THICK ATTRACTORS WITH INTERMINGLED BASINS

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Abstract. We construct various novel and elementary examples of dynamics with metric attractors that have intermingled basins. A main ingredient is the introduction of random walks along orbits of a given dynamical system. We develop theory for it and use it in particular to provide examples of thick metric attractors with intermingled basins.

1. Introduction

Consider a continuous map $F : X \to X$ on a metric space X, equipped with a probability (reference) measure μ. A closed subset $A \subset X$ will be called a *metric attractor* of F if it satisfies two conditions:

(1) The *basin of attraction*
$$\rho(A) := \{x \in X \; ; \; \omega(x) \subset A\}$$
has positive measure $\mu(\rho(A)) > 0$.

(2) There is no strictly smaller closed set $A' \subset A$ so that $\rho(A')$ coincides with $\rho(A)$ up to a set of measure zero.

The basin of attraction is not required to be an open set. This notion of attractor was coined by Milnor [27] to have a definition that is less restrictive than topological definitions making use of asymptotic stability. One also finds the term Milnor attractor in the literature. A metric attractor is called a *minimal (metric) attractor* if moreover

(3) There is no strictly smaller closed set $A' \subset A$ for which $\rho(A')$ has positive measure.

We also adopt from [27] the definition of *likely limit set* A of F as the smallest closed subset of X with the property that $\omega(x) \subset A$ for every point $x \in X$ outside of a set of measure zero.

The basins of attraction of two attractors A_1 and A_2 are said to be *intermingled basins of attraction* if they are measure theoretically dense in each other: if one basin meets an open set U in a set of positive measure then also the other basin meets U in a set of positive measure [1]. A key example of a system with intermingled metric attractors is due to Kan [24]. It involves a smooth map on the annulus $\mathbb{T} \times [0, 1]$, that fixes the boundaries $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{1\}$. The boundaries in this example are metric attractors. An explicit expression for such a map is
$$(u, x) \mapsto \left(3u \pmod 1, x + \frac{\cos(2\pi u)}{32} x(1 - x)\right).$$
Various other constructions have been provided in the literature, for instance [6, 7, 12, 26]. Examples of the occurrence of attractors with intermingled basins in models are in [8, 16, 17].

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To address measure theoretical aspects of Kan's example, we need two more notions. Recall first that for an ergodic invariant measure ν of F, the basin of ν is defined by

$$\rho(\nu) := \{x \in X \; ; \; \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^i(x)} \to \nu\},$$

with convergence in the weak star topology and where δ_x stands for the atomic measure at x. For the second notion, we need that X is a smooth manifold and F is a smooth map defined on it. By the classical Oseledec's theorem, if ν is an ergodic F-invariant measure then for ν-a.e $x \in X$ and any vector v belonging to the tangent space T_xX, the limits

$$\lim_{n\to\infty} \frac{1}{n} \log \|DF^n(x)v\|$$

exist. The measure ν is called *hyperbolic* if the above limits are all non-zero.

In Kan's example, the Lebesgue measures on the two boundary circles are ergodic invariant measures with positive horizontal and negative vertical Lyapunov exponents. The negative Lyapunov exponents guarantee that the basins have positive volume. The supports of both measures have zero volume, as they are supported on the boundary circles. In particular, these two measures are singular with respect to the volume on $\mathbb{T} \times [0, 1]$. Now, a question arises.

Question. Is there a dynamical system with two attractors with hyperbolic ergodic absolutely continuous invariant measures whose basins are intermingled in some open set?

There are obstacles for an example in the context of partially hyperbolic dynamics. For instance, in [31] the authors prove that each ergodic invariant measure of a partially hyperbolic diffeomorphism on $\mathbb{T}^3$ with intermingled basins is supported on a two dimensional torus and so it can not be absolutely continuous. In this context, we also have the stable and unstable saturation of dynamics on the support of an absolutely continuous measure as a crucial obstacle (see [34]). In contrast, using the Anosov-Katok method, the author in [13] provides an example of a C^∞ diffeomorphism with two intermingled basins but with one hyperbolic component (open and dense basin) and the other having zero exponent.

The above question is a motivation for us to consider a class of attractors, called *thick metric attractors*, that have the potential to be the support of absolutely continuous measures. A metric attractor A is thick if $0 < \mu(A) < 1$, so A has positive but not full measure. This phenomenon of a thick attractor has been described by Ilyashenko in [20] who gave constructions for skew product systems over shifts, arising from iterated function systems, and later in [21] also for diffeomorphisms.

Here, we begin by presenting fundamental ideas for creating attractors with intermingled basins. One of the main yet simple idea is random walks along orbits of a given dynamical system. Random walks along orbits appeared in [10,22,23,30] that looked at contexts of ergodic measure-preserving automorphisms. One setting is that of a diffeomorphism f, where random walks along orbits amounts to

$$x_{n+1} := f^{\eta_n}(x_n)$$

for steps $\eta_n = \pm 1$. We will allow the distribution of the steps η_n to be position dependent, that is, depending on x_n. For a flow $x(t) = \varphi(t, x_0)$ of a differential equation we will look at

$$x(t) = \varphi(s_t, x_0)$$

where s_t is a random process on $\mathbb{R}$.

We rewrite such random systems as deterministic systems to give novel and elementary constructions of attractors with intermingled basins. Then, we will use the large freedom the constructions allow for to provide examples of thick metric attractors with intermingled basins. Finally, we describe how this strategy can lead to the creation of multiple metric attractors with intermingled basins.

2. Elementary examples of intermingled basins

We present different but related constructions of dynamics with metric attractors and intermingled basins. We start with definitions of spaces and maps, also in order to fix notation. Random binary choices are modeled by shifts on sequences of symbols. For this we denote

$$\Sigma_2^+ := \{0,1\}^{\mathbb{N}}, \tag{2.1}$$

equipped with the product topology. Elements of Σ_2^+ are written as $\omega = (\omega_i)_{i\in\mathbb{N}}$. The left shift operator $\sigma : \Sigma_2^+ \to \Sigma_2^+$ is given by $(\sigma\omega)_i = \omega_{i+1}$. We write $[a_0, \dots, a_k]$ for the cylinder

$$[a_0, \dots, a_k] := \{\omega \in \Sigma_2^+ \,;\, \omega_i = a_i, 0 \leqslant i \leqslant k\}.$$

The number $k+1$ of fixed symbols is the depth of the cylinder. For two cylinders $C := [a_0, \dots, a_k]$ and $D := [b_0, \dots, b_l]$ we write $C\,D := [a_0, \dots, a_k, b_0, \dots, b_l]$. Bernoulli measure ν_p corresponding to a probability p for the symbol 0 and $1-p$ for the symbol 1 is determined by defining it on cylinder sets as

$$\nu_p\left([a_0, \dots, a_k]\right) := p^{\#\{i\,;\,0\leqslant i\leqslant k, a_i=0\}}(1-p)^{\#\{i\,;\,0\leqslant i\leqslant k, a_i=1\}}.$$

For two-sided sequences we write $\Sigma_2 := \{0,1\}^{\mathbb{Z}}$, taking away the superscript "+".

Recall that two maps $F : X \circlearrowleft$ and $G : Y \circlearrowleft$ that preserve probability measures μ and ν are measurably isomorphic if there is a measure preserving bijection $H : X' \to Y'$ between invariant subsets $X' \subset X$ and $Y' \subset Y$ of full measure so that $H \circ F = G \circ H$, see [11, Definition 2.7]. From the correspondence between binary expansions of numbers in the unit interval and symbol sequences in Σ_2, one gets that the shift on Σ_2^+ with Bernoulli measure ν_p is measurably isomorphic to the piecewise linear map $E_p : [0,1] \to [0,1]$ with Lebesgue measure, where E_p is given by

$$E_p(u) := \begin{cases} \dfrac{u}{p}, & 0 \leqslant u < p, \\ \dfrac{u-p}{1-p}, & p \leqslant u \leqslant 1. \end{cases}$$

Note that indeed E_p leaves Lebesgue measure on $[0,1]$ invariant. Likewise, the shift on Σ_2 endowed with Bernoulli measure ν_p is measurably isomorphic to the two-dimensional baker

map B_p on $[0,1)^2$ given by

$$B_p(w,y) := \begin{cases} \left(\frac{w}{p}, py\right), & 0 \leqslant w < p, \\ \left(\frac{w}{1-p} - \frac{p}{1-p}, (1-p)y + p\right), & p \leqslant w < 1, \end{cases} \tag{2.2}$$

and endowed with Lebesgue measure.

We will use the single symbol λ to denote Lebesgue measure on the interval $[0,1]$ or the circle $\mathbb{T}$. We also write λ for Lebesgue measure on higher dimensional intervals $[0,1]^d$ and tori $\mathbb{T}^d$.

2.1. **Random walks on orbits of diffeomorphisms: position dependent probabilities.** Consider a compact manifold N and a diffeomorphism $f : N \circlearrowleft$. Let $p : N \to (0,1)$ be a continuous function. For a given initial point $x_0 \in N$, take a random walk

$$x_{n+1} := f^{\eta_n}(x_n) \tag{2.3}$$

along the orbit of x_0, where $\eta_n = \pm 1$ for $n \in \mathbb{Z}$ is taken independently with probabilities $p(x_n)$ for the value $+1$ and $1 - p(x_n)$ for the value -1. One can view this as random walks in environments on $\mathbb{Z}$, where an environment stands for a choice of probabilities on points in $\mathbb{Z}$ with which possible steps are taken [9]. Perturbations of initial points can then give different environments. We get $x_n = x_n(\eta)$ as a function of $\eta = (\eta_i)_{i\in\mathbb{Z}}$.

