(x) - f_\omega^n(x_0)| < e^{n\Lambda_n}|x - x_0| < e^{n(L_\varepsilon + \varepsilon')}|x - x_0| < \lambda^n |x - x_0|.$$

Define

$$C'(\omega) = \max_{0\leqslant n\leqslant N_{\varepsilon'}} \{e^{\sum_{i=0}^{n-1} a(\omega,i)}\},$$

and

$$C(\omega) = \max\{1, \frac{C'(\omega)}{e^{N_{\varepsilon'}(L_\varepsilon+\varepsilon')}}\}.$$

Then for ν^+-almost every $\omega \in \Sigma_k^+$ and every $n \in \mathbb{N}$ we have

$$|f_\omega^n(x) - f_\omega^n(y)| \leqslant C(\omega)\lambda^n|x - x_0|.$$

The function $C(\omega)$ depends measurably on ω. So for every small $\zeta > 0$, by Lusin's theorem there is a compact set $\mathcal{A}^+ \subset \Sigma_k^+$ of measure $\nu^+(\mathcal{A}^+) > 1 - \zeta$ so that B_ω^s is of length at least $0 < \eta < \delta$ and the function C is continuous on $\mathcal{A}^+$. Therefore it is bounded by a constant C and the proposition is proved. □

By taking, in the proof of Proposition 3.1, $x_0 \in \mathrm{supp}(m)$, we find

$$\nu^+ \times m \left(\bigcup_{\omega\in\mathcal{A}^+} \{\omega\} \times B_\omega^s \right) > 0. \tag{8}$$

The following corollary is just a reformulation of Proposition 3.1 for the two sided skew product system $F : \Sigma^k \times I \to \Sigma^k \times I$, obtained by noting that the fiber coordinates of $F(\omega, x)$ do not depend on the past ω^- of $\omega = (\omega^-, \omega^+) \in \Sigma^- \times \Sigma^+$.

Corollary 3.1. *Take the assumptions of Proposition 3.1. For each $\zeta > 0$, there are $\eta > 0$, a set $\mathcal{A} = \Sigma_k^- \times \mathcal{A}^+ \subset \Sigma_k$ of positive measure $\nu(\mathcal{A}) > 1 - \zeta$ and constants $C > 0, 0 < \lambda < 1$, so that for $\omega \in \mathcal{A}$, there is an interval $B^s_\omega \subset I$ of length at least η with*

$$|f^n_\omega(x) - f^n_\omega(y)| \leqslant C\lambda^n \, |x - y| \tag{9}$$

for $x, y \in B^s_\omega$.

3.2 Proof of Theorem 3.1

Let us start with some comments on the proof strategy of Theorem 3.1. Proposition 3.1 gives local stable sets in fibers $\{\omega\} \times I$. Typical orbits enter these stable sets by ergodicity, resulting in local contraction properties. This will be combined with the existence of a large stable set for one of the logistic maps, given by item (i) in Theorem 3.1. We use a pullback argument so that we can apply the convergence (3) in Proposition 2.1. For this the extension to the two sided shift on Σ_k is needed. We refer to Section 3.5 for an argument that avoids this pullback argument, valid under stronger assumptions.

Proof of Theorem 3.1. We will deduce Theorem 3.1 from the existence of an invariant measurable graph Γ for the two sided system $F : \Sigma_k \times I \to \Sigma_k \times I$. The graph Γ is the graph of a measurable function $X : \Sigma_k \to I$ constructed in the following proposition. To prove its existence we follow the approach in [16].

Proposition 3.2. *Assume the conditions from Theorem 3.1. Let μ be the invariant measure for F corresponding to m as in Proposition 2.1. Then the conditional measure μ_ω on $\{\omega\}\times I$ is a delta measure for ν-almost all ω: there exists a measurable function $X : \Sigma_k \to I$ so that*

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} m = \delta_{X(\omega)},$$

for ν-almost all ω, with convergence in the weak star topology.

Proof. Consider the map f_{i_1} from item (i) and its attracting fixed point at q_{i_1}. Note that

$$m(W^s(q_{i_1})) = m((0, 1)) = 1.$$

Hence, for any $\varepsilon > 0$ there is a closed interval $R_\varepsilon \subset (0, 1)$ with $m(R_\varepsilon) > 1-\varepsilon$. For an $\varepsilon > 0$ let $\Delta_\varepsilon \subset \mathcal{M}_I$ be the subset of probability measures on I that assign at least mass $1 - \varepsilon$ to some point:

$$\Delta_\varepsilon = \{m \in \mathcal{M}_I;\, m(x) \geqslant 1 - \varepsilon \,\text{for some}\, x \in I\}.$$

Note that Δ_ε's are closed subsets of $\mathcal{M}_I$.

Fix a small $\varepsilon > 0$ and take R_ε and Δ_ε as above. Consider $\mathcal{A} \subset \Sigma_k$, B^s_ω and $\eta > 0$ provided by Corollary 3.1.

Lemma 3.1. *There exists $L \in \mathbb{N}$ so that for each $\omega \in \mathcal{A}$, there exists $\mathcal{B}_{\omega^+} \subset \Sigma_k^-$ so that for $\gamma \in \mathcal{B}_{\omega^+} \times \{\omega^+\}$, $f^L_{\sigma^{-L}\gamma}$ maps R_ε into B^s_ω.*

Proof. For any small $r > 0$, there is a sufficiently large L_1 iterate of f_{i_1} that maps R_ε into a neighborhood $(q_{i_1} - r, q_{i_1} + r)$ of q_{i_1}. Since IFS $(\mathbb{F})$ is minimal on $\operatorname{supp}(m) \setminus \{0\}$, recall item (iii), every open set in $\operatorname{supp}(m)$ has intersection with the set $\bigcup_{\omega \in \Sigma_k} f^n_\omega(q_{i_1})$ for some $n \geqslant 0$. Hence for $\eta > 0$ there is an integer L_2 so that for any open interval B of diameter η with positive measure, there are symbols $j_{L_2}, \ldots, j_1$ so that $f_{j_{L_2}} \circ \cdots \circ f_{j_1}(q_{i_1}) \in B$. Combining the above statements, there is a composition $f_{j_{L_2}} \circ \cdots \circ f_{j_1} \circ f^{L_1}_{i_1}$ that maps R_ε into B^s_ω. We let $\mathcal{B}_{\omega^+}$ consist of the sequences in Σ_k^- that end with symbols ${i_1}^{L_1} j_1 \ldots j_{L_2}$. We have that L_1, L_2 are uniformly bounded but need not be constant in ω. We can get $L_1 + L_2$ to be constant on $\mathcal{A}$ by adjusting the number of iterates L_1. This proves the lemma with $L = L_1 + L_2$. □

Observe that $\nu_-(\mathcal{B}_{\omega^+})$ is uniformly bounded away from zero. Consequently the union

$$\mathcal{B} = \cup_{\omega \in \mathcal{A}} \mathcal{B}_{\omega^+} \times \{\omega^+\} \tag{10}$$

has positive measure: $\nu(\mathcal{B}) > 0$. By ergodicity of ν, for ν-almost all ω, its orbit under σ^{-1} intersects $\mathcal{B}$. So for such ω and every small $\varepsilon > 0$, Lemma 3.1 and Corollary 3.1 yield

$$\liminf_{n \to \infty} d_{\mathcal{M}_I}\left(f^n_{\sigma^{-n}\omega} m, \Delta_\varepsilon\right) = 0 \tag{11}$$

(recall from Section 2.1 that $d_{\mathcal{M}_I}$ is a metric on $\mathcal{M}_I$ generating the weak star topology). Letting $\varepsilon \to 0$, we observe that Δ_ε converges to the set of δ-measures in $\mathcal{M}_I$. Therefore, by (11) and Proposition 2.1,

$$\lim_{n \to \infty} f^n_{\sigma^{-n}\omega} m = \delta_{X(\omega)}$$

for a measurable function $X : \Sigma_k \to I$. □

The synchronization property (6) will be obtained as a consequence of the existence of the invariant graph Γ and the negative sign of the Lyapunov exponents. For ν-almost all $\omega \in \Sigma_k$, the Lyapunov exponents at $(\omega, X(\omega))$ exist and are strictly negative. Write $W^s(X(\omega))$ for the stable set of $X(\omega)$ inside the fiber $\{\omega\} \times I$;

$$W^s(X(\omega)) = \{y \in I \; ; \; \lim_{n \to \infty} |f^n_\omega(y) - X(\sigma^n \omega)| = 0\}.$$

The theory of nonuniform hyperbolicity, as in Proposition 3.1 and Corollary 3.1, yields the following. Write $D_\delta(X(\omega))$ for the δ-ball around $X(\omega)$. Then for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$S(\delta) = \{\omega \in \Sigma_k \; ; \; D_\delta(X(\omega)) \subset W^s(X(\omega))\}$$

satisfies

$$\nu(S(\delta)) > 1 - \varepsilon. \tag{12}$$

Once orbits are in a δ-ball $D_\delta(X(\omega))$ for $\omega \in S(\delta)$, distances to the orbit of $X(\omega)$ decrease to zero, which we may assume to happen at a uniform rate as in (9).

Proposition 3.3. *For ν-almost all $\omega \in \Sigma_k$, $W^s(X(\omega))$ is open with $m(W^s(X(\omega))) = 1$.*

Proof. We follow [16]. For ν-almost all $\omega \in \Sigma_k$, $W^s(X(\omega))$ is open. Indeed, take $y \in W^s(X(\omega))$. For ν-almost all $\omega \in \Sigma_k$, $\sigma^n \omega \in S(\delta)$ for infinitely many positive integers n. We may take n large so that $\sigma^n \omega \in S(\delta)$ and $f^n_\omega(y) \in D_\delta(X(\sigma^n \omega)) \subset W^s(X(\sigma^n \omega))$. By continuity of $f_1, \ldots, f_k$, a small neighborhood of y lies in $W^s(X(\omega))$.

We have that $f^n_{\sigma^{-n}\omega} m$ converges to $\delta_{X(\omega)}$, ν-almost surely. This implies convergence in measure, and since σ leaves ν invariant, also that $f^n_\omega m$ converges to $\delta_{X(\sigma^n \omega)}$ in measure. That is, for any $\varepsilon > 0$,

$$\nu\{\omega \in \Sigma_k \; ; \; d_{\mathcal{M}_I}(f^n_\omega m, \delta_{X(\sigma^n\omega)}) > \varepsilon\} \to 0$$

as $n \to \infty$. This in turn implies that for some subsequence $n_k \to \infty$,

$$\nu\{\omega \in \Sigma_k \; ; \; \lim_{k\to\infty} d_{\mathcal{M}_I}(f^{n_k}_\omega m, \delta_{X(\sigma^{n_k}\omega)}) = 0\} = 1 \tag{13}$$

(see e.g. [27, Theorem II.10.5]). We combine this with the existence of stable sets around $X(\sigma^n\omega)$ to prove that $d_{\mathcal{M}_I}(f^n_\omega m, \delta_{X(\sigma^n\omega)}) \to 0$ almost surely. In more detail, let

$$\Gamma(\hat{\delta}, N) = \{\omega \in \Sigma_k \; ; \; d_{\mathcal{M}_I}(f^N_\omega m, \delta_{X(\sigma^N\omega)}) < \hat{\delta}\}.$$

Now (13) implies that for any given $\varepsilon > 0$, $\hat{\delta} > 0$, there is $N > 0$ with

$$\nu(\Gamma(\hat{\delta}, N)) > 1 - \varepsilon. \tag{14}$$

A measure is close to a delta measure if most of the measure is in a small ball: for any ε, δ there is $\hat{\delta} > 0$ so that $d_{\mathcal{M}_I}(\mu, \delta_x) < \hat{\delta}$ implies $\mu(D_\delta(x)) > 1 - \varepsilon$. So (14) gives that for any $\varepsilon > 0, \delta > 0$ there exists $N > 0$ so that

$$\nu\{\omega \in \Sigma_k \; ; \; f^N_\omega m(D_\delta(X(\sigma^N\omega))) > 1 - \varepsilon\} > 1 - \varepsilon.$$

Combining this with (12) we get that for all $\varepsilon > 0$, $\delta > 0$ there exists $N > 0$ so that the set

$$T_\varepsilon = \{\omega \in \Sigma_k \; ; \; \text{for all } n \geqslant N, f^n_\omega m(D_\delta(X(\sigma^n\omega))) > 1 - \varepsilon\}$$

satisfies $\nu(T_\varepsilon) > 1 - \varepsilon$. We thus find that for all $\delta > 0$,

$$\nu\left(\{\omega \in \Sigma_k \; ; \; \lim_{n\to\infty} f^n_\omega m(D_\delta(X(\sigma^n\omega))) = 1\}\right) = 1.$$

Consequently, for ν-almost all $\omega \in \Sigma_k$,

$$\lim_{n\to\infty} d_{\mathcal{M}_I}(f^n_\omega m, \delta_{X(\sigma^n\omega)}) = 0$$

and

$$m(W^s(X(\omega))) = 1.$$

□

The synchronization property (6) holds for $x, y \in W^s(X(\nu))$ with $\pi\nu = \omega$. □

Observe that as a consequence of Proposition 3.3,

$$\operatorname{supp}(m) \subset \overline{W^s(X(\omega))}$$

for ν-almost all $\omega \in \Sigma_k$.

3.3 Stationary measures with negative Lyapunov exponents

In this section we investigate stationary measures with full measure in $(0,1)$ and in particular stationary measures for which the iterated function system has negative Lyapunov exponents. Some specific cases of the existence of stationary measures, in particular when a reduction to iterated function systems of monotone interval maps is possible, are contained in [6], see also [5, Section 4.4], and further [8].The existence of a stationary measure with $m(\{0\}) = 0$ under our assumptions is proved in [2, Theorem 2]; following [12, 13] we provide an alternative argument and a bound on the measure near the boundary point 0.

Proposition 3.4. *Consider an iterated function system IFS*$(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$. *Assume* $L(0) > 0$. *Then there exists an ergodic stationary measure* m *with* $m(\{0\}) = 0$. *Moreover, for some* $c > 0$ *and* $\alpha > 0$, $m\big([0,x)\big) \leqslant cx^\alpha$.

Proof. Recall from Section 2.1 that $\mathcal{T}$ defines a map from the space of probability measures $\mathcal{M}_I$ to itself and that a measure m on I is a stationary measure for the iterated function system precisely if $\mathcal{T}m = m$. The construction of stationary measures here is similar to the construction of stationary measures in [13, Lemma 3.2] and [12, Proposition 4.1] for interval diffeomorphisms.

Note that $\rho_i \in (0,4)$ implies that $\max_{1\leqslant i\leqslant k} \rho_i < 4$. Since $f_i \leqslant \rho_i/4$ on I we find that any stationary measure has support contained in $[0, \max_{1\leqslant i\leqslant k} \rho_i/4] \subset [0,1)$. For small $0 < \alpha < 1$ and $q > 0$ define

$$\mathcal{N}_c = \{m \in \mathcal{M}_I;\ \mathrm{supp}\,(m) \subset [0, \max_{1\leqslant i\leqslant k} \rho_i/4] \text{ and } \forall\ 0 \leqslant x \leqslant q,\ m\big([0,x)\big) \leqslant cx^\alpha\}. \tag{15}$$

This defines a closed subset of $\mathcal{M}_I$. The condition on the measure of small intervals $[0,x)$ excludes stationary measures that assign positive measure to $\{0\}$. Note that $\mathcal{N}_c$ depends on α and q; but this dependence is not included in the notation. Similar to the proof in [13] and [12] one can show that there exist positive α and q close to 0 such that $\mathcal{T}(\mathcal{N}_c) \subset \mathcal{N}_c$. By the Krylov-Bogolyubov averaging method, for a measure $m \in \mathcal{N}_c$ there is a subsequence of $\{\frac{1}{n}\sum_{r=0}^{n-1} \mathcal{T}^r m\}_{n\in\mathbb{N}}$ that is convergent to a probability measure $\hat{m} \in \mathcal{N}_c$ such that $\mathcal{T}\hat{m} = \hat{m}$. It is proved in [13] and [12] that also an ergodic stationary measure in $\mathcal{N}_c$ exists. □

Proposition 3.5. *Consider an iterated function system IFS*$(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I. *Assume* $L(0) > 0$ *and for all* $1 \leqslant i \leqslant k$, $\rho_i \in (0, 8/3)$. *Then there is a stationary measure* m *with* $m(\{0\}) = 0$ *and so that IFS*$(\mathbb{F})$ *has negative Lyapunov exponents with respect to* m.

Proof. We will demonstrate that with respect to the ergodic stationary measure m from Proposition 3.4, IFS$(\mathbb{F})$ has negative Lyapunov exponents. Write $\mathfrak{I} = \sum_{i=1}^k p_i \ln(\rho_i)$. We first derive the identity

$$\mathfrak{I} = -\int_0^1 \ln(1-x)\ dm(x), \tag{16}$$

also contained in [2, Theorem 1]. For an integrable function φ on I,

$$\int_0^1 \sum_{i=1}^k p_i \varphi \circ f_i\, dm = \int_0^1 \varphi\, dm. \tag{17}$$

One obtains from Proposition 3.4 that the function $x \mapsto \ln(x)$ is integrable. For completeness we provide a proof of this elementary fact.

Lemma 3.2. *The function $x \mapsto \ln(x)$ is integrable for the measure m.*

Proof. Let $h(x) = cx^\alpha$, thus $h^{-1}(x) = \frac{1}{c^{1/\alpha}} x^{1/\alpha}$, and consider $\tilde{m} = hm$. Observe that

$$\tilde{m}([0,x)) \leqslant x. \tag{18}$$

Then $\int_J \ln(x)\, dm(x) = \int_{h^{-1}(J)} \ln(h^{-1}(x))\, d\tilde{m}(x)$ for suitable Borel sets J. It suffices to check that $\ln(h^{-1}(x))$ is integrable for the measure $\tilde{m}$. By integrability of $\ln(h^{-1}(x))$ for Lebesgue measure, we have that for any $\varepsilon > 0$ one can find $0 = a_{N+1} < a_N < \ldots < a_1$, so that

$$\sum_{i=1}^N |\ln(h^{-1}(a_i))||a_i - a_{i+1}| \leqslant \int_0^{a_1} |\ln(h^{-1}(x))|\, dx + \varepsilon.$$

Denote $I_i = [a_{i+1}, a_i)$ and $|I_i| = |a_i - a_{i+1}|$. If $\tilde{m}(I_1) - |I_1| \leqslant 0$, then

$$|\ln(h^{-1}(a_1))|\tilde{m}(I_1) \leqslant |\ln(h^{-1}(a_1))||I_1|.$$

If $\tilde{m}(I_1) - |I_1| > 0$, then

$$\begin{aligned} |\ln(h^{-1}(a_1))|\tilde{m}(I_1) &\leqslant |\ln(h^{-1}(a_1))||I_1| + |\ln(h^{-1}(a_1))|(\tilde{m}(I_1) - |I_1|) \\ &\leqslant |\ln(h^{-1}(a_1))||I_1| + |\ln(h^{-1}(a_2))|(\tilde{m}(I_1) - |I_1|). \end{aligned}$$

If also $\tilde{m}(I_2) - |I_2| + \tilde{m}(I_1) - |I_1| > 0$, then

$$\sum_{i=1}^2 |\ln(h^{-1}(a_i))|\tilde{m}(I_i) \leqslant \sum_{i=1}^2 |\ln(h^{-1}(a_i))||I_i| + |\ln(h^{-1}(a_2))| \sum_{i=1}^2 (\tilde{m}(I_i) - |I_i|).$$

Continuing this reasoning, employing that $\sum_{i=1}^N \tilde{m}(I_i) - |I_1| \leqslant 0$ holds by (18), shows

$$\begin{aligned} \sum_{i=1}^N |\ln(h^{-1}(a_i))|\tilde{m}(I_i) &\leqslant \sum_{i=1}^N |\ln(h^{-1}(a_i))||I_i| \\ &\leqslant \int_0^{a_1} |\ln(h^{-1}(x))|\, dx + \varepsilon. \end{aligned}$$

This estimate proves integrability of $\ln(h^{-1}(x))$ for the measure $\tilde{m}$. □

Applying identity (17) to $\varphi(x) = \ln(x)$,

$$\int_0^1 \sum_{i=1}^k p_i \ln(\rho_i)\, dm(x) + \int_0^1 \sum_{i=1}^k p_i \ln(x)\, dm(x) + \int_0^1 \sum_{i=1}^k p_i \ln(1-x)\, dm(x) = \int_0^1 \ln(x)\, dm(x).$$

This computation proves (16).

On the other hand, we have

$$\begin{aligned}\int_0^1 \sum_{i=1}^k p_i \ln|f_i'(x)|\, dm(x) &= \int_0^1 \sum_{i=1}^k p_i \ln|\rho_i(1-2x)|\, dm(x)\\ &= \int_0^1 \sum_{i=1}^k p_i \ln(\rho_i)\, dm(x) + \int_0^1 \sum_{i=1}^k p_i \ln|1-2x|\, dm(x)\\ &= \int_0^1 \mathfrak{I}\, dm(x) + \int_0^1 \ln|1-2x|\, dm(x)\\ &= \mathfrak{I} + \int_0^1 \ln|1-2x|\, dm(x). \end{aligned} \tag{19}$$

Combining (16) and (19), the condition $\int_0^1 \sum_{i=1}^k p_i \ln|f_i'(x)|\, dm(x) < 0$ yields

$$\int_0^1 \ln|1-2x|\, dm(x) - \int_0^1 \ln(1-x)\, dm(x) = \int_0^1 \ln\left(\frac{|1-2x|}{1-x}\right) dm(x) < 0.$$

A sufficient condition for this inequality to hold is $\ln\left(\frac{|1-2x|}{1-x}\right) < 0$ on $\mathrm{supp}(m)$; i.e. $|1-2x| < 1-x$ for every $x \in \mathrm{supp}(m)$. This means that $\mathrm{supp}(m) \subset [0, \frac{2}{3})$ (note that we have $m(\{0\}) = 0$). Since f_i assumes its maximum at $f_i(1/2) = \rho_i/4$, negative Lyapunov exponents occur when $\rho_i < \frac{8}{3}$ for all $1 \leqslant i \leqslant k$. □

Akin to Singer's theorem [22, Chapter II, Theorem 6.1], stating that a logistic map can have at most one periodic attractor, we find that an iterated function system of logistic maps can have at most one stationary measure with negative Lyapunov exponents.