We will write the above random walk as a skew product system over a shift. Instead of taking two symbols 0 and 1 as in (2.1) we find it convenient to use symbols -1 and $+1$ and write

$$\Omega^+ = \{-1, +1\}^{\mathbb{N}}.$$

Other notation such as for the shift operator and cylinders will be as before. Define the skew product map $G : \Omega^+ \times N \circlearrowleft$ by

$$G(\eta, x) := (\sigma\eta, f^{\eta_0}(x)). \tag{2.4}$$

For iterates we use notation

$$G^n(\eta, x) = \left(\sigma^n\eta, f^{S_n(\eta)}(x)\right), \text{ with } S_n(\eta) := \sum_{i=0}^{n-1} \eta_i. \tag{2.5}$$

Write $p_{-1}(x) := 1 - p(x)$ and $p_1(x) := p(x)$. Let ζ_x be the measure on Ω^+ which is defined on cylinders by

$$\zeta_x([a_0 \cdots a_k]) := \prod_{i=0}^{k} p_{a_i}(f^{S_i(a)}(x)). \tag{2.6}$$

In words, $\zeta_x([a_0 \cdots a_k])$ equals the probability for the random walk starting at x to walk through $x, f^{a_0}(x), f^{a_0+a_1}(x), \ldots, f^{a_0+\cdots+a_k}(x)$. By the Hahn-Kolmogorov extension theorem,

(2.6) defines the measure ζ_x on the Borel σ-algebra on Ω^+. For a measure m on N, define the measure μ_m on $\Omega^+ \times N$ by

$$\mu_m(A) := \iint\limits_{\Omega^+ \times N} \mathbb{1}_{A_x}(\omega)\, d\zeta_x(\omega) dm(x), \tag{2.7}$$

where $A_x := A \cap (\Omega^+ \times \{x\})$. As p is continuous we find that ζ_x depends continuously on x in the weak star topology.

A Borel measure m on N is stationary if

$$m(I) = \int_{f^{-1}(I)} p_1(x)\, dm(x) + \int_{f(I)} p_{-1}(x)\, dm(x). \tag{2.8}$$

A theory of deterministic representations for the random system (2.3) is developed in [3], relating stationary measures to invariant measures of a certain deterministic system defined on $[0,1] \times N$. Our approach centers around the skew product system G and has the following correspondence result between stationary measures and invariant measures for G. Although formulated for the specific setup of iterated function systems generated by f and f^{-1}, it is not restricted to this setup and works more generally for iterated function systems with position dependent probabilities. Although stationary measures do not play a vital role in the constructions in this paper, the following result is included as it clarifies the setup.

Proposition 2.1. *A probability measure m is stationary for the Markov process if and only if μ_m is invariant for G.*

Proof. We first prove that if μ_m is invariant for G, then m is stationary. Consider a product set $A = C \times I$ of a cylinder C and a Borel set I. By definition of the measures ζ_x we have

$$\begin{aligned} \zeta_x(-1\, C) &= p_{-1}(x)\zeta_{f^{-1}(x)}(C), \\ \zeta_x(1\, C) &= p_1(x)\zeta_{f(x)}(C). \end{aligned} \tag{2.9}$$

Calculate

$$\begin{aligned} \mu_m(G^{-1}(A)) &= \int_{f^{-1}(I)} \zeta_x(1\, C)\, dm(x) + \int_{f(I)} \zeta_x(-1\, C)\, dm(x) \\ &= \int_{f^{-1}(I)} p_1(x)\zeta_{f(x)}(C)\, dm(x) + \int_{f(I)} p_{-1}(x)\zeta_{f^{-1}(x)}(C)\, dm(x) \end{aligned}$$

and note

$$\mu_m(A) = \int_I \zeta_x(C)\, dm(x).$$

By invariance of μ_m, these expressions are equal. Apply this to $C = \Omega^+$. Then

$$m(I) = \int_{f^{-1}(I)} p_1(x)\, dm(x) + \int_{f(I)} p_{-1}(x)\, dm(x)$$

which means that m is stationary.

For the other direction, let $A := C \times I$ be a product set in $\Omega^+ \times N$ as above and suppose m is a stationary probability measure. Write (2.8) as

$$\int \mathbb{1}_I(x)\, dm(x) = \int \mathbb{1}_I(f(x))p_1(x)\, dm(x) + \int \mathbb{1}_I(f^{-1}(x))p_{-1}(x)\, dm(x)$$

Approximating an arbitrary integrable function ψ by step functions we get from this

$$\int \psi(x)\, dm(x) = \int \psi(f(x))p_1(x)\, dm(x) + \int \psi(f^{-1}(x))p_{-1}(x)\, dm(x). \tag{2.10}$$

Using (2.9) calculate

$$\begin{aligned}
\mu_m(G^{-1}(A)) &= \iint \mathbb{1}_{1\,C}(\omega)\mathbb{1}_I(f(x))\, d\zeta_x(\omega)dm(x) + \iint \mathbb{1}_{-1\,C}(\omega)\mathbb{1}_I(f^{-1}(x))\, d\zeta_x(\omega)dm(x) \\
&= \int \zeta_x(1\,C)\mathbb{1}_I(f(x))\, dm(x) + \int \zeta_x(-1\,C)\mathbb{1}_I(f^{-1}(x))\, dm(x) \\
&= \int p_1(x)\zeta_{f(x)}(C)\mathbb{1}_I(f(x))\, dm(x) + \int p_{-1}(x)\zeta_{f^{-1}(x)}(C)\mathbb{1}_I(f^{-1}(x))\, dm(x).
\end{aligned}$$

Choosing $\psi(x) := \zeta_x(C)\mathbb{1}_I(x)$, we get from (2.10) that the last line equals

$$\int \zeta_x(C)\mathbb{1}_I(x)\, dm(x),$$

which is $\mu_m(A)$. So

$$\mu_m(G^{-1}(A)) = \mu_m(A),$$

which proves invariance of μ_m since these product sets generate the Borel σ-algebra. □

2.1.1. *North pole/south pole diffeomorphisms on the circle.* We specialize the above construction to a north pole/south pole diffeomorphism f on a circle $\mathbb{S}^1$. This gives a prototype example of metric attractors with intermingled basins. The north pole p_N is a repelling fixed point for f and the south pole p_S an attracting fixed point. The basin of attraction of p_S is $\mathbb{S}^1\backslash\{p_N\}$.

Assume the function p satisfies

$$p(p_S) > 1/2, \qquad p(p_N) < 1/2. \tag{2.11}$$

As before, denote Lebesgue measure on the circle by λ. The measure μ_λ on $\Omega^+ \times \mathbb{S}^1$ is then defined as in (2.7).

Theorem 2.2. *Let the continuous function $p : \mathbb{S}^1 \to (0,1)$ satisfy (2.11) and take $\Omega^+ \times \mathbb{S}^1$ endowed by the reference measure μ_λ. Consider the dynamical system on $\Omega^+ \times \mathbb{S}^1$ given by (4.4). Then $\Omega^+ \times \{p_N\}$ and $\Omega^+ \times \{p_S\}$ are metric attractors with intermingled basins. The union $\Omega^+ \times \{p_N, p_S\}$ is the likely limit set.*

Proof. Consider an orbit $x_{n+1} := f(x_n)$. Identifying x_n with n, the random walk (2.3) gives a random walk on $\mathbb{Z}$. The statements follow from the theory of random walks in environments, see [9, Section I.12] and [25, Section 2.2]. For every $x \in \mathbb{S}^1\backslash\{p_S, p_N\}$, there is a $P(x) \in (0,1)$ so

that the probability of converging to p_S equals $P(x)$ and the probability of converging to p_N is $1 - P(x)$. This means that

$$\zeta_x\left(\left\{\eta \in \Omega^+ \,;\, \lim_{n\to\infty} f^{S_n(\eta)}(x) = p_S\right\}\right) = P(x),$$
$$\zeta_x\left(\left\{\eta \in \Omega^+ \,;\, \lim_{n\to\infty} f^{S_n(\eta)}(x) = p_N\right\}\right) = 1 - P(x),$$

where, S is defined as in (2.5). So the basins of attraction $\rho\left(\Omega^+ \times \{p_S\}\right)$ and $\rho\left(\Omega^+ \times \{p_N\}\right)$ satisfy

$$\zeta_x\left(\rho\left(\Omega^+ \times \{p_S\}\right) \cap \left(\Omega^+ \times \{x\}\right)\right) = P(x),$$
$$\zeta_x\left(\rho\left(\Omega^+ \times \{p_N\}\right) \cap \left(\Omega^+ \times \{x\}\right)\right) = 1 - P(x).$$

Let U be an open set in $\Omega^+ \times \mathbb{S}^1$. Take a product set $C \times I \subset U$ of a cylinder C and an open interval $I \subset S^1$. If C has depth k, then the iterate $G^k(C \times I)$ will be of the form $\Omega^+ \times I_k$ for an open interval I_k. This gives

$$\mu_\lambda(\rho\left(\Omega^+ \times \{p_S\}\right) \cap \Omega^+ \times I_k) = \int_{I_k} P(x)\, d\lambda(x) > 0,$$
$$\mu_\lambda(\rho\left(\Omega^+ \times \{p_N\}\right) \cap \Omega^+ \times I_k) = \int_{I_k} 1 - P(x)\, d\lambda(x)(x) > 0.$$

So also both $\mu_\lambda(U \cap \rho\left(\Omega^+ \times \{p_S\}\right)) > 0$ and $\mu_\lambda(U \cap \rho\left(\Omega^+ \times \{p_N\}\right)) > 0$. This proves the statements. □

2.2. **Skew product systems.** We continue with elementary constructions of intermingled basins somewhat in the spirit of Kan's example [24], but using random walks on orbits of flows on compact manifolds.