Proposition 3.6. *Consider an iterated function system IFS*$(\mathbb{F})$*,* $\mathbb{F} = \{f_1, \dots, f_k\}$*, of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$*. The iterated function system IFS*$(\mathbb{F})$ *possesses at most one stationary measure with negative Lyapunov exponents.*

Proof. Suppose m is an ergodic stationary measure with negative Lyapunov exponents, so that $\nu^+ \times m$ is an invariant measure for F^+. Consider the extension F, and an ergodic invariant measure μ, with marginal ν on Σ_k, corresponding to m as provided by Proposition 2.1. Let $\mathcal{A} \subset \Sigma_k$ be the set of positive measure from Corollary 3.1.Write $\mathbb{X} \subset \Sigma_k \times I$ for the collection

$$\mathbb{X} = \bigcup_{\omega \in \mathcal{A}} \{\omega\} \times B^s_\omega$$

of stable sets. Similarly to (8) we have $\mu(\mathbb{X}) > 0$. For μ-almost all (ω, x), (2) holds, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i_\omega(x)} = m.$$

Further, by ergodicity, for μ-almost all (ω, x), the orbit under F intersects $\mathbb{X}$ infinitely often. Consider two iterates $F^p(\omega, x)$ and $F^n(\omega, x)$, $n > p$, that lie in $\mathbb{X}$.

Now for $n-p$ large enough, $f^{n-p}_{\sigma^p\omega}$ maps $B^s_{\sigma^p\omega}$ into $B^s_{\sigma^n\omega}$ and has small diameter since

$$\left| f^{n-p}_{\sigma^p\omega}(B^s_{\sigma^p\omega}) \right| \leqslant C\lambda^{n-p} \tag{20}$$

for some $C > 0, 0 < \lambda < 1$.

Each composition f^i_ω has negative Schwarzian derivative. By the minimum principle, see [22, Chapter II, Lemma 6.1], if $f^{n-p}_{\sigma^p\omega}$ does not have a critical point on $B^s_{\sigma^p\omega}$, then the minimum of $|\left(f^{n-p}_{\sigma^p\omega}\right)'|$ on $B^s_{\sigma^p\omega}$ is assumed at a boundary point. We may write $B^s_{\sigma^p\omega} = (b_l, b_r)$, and assume then that b_r is this boundary point. Let $\hat{B}_{\sigma^p\omega} = (b_l, c_r)$ be the interval, extended maximally on the side of b_r, so that $f^{n-p}_{\sigma^p\omega}$ is monotone on $\hat{B}_{\sigma^p\omega}$. So $f^i_{\sigma^p\omega} c_r = 1/2$ for some $0 \leqslant i < n-p$. Because

$$\left| \left(f^{n-p}_{\sigma^p\omega}\right)'\Big|_{\hat{B}_{\sigma^p\omega}} \right| \leqslant \left| \left(f^{n-p}_{\sigma^p\omega}\right)'\Big|_{B^s_{\sigma^p\omega}} \right|$$

and $|B^s_{\sigma^p\omega}| \geqslant \eta$, we find that there is $\tilde{C} \geqslant 1$ so that

$$\left| f^{n-p}_{\sigma^p\omega}(\hat{B}_{\sigma^p\omega}) \right| \leqslant \tilde{C} \left| f^{n-p}_{\sigma^p\omega}(B^s_{\sigma^p\omega}) \right|.$$

Because of this and (20), we find that for $n-p$ large enough,

$$f^{n-p}_{\sigma^p\omega}(\hat{B}_{\sigma^p\omega}) \subset B^s_{\sigma^n\omega}.$$

We conclude that always there is $r \geqslant 0$ so that

$$\lim_{i\to\infty} \left| f^i_{\sigma^r\omega}(f^r_\omega(x)) - f^i_\omega(1/2) \right| = 0,$$

and so

$$\lim_{i\to\infty} \left| f^i_\tau(y)) - f^i_\tau(1/2) \right| = 0,$$

for $(\tau, y) = F^r(\omega, x)$. Observe that (2) holds when replacing (ω, x) by (τ, y):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i_\tau}(y) = m.$$

Take $\varepsilon > 0$ and let $\psi : I \to \mathbb{R}$ be a continuous function. By uniform continuity, there is $\delta > 0$ so that $|\psi(y_1) - \psi(y_2)| < \varepsilon$ whenever $|y_1 - y_2| < \delta$. Let $N > 0$ be large enough so that $|f^i_\tau(1/2) - f^i_\tau(y)| < \delta$ for $i \geqslant N$, and let then n be large enough so that $\frac{1}{n} 2N \max_{z\in[0,1]} |\psi(z)| < \varepsilon$. Then

$$\begin{aligned}
\left| \frac{1}{n} \sum_{i=0}^{n-1} \left(\psi(f^i_\tau(1/2)) - \psi(f^i_\tau(y))\right) \right| &\leqslant \frac{1}{n} \sum_{i=0}^{N-1} \left|\left(\psi(f^i_\tau(1/2)) - \psi(f^i_\tau(y))\right)\right| \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \sum_{i=N}^{n-1} \left|\left(\psi(f^i_\tau(1/2)) - \psi(f^i_\tau(y))\right)\right| \\
&\leqslant \frac{2N \max_{z\in[0,1]} |\psi(z)|}{n} + \varepsilon \\
&< 2\varepsilon.
\end{aligned}$$

As ε is arbitrary, this shows

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i_\tau(1/2)) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i_\tau(y)),$$

for each continuous function $\psi : I \to \mathbb{R}$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i_\tau}(1/2) = m.$$

This proves the proposition. □

The argument in the above proof makes clear that for ν^+-almost all $\omega \in \Sigma_k^+$, $(\omega, 1/2)$ is a generic point for an ergodic invariant measure $\nu^+ \times m$ (i.e. (2) holds for $x = 1/2$) when one assumes negative Lyapunov exponents.

Write

$$\Delta^\ell = \left\{ (x_1, \ldots, x_\ell) \in [0,1]^\ell \; ; \; \sum_{i=1}^{\ell} x_i = 1 \right\}$$

for the standard ℓ-simplex.

Proposition 3.7. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$. *Suppose IFS* $(\mathbb{F})$ *admits a unique stationary measure* m *with* $m(\{0\}) = 0$. *Assume that* $L(0) > 0$ *and that IFS* $(\mathbb{F})$ *has negative Lyapunov exponents with respect to* m. *Let* $\ell \geqslant k$ *and* $\rho_{k+1}, \ldots, \rho_\ell \in (0,4)$. *There are neighborhoods* V *of* $(\rho_1, \ldots, \rho_\ell)$ *in* $(0,4)^\ell$ *and* U *of* $(p_1, \ldots, p_k, 0, \ldots, 0)$ *in* Δ^ℓ, *so that for elements of* U, V, *the corresponding iterated function system has a unique stationary measure that assigns measure* 0 *to* $\{0\}$ *and has negative Lyapunov exponents.*

Proof. Take a sequence $f^i_1, \ldots, f^i_\ell$ of logistic maps converging to $f_1, \ldots, f_\ell$ as $i \to \infty$. That is, with $f^i_j(x) = \rho^i_j x(1-x)$ and $f_j(x) = \rho_j x(1-x)$, we assume $\rho^i_j \to \rho_i$ as $i \to \infty$. Let also the probabilities p^i_j with which f^i_j is chosen, converge to p_j.

By Proposition 3.4 there is a stationary measure m_i for IFS $\left(\{f^i_1, \ldots, f^i_\ell\}\right)$ with $m_i(\{0\}) = 0$. By Lemma 2.1, any limit point of m_i in $\mathcal{M}_I$ is a stationary measure for IFS $(\mathbb{F})$. Further, there is a fixed space $\mathcal{N}_c$ as in (15) so that $m_i \in \mathcal{N}_c$ for all i large enough. Therefore m_i can not converge to a measure that assigns positive measure to $\{0\}$. We conclude that $m_i \to m$ as $i \to \infty$. The Lyapunov exponents with respect to m_i are therefore negative for large i. Proposition 3.6 implies that for i large, m_i is the unique stationary measure for IFS $(\{f^i_1, \ldots, f^i_\ell\})$ that assigns measure 0 to $\{0\}$. □

The following corollary, together with the statements on minimality in Section 3.4, allows the construction of iterated function systems for which Theorem 3.1 holds and that include any given logistic map $f_i(x) = \rho_i x(1-x)$ with $0 < \rho_i < 4$.

Corollary 3.2. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 8/3$. *Suppose* $L(0) > 0$. *Let* $\ell \geqslant k$ *and* $\rho_{k+1}, \ldots, \rho_\ell \in (0,4)$. *There are neighborhoods* V *of* $(\rho_1, \ldots, \rho_\ell)$ *in* $(0,4)^\ell$ *and* U *of* $(p_1, \ldots, p_k, 0, \ldots, 0)$ *in* Δ^ℓ, *so that for elements of* U, V, *the corresponding iterated function system has a unique stationary measure that assigns measure* 0 *to* $\{0\}$ *and has negative Lyapunov exponents.*

Proof. By Proposition 3.5, IFS$(\{f_1, \ldots, f_k\})$ admits a stationary measure m with $m(\{0\}) = 0$ and with negative Lyapunov exponents. Take a sequence $f^i_1, \ldots, f^i_\ell$ of logistic maps as in the proof of Proposition 3.7. As in the proof of Proposition 3.7 there are stationary measures m_i of IFS$(\{f_1^i, \ldots, f_\ell^i\})$ and there is a fixed space $\mathcal{N}_c$ as in (15) so that $m \in \mathcal{N}_c$ and $m_i \in \mathcal{N}_c$ for all i large enough. By Proposition 3.6, m is the unique stationary measure for IFS$(\mathbb{F})$ with negative Lyapunov exponents. The proof of Proposition 3.5 shows that any stationary measure for IFS$(\mathbb{F})$ in $\mathcal{N}_c$ has negative Lyapunov exponents, so that m is the unique stationary measure in $\mathcal{N}_c$. By Lemma 2.1, $m_i \to m$ as $i \to \infty$. □

3.4 Minimal iterated function systems

Sufficient conditions for minimality of the iterated function system on supp(m) are in the following result.

Proposition 3.8. *Consider an iterated function system IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, of logistic maps $f_i(x) = \rho_i x(1-x)$ on I with $0 < \rho_i < 4$. Assume $L(0) > 0$. Under any of the following conditions, the iterated function system IFS$(\mathbb{F})$ admits an ergodic stationary measure m with $m(\{0\}) = 0$ so that it acts minimally on supp$(m) \setminus \{0\}$.*

(a) There exists j, $1 \leqslant j \leqslant k$, with $\rho_j \in (0,1)$;

(b) There exists j_1, j_2, $1 \leqslant j_1, j_2 \leqslant k$, with $6/5 < \rho_{j_1} < \rho_{j_2} < 3/2$.

Under condition (a),

$$supp(m) = [0, M],$$

where $M = \max_{1 \leqslant i \leqslant k} f_i(1/2) = \max_{1 \leqslant i \leqslant k} \frac{\rho_i}{4}$.

Proof. Take a stationary measure m as provided by Proposition 3.4. Since $[0, M]$ is invariant for IFS$(\mathbb{F})$ and $f_i(x) \in [0, M]$ for every $x \in I$, $1 \leqslant i \leqslant k$, we have supp$(m) \subset [0, M]$.

We begin the proof with item (a) and will prove that the iterated function system is minimal on $(0, M]$ and that supp$(m) = [0, M]$. Consider f_j with $\rho_j \in (0,1)$. Since it is attracting at $x = 0$ the orbits can get arbitrarily close to 0. Hence, for every small $\delta > 0$ we have $m\big((0,\delta)\big) > 0$ and $\min \mathrm{supp}(m) = 0$. Applying the following lemma, which relates to Il′yashenko's work [18], we will find that the iterated function system is minimal on $(0, M]$.

Lemma 3.3. *For diffeomorphisms $f, g : I \to I$ fixing the boundary point 0, assume that $\lambda = f'(0) < 1$, $\mu = g'(0) > 1$, and*

$$\frac{f''(0)}{\lambda^2 - \lambda} \neq \frac{g''(0)}{\mu^2 - \mu}. \tag{21}$$

Then there is an interval $(0, u] \subset (0, 1)$ such that all orbits of the iterated function system generated by f, g restricted to $(0, u]$ are dense in it.

Proof. See [12, Proposition 2.1]. □

Recall the logistic map f_{i_1} with $\rho_{i_1} \in (1,3)$ from item (i) in Theorem 3.1. We know that $f_j'(0) < 1$, $f_{i_1}'(0) > 1$. The inequality (21) for $f = f_{i_1}$ and $g = f_j$ reads

$$\frac{-2\rho_j}{\rho_j^2 - \rho_j} \neq \frac{-2\rho_{i_1}}{\rho_{i_1}^2 - \rho_{i_1}}$$

and is satisfied. Hence Lemma 3.3 holds for $f = f_j$ and $g = f_{i_1}$ and some $(0, u] \subset (0, \frac{1}{2}]$.

In order to prove minimality of IFS$(\mathbb{F})$ on $(0, M]$ we need to show that for every $x \in (0, M]$ and every open interval $J' \subset (0, M]$ there is a composition h_1 of the f_i's, $1 \leqslant i \leqslant k$, such that $h_1(x) \in J'$. Let us assume that ρ_t is the maximal parameter value, so that $M = \rho_t/4$.

Note that there is an interval $K \subset (0,u]$ such that a finite number of iterations of K by f_t covers $(0,M]$. Consider a small interval $J \subset K$ that is mapped inside J' by an iterate of f_t, say $f_t^{n_1}(J) \subset J'$. Since 0 is attracting for f_j, there is $n_2 \in \mathbb{N}$ such that $f_j^{n_2}(x) \in (0,u]$. By Lemma 3.3 there is a map $h \in \text{IFS}(\{f_j, f_{i_1}\})$ so that $h(f_j^{n_2}(x)) \in J$. Hence, $f_t^{n_1}(h(f_j^{n_2}(x))) \in J'$ and we can take $h_1 = f_t^{n_1} \circ h \circ f_j^{n_2}$. We conclude that IFS $(\mathbb{F})$ is minimal on $(0,M]$. This implies that supp $(m) = [0,M]$, compare [20, Proposition 5].

Now we establish the statement for item (b). Recall that f_i has a fixed point at $q_i = \frac{\rho_i - 1}{\rho_i}$. Note that the interval $R = [q_{j_1}, q_{j_2}]$ is invariant, since q_{j_1} and q_{j_2} are only attracting fixed points for f_{j_1} and f_{j_2} respectively. Also, the maps f_{j_1} and f_{j_2} are strictly increasing on R. We claim that

(I) $f_{j_1}(R) \cup f_{j_2}(R) = R$,
(II) f_{j_1} and f_{j_2} are contractions on R.

If

$$f_{j_1}(q_{j_2}) > f_{j_2}(q_{j_1}),$$

then $f_{j_1}(R) \cup f_{j_2}(R) = R$. Working out gives

$$\rho_{j_1} \frac{1}{\rho_{j_2}} \frac{\rho_{j_2} - 1}{\rho_{j_2}} > \rho_{j_2} \frac{1}{\rho_{j_1}} \frac{\rho_{j_1} - 1}{\rho_{j_1}},$$

that is

$$\frac{\rho_{j_1}^3}{\rho_{j_1} - 1} > \frac{\rho_{j_2}^3}{\rho_{j_2} - 1}.$$

This inequality holds for $1 < \rho_{j_1} < \rho_{j_2} < 3/2$.

Note that $0 < f'_{j_1} < 1$ on R. The minimum of f'_{j_2} on R is assumed at q_{j_1} and equals $\rho_{j_2}(\frac{2}{\rho_{j_1}} - 1)$. Assuming this to be smaller than 1, assures that f_{j_1} and f_{j_2} are contractions on R. This holds for $6/5 < \rho_{j_1} < \rho_{j_2} < 3/2$. It follows that IFS $(\{f_{j_1}, f_{j_2}\})$ acts minimally on the interval R. One necessarily has $R \subset \text{supp}\,(m)$. From this we can conclude that IFS $(\mathbb{F})$ acts minimally on supp $(m) \setminus \{0\}$. □

3.5 Forward convergence

We provide a simpler proof of synchronization, valid for specific cases, without using extensions to two sided time and pullback convergence techniques.

Theorem 3.2. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* $I = [0,1]$ *with* $1 < \rho_i < 4$. *Suppose*

(i) For some $1 \leqslant i_1 \leqslant k$, $\rho_{i_1} \in (1,3)$*: the map* f_{i_1} *possesses an attracting fixed point in* $(0,1)$ *with basin of attraction equal to* $(0,1)$*;*
(ii) There is an ergodic stationary probability measure m *such that*
  (a) with respect to m*, the iterated function system has negative Lyapunov exponents;*
  (b) the iterated function system is minimal on supp (m).

For ν^+*-almost all* $\omega \in \Sigma_k^+$,

$$\lim_{n\to\infty} |f_\omega^n(x) - f_\omega^n(y)| = 0 \tag{22}$$

for $x, y \in (0,1)$.

Proof. It is easily seen that the interval $J = [j_l, j_r]$ with

$$j_r = \max_{1 \leqslant i \leqslant k} f_i(1/2),$$
$$j_l = \min_{1 \leqslant j \leqslant k} f_j(j_r),$$

is forward invariant for IFS$(\mathcal{F})$. Every $x \in (0,1)$ is eventually mapped into J. It follows that

$$\mathrm{supp}\,(m) \subset J.$$

So the support $\mathrm{supp}\,(m)$ of the stationary measure m is disjoint from 0 and 1.

Lemma 3.4. *There are a set $\mathcal{C}^+ \subset \Sigma_k$ of positive measure $\nu^+(\mathcal{C}) > 0$ and constants $C > 0, 0 < \lambda < 1$, so that for $\omega \in \mathcal{C}^+$, there is an interval $B^s_\omega \subset I$ with $J \subset B^s_\omega$ and*

$$|f^n_\omega(x) - f^n_\omega(y)| \leqslant C\lambda^n\,|x - y| \tag{23}$$

for $x, y \in B^s_\omega$.

Proof. This follows by combining Proposition 3.1 and Lemma 3.1. Indeed, $\mathcal{C}^+ = \pi\sigma^{-L}\mathcal{B}$ with $\mathcal{B}$ given by (10). □

By ergodicity of ν^+ we find that for ν^+-almost all $\omega \in \Sigma^+_k$, the positive orbit under σ intersects $\mathcal{C}^+$ infinitely often. For such ω and for any $x, y \in (0,1)$, there is $n \geqslant 0$ so that $f^n_\omega(x)$ and $f^n_\omega(y)$ are contained in $B^s_{\sigma^n\omega}$. Thus (22) holds. □

The condition in Theorem 3.2 on minimality of the iterated function system on $\mathrm{supp}\,(m)$ is needed as the example with nonunique stationary measures in Section 4.3 makes clear. In contrast to Theorem 3.1, synchronization is shown for all $x, y \in (0,1)$.

4 Intermittency

As before, consider logistic maps

$$f_i(x) = \rho_i x(1 - x),$$

$1 \leqslant i \leqslant k$, $0 < \rho_i \leqslant 4$, on $I = [0,1]$. We say that IFS $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, displays intermittency if the following holds for any small neighborhood U of 0 and Lebesgue almost any $x \in (0,1)$: for ν^+-almost all $\omega \in \Sigma^+_k$,

(a) $f^n_\omega(x) \notin U$ for infinitely many n;
(b) $\lim_{N\to\infty} \frac{1}{N}\,|\{0 \leqslant n < N\ ;\ f^n_\omega(x) \in U\}| = 1$.

Here, for a finite set S, we write $|S|$ for its cardinality. Compare also [13], where intermittency is studied in a context of interval diffeomorphisms and zero Lyapunov exponents. We establish intermittency in a set-up of iterated function systems generated by k logistic maps $f_1, \ldots, f_k$ that include the two logistic maps $f_{i_1}(x) = 2x(1-x)$ and $f_{i_2}(x) = 4x(1-x)$. The following theorem includes a condition on $L(0)$, the Lyapunov exponent at 0 given by (5). A result in this direction, focusing on null recurrence, is in [8] for the case of the iterated function system with two maps $f_1(x) = 2x(1-x)$ and $f_2(x) = 4x(1-x)$.

Theorem 4.1. *Let IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, , \ldots, f_k\}$, be an iterated function system of logistic maps $f_i(x) = \rho_i x(1 - x)$, taken with probabilities $p_i > 0$, with the following conditions.*

(i) There is i_1 with $f_{i_1}(x) = 2x(1-x)$;
(ii) There is i_2 with $f_{i_2}(x) = 4x(1-x)$;
(iii) $L(0) > 0$.

If $p_{i_1} > \frac{1}{2}$, then the delta measure at zero is the unique stationary measure. Moreover, IFS$(\mathbb{F})$ displays intermittency.

This theorem will be proved in Section 4.1. Other cases of intermittent time series are considered in Section 4.3. Contrasting with the above result is the following topological characterization.