Consider the flow of a Morse-Smale gradient differential equation

$$\dot{x} = g(x) \tag{2.12}$$

on a compact manifold N. We have

$$g = -\text{grad}\,(h) \tag{2.13}$$

with a height function $h : N \to [0, 1]$. See for instance [29] for background. We assume that (2.12) has a unique attracting equilibrium p_S and a unique repelling equilibrium p_N. Such flows exist on any compact manifold. We can assume $h(p_S) = 1$ and $h(p_N) = 0$. Note that flows that provide north pole/south pole diffeomorphisms on the circle or on a sphere are possible examples. Write $\varphi_t : N \to N$ for the flow of (2.12). We will also write $\varphi(t, x) := \varphi_t(x)$.

Let $E : \mathbb{T} \to \mathbb{T}$ (here $\mathbb{T} := \mathbb{R}/\mathbb{Z}$) be an expanding map $E(u) := Lu \pmod 1$, for some integer $L > 2$. Let $s : \mathbb{T} \times N \to \mathbb{R}$ be a smooth scalar function satisfying the following properties.

(1) There are fixed points q_1, q_2 for E so that for all $x \in N$,

$$s(q_1, x) < 0, \qquad s(q_2, x) > 0.$$

(2) We have

$$\int_{\mathbb{T}} s(u, p_N)\, du < 0, \qquad \int_{\mathbb{T}} s(u, p_S)\, du > 0.$$

By compactness of N there is $C > 0$ so that for all $x \in N$, $s(q_1, x) < -C < 0$ and $s(q_2, x) > C > 0$. An example of a system with the above conditions is obtained by taking a smooth scalar function $\eta : \mathbb{T} \to \mathbb{R}$ satisfying

$$\int_{\mathbb{T}} \eta(u)\, du = 0, \qquad \eta(q_1) < 0, \qquad \eta(q_2) > 0,$$

and then letting

$$s(u, x) := \eta(u) + \delta \left(h(x) - \frac{1}{2} \right)$$

for some small $\delta > 0$.

Consider the skew product system $F : \mathbb{T} \times N \circlearrowleft$ given by

$$F(u, x) := \left(E(u), \varphi_{s(u,x)}(x) \right). \tag{2.14}$$

Write this as

$$F(u, x) = \left(E(u), f_u(x) \right),$$

and denote iterates as

$$F^n(u, x) = \left(E^n(u), f_u^n(x) \right).$$

Note $f_u^n(x) = \varphi_{s_n(u,x)}(x)$ with

$$s_n(u, x) := \sum_{i=0}^{n-1} s(E^i(u), f_u^i(x)).$$

The top fiber Lyapunov exponent $L_{p_S} := \lim_{n\to\infty} \frac{1}{n} \ln \left(\| Df_u^n(p_S) \| \right)$ at p_S exists for Lebesgue almost all $u \in \mathbb{T}$. Applying Birkhoff's ergodic theorem, one gets

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} s(E^i(u), p_S) = \int_{\mathbb{T}} s(u, p_S)\, du > 0.$$

Using this in the linearized flow $v \mapsto D\varphi_{s(u,p_S)}(p_S)v = e^{Dg(p_S)s(u,p_S)}v$ shows

$$L_{p_S} = \lim_{n\to\infty} \frac{1}{n} \ln \left(\left\| e^{Dg(p_S)n} \right\| \right) \int_{\mathbb{T}} s(u, p_S)\, du < 0.$$

Here $\lim_{n\to\infty} \frac{1}{n} \ln \left(\left\| e^{Dg(p_S)n} \right\| \right)$ is the top Lyapunov exponent of the flow $v \mapsto D\varphi_t(p_S)v = e^{Dg(p_S)t}v$. Likewise we have a negative top fiber Lyapunov exponent at p_N.

Theorem 2.3. *Consider the skew product system F from (2.14) on $\mathbb{T} \times N$. Take volume on $\mathbb{T} \times N$ as reference measure. The sets $\mathbb{T} \times \{p_N\}$ and $\mathbb{T} \times \{p_S\}$ are minimal metric attractors with intermingled basins. The union $\mathbb{T} \times \{p_N, p_S\}$ is the likely limit set.*

Proof. For δ positive, write $B_\delta(p_S)$ for the δ-neighborhood of p_S. For δ sufficiently small, say $\delta \leqslant \delta_0$, $B_\delta(p_S)$ will be a ball. Let $\lambda < 0$ be such that the spectrum of $Dg(p_S)$ is contained in $\{z \in \mathbb{C} \ ; \ \mathrm{Re}\,(z) < \lambda\}$. Standard estimates on solutions of differential equations near equilibria show that for orbits $f_u^n(x)$ that stay inside $B_\delta(p_S)$, so $f_u^i(x) \in B_\delta(p_S)$ for $0 \leqslant i \leqslant n$, we have a bound

$$\begin{aligned} d(f_u^n(x), p_S) &= d\left(\varphi_{s\left(E^n(u), f_{E^n(u)}^{n-1}(x)\right)}\left(f_{E^n(u)}^{n-1}(x)\right) \circ \cdots \circ \varphi_{s(u,x)}(x), p_S\right) \\ &\leqslant Ce^{\lambda\, s_n(u,p_S)} d\,(x, p_S) \end{aligned}$$

for some $C > 0$. The function

$$r(u) = \sup\left\{\delta \ ; \ \delta \leqslant \delta_0, f_u^n(x_0) \in B_{\delta_0}(p_S) \text{ for } n \geqslant 0, \lim_{n\to\infty} f_u^n(x_0) = p_S\right\}$$

is therefore positive for Lebesgue almost all u. Compare the exposition of Pesin theory in [4], and the direct estimates as in [14, Lemma 3.1], [24, Lemma 2.2] or [7, Lemma A.1]. We thus find local stable manifolds $\{u\} \times U_u \subset \{u\} \times N$ of (u, p_S), for Lebesgue almost all u. More precisely, $x_0 \in U_u$ means that $F^n(u, x_0) \in \mathbb{T} \times B_{\delta_0}(p_S)$, for all $n \geqslant 0$ and hence the ω-limit set of (u, x_0) is contained in $\mathbb{T} \times \{p_S\}$. Likewise there are local stable manifolds $\{u\} \times V_u$ of (u, p_N), for which (u, x_0), $x_0 \in V_u$, has its ω-limit set in $\mathbb{T} \times \{p_N\}$.

Take an open set $U \subset \mathbb{T} \times N$. Because E is an expanding map on $\mathbb{T}$, there is an iterate $F^j(U)$ so that the projection to the first coordinate in $\mathbb{T}$ of $F^j(U)$ is surjective. In particular $F^j(U)$ intersects the fiber $\{q_2\} \times N$ with q_2 as in property (1). Using $s(q_2, x) > 0$, for all $x \in N$, shows that there are iterates $F^n(U)$ that intersect the union

$$W^s(\mathbb{T} \times \{p_S\}) := \bigcup_{u\in\mathbb{T}} \{\omega\} \times U_u$$

of local stable manifolds of (u, p_S) in a set of positive measure. Likewise there are iterates $F^n(U)$ that intersect the union

$$W^s(\mathbb{T} \times \{p_N\}) := \bigcup_{u\in\mathbb{T}} \{\omega\} \times V_u$$

of local stable manifolds of (u, p_N) in a set of positive measure. This means that the basins of attraction are intermingled.

To prove that $\mathbb{T} \times \{p_N, p_S\}$ is the likely limit set, suppose it is not and consider the set Λ of positive measure of points whose ω-limit set is not contained in $\mathbb{T} \times \{p_N, p_S\}$. By Fubini there is a set $\mathbb{T} \times \{q\}$ that intersects Λ in a set of positive measure. Take a Lebesgue density point (v, q) of $\Lambda \cap (\mathbb{T} \times \{q\})$. We may take q inside both the basin of p_S and the unstable set of p_N for φ_t. Consider a small interval J around (v, q) in $\mathbb{T} \times \{q\}$. There exists n depending on J so that $E^n(J)$ covers $\mathbb{T}$.

Note that $f_u^n(q)$, $u \in J$, are contained in the orbit of q. Removing δ_0-balls around p_S and p_N, a compact part of the orbit of q remains. There is therefore a constant $c > 0$ and a positive integer m so that $F^{n+m}(J)$ intersects either $W^s(\mathbb{T} \times \{p_N\})$ or $W^s(\mathbb{T} \times \{p_S\})$ in a set of measure at least c. Here m and c do not depend on n. Since E is a piecewise linear map, the proportion $\Lambda \cap J$ in J remains unchanged under iteration. Shrinking J and increasing n makes that

$F^{n+m}(J) \cap \Lambda$ goes to one. In particular $F^{n+m}(J) \cap \Lambda > 1 - c$ for n large, which gives a contradiction. □

One can replace the base map E by an invertible map such as a hyperbolic torus automorphism. We leave this to the reader, compare the discussion of Kan's example in [5].