Proposition 4.1. *Let IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, , \ldots, f_k\}$, be an iterated function system of logistic maps $f_i(x) = \rho_i x(1-x)$, taken with probabilities $p_i > 0$. Suppose there is i_2 with $f_{i_2}(x) = 4x(1-x)$. Then the skew product system $F^+ : \Sigma_k^+ \times I \to \Sigma_k^+ \times I$ is topologically mixing.*

Proof. Let U, V be open sets in $\Sigma_k^+ \times I$. Write $P : \Sigma_k^+ \times I \to \Sigma_k^+$ for the coordinate projection. Since σ on Σ_k^+ is an expanding map, $P(F^+)^{t_1}(U) = \Sigma_k^+$ for some t_1 large enough. The set U therefore contains a point (ω, x) with $\omega_i = i_2$ for all $i \geqslant t_1$. The map f_{i_2} has the expansion property that for any nontrivial interval $J \subset I$, there is an $n > 0$ so that $I \subset f_{i_2}^n(J)$. From this property of f_{i_2} one gets that some further iterate $(F^+)^{t_2}((F^+)^{t_1}(U))$ contains $\{\sigma^{t_1}\omega\} \times I = \{(i_2)^\infty\} \times I$, where $(i_2)^\infty$ stands for the sequence of only symbols i_2 in Σ_k^+. Again using that σ on Σ_k^+ is an expanding map, there exists $t > t_1 + t_2$ so that $(F^+)^t(U)$ covers $\Sigma_k^+ \times I$ and hence $(F^+)^n(U)$ intersects V for all $n \geqslant t$. □

In more detail we will look at the iterated function system generated by just the two logistic maps

$$f_1(x) = 2x(1-x), \tag{24}$$

$$f_2(x) = 4x(1-x). \tag{25}$$

For the iterates, pick f_1 with probability p_1, $0 < p_1 < 1$, and f_2 with probability $p_2 = 1-p_1$. Reminiscent of results for the Pomeau-Manneville map [24], we discuss σ-finite stationary measures in the following theorem. Here a σ-finite stationary measure is a σ-finite measure that satisfies the same identity that defines a stationary measure.

Theorem 4.2. *The iterated function system IFS$(\{f_1, f_2\})$ with f_1, f_2 given by (24), (25), admits an absolutely continuous σ-finite stationary measure, which is not finite for $p_1 > 1/2$.*

We do not have a proof that this absolutely continuous σ-finite stationary measure is finite for $p_1 < 1/2$. The construction used to prove Theorem 4.2 gives the following, irrespective of the value of p_1. Denote Lebesgue measure on I by λ.

Proposition 4.2. *Consider the iterated function system IFS$(\{f_1, f_2\})$ with f_1, f_2 given by (24), (25). For $\nu^+ \times \lambda$-almost all (ω, x), its orbit under F^+ lies dense in $\Sigma_2^+ \times I$.*

Before starting proofs we include a remark on the discontinuous dependence of stationary measures on parameters. Consider an iterated function system IFS$(\{f_1, \ldots, f_{k-1}\})$, with $f_{i_1}(x) = 2x(1-x)$ and $p_{i_1} > 1/2$. Assume none of the maps f_i, $1 \leqslant i < k$, equals $x \mapsto 4x(1-x)$. If further $L(0) > 0$, then by Proposition 3.4 there is a stationary measure m with $m(\{0\}) = 0$. Now include the map $f_k(x) = 4x(1-x)$ with probability $p_k = \varepsilon$, multiplying the other probabilities p_i with $1-\varepsilon$ to ensure that the sum of probabilities stays 1. For ε small, by Theorem 4.1 the only stationary measure for IFS$\,(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, is the delta measure δ_0 at 0. We conclude that the set of stationary measures for IFS$\,(\mathbb{F})$ changes discontinuously in ε, at $\varepsilon = 0$.

4.1 Proof of Theorem 4.1

Our proof of Theorem 4.1 is a direct study of time series: in typical time series one expects a frequent occurrence of compositions $f_{i_2} \circ f_{i_1}^{\ell}$ with large $\ell > 0$. In such a composition, a point which is not too close to 0 or 1 is first mapped by $f_{i_1}^{\ell}$ to a point very close to $1/2$ (since $1/2$ is a superstable fixed point for f_{i_1}) and then by f_{i_2} to a point very close to 1. The next iterates bring the point first very close to 0 after which a very large number of iterates is needed to map the point outside a neighborhood of 0. In the proof we supply the estimates making this explicit.

Proof of Theorem 4.1. We claim that for all $x \in I$ there is a set $\Omega_x \subset \Sigma_k^+$ of full ν^+ measure so that for $\omega \in \Omega_x$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_\omega^n}(x) = \delta_0, \tag{26}$$

with convergence in the weak star topology.

It follows from this claim that δ_0 is the unique stationary measure. To prove uniqueness: suppose there is another ergodic stationary measure m. Then $\nu^+ \times m$ is an invariant measure for F^+. Recall from (2) that for $\nu^+ \times m$-almost every (ω, x), we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_\omega^n(x)} = m. \tag{27}$$

By Fubini's theorem there is a subset of I of full m-measure, so that in any $\Sigma_k^+ \times \{x\}$ with x from this subset, there is a set of full ν^+-measure for which (27) holds. This however contradicts (26), since that applies to all x in I.

For $\varepsilon > 0$ write

$$U_\varepsilon = [0, \varepsilon) \cup (1-\varepsilon, 1].$$

We prove (26) by establishing that for all small $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \; ; \; f_\omega^n(x) \notin U_\varepsilon\}| = 0. \tag{28}$$

This is equivalent to (26).

We will however first show that for Lebesgue almost all $x \in (0,1)$, ν^+-almost all $\omega \in \Sigma_k^+$ and ε small enough, $f_\omega^n(x) \notin U_\varepsilon$ for infinitely many $n \in \mathbb{N}$. It is a consequence of the following lemma. This lemma is redundant in case $f_i'(0) > 1$ for all $1 \leqslant i \leqslant k$, for instance the iterated function system with only $x \mapsto 2x(1-x)$ and $x \mapsto 4x(1-x)$ included.

Lemma 4.1. *For every small $\varepsilon > 0$ and ν^+-almost all $\omega \in \Sigma_k^+$, for any $x \in (0, \varepsilon]$ there is $N > 0$ with $f_\omega^N(x) > \varepsilon$.*

Proof. Recall the assumption $L(0) = \sum_{i=1}^{k} p_i \ln(f_i'(0)) > 0$. For every $\delta > 0$ there is $\varepsilon_0 > 0$ such that for every $0 < x \leqslant \varepsilon_0$ and $1 \leqslant i \leqslant k$,

$$|f_i'(x) - f_i'(0)| < \delta. \tag{29}$$

By Birkhoff's ergodic theorem, for ν^+-almost all $\omega \in \Sigma_k^+$,

$$\lim_{N\to\infty} \frac{1}{N} \left|\{0 \leqslant n < N \,;\, \sigma^n\omega \in C_0^i\}\right| = p_i,\ \forall\, 1 \leqslant i \leqslant k. \tag{30}$$

For $N \in \mathbb{N}$ and $1 \leqslant i \leqslant k$, denote

$$q_i(N) = \frac{1}{N} \left|\{0 \leqslant n < N \,;\, \sigma^n\omega \in C_0^i\}\right|.$$

For given ω for which (30) holds, we have that for every $\delta > 0$ there exists $N_0 > 0$ such that for every $N > N_0$,

$$|q_i(N) - p_i| < \delta,\ \forall\, 1 \leqslant i \leqslant k. \tag{31}$$

Take $\delta > 0$ small so that $p_i - \delta > 0$, $f_i'(0) - \delta > 0$ for every $1 \leqslant i \leqslant k$ and

$$\sum_{i=1}^{k} (p_i - \delta) \ln(f_i'(0) - \delta) > 0.$$

Take $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ large such that (29) and (31) are satisfied. Take $0 < \varepsilon \leqslant \varepsilon_0$ and $x \in (0, \varepsilon]$. Also take ω for which (30) holds and suppose that for every $N \geqslant 1$, $x, \ldots, f_\omega^N(x) \in (0, \varepsilon]$. Calculate

$$\begin{aligned}
\frac{1}{N} \ln \left| f_{\omega_{N-1}}'(f_\omega^{N-1}(x)) \cdots f_{\omega_0}'(x) \right| &= \frac{1}{N} \sum_{j=0}^{N-1} \ln |f_{\sigma^j\omega}'(f_\omega^j(x))| \\
&> \sum_{i=1}^{k} q_i(N) \ln(f_i'(0) - \delta) \\
&> \sum_{i=1}^{k} (p_i - \delta) \ln(f_i'(0) - \delta),
\end{aligned}$$

for every $N > N_0$. Since the right hand side is positive, it follows that for $N > N_0$ the map f_ω^N is expanding at x, which is a contradiction. So $f_\omega^N(x) > \varepsilon$ for some $N \geqslant 1$. □

Fix $\varepsilon > 0$ small for which Lemma 4.1 holds. In order to obtain (28), we need information on the numbers of consecutive iterates spend in the different sets U_ε and $I \setminus U_\varepsilon$. We focus on orbit pieces that start at a point in $I \setminus U_\varepsilon$, contain a sufficiently large number of iterates of f_{i_1} (bringing the point close to 1/2) and then one iterate of f_{i_2} (bringing the point close to 1). The following lemma provides necessary estimates.

Lemma 4.2. *Assume all conditions of Theorem 4.1 hold. There exist constants $L, T > 0$ so that*

(i) For every $y \in I \setminus U_\varepsilon$, $y' = f_{i_2} \circ f_{i_1}^\ell(y) \in U_\varepsilon$ for every $\ell \geqslant L$.

(ii) Let $y \in I \setminus U_\varepsilon$ and $y' \in U_\varepsilon$ as in item (i). For $\eta \in \Sigma_k^+$, let $h = h(\eta, y) > 0$ be the smallest integer with $f_\eta^h(y') \in I \setminus U_\varepsilon$. Then $h \geqslant T2^\ell$.

Proof. With $c = 2$ we have that on small closed neighborhoods V of the critical point,

$$|f_{i_1}(y) - \frac{1}{2}| \leqslant c|y - \frac{1}{2}|^2, \tag{32}$$

for every $y \in V$. Write $V = [\frac{1}{2} - \kappa, \frac{1}{2} + \kappa]$ for $\kappa > 0$ and take V small so that $c\kappa < 1$. By (32) for every $\ell \in \mathbb{N}$ and $y \in V$

$$|f_{i_1}^{\ell}(y) - 1/2| \leqslant c^{2^{\ell}-1}|y - 1/2|^{2^{\ell}},$$

and

$$|f_{i_2} \circ f_{i_1}^{\ell}(y) - 1| \leqslant 4c^{2(2^{\ell}-1)}|y - 1/2|^{2^{\ell+1}}.$$

Therefore, for every $y \in V$ the distance of $f_{i_2} \circ f_{i_1}^{\ell}(y)$ to 1 is smaller than $(\frac{2}{c})^2(c\kappa)^{2^{\ell+1}}$. Since $c\kappa < 1$, there is $L > 0$ such that $(\frac{2}{c})^2(c\kappa)^{2^{\ell+1}} < \varepsilon$ for every $\ell \geqslant L$. This gives item (i) for $y \in V$. Since $f_i(x) \leqslant 4x$ for all $1 \leqslant i \leqslant k$, we have $f_{\eta}^{h}(x) \leqslant 4^h x$. The number h of iterates needed to map $f_{i_2} \circ f_{i_1}^{\ell}(y)$ outside of U_ε satisfies $4^h(\frac{2}{c})^2(c\kappa)^{2^{\ell+1}} \geq \varepsilon$ and thus $h \geq T2^{\ell}$, for some $T > 0$ (depending only on ε). This gives item (ii) for $y \in V$. □

Write

$$\mathbb{A} = \Sigma_k^+ \times (I \setminus U_\varepsilon)$$

and let $F_{\mathbb{A}}^+ : \mathbb{A} \to \mathbb{A}$ be the first return map on $\mathbb{A}$.

Note that $F_{\mathbb{A}}^+$ is not defined everywhere, in particular not on $C_0^{i_2} \times \{1/2\}$. The set $\mathbb{H} = \mathbb{A} \cap \bigcup_{n \geqslant 0}(F^+)^{-n}(C_0^{i_2} \times \{1/2\})$ is a countable union of segments $C_{0,\ldots,n-1}^{\upsilon_0,\ldots,\upsilon_{n-1}} \times \{y\}$ with $y \in I \setminus U_\varepsilon$ and $f_{\upsilon}^{n}(y) = 1/2$. In particular, $\mathbb{H}$ involves a countable set Z of points $y \in \mathbb{A}$. Note that (28) holds for $(\omega, x) \in \mathbb{H}$. We find it convenient to artificially alter $F_{\mathbb{A}}^+$ on $C_0^{i_2} \times \{1/2\}$. For this purpose, for $(\omega, x) \in \mathbb{H}$, we define $F_{\mathbb{A}}^+(\omega, x) = (\sigma\omega, y)$ for some $y \notin Z$.

We claim that, after this alteration, there is a set $\Omega \subset \Sigma_k^+$ with $\nu^+(\Omega) = 0$ so that $F_{\mathbb{A}}^+$ is defined on $\Sigma_k^+ \setminus \Omega \times (0, 1)$. By Lemma 4.1, $\nu^+(\Omega_0) = 0$ for

$$\Omega_0 = \{\omega \in \Sigma_k^+ \, ; \, \exists 0 < x < \varepsilon, \forall n \in \mathbb{N}, f_{\omega}^{n}(x) < \varepsilon\}.$$

Then also $\nu^+(\Omega) = 0$ with $\Omega = \cup_{n \geqslant 0}\sigma^{-n}(\Omega_0)$. For $(\omega, x) \in (\Sigma_k^+ \setminus \Omega) \times (0, 1)$, $(F^+)^n(\omega, x)$ is defined for all $n \in \mathbb{N}$ and $f_{\omega}^{n}(x) \in I \setminus U_\varepsilon$ infinitely often.

Let

$$\mathbb{A}_1 = \{(\eta, y) \in \mathbb{A} \, ; \, (\eta, y) \in C_{0,\ldots,\ell}^{i_1^{\ell}, i_2} \times I \setminus U_\varepsilon, \; \ell \geqslant L\},$$

corresponding to points in $I \setminus U_\varepsilon$ that are first iterated by $f_{i_2} \circ f_{i_1}^{\ell}$ for sufficiently large values of ℓ. Write

$$\mathbb{A}_2 = \mathbb{A} \setminus \mathbb{A}_1.$$

Consider $y' \in U_\varepsilon$ obtained from item (i) of Lemma 4.2, an image of some $y \in I \setminus U_\varepsilon$. Remember that $h(\eta, y)$ is the minimal positive integer that $f_{\eta}^{h(\eta,y)}(y') \in I \setminus U_\varepsilon$. For a time series $\{(\omega, x), F_{\mathbb{A}}^+(\omega, x), \ldots\}$ for $F_{\mathbb{A}}^+$, we distinguish different orbit pieces. A point $(\eta, y) = (F_{\mathbb{A}}^+)^n(\omega, x)$ in $\mathbb{A}_1$ for $\ell \geqslant L$ yields an orbit piece

$$(\eta, y), (\sigma\eta, f_\eta(y)), \ldots, (\sigma^{\ell}\eta, f_{\eta}^{\ell}(y)), \ldots, (\sigma^{\ell+h(\eta,y)}\eta, f_{\eta}^{\ell+h(\eta,y)}(y))$$

for F^+. The first $\ell + 1$ points (η, y) with $\eta = (i_1 \ldots i_1 \, i_2 \ldots)$ up to $(\sigma^{\ell}\eta, f_{\eta}^{\ell}(y))$ with $\sigma^{\ell}\eta = (i_2 \ldots)$ are in $\mathbb{A}$, as is the final point $(\sigma^{\ell+h(\eta,y)}\eta, f_{\eta}^{\ell+h(\eta,y)}(y))$. The time series for $F_{\mathbb{A}}^+$ is union of all F^+ orbit pieces starting in $\mathbb{A}$, where the final point of an orbit piece is the initial point of the next orbit piece.

Consider an orbit piece of length N for F^+ with an initial point $(\omega, x) \in \mathbb{A}$ for N large. Let M be the length of the corresponding orbit piece for $F^+_{\mathbb{A}}$. Denote

$$\alpha_1 = \left|\{0 \leqslant n < M \; ; \; (F^+_{\mathbb{A}})^n(\omega, x) \in \mathbb{A}_1\}\right|,$$
$$\alpha_2 = \left|\{0 \leqslant n < M \; ; \; (F^+_{\mathbb{A}})^n(\omega, x) \in \mathbb{A}_2\}\right|.$$

We have $M = \alpha_1 + \alpha_2$. The probability that points in $I \setminus U_\varepsilon$ are first iterated by $f_{i_2} \circ f^\ell_{i_1}$ for some $\ell \geqslant L$ is $\nu^+(C^{i_1^\ell, i_2}_{0,\dots,\ell}) = p^\ell_{i_1} p_{i_2}$. Denote $\pi_0(\omega, x) = \omega_0$ and consider the sequence of random variables

$$\eta_n = \pi_0 (F^+_{\mathbb{A}})^n(\omega, x).$$

We formulate the following, perhaps intuitively obvious, statement.

Lemma 4.3. *The sequence of stochastic variables η_n are independently distributed with identical distribution, in which values $1, \dots, k$ have probabilities $p_1, \dots, p_k$.*

Proof. It is clear that each η_n is distributed with probabilities $p_1, \dots, p_k$ for the values $1, \dots, k$, because each η_n equals some ω_i. For instance the probability $P(\eta_i = k)$ equals the sum $\sum_{s \geqslant 0} P(\eta_i = \omega_s, \omega_s = k)$. Since the ω_m's are independent and for ν^+-almost all $\omega \in \Sigma^+_k$, there are infinitely many η_n's, this probability equals $P(\omega_s = k)$ times the total probability of the set of sequences that give $\eta_i = \omega_s$ for some $s \geqslant 0$. This total probability is one, so that $P(\eta_i = k) = p_k$.

For independence we must show that the probability $P(\eta_{i_1} = k_1, \dots, \eta_{i_h} = k_h)$, $h \geqslant 2$, equals the product of probabilities $P(\eta_{i_j} = k_j)$,$1 \leqslant j \leqslant h$. For simplicity we consider $h = 2$, a higher number of events goes similarly. We have $P(\eta_i = k, \eta_j = l) = \sum_{0 \leqslant s < t} P(\eta_i = \omega_s, \eta_j = \omega_t, \omega_s = k, \omega_t = l)$. Again by independence of the ω_m's and since for ν^+-almost all $\omega \in \Sigma^+_k$, there are infinitely many η_n's, this probability equals $p_k p_l$. □

By Lemma 4.3 and Kolmogorov's strong law of large numbers, see e.g. [27, Chapter IV, §3], for ν^+-almost all $\omega \in \Sigma^+_k$,

$$\lim_{M \to \infty} \frac{\alpha_1}{M} = p^\ell_{i_1} p_{i_2}. \tag{33}$$

By item (ii) we know that for every $\ell \geqslant L$ and M large, $N \geqslant T 2^\ell \alpha_1 + M$. From (33) we see that that $T 2^\ell \frac{\alpha_1}{M}$ is arbitrary large if $2p_{i_1} > 1$ and ℓ, M large. Thus we can calculate

$$\lim_{N \to \infty} \frac{1}{N} \left|\{0 \leqslant n < N \; ; \; f^n_\omega(x) \notin U_\varepsilon\}\right| = \lim_{M \to \infty} \frac{M}{N} \leqslant \lim_{M \to \infty} \frac{M}{T 2^\ell \alpha_1 + M},$$

where the last value is arbitrary close to 0 for ℓ large. This proves (28) and the theorem is proved. □

4.2 Proof of Theorem 4.2

In this section we prove Theorem 4.2 and obtain Proposition 4.2 as a corollary. So we continue with the iterated function system generated by the two maps $f_1(x) = 2x(1-x)$ and $f_2(x) = 4x(1-x)$.

To prove Theorem 4.2 we adapt the line of thought that is used in the study of invariant measures for interval maps such as Pomeau-Manneville maps [24] or Misiurewicz maps [22, Chapter V, Section 3]. Here one proceeds through the construction of a first return map on a subinterval. The first return map is a Markov map for which one constructs an invariant measure by finding its density as the fixed point of a Perron-Frobenius operator. This then provides an invariant measure for the original system, whose finiteness depends on integrability of the return times.

Proof of Theorem 4.2. Take the two critical points of f_2^3 in $(0,1/2)$ that are closest to zero. Let J be the open interval between these two points. Note that J is of the form $(r, f_2(r))$ with $0 < r < f_2(r) < 1/2$. Define

$$\mathbb{B} = C_{0,1,2}^{2,2,2} \times J.$$

The boundary $\partial\mathbb{B}$ of $\mathbb{B}$ equals $C_{0,1,2}^{2,2,2} \times \{r, f_2(r)\}$. It is immediate that the following lemma holds.

Lemma 4.4.

$$\left(F^+\right)^n(\partial\mathbb{B}) \cap \mathbb{B} = \emptyset,$$

for all $n \geqslant 0$.

Consider the first return map $F_{\mathbb{B}}^+ : \mathbb{B} \to \mathbb{B}$ associated to F^+. That is,

$$F_{\mathbb{B}}^+(\omega, x) = (\sigma^s\omega, f_\omega^s(x)),$$

with $s = s(\omega, x)$ the smallest positive natural number with $\sigma^s\omega \in C_{0,1,2}^{2,2,2}$ and $f_\omega^s(x) \in J$. Each map f_ω^s has its set of critical values equal to either $\{0, 1/2\}$ (if $\omega_{s-1} = 1$) or $\{0, 1\}$ (if $\omega_{s-1} = 2$). The domains on which $F_{\mathbb{B}}^+$ is continuous, that is on which $s(\omega, x)$ is constant, are products of cylinders $C_{0,\ldots,s(\omega,x)-1}^{\eta_0,\ldots,\eta_{s(\omega,x)-1}}$ with $\eta_0, \eta_1, \eta_2 = 2$ and open intervals in J. Let $\mathbb{P} = \{\mathbb{P}_i\}$ be the countable partition consisting of these domains. Thus each $\mathbb{P}_i$ is a product $C_i \times K_i$ of a cylinder and an open interval. By Lemma 4.4,

$$F_{\mathbb{B}}^+(\mathbb{P}_i) = \mathbb{B} \tag{34}$$

for each $i \geqslant 1$. Write

$$\mathbb{E} = \cup_i \mathbb{P}_i$$

for the domain of $F_{\mathbb{B}}^+$.