2.2.1. *Multiple metric attractors.* We continue with a construction, in the vein of the setting of Theorem 2.3, of skew product systems with multiple metric attractors and mutually intermingled basins of attraction. In the following $k \geqslant 4$ will be a positive even integer. Let φ_t be the flow of gradient Morse-Smale vector field on the torus $\mathbb{T}^2$, with a sink s_1 with open and dense basin, and further equilibria $s_2, \ldots, s_k$ that are saddles or sources. It is easy to see that such vector fields exist, for any even $k \geqslant 4$. For each integer j, $2 \leqslant j \leqslant k$, let $h_j : \mathbb{T}^2 \to \mathbb{T}^2$ be a smooth diffeomorphism that permutes equilibria by

$$h_j(s_i) := s_{i+j-1 \mod k+1}.$$

For notational convenience, take h_1 to be the identity map. Write $\psi_j : \mathbb{T}^2 \to \mathbb{T}^2$ for maps $h_j \circ \varphi_1 \circ h_j^{-1}$. The maps ψ_j, $1 \leqslant j \leqslant k$ will then be conjugate to each other by a smooth conjugacy. We find that s_i is the unique attracting fixed point for ψ_i with open and dense basin $W^s(s_i)$. Note also that the spectrum of $D\psi_i(s_{i+j \mod k+1})$ is equal to the spectrum of $D\psi_1(s_j)$.

As before, let E be an expanding map defined on $\mathbb{T}$ by $E(x) := Lx \pmod 1$, where $L > 1$ is a large enough natural number ensuring that E has k fixed points $q_1, \ldots, q_k$. Let $J_1, \ldots, J_k$ be disjoint open intervals in $\mathbb{T}$, with $q_i \in J_i$. Let $t : \mathbb{T} \to [0, 1]$ be a smooth function that is positive on $\bigcup_{i=1}^k J_i$, vanishes outside $\bigcup_{i=1}^k J_i$, and satisfies $t(q_i) = 1$. Finally, for a parameter $u \in \mathbb{T}$, take $f_u : \mathbb{T}^2 \to \mathbb{T}^2$ to be a diffeomorphism on $\mathbb{T}^2$ with the following properties:

(1) For $u \in J_i$ and $1 \leqslant j \leqslant k$,

$$f_u := h_j \circ \varphi_{t(u)} \circ h_j^{-1},$$

(2) For each $i = 1, \ldots, k$,

$$\int_{\mathbb{T}} \log \|Df_u(s_i)\| du < 0.$$

Item (2) can be achieved by choosing the flow φ_t so that the eigenvalues of $D\psi_i(s_i)$ (which equal those at $D\psi_1(s_1)$) are strongly contracting relative to the eigenvalues at the other equilibria. Observe that f_u is a diffeomorphism that depends smoothly on u. For u outside $\bigcup_{i=1}^k J_i$, f_u is the identity map. For each $u \in \bigcup_{i=1}^k J_i$,

$$\Omega(f_u) = \{s_1, \ldots, s_k\}.$$

For comparison and inspiration for further constructions we refer to [28].

Define the skew product F on $\mathbb{T} \times \mathbb{T}^2$ by

$$F(u, x) := (E(u), f_u(x)).$$

Theorem 2.4. *The mapping F has k metric attractors $\mathbb{T} \times \{s_i\}$, $i = 1, \ldots, k$, whose basins are dense in $\mathbb{T} \times \mathbb{T}^2$ and mutually intermingled. The union $\mathbb{T} \times \{s_1, \ldots, s_k\}$ is the likely limit set.*

Proof. By item (2) above, the measures $\lambda \times \delta_{s_i}$, $i = 1, \dots, k$, are invariant measures for F supported on $\Lambda_i := \mathbb{T} \times \{s_i\}$, $i = 1, \dots, k$ whose fiber Lyapunov exponents are negative. In particular, by Pesin theory, see also the proof of Theorem 2.3, each Λ_i is a metric attractor of F. To prove that their basins are intermingled, suppose that $U \subset \mathbb{T} \times S$ is an open set. Choose a natural number N such that

$$F^N(U) \bigcap (\{q_i\} \times W^s(s_i)) \neq \varnothing. \tag{2.15}$$

Any point in the intersection (2.15) tends to Λ_i under the iteration of F. The rest proceeds as in the proof of Theorem 2.3. □

2.2.2. *Iterated function systems.* Reference [14] contains elementary constructions of skew products of interval diffeomorphisms over shifts, arising from iterated function systems, admitting metric attractors with intermingled basins. Here we indicate analogous constructions for surface diffeomorphisms with more attractors.

Consider the iterated function system generated by ψ_i, $1 \leqslant i \leqslant k$ (defined in Section 2.2.1), and using equal probability $1/k$ for all of them. Assume that for all j between 1 and k,

$$\frac{1}{k} \sum_{i=1}^{k} \ln \|D\psi_i(s_j)\| < 0. \tag{2.16}$$

Define $F : \Sigma_k \times \mathbb{T}^2 \circlearrowleft$ by

$$F(\omega, x) := (\sigma\omega, \varphi_{\omega_0}(x)) \,. \tag{2.17}$$

On Σ_k we take Bernoulli measure ν corresponding to equal probabilities for the symbols. The inequality (2.16) means that the fiber Lyapunov exponents at s_j, for any $1 \leqslant j \leqslant k$, are negative. As in Theorem 2.4 we get the following result.

Theorem 2.5. *Consider the skew product system F from (2.17) on $\Sigma_k \times \mathbb{T}^2$. Take the product of Bernoulli measure ν and Lebesgue measure λ on $\mathbb{T}^2$ as reference measure. The sets $\Sigma_k \times \{s_i\}$, $1 \leqslant i \leqslant k$ are minimal metric attractors with intermingled basins. The union $\Sigma_k \times \{s_1, \dots, s_k\}$ is the likely limit set.*

2.2.3. *Skew product systems over minimal torus diffeomorphisms.* We provide a construction using a skew product system over minimal torus diffeomorphisms, motivated by [12,26,33]. Let g be a conservative minimal C^∞ diffeomorphism on T^2 with two ergodic measures $\mu_{\pm 1}$ each of which are absolutely continuous with respect to Lebesgue measure. Such a diffeomorphism can be built using Anosov-Katok method from [2] (see [26,33] for details). Put

$$\Phi := \left\{ \varphi : \mathbb{T}^2 \to \mathbb{R} \,;\; \int_{\mathbb{T}^2} \phi \, d\mu_{-1} < 0 < \int_{\mathbb{T}^2} \varphi \, d\mu_1 \right\}.$$

It is not difficult to see that Φ is nonempty.

Lemma 2.6. *For any $\varphi \in \Phi$, there is a constant $c \in (0,1)$ such that the mapping $H_c : \mathbb{T}^2 \times \mathbb{T} \circlearrowleft$ defined by*

$$H_c(x,t) := (g(x), \varphi(x) + c + t \pmod 1)$$

is minimal.

Proof. First note that

$$H_c^n(x,t) = (g^n(x), \varphi^n(x) + nc + t \pmod 1), \tag{2.18}$$

where by slight abuse of notation we have written

$$\varphi^n(x) := \sum_{i=0}^{n-1} \varphi(g^i(x)).$$

Let $\{V_k\}_{k\in\mathbb{N}}$ be an open basis of the topology on $\mathbb{T}^2$ and $x \in \mathbb{T}^2$ an arbitrary point. For any $k \in \mathbb{N}$, choose a sequence $\{n_i^{(k)}\}$ of natural numbers such that $g^{n_i^{(k)}}(x) \in V_k$. Now, consider the mapping $\Theta_{(x,k)}$ from $[0,1]$ to the set of all closed subsets of $[0,1]$ (endowed with the Hausdorff distance) defined by

$$\Theta_{(x,k)}(c) := \mathrm{Cl}\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}c \pmod 1\right\}_{i=0}^{\infty}\right).$$

Claim. The mapping $\Theta_{(x,k)}$ is lower semi-continuous.

Proof. For any given $\varepsilon > 0$, choose $i_0 \in \mathbb{N}$ such that

$$\mathrm{Cl}\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}c \pmod 1\right\}_{i=0}^{\infty}\right) \subseteq U_{\varepsilon/2}\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}c \pmod 1\right\}_{i=0}^{i_0}\right).$$

Here $U_{\varepsilon/2}(\cdot)$ stands for the open $\varepsilon/2$-neighborhood of the given closed set. Now, if $\tilde{c}$ is sufficiently close to c then

$$\begin{aligned} U_\varepsilon\left(\mathrm{Cl}\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}\tilde{c} \pmod 1\right\}_{i=0}^{\infty}\right)\right) &\supseteq U_\varepsilon\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}\tilde{c} \pmod 1\right\}_{i=0}^{i_0}\right) \\ &\supseteq U_{\varepsilon/2}\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}c \,(\mathrm{mod}\,1)\right\}_{i=0}^{i_0}\right) \\ &\supseteq \mathrm{Cl}\left(\left\{\varphi^{n_i^{(k)}}(x) + n_i^{(k)}c \pmod 1\right\}_{i=0}^{\infty}\right). \end{aligned}$$

□

Let $\mathcal{R}_{(x,k)}$ be the set of continuity points of $\Theta_{(x,k)}$, which is known to be a residual set, and put $\mathcal{R}_x = \bigcap_k \mathcal{R}_{(x,k)}$, a residual set again. We prove that for any $c \in \mathcal{R}_x$, $\{H_c^n(x,t)\}_{n\in\mathbb{N}}$ is dense in $\mathbb{T}^2 \times \mathbb{T}$. To prove it, let $U \times I$ be an arbitrary open set in $\mathbb{T}^2 \times \mathbb{T}$ and choose k such that $V_k \subset U$. Suppose by contradiction,

$$H_c^{n_i^{(k)}}(x,t) \notin U \times I, \text{ for any } i. \tag{2.19}$$

On the other hand, recall (2.18),

$$H_c^{n_i^{(k)}}(x,t) = \left(g^{n_i^{(k)}}(x), \varphi^{n_i^{(k)}}(x) + n_i^{(k)}c + t \pmod 1\right). \tag{2.20}$$

As $g^{n_i^{(k)}}(x) \in V_k$, (2.19) implies that $\varphi^{n_i^{(k)}}(x) + n_i^{(k)}c + t \pmod 1 \notin I$, for any i. That is, $\Theta_{(x,k)}(c) \cap I - t = \varnothing$. However, (2.20) makes clear that there are arbitrarily small $\delta > 0$ so that

$$\Theta_{(x,k)}(c+\delta) \cap I - t \neq \varnothing.$$

This contradicts the fact that c is a continuity point of $\Theta_{(x,k)}$.

Having established the existence of a point (x,t) with dense positive orbit, proving minimality follows from a standard argument using expression (2.18) and minimality of g. Namely, for any $\varepsilon > 0$, there is N_0 so that $\left\{(x,t), H_c(x,t), \ldots, H_c^{N_0}(x,t)\right\}$ is ε-dense in $\mathbb{T}^2 \times \mathbb{T}$. By expression (2.18) the same is true for any point from $\{x\} \times \mathbb{T}$ replacing (x,t). For an arbitrary point $(y,s) \in \mathbb{T}^2 \times \mathbb{T}$, since g is minimal, its positive orbit accumulates on a point (x,u). It is therefore ε-dense. Since ε is arbitrary, $\{H_c^n(y,u)\}_{n\in\mathbb{N}}$ is dense for any $(y,s) \in \mathbb{T}^2 \times \mathbb{T}$. □

Let F be a north pole (p_N)/south pole (p_S) diffeomorphism on $\mathbb{T}$ and consider its suspension flow defined by $F_s(y,t) = (y, t+s)$ on the suspended manifold M given by $(y,t+1) \sim (F(y),t)$ (see for instance [29, Chapter 3, Proposition 3.7] and [32, Section 3.4.1]). The flow F_s has invariant sets

$$\Lambda_S := \{p_S\} \times \mathbb{T}, \qquad \Lambda_N := \{p_N\} \times \mathbb{T}.$$

Choose $\varphi \in \Phi$ and a constant c such that the mapping H_c defined in Lemma 2.6 is minimal. For simplicity denote $\varphi + c$ by φ again. Define $H : \mathbb{T}^2 \times M \circlearrowleft$ by

$$H\left(x,(y,t)\right) := \left(g(x), F_{\varphi(x)}(y,t)\right). \tag{2.21}$$

Theorem 2.7. *The mapping H defining by (2.21) has two metric attractors $\mathbb{T}^2 \times \Lambda_S$ and $\mathbb{T}^2 \times \Lambda_N$ whose basins are intermingled.*

Proof. Note that

$$\begin{aligned} H^n\left(x,(y,t)\right) &= \left(g^n(x), F_{\varphi^n(x)}(y,t)\right) \\ &= \left(g^n(x), F^{[t+\varphi^n(x)]}(y), t + \varphi^n(x) \pmod 1\right), \end{aligned} \tag{2.22}$$

where $[\cdot]$ denotes the integer part and as before, $\varphi^n(x) = \sum_{i=0}^{n-1} \varphi(g^i(x))$.

In view of Lemma 2.6, $\mathbb{T}^2 \times \Lambda_S$ and $\mathbb{T}^2 \times \Lambda_N$ are metric attractors of H. We prove that the basins of attraction are intermingled. For this, let $\left(x,(y,t)\right)$ be an arbitrary point in $\mathbb{T}^2 \times M$. Take $\tilde{x}$ close to x to be a generic point with respect to μ_1 and $\tilde{y} \neq p_N$ close to y. It is not difficult to see that by (2.22),

$$H^n\left(\tilde{x},(\tilde{y},t)\right) \to \mathbb{T}^2 \times \Lambda_S$$

as $n \to \infty$. Likewise take $\hat{x}$, a generic point with respect to μ_{-1}, and $\hat{y} \neq p_S$, sufficiently close to x and y respectively. Then

$$H^n\left(\hat{x},(\hat{y},t)\right) \to \mathbb{T}^2 \times \Lambda_N$$

as $n \to \infty$. □

3. Flows of smooth vector fields

We construct similar examples for flows of smooth vector fields. Let us first remark that the definitions of metric attractor, likely limit set, and intermingled basin given in the introduction for continuous maps transfer to flows $x \mapsto \phi_t(x)$ generated by differential equations.

We will consider skew product systems over an ergodic volume preserving flow. To be concrete, let g be a smooth vector field that generates the suspension of a hyperbolic torus automorphism. The flow $\psi_t : M \to M$ of

$$\dot{u} = g(u) \tag{3.1}$$

on $M := \mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ preserves Lebesgue measure λ and is ergodic. Consider the flow $\varphi_t : N \to N$ of a Morse-Smale gradient differential equation $\dot{x} = f(x) = -\mathrm{grad}(h)$ on a compact manifold N as in (2.12), (2.13). We will also write $\varphi(t, x) = \varphi_t(x)$. Take a skew product systems of the form

$$\begin{aligned} \dot{u} &= g(u), \\ \dot{x} &= \zeta(u, x) f(x), \end{aligned} \tag{3.2}$$

with $(u, x) \in M \times N$ and with $\zeta : M \times N \to \mathbb{R}$ a smooth scalar function. Take volume on $M \times N$ as reference measure. Write $\Phi_t(u, x)$ for the flow of (3.2).

Example 3.1. A special case is where $\zeta(u, x) = \eta(u)$ depends only on u. Let $\eta : M \to \mathbb{R}$ be a smooth scalar function taking positive and negative values and satisfying

$$\int_M \eta(u)\, d\lambda(u) > 0. \tag{3.3}$$

Consider

$$\begin{aligned} \dot{u} &= g(u), \\ \dot{x} &= \eta(u) f(x). \end{aligned} \tag{3.4}$$

For different values of η, the flow $\varphi(t, x_0)$ of f is followed in different time directions. Observe that

$$\Phi_t(u_0, x_0) = (\psi_t(u_0), \varphi(\tau(t), x_0))$$

with

$$\tau(t) = \int_0^t \eta(u(s))\, ds.$$

By Birkhoff's ergodic theorem we find $\tau(t) \to \infty$ as $t \to \infty$ for almost all $u_0 \in M$. Thus $M \times \{p_S\}$ is a metric attractor and also the likely limit set. ■

Analogous to the choice of the function s in Section 2.2, take the smooth function $\zeta : M \times N \to \mathbb{R}$ with the following properties:

(1) There are hyperbolic periodic orbits q_1, q_2 for g for which

$$\zeta|_{\{q_1\} \times N} > 0, \qquad \zeta|_{\{q_2\} \times N} < 0.$$

(2) We have

$$\int_M \zeta(u, p_S)\, d\lambda(u) > 0, \qquad \int_M \zeta(u, p_N)\, d\lambda(u) < 0.$$

Now

$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\left\|D\varphi_{\tau(t)}(p_S)\right\|\right), \qquad \text{with } \tau(t) = \int_0^t \zeta(u(s), p_S)\, ds,$$

is negative, which means as in Section 2.2 that the fiber Lyapunov exponents at p_S are negative. Likewise we have only negative fiber Lyapunov exponents at p_N.

Theorem 3.2. *Consider the differential equation (3.2) on $M \times N$ and suppose ζ satisfies (1), (2) above. Take volume on $M \times N$ as reference measure. Then $M \times \{p_N\}$ and $M \times \{p_S\}$ are metric attractors for the flow Φ_t of (3.2) with intermingled basins. The union $M \times \{p_N, p_S\}$ is the likely limit set.*

Proof. Take a neighborhood U of p_S and a smooth scalar function $\eta : M \to \mathbb{R}$ so that

$$\eta(u) < \zeta(u, x), \qquad u \in M, x \in U, \tag{3.5}$$

and so that (3.3) holds. Let $z(t)$ be a solution to $\dot{z} = f(z)$ contained in the basin of attraction of p_S for the flow φ_t. Take a solution $(u(t), x(t))$ to (3.4) with $u(0) = u_0$ and $x(0) = z(T)$ for some T. Write $x(t) = z(\tau(t))$, where $\tau(0) = T$.

For almost all $u_0 \in M$, $\lim_{t\to\infty} \tau(t) = \infty$ (see Example 3.1). For such u_0, $\tau(t)$ for $t \geqslant 0$ has a minimum value. For a larger starting point $\tau(0) = T$ the minimum value increases. Now T large corresponds to an initial point close to p_S. So for T large enough, $x(t)$, $t \geqslant 0$, stays in a neighborhood of p_S and converges to p_S. This shows that for almost all $u \in M$, there are local stable manifolds $\{u\} \times U_u$ of (u, p_S). By (3.5), these manifolds are local stable manifolds of (u, p_S) also for (3.2). Likewise there are local stable manifolds $u \times V_u$ of (u, p_N), for which $\Phi_t(u, z)$, $z \in V_u$, converges to $M \times \{p_N\}$ as $t \to \infty$. We conclude that $M \times \{p_N\}$ and $M \times \{p_S\}$ are metric attractors.