Take a reference probability measure $\nu^+ \times \lambda$ on $\Sigma_2^+ \times J$, where λ stands for a multiple of Lebesgue measure. The next step is to prove that $\mathbb{E}$ has full measure in $\mathbb{B}$ (for this reference measure) as stipulated by Lemma 4.5 below.

Given a measurable set $\mathbb{S} \subset \Sigma_2^+ \times I$, a property is said to hold for $\nu^+ \times \lambda$-almost all $(\omega, x) \in \mathbb{S}$, if the set of points in $\mathbb{S}$ for which it does not hold has zero measure. We call (ω, x) a fiber density point of $\mathbb{S}$ if it is a Lebesgue density point in the fiber $\{\omega\} \times I$:

$$\lim_{D \longrightarrow \{x\}} \frac{\lambda(\mathbb{S} \cap (\{\omega\} \times I) \cap D)}{\lambda(D)} = 1,$$

where the limit is over decreasing intervals $D \ni x$.

In the next lemma we will also use the notion of horizontal density points in sets $\left(\Sigma_2^+ \times \{x\}\right) \cap \mathbb{S}$, where the notion is defined through the isomorphism of the shift map on Σ_2^+ and a piecewise expanding map on I. This isomorphism arises as follows. Consider the expanding interval map $g : I \to I$ that expands intervals $I_1 = [0, p_1]$, $I_2 = [p_1, 1]$ by factors $\frac{1}{p_1}$, $\frac{1}{p_2}$ respectively: writing $p_0 = 0$,

$$g(y) = \frac{y - p_{i-1}}{p_i},\ y \in I_i.$$

Note that g preserves Lebesgue measure λ. An itinerary $\omega \in \Sigma_2^+$ corresponds to a point $y \in I$ via

$$y = \sum_{i=0}^{\infty} l_{\omega(i)} \prod_{j=0}^{i-1} p_{\omega(j)}.$$

This formula defines a map $h : \Sigma_2^+ \to I$ that provides a topological semi-conjugacy $g \circ h = h \circ \sigma$. Now h defines an isomorphism because the Bernoulli measure on Σ_2^+ is pushed forward to Lebesgue measure on I by h. Through this isomorphism of Σ_2^+ with I, we speak of horizontal density points of $\mathbb{S}$ in $\Sigma_2^+ \times \{x\}$.

Lemma 4.5. $\nu^+ \times \lambda(\mathbb{B} \setminus \mathbb{E}) = 0$.

Proof. Suppose by contradiction that there is a set $\mathbb{S}$ of points $(\omega, x) \in \mathbb{B}$, of positive measure, for which the orbit stays outside of $\mathbb{B}$. By Fubini's theorem and the Lebesgue density theorem, $\nu^+ \times \lambda$-almost all points $(\omega, x) \in \mathbb{S}$ are horizontal density points of $\mathbb{S} \cap (C^{2,2,2}_{0,1,2} \times \{x\})$.

We first claim that for $\nu^+ \times \lambda$-almost all $(\omega, x) \in \mathbb{B}$, $f^n_\omega(x) \in J$ infinitely often. Recall from item (i) of Lemma 4.2 that for ν^+-almost all $\omega \in \Sigma_2^+$ and $\ell \geqslant L$, we have $\sigma^i\omega \in C^{1,\dots,1,2}_{0,\dots,\ell}$ infinitely often. For such ω and $x \in I$ there can not be $m \in \mathbb{N}$ so that $f^i_\omega(x) \in I \setminus U_\varepsilon$ for all $i > m$. Thus for such ω we have $f^n_\omega(x) \in U_\varepsilon$ for infinitely many values of n. Remove the set $\cup_{n\in\mathbb{N}} (F^+)^{-n}(C^2_0 \times \{1/2\})$ of points that are mapped onto 0 by some iterate. This set has zero measure for $\nu^+ \times \lambda$. The claim now follows since each point in $(0, \varepsilon)$ will pass through J under iteration.

By altering $\mathbb{S}$ we may thus assume that for every point $(\omega, x) \in \mathbb{S}$, $f^n_\omega(x) \in J$ infinitely often. Now take $(\omega, x) \in \mathbb{S}$ such that (ω, x) is a horizontal density point of $\mathbb{S}$ in $C^{2,2,2}_{0,1,2} \times \{x\}$. Suppose $f^{n(j)}_\omega(x) \in J$ for infinitely many positive values $n(j)$. Let $\chi_{n(j)}$ be the smallest neighborhood of ω with $\sigma^{n(j)}(\chi_{n(j)}) = \Sigma_2^+$. Observe that for every $\eta \in \chi_{n(j)}$, we have $f^{n(j)}_\eta(x) = f^{n(j)}_\omega(x) \in J$. Observe that for every j, $\chi_{n(j+1)} \subset \chi_{n(j)}$ and $\chi_{n(j)} \to \{\omega\}$ as $j \to \infty$.

Since ω is a horizontal density point of $\mathbb{S} \cap (C^{2,2,2}_{0,1,2} \times \{x\})$, for every j there exists a set $\mathcal{D}_{n(j)} \subset \chi_{n(j)}$ of positive measure such that $\mathcal{D}_{n(j)} \times \{x\} \subset \mathbb{S}$ and

$$\lim_{j\to\infty} \frac{\nu^+(\mathcal{D}_{n(j)})}{\nu^+(\chi_{n(j)})} = 1. \tag{35}$$

Since $\sigma^{n(j)}(\chi_{n(j)}) = \Sigma_2^+$ there exists a set $\mathcal{E}_{n(j)} \subset \chi_{n(j)}$ such that $\sigma^{n(j)}(\mathcal{E}_{n(j)}) = C^{2,2,2}_{0,1,2}$. For every j and $\eta \in \mathcal{E}_{n(j)}$, $F^{n(j)}(\eta, x) \in \mathbb{B}$. For every j, $\frac{\nu^+(\mathcal{E}_{n(j)})}{\nu^+(\chi_{n(j)})}$ is the positive constant $\frac{\nu^+(C^{2,2,2}_{0,1,2})}{\nu^+(\Sigma_2^+)}$, while $\mathcal{E}_{n(j)} \cap \mathcal{D}_{n(j)} = \emptyset$. This is a contradiction with (35). □

We find an invariant measure for $F_{\mathbb{B}}^{+}$ by pushing forward $\nu^{+} \times \lambda$ under iterates of $F_{\mathbb{B}}^{+}$;

$$F_{\mathbb{B}}^{+} (\nu^{+} \times \lambda)(U) = \sum_{i=1}^{\infty} \left(F_{\mathbb{B}}^{+}|_{\mathbb{P}_i}\right)^{-1} (U)$$

for Borel sets $U \subset \mathbb{B}$.

Because of (34) and Lemma 4.5,

$$(F_{\mathbb{B}}^{+})^n (\nu^{+} \times \lambda)(\mathbb{B}) = \sum_{i=1}^{\infty} (\nu^{+} \times \lambda)(\mathbb{P}_i) = 1.$$

Since $F_{\mathbb{B}}^{+}$ restricted to a partition element is a product map and each $\left(F_{\mathbb{B}}^{+}|_{\mathbb{P}_i}\right)$ maps $\nu^{+} \times \lambda$ on $\mathbb{P}_i$ to a measure that projects to a multiple of ν^{+} on Σ_2^{+}, also $(F_{\mathbb{B}}^{+})^n (\nu^{+} \times \lambda)$ is a product measure. So

$$(F_{\mathbb{B}}^{+})^n (\nu^{+} \times \lambda) = \nu^{+} \times m_n$$

for some finite measure m_n on I, with $\nu^{+} \times m_n$ a probability measure. In order to investigate the densities of m_n, define a Perron-Frobenius operator $\mathcal{P} : L^1(J, \lambda) \to L^1(J, \lambda)$ by

$$\mathcal{P}\varphi(x) = \sum_{i=1}^{\infty} \nu^{+}(C_i) \frac{\varphi(g_i^{-1}(x))}{|g_i'(g_i^{-1}(x))|}.$$

Here $F_{\mathbb{B}}^{+}|_{\mathbb{P}_i} = \sigma^{s(i)} \times g_i$. Now observe that, if $\varphi_n \in L^1(J, \lambda)$ denotes the density $dm_n/d\lambda$ of m_n, we get

$$\varphi_n = \mathcal{P}^n 1. \tag{36}$$

With this formula in place, we obtain the following result formulated as a lemma.

Lemma 4.6. *$F_{\mathbb{B}}^{+}$ admits an invariant measure $\nu^{+} \times m_{\mathbb{B}}$ where $m_{\mathbb{B}}$ is absolutely continuous with respect to Lebesgue measure. Moreover, there are positive constants c_1, c_2 so that*

$$c_1 < \frac{dm_{\mathbb{B}}}{d\lambda} < c_2.$$

Proof. The proof of the folklore theorem as in [22, Chapter V, Theorem 2.2] studies iterates (36) for a Perron-Frobenius operator obtained from a Markov map as defined in [22, Chapter V, Section 2]. We can follow the proof of the folklore theorem [22, Chapter V, Theorem 2.2] verbatim to obtain the result.

Indeed, the essential uniform bounded distortion estimates for branches of Markov maps (see Koebe's principle [22, Chapter IV, Theorem 1.2]) also hold for the branches of $F_{\mathbb{B}}^{+}$ (the maps g_i) and for branches of iterates $(F_{\mathbb{B}}^{+})^n$, because each g_i has negative Schwarzian derivative. □

Finally, an invariant measure μ of F^{+} is as usual obtained from $\nu^{+} \times m_{\mathbb{B}}$ by

$$\mu = \sum_{j=1}^{\infty} \sum_{i=0}^{s(j)-1} (F^{+})^i \left(\nu^{+} \times m_{\mathbb{B}}|_{C_j \times K_j}\right)$$

Invariance follows from

$$\begin{aligned}
F^+ \mu &= F^+ \sum_{j=1}^{\infty} \sum_{i=0}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) + \sum_{j=1}^{\infty} (F^+)^{s(j)} \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) + F^+_{\mathbb{B}} \left(\nu^+ \times m_{\mathbb{B}}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) + \nu^+ \times m_{\mathbb{B}} \\
&= \mu.
\end{aligned}$$

Because

$$\mu(\mathbb{B}) = \mu(C^{2,2,2}_{0,1,2} \times J) = \sum_{j=1}^{\infty} \sum_{i=0}^{s(j)-1} (\nu^+ \times m_{\mathbb{B}})(C_j \times K_j) = \sum_{j=1}^{\infty} s(j) \nu^+(C_j) m_{\mathbb{B}}(K_j),$$

μ is finite if the last sum is finite. From Theorem 4.1 it follows that μ is not finite if $p_1 > 1/2$.