Write

$$W^s(M \times \{p_S\}) := \bigcup_{u\in M} \{u\} \times U_u$$

for the union of local stable manifolds of (u, p_S), and

$$W^s(M \times \{p_N\}) := \bigcup_{u\in M} \{u\} \times V_u$$

for the union of local stable manifolds of (u, p_N). Now take an open set $U \subset M \times N$. Since stable manifolds of periodic orbits of (3.1) lie dense in M, any open set will intersect the stable manifolds of q_1 and q_2 from (1). This implies that for large enough t, $\Phi_t(U)$ intersects both $W^s(M \times \{p_S\})$ and $W^s(M \times \{p_N\})$ in sets of positive measure. The metric attractors $M \times \{p_N\}$ and $M \times \{p_S\}$ therefore have intermingled basins of attraction, where both basins lie dense in $M \times N$.

The proof that $M \times \{p_N, p_S\}$ is the likely limit set goes as in Theorem 2.3. We give a brief account. As in that proof, consider the set $\Lambda \subset M \times B$ consisting of points whose ω-limit sets are not contained in $M \times \{p_N, p_S\}$. Suppose Λ has positive volume. Then by Fubini's theorem

there is a slice $M \times \{y\}$ for some $y \in B$ that intersects Λ in a set of positive volume. Take a Lebesgue density point (v, y) of $\Lambda \cap (M \times \{y\})$. Consider the flow of a small ball Σ around (v, y) in $M \times \{y\}$. For high enough t, $\Phi_t(\Sigma)$ will intersect the union of local stable manifolds of (u, p_N), and the union of local stable manifolds of (u, p_S). These intersections are of positive measure. Since the dynamics in the base is a suspension of a linear hyperbolic torus map, the proportion $\Lambda \cap \Sigma$ in Σ remains unchanged under iteration. This gives a contradiction. □

Corollary 3.3. *Consider the time-one map Φ_1 on $\mathbb{T}^2 \times N$. Then $\mathbb{T}^2 \times \{p_N\}$ and $\mathbb{T}^2 \times \{p_S\}$ are metric attractors for Φ_1 with intermingled basins.*

Proof. Take $(u_0, 0) \in \mathbb{T}^2 \times \mathbb{T}$. The orbit $\Phi_t(u_0, 0, x_0)$ converges to $M \times \{p_S\}$ if and only if the sequence $(u_n, x_n) = \Phi_n(u_0, x_0)$, $n \in \mathbb{N}$, converges to $\mathbb{T}^2 \times \{p_S\}$. □

4. Intermingled basins of two thick attractors

This section contains a construction of a skew product system, arising from an iterated function system, possessing a pair of thick attractors with intermingled basins. We start with the construction of an iterated function system with both a thick attractor and a thick repeller, after which a random walk on orbits is introduced. We continue with sketching alternative constructions, leading to skew product systems with several thick metric attractors and mutually intermingled basins. We will restrict to skew product systems over symbolic dynamics, and will not discuss extensions to smooth maps.

4.1. **Iterated function system with a thick attractor/thick repeller pair.** In building a model we start with the introduction of an iterated function system on the unit interval $[0, 1]$ generated by two diffeomorphisms f_0, f_1. The graphs of f_0 and f_1 are sketched in Figure 1.

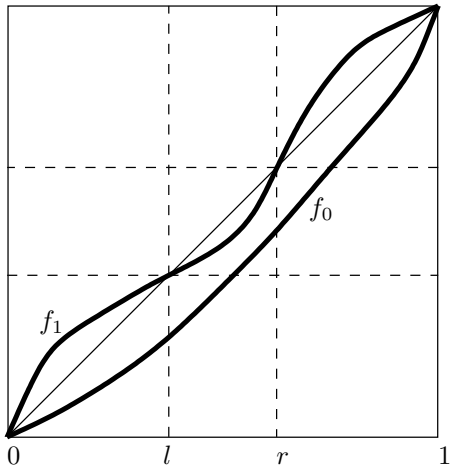

Figure 1. We consider an iterated function system generated by diffeomorphisms f_0 and f_1 on $[0, 1]$ with graphs as depicted. There is an invariant interval $I_l = [0, l]$. For the inverse maps f_0^{-1} and f_1^{-1}, the interval $I_r = [r, 1]$ is invariant.

Both maps fix the end points 0 and 1. For points $0 < l < r < 1$ we have

$$\begin{aligned} f_1(l) &= l, \qquad & f_1(x) &> x \text{ for } x \in (0,l), \\ f_1(r) &= r, \qquad & f_1(x) &> x \text{ for } x \in (r,1). \end{aligned}$$

We further have

$$f_0(x) < x \text{ for } x \in (0,1), \qquad f_1(x) < x \text{ for } x \in (l,r).$$

The interval $I_l = [0,l]$ is mapped into itself by f_0 and f_1. The interval $I_r = [r,1]$ is mapped into itself by f_0^{-1} and f_1^{-1}. We take conditions on derivatives at 0 and 1, namely

$$f_0'(0)f_1'(0) > 1, \qquad (f_0^{-1})'(1)(f_1^{-1})'(1) > 1,$$

and generic conditions on second order derivatives, see [21, Section 3.1] or [14, Proposition 2.1].

The iterated function system generated by f_0 and f_1 has a representation as a skew product system $F : \Sigma_2 \times [0,1] \circlearrowleft$ given by

$$F(\omega, x) := (\sigma\omega, f_{\omega_0}(x)) .$$

This defines a homeomorphism on $\Sigma_2 \times [0,1]$. Denote iterates of F by

$$F^n(\omega, x) = (\sigma^n, f_\omega^n(x)).$$

On Σ_2 we take Bernoulli measure $\nu_{1/2}$. By [20] the map F restricted to $\Sigma_2 \times I_l$ admits a thick metric attractor

$$\Lambda_l := \bigcap_{n=0}^{\infty} F^n(\Sigma_2 \times I_l)$$

with $\nu_{1/2} \times \lambda(\Lambda_l) > 0$. The attractor is characterized by

$$\Lambda_l = \{(\omega, [0, X_l(\omega)]) \; ; \; \omega \in \Sigma_2\}$$

for an invariant measurable function $X_l : \Sigma_2 \to [0,l]$ with $X_l > 0$ almost everywhere. The values $X_l(\omega)$ are obtained from a pullback construction

$$X_l(\omega) := \lim_{n\to\infty} f_{\sigma^{-n}\omega}^n(l), \tag{4.1}$$

where $f_{\sigma^{-n}\omega}^n(l)$ is a monotone decreasing sequence. For any $(\omega, x) \in \Lambda_l$ with $0 < x < X_l(\omega)$ and any neighborhood U of (ω, x) there is a point $(\omega', X_l(\omega')) \in U$.

The skew product F likewise admits a thick metric repeller

$$\Lambda_r := \bigcap_{i=0}^{\infty} F^{-i}(\Sigma_2 \times I_r)$$

in $\Omega^+ \times I_r$. That is, $\nu_{1/2} \times \lambda(\Lambda_r) > 0$ and Λ_r is a thick attractor for F^{-1}. The argument of [20, Remark 1] and (4.1) shows that every open ball $U \subset \Sigma_2 \times [0,1]$ has nonempty intersection with the complement of $\Lambda_l \cup \Lambda_r$.

4.2. **Thick metric attractors with intermingled basins.** The next step is to define a random walk on orbits of F. As before, With $\Omega := \{-1,+1\}^{\mathbb{Z}}$, define $G : \Omega \times \Sigma_2 \times [0,1] \circlearrowleft$ by

$$G(\eta,\omega,x) := (\sigma\eta, F^{\eta_0}(\omega,x)).$$

Take a continuous function $p : [0,1] \to (0,1)$ with

$$p(x) = p_l > 1/2, \text{ for } x \in I_l, \quad p(x) = p_r < 1/2, \text{ for } x \in I_r.$$

Let $\zeta_{\omega,x}$ be the measure on Ω which is defined on cylinders by

$$\zeta_{\omega,x}([a_0 \cdots a_k]) := \prod_{i=0}^{k} p_{a_i}(f_\omega^{S_i(a)}(x)),$$

where, as before, $p_{-1}(x) := 1-p(x)$ and $p_1(x) := p(x)$ and $S_i(a) = \sum_{i=0}^{i-1} a_i$. Define the measure μ on $\Omega \times \Sigma_2 \times [0,1]$ by

$$\mu(A) := \int \mathbb{1}_{A_{(\omega,x)}} \, d\zeta_{\omega,x} d(\nu_{1/2} \times \lambda)(\omega,x),$$

where $A_{(\omega,x)} = A \cap (\Omega^+ \times \{(\omega,x)\})$. Iterates of G are of the form

$$G^n(\eta,\omega,x) = \left(\sigma^n\eta, \sigma^{S_n(\eta)}\omega, f_\omega^{S_n(\eta)}(x)\right).$$

As in the proof of Theorem 2.2, we find that for μ-almost all (η,x), $S_n(\eta)$ goes to ∞ or $-\infty$ as $n \to \infty$.

With $\chi : \Omega \times \Sigma_2 \circlearrowleft$ given by

$$\chi(\eta,\omega) := (\sigma\eta, \sigma^{\eta_0}\omega),$$

we can consider G as a skew product of interval diffeomorphisms over χ. Namely,

$$G(\eta,\omega,x) = \begin{cases} (\chi(\eta,\omega), f_0(x)), & (\eta_0,\omega_0) = (+,0), \\ (\chi(\eta,\omega), f_1(x)), & (\eta_0,\omega_0) = (+,1), \\ (\chi(\eta,\omega), f_0^{-1}(x)), & (\eta_0,\omega_{-1}) = (-,0), \\ (\chi(\eta,\omega), f_1^{-1}(x)), & (\eta_0,\omega_{-1}) = (-,1). \end{cases}$$

For (positive) orbits contained in $\Omega \times \Sigma_2 \times I_l$, the measure we consider on Ω is ν_{p_l}. For such orbits the following lemma becomes useful.

Lemma 4.1. *The measure $P := \nu_{p_l} \times \nu_{1/2}$ is an invariant ergodic measure for χ.*

Proof. Invariance of P is clear and so we only have to prove ergodicity. In the same way as the shift on Σ_2 endowed with Bernoulli measure ν_p is measurably isomorphic to the two-dimensional baker map B_p (from (2.2)), one has that χ, endowed with P, is measurably isomorphic to $J : [0,1]^2 \times [0,1]^2 \circlearrowleft$ given by

$$J(x,y,u,v) := \begin{cases} (B_p(x,y), B_{1/2}(u,v)), & 0 \leqslant x < p, \\ \left(B_p(x,y), B_{1/2}^{-1}(u,v)\right), & p \leqslant x \leqslant 1, \end{cases}$$

and endowed with Lebesgue measure. Lebesgue measure is indeed invariant for J. For Lebesgue almost all points, there are local stable and local unstable manifolds. Namely, for Lebesgue

almost all x_0, $W^s := \{x = x_0, 0 \leqslant y, u, v < 1\}$ has the property that $G^n(x_0, y, u, v) - G^n(x_0, y', u', v')$ goes to zero as $n \to \infty$. Similar for $W^u := \{y = y_0, u = u_0, v = v_0, 0 \leqslant x < 1\}$ and iterates under G^{-n}. A standard Hopf argument (see [32, Section 4.2.6]) shows ergodicity, just as for hyperbolic diffeomorphisms. We note that [18, Theorem 3.2] treats a similar situation of piecewise linear maps. □

Theorem 4.2. *Consider the skew product system F from (2.14) on $\Omega \times \Sigma_2 \times [0,1]$. Take μ as reference measure. The sets $\Omega \times \Lambda_l$ and $\Omega \times \Lambda_r$ are thick metric attractors for G with intermingled basins. The union $\Omega \times (\Lambda_l \cup \Lambda_r)$ is the likely limit set.*

Proof. The identity

$$G(\eta, \omega, X_l(\omega)) = (\sigma\eta, \sigma^{\eta_0}\omega, X_l(\sigma^{\eta_0}\omega))$$

implies that $\Omega \times \Lambda_l$ is invariant for G. The same is true for $\Omega \times \Lambda_r$. As in the proof of Theorem 2.2 we find that μ-almost all orbits converge to either $\Omega \times \Lambda_l$ or $\Omega \times \Lambda_r$. It remains to show that these sets are thick metric attractors.

To show that $\Omega \times \Lambda_l$ is a thick metric attractor we follow [20, Lemma 2]. Let us summarize this approach for the skew product system F on $\Sigma_2 \times I_l$ over the shift σ, and then discuss how to adapt it to the setting of a skew product over χ.

An invariant measure for F is called good if it a weak limit of $\frac{1}{n}\sum_{k=0}^{n-1} F_*^k(\nu_{1/2} \times \lambda)$. Put

$$A_{\min} := \mathrm{Cl}\Big(\bigcup_{\mu \text{ is a good measure}} \mathrm{Supp}(\mu)\Big).$$

Following the notation of [20], we call $A_{\min}$ the minimal attractor of F. By [15, 19], $A_{\min} \subseteq \Lambda_r$ and furthermore, as shown in [20],

$$\nu_{1/2}\left(\{\omega \in \Sigma_2 \,;\, (\omega, X_l(\omega)) \in A_{\min}\}\right) = 1,$$

after which the proof can be concluded by studying properties of X_l.

Now, we turn to the skew product G. Put

$$E := \left\{ \eta \in \Omega \,;\, \sum_{i=0}^{\infty} \eta_i = \infty \right\}$$

and note that $\nu_{p_l}(E) = 1$. For a point $p \in \Omega \times \Sigma_2 \times [0,1]$, write $x(p)$ for the coordinate in $[0,1]$. For $\eta \in E$ and $\omega \in \Sigma_2$ there exists $y := y(\eta, \omega) \in (X_l(\omega), l)$ so that $x(G^n(\eta, \omega, u)) \leqslant l$ for all $u \leqslant y$ and $n \geqslant 0$. In fact $G^n(\eta, \omega, y))$ converges to the graph of X_l as $n \to \infty$. The set

$$\mathcal{K} := \{(\eta, \omega, u) \in \Omega \times \Sigma_2 \times I_l \,;\, \eta \in E, X_l(\omega) < u < y(\eta, \omega)\}$$

forms a set of positive measure $\mu(\mathcal{K}) > 0$. Since the graph(X_l) is G-invariant, $G_*^k(\mu|_{\mathcal{K}})(\mathcal{K}) = \mu(\mathcal{K}) > 0$, for any k. On the other hand, $G^k(\mathcal{K})$ tends to graph(X_l). If μ_∞ is a weak limit point of the sequence $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} G_*^k(\mu|_{\mathcal{K}})$, a good measure, then we get

$$\mu_\infty(\mathrm{graph}\,(X_l)) = \lim_{n\to\infty} \mu_n(\mathrm{graph}\,(X_l)) \geqslant \mu(\mathcal{K}) > 0.$$

Hence (denoting the minimal attractor of G by A_{min})

$$P\{(\eta, \omega) \in \Omega \times \Sigma_2 \,;\, (\eta, \omega, X_l(\omega)) \in A_{min}\} > 0.$$

Ergodicity of χ implies that this measure is 1:

$$P\left(\{(\eta,\omega)\in\Omega\times\Sigma_2\ ;\ (\eta,\omega,X_l(\omega))\in A_{min}\}\right)=1.$$

The reasoning of [20, Lemma 4] can be followed to conclude that the metric attractor equals $\Omega\times(\Lambda_l\bigcup\Lambda_r)$. This can be concluded from the next property which we claim to hold: for any $(\eta,\omega,x)\in\Omega\times\Sigma_2\times I_l$ with $0<x<X_l(\omega)$ and any neighborhood U of (η,ω,x) there is set $S\subset\Omega\times\Sigma_2$ with $P(S)>0$ and $(\eta',\omega',X_l(\omega'))\in U$ for $(\eta',\omega')\in S$.

To see this, take $(\hat\eta,\omega,\hat x)\in U$ with $\hat\eta\in E$. Take an open neighborhood $V\subset U$ of $(\hat\eta,\omega,\hat x)$. We may assume V is of the form $C_\Omega\times C_\Sigma\times J$ for cylinders $C_\Omega=[\hat\eta_{-m},\dots,\hat\eta_m]$, $C_\Sigma=[\omega_{-n},\dots,\omega_n]$ and an open interval J. For n given and large we can take $m\geqslant n$ so that $\chi^{-m}(\hat\eta,\omega)=(\sigma^{-m}\hat\eta,\sigma^{-n}\omega)$. We can moreover take m so that $\chi^{-i}(\hat\eta,\omega)$, $-m\leqslant i\leqslant 0$, only involves ω_j with $-n\leqslant j$. Let J_n be such that

$$f^n_{\sigma^{-n}\omega}(J_n)=J.$$

We will find points $(\eta',\omega',X_l(\omega'))\in V$. Consider the distribution function

$$\Phi(x)=P(\{(\eta,\omega)\in\Omega\times\Sigma_2\ ;\ X_l(\omega)\leqslant x\})=\nu_{1/2}(\{\omega\in\Sigma_2\ ;\ X_l(\omega)\leqslant x\}).$$

As this is a monotone function, its derivative ψ exists almost everywhere. Let x_0 be such that $\psi(x_0)>0$. Then $P(S_\delta)>0$ for $S_\delta=\{(\eta,\omega)\ ;\ |X_l(\omega)-x_0|\leqslant\delta\}$. By [20] there exists a word β with $f^k_\beta(x_0)\in J_n$. Let $\omega'_i=\omega_i$ for $-n\leqslant i\leqslant n$ and $\omega'_{-n+i}=\beta_i$ for $-k\leqslant i\leqslant -1$. Let $\eta'_i=\hat\eta_i$ for $-m\leqslant i\leqslant m$ and $\eta'_{-m+i}=$ "$+1$" for $-k\leqslant i\leqslant -1$. We have $x(G^{m+k}(\chi^{-(m+k)}(\eta',\omega'),x_0))\in J$. For $(\bar\eta,\bar\omega,x)$ with $x=X_l(\bar\omega)$ within distance δ of x_0 concatenate $(\tilde\eta,\omega)$ on the left with the negative part of $(\bar\eta,\bar\omega)$: $\eta'_{-m+k+i}=\bar\eta_i$ for $i<0$ and $\omega'_{-n+k+i}=\bar\omega_i$ for $i<0$. This provides the required positive measure set of sequences. □

4.3. **Multiple thick metric attractors with intermingled basins.** We discuss a slightly different approach and a possible extension leading to examples with multiple thick metric attractors. We will only sketch the constructions and will not provide complete arguments.