To affirm that μ is a product measure that projects to ν^+ on Σ_2^+, note that it can alternatively be written as $\mu = \sum_{i=0}^{\infty} \mu_i$ with $\mu_0 = \nu^+ \times m_{\mathbb{B}}$ and

$$\mu_i = F^+ \mu_{i-1} - \left(F^+ \mu_{i-1}\right)|_{\mathbb{B}}$$

for $i \geq 1$. So e.g. μ_1 has its support on $F^+(\mathbb{B})$ and μ_3 is a product measure that projects to (a multiple of) ν^+ on Σ_2^+. It follows that μ is a product measure that projects to ν^+ on Σ_2^+. Moreover, μ restricted to $\Sigma_2^+ \times J$ equals $\nu^+ \times m_{\mathbb{B}}$. We see that μ assigns finite measure to sets $\Sigma_2^+ \times V$ with the closure of V inside $(0,1)$, since it contains cylinders $C \times V$ that are mapped inside $\Sigma_2^+ \times J$ in a bounded number of iterates. □

Proposition 4.2 is obtained as a corollary to Theorem 4.2.

Proof of Proposition 4.2. The statement is a consequence of the invariant probability measure $\nu^+ \times m_{\mathbb{B}}$ for the first return map occurring in the above proof of Theorem 4.2. This is an ergodic measure (see [22, Chapter V, Theorem 2.2]). Therefore by Birkhoff's ergodic theorem and absolute continuity of $m_{\mathbb{B}}$, for $\nu^+ \times \lambda$-almost all initial points, its orbit under $F^+_{\mathbb{B}}$ intersects any open set in $\mathbb{B}$ infinitely often. This then also holds for the orbit under F^+ and any open set in $\Sigma_2^+ \times I$ by appealing to Proposition 4.1. □

4.3 Other cases of intermittency

In the previous sections we treated intermittency in iterated function systems generated by a logistic map with a superstable fixed point and $x \mapsto 4x(1-x)$. In this section we address some other cases leading to intermittent time series. We will not provide detailed proofs.

The arguments to prove Theorem 4.1 can be used to prove intermittency involving superstable periodic orbits of higher period. We provide a statement in the following result. The left frame of Figure 3 illustrates a time series, where we note that Theorem 4.3 does formally not cover the chosen iterated function system. We remark that Proposition 4.1 applies under the assumptions of the following theorem.

Theorem 4.3. *Let IFS*$(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *be an iterated function system of logistic maps* $f_i(x) = \rho_i x(1-x)$, *taken with probabilities* $p_i > 0$, *with the following conditions.*

- *(i) There is* i_1 *with* f_{i_1} *possessing a superstable periodic orbit of period* t*:* $f^t_{i_1}(1/2) = 1/2$*;*
- *(ii) There is* i_2 *with* $f_{i_2}(x) = 4x(1-x)$*;*
- *(iii) There is* i_3 *with* $f_{i_3}(x) = \rho_{i_3} x(1-x)$ *with* $1 < \rho_{i_3} < 3$*;*
- *(iv) There is* i_4 *with* $f_{i_4}(x) = \rho_{i_4} x(1-x)$ *with* $0 < \rho_{i_4} < 1$*;*
- *(v)* $L(0) > 0$.

If $p_{i_1} > \sqrt[t]{\frac{1}{2}}$, *then the delta measure at zero is the unique stationary measure. Moreover, IFS*$(\mathbb{F})$ *displays intermittency.*

Sketch of proof. The arguments to prove Theorem 4.1 can be largely followed. Lemma 4.2 is replaced by the following.

Lemma 4.7. *Assume all conditions of Theorem 4.1 hold. There exist constants* $L, T > 0$ *and a finite word* τ *of symbols in* $\{1, \dots, k\}$ *such that the following items hold:*

- *(i) For every* $y \in I \setminus U_\varepsilon$, $y' = f_{i_2} \circ f^{t\ell}_{i_1} \circ f_\tau(y) \in U_\varepsilon$ *for every* $\ell \geqslant L$.
- *(ii) Let* $y \in I \setminus U_\varepsilon$ *and* $y' \in U_\varepsilon$ *obtained from item (i). Assume that for* $\eta \in \Sigma^+_k$, $h = h(\eta, y) > 0$ *is the smallest integer such that* $f^h_\eta(y') \in I \setminus U_\varepsilon$*. Then* $h \geqslant T 2^\ell$*.*

Proof. By Proposition 3.8 there is $K \in \mathbb{N}$ and $\tau_0, \dots, \tau_{K-1}$ so that $f^K_\tau(I \setminus U_\varepsilon) \subset W^s_{loc}(1/2)$, the immediate basin of $1/2$ for f_{i_1} (the immediate basin of $1/2$ is the maximal interval containing $1/2$ that is in the basin of attraction of $1/2$ for $f^t_{i_1}$). The reasoning of Lemma 4.2 can now be followed almost verbatim with $f^t_{i_1}$ replacing f_{i_1}. □

Equation (33) is replaced by

$$\lim_{M \to \infty} \frac{\alpha_1}{M} = p(\tau) p^{t\ell}_{i_1} p_{i_2},$$

for ν^+-almost all $\omega \in \Sigma^+_k$. Here $p(\tau) = \nu^+(C^{\tau_0, \dots, \tau_{K-1}}_{0, \dots, K-1})$. The rest of the reasoning can be followed to conclude the proof. □

Now we consider a case of intermittency near an invariant set with zero Lyapunov exponents, akin to [4, 13]. Reference [3] provides examples of nonunique stationary measures that assign measure 0 to $\{0\}$. Indeed, taking two logistic maps

$$f_1(x) = \rho_1 x(1-x),\ \rho_1 \in (2, 4),$$
$$f_2(x) = \rho_2 x(1-x),\ \rho_2 = \frac{\rho_1}{\rho_1 - 1} \in \left(\frac{4}{3}, 2\right),$$

we find that

$$S = \{1/\rho_1, 1 - 1/\rho_1\} = \{1 - 1/\rho_2, 1/\rho_2\}$$

is an invariant set for IFS $(\{f_1, f_2\})$. These points are the fixed point $1-1/\rho_1$ of f_1 and the fixed point $1-1/\rho_2$ of f_2, see Figure 4. If f_1 and f_2 are picked with probabilities p_1 and p_2, then the measure m given by

$$m(\{1-1/\rho_1\}) = p_1,\ m(\{1-1/\rho_2\}) = p_2 \tag{37}$$

is a stationary measure supported on S. The Lyapunov exponent with respect to m equals

$$\begin{aligned} L &= \int_I p_1 \ln |f_1'(x)| + p_2 \ln |f_2'(x)| \, dm(x) \\ &= p_2 \left[p_1 \ln |f_1'(1/\rho_1)| + p_2 \ln |f_2'(1/\rho_1)|\right] + p_1 \left[p_1 \ln |f_1'(1/\rho_2)| + p_2 \ln |f_2'(1/\rho_2)|\right] \\ &= p_1 \ln(f_1'(1/\rho_1)) + p_2 \ln(f_2'(1/\rho_1)) \\ &= p_1 \ln(\rho_1(1-2/\rho_1)) + p_2 \ln(\rho_2(1-2/\rho_1)) \end{aligned} \tag{38}$$

and is positive for $\rho_1 > 3$ and p_2 small enough. Note that

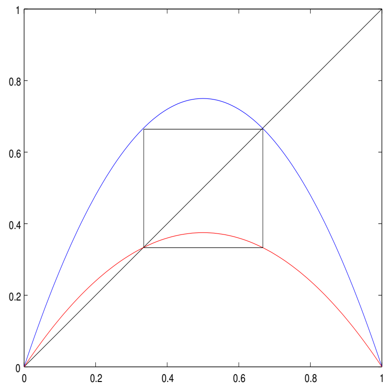

Figure 4. The graphs of two logistic maps, for which the corresponding iterated function system has an invariant set S consisting of the two positive fixed points.

$$\mathrm{supp}\,(m) \subset [f_2 \circ f_1(1/2), f_1(1/2)]$$

for any stationary measure m with support in $(0,1)$. In [3] it is shown that if in addition to the positivity of the Lyapunov exponent (38), one assumes that $f_1^{-1}(1/\rho_1)$ is disjoint from $[f_2 \circ f_1(1/2), f_1(1/2)]$, there exists another stationary measure that also assigns measure 0 to $\{0\}$.

The following theorem finds intermittency near the invariant set S, meaning the following: for any small neighborhood U of S and any $x \in (0,1) \setminus S$, for ν^+-almost all $\omega \in \Sigma_2^+$,

(a) $f_\omega^n(x) \notin U$ for infinitely many n;
(b) $\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \; ; \; f_\omega^n(x) \in U\}| = 1$.

The right frame of Figure 3 illustrates a time series.

Theorem 4.4. *Consider the above families of logistic maps f_1 and f_2, with $3 < \rho_1 < 1+\sqrt{5}$. Take f_1 with probability p_1 and f_2 with probability $p_2 = 1-p_1$. Let the stationary measure m be given by (37). There is a value of p_1 so that IFS$(\{f_1, f_2\})$ has zero Lyapunov exponents with respect to m. For this value, intermittency near the invariant set S occurs.*

Sketch of proof. A further calculation on (38), using $\rho_2 = \rho_1/(\rho_1 - 1)$ and $p_2 = 1 - p_1$, shows $L = \ln(\rho_1 - 2) - (1 - p_1)\ln(\rho_1 - 1)$. We get $L = 0$ for

$$p_1 = 1 - \frac{\ln(\rho_1 - 2)}{\ln(\rho_1 - 1)}.$$

Note that the fixed point in $(0, 1)$ of $x \mapsto ax(1 - x)$ is unstable for $a > 3$. At $a = 1 + \sqrt{5}$, $x \mapsto ax(1-x)$ possesses a superstable period two orbit. Under the conditions of the theorem, f_1 possesses a stable period two orbit $\{q, f_1(q)\}$ with $1/2 < q < f_1(q)$. Write $J = [q, f_1(q)]$. The set $f_2 J \cup J$ is the union of two disjoint intervals, disjoint also from the critical point at $1/2$, and is invariant for IFS $(\{f_1, f_2\})$.

Consider the involution $R : [0, 1] \to [0, 1]$, $Rx = 1 - x$. Identify x with Rx. On J we have an iterated function system generated by f_1 and Rf_2. Observe that f_1 is monotone decreasing and Rf_2 is monotone increasing on J. Compare Figure 5. Using the methods of

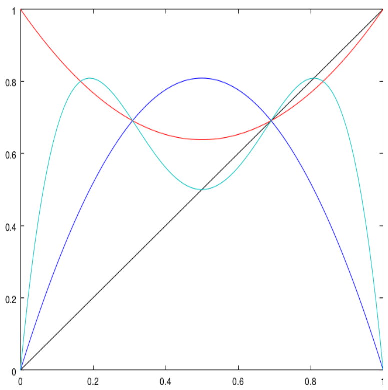

Figure 5. The graphs of $f_1(x) = rx(1 - x)$ with $r = 1 + \sqrt{5}$, $f_1^2(x)$ and $Rf_2(x)$ in one figure.

[13, Theorem 5.1 and Theorem 5.2] proves intermittency. Details are left to the reader. □

It would be interesting to consider the bifurcation scenario in which the probabilities and the logistic maps of Theorem 4.4 are varied.

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