4.3.1. *An alternative construction.* Take two diffeomorphisms f_0,f_1 on $[0,1]$ with graphs as depicted in the left panel of Figure 2. The iterated function system generated by f_0 and f_1 provides a skew product system $F:\Sigma_2\times[0,1]\circlearrowleft$ given by $F(\omega,x)=(\sigma\omega,f_{\omega_0}(x))$. Take on $\Sigma_2\times[0,1]$ a reference measure $\nu_{1/2}\times\lambda$. The map F has two thick attractors, a thick attractor $\Lambda_l\subset\Sigma_2\times[0,l]$ and a thick attractor $\Lambda_r\subset\Sigma_2\times[r,1]$. With ϕ_0,ϕ_1 diffeomorphisms on $[0,1]$ as shown in the right panel of Figure 2, let $H:\Sigma_2\times\Omega\times\Sigma_2\times[0,1]\circlearrowleft$ be given by

$$H(\xi,\eta,\omega,x)=\begin{cases}\Big(\sigma\xi,\sigma\eta,\sigma\omega,\phi_{\xi_0}\circ f_{\omega_0}(x)\Big), & \eta_0=+,\\ \Big(\sigma\xi,\sigma\eta,\sigma^{-1}\omega,\phi_{\xi_0}\circ f^{-1}_{\omega_{-1}}(x)\Big), & \eta_0=-.\end{cases}$$

Without the maps ϕ_0,ϕ_1 this is the skew product corresponding to a random walk along orbits of F. The additional compositions with ϕ_0,ϕ_1 provide a random walk between basins of attraction of the two thick metric attractors.

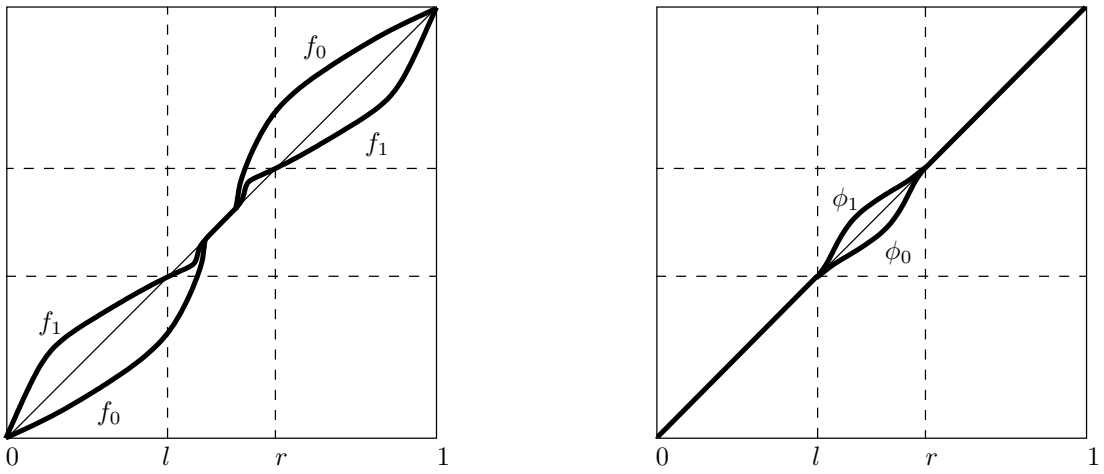

FIGURE 2. The skew product system corresponding to the iterated function system generated by maps f_0, f_1 in left panel, admits two thick attractors. Taking a random walk on its orbits, and adding a composition with additional random choice from maps ϕ_0, ϕ_1 as in the right panel, creates thick metric attractors with intermingled basins.

Theorem 4.3. *Consider the skew product system H on $\Sigma_2 \times \Omega \times \Sigma_2 \times [0,1]$. Take $\nu_{1/2} \times \nu_p \times \nu_{1/2} \times \lambda$ with $1/2 < p < 1$ as reference measure. The sets $\Sigma_2 \times \Omega \times \Lambda_l$ and $\Sigma_2 \times \Omega \times \Lambda_r$ are thick metric attractors for $H$with intermingled basins.*

Sketch of proof. Write $\Pi : \Sigma_2 \times \Omega \times \Sigma_2 \times [0,1] \to [0,1]$ for the coordinate projection to the last coordinate in $[0,1]$. As before in Section 4.2 we find that $\Sigma_2 \times \Omega \times \Lambda_l$ and $\Sigma_2 \times \Omega \times \Lambda_r$ are thick metric attractors.

For any $x \in [0,1]$,

$$\nu_{1/2} \times \nu_p \times \nu_{1/2} \{(\xi, \eta, \omega) \, ; \, \Pi H^n(\xi, \eta, \omega, x) \in [0, l]\} > 0, \tag{4.2}$$

since one can find a cylinder $C \subset \Sigma_2 \times \Omega \times \Sigma_2$ so that $\Pi H^n(\xi, \eta, \omega, x) \in [0, l]$ for $(\xi, \eta, \omega) \in C$. Likewise

$$\nu_{1/2} \times \nu_p \times \nu_{1/2} \{(\xi, \eta, \omega) \, ; \, \Pi H^n(\xi, \eta, \omega, x) \in [r, 1]\} > 0. \tag{4.3}$$

From this we get that for any open set $U \subset \Sigma_2 \times \Omega \times \Sigma_2 \times [0,1]$, the basin of attraction of $\Sigma_2 \times \Omega \times \Lambda_l$ and of $\Sigma_2 \times \Omega \times \Lambda_r$ intersect U in a set of positive reference measure. So H admits two thick metric attractors with intermingled basins. □

4.3.2. *Multiple thick metric attractors with intermingled basins.* We point out possible extensions to skew product systems with multiple thick metric attractors and also multiple thick metric attractors with mutually intermingled basins. Consider an iterated function system on $[0,1]$ generated by functions f_0, f_1 with graphs as depicted in Figure 3. There is an invariant interval $I = [l, r]$ (mapped into itself by both maps). Outside a larger interval $\tilde{I}$ the maps are the identity maps.

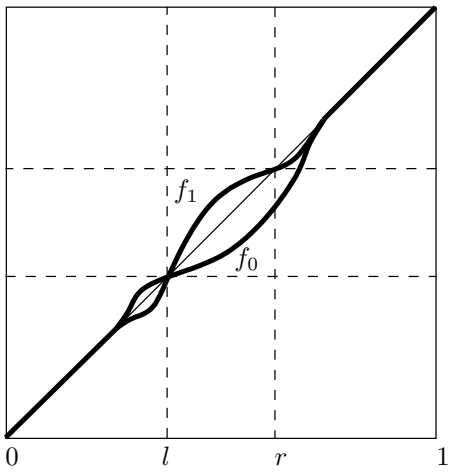

FIGURE 3. We consider an iterated function system generated by diffeomorphisms f_0 and f_1 on $[0,1]$ with graphs as depicted. There is an invariant interval $[l,r]$.

For a positive integer K_1, take K_1 mutually disjoint intervals $\tilde{I}_1,\dots,\tilde{I}_{K_1}$ inside $\mathbb{T}$, with subintervals $I_i \subset \tilde{I}_i$. Let f_0, f_1 be diffeomorphisms on $\mathbb{T}$ that are identity maps outside $\tilde{I}_1 \cup \cdots \cup \tilde{I}_{K_1}$ and on $\tilde{I}_i$ have graphs as depicted in Figure 3. For a second set of intervals $J_1 \subset \tilde{J}_1,\dots,J_{K_2} \subset \tilde{J}_{K_2}$ with the $\tilde{J}_i$'s mutually disjoint, take a similar set of diffeomorphisms g_0, g_1 on $\mathbb{T}$. Both iterated function systems, generated by $\{f_0,f_1\}$ and $\{g_0,g_1\}$ have corresponding skew product systems. Take the product of these. That is, consider the skew product system $F : \Sigma_2 \times \Sigma_2 \times \mathbb{T}^2 \circlearrowleft$ defined by

$$F(\omega,\zeta,x,y) = (\sigma\omega,\sigma\zeta,f_{\omega_0}(x),g_{\zeta_0}(y))\,. \tag{4.4}$$

On both of the Σ_2's we take $(1/2,1/2)$-Bernoulli measure $\nu_{1/2}$. The skew product system $(\omega,x)\mapsto(\sigma\omega,f_{\omega_0}(x))$ has K_1 thick attractors $\Lambda_i \subset \Sigma_2 \times I_i$. The skew product system $(\zeta,y)\mapsto(\sigma\zeta,g_{\zeta_0}(y))$ has K_2 thick attractors $\Xi_j \subset \Sigma_2 \times J_j$. As in [20] one can show that F admits K_1K_2 thick attractors $\Lambda_i \times \Xi_j \subset \Sigma_2 \times \Sigma_2 \times I_i \times J_j$, $1 \leqslant i \leqslant K_1, 1 \leqslant j \leqslant K_2$.

To construct examples of systems with thick metric attractors and mutually intermingled basins, we use the above strategy of introducing random walks along orbits, and compose with additional random walks outside the squares $I_i \times J_j$, $1 \leqslant i \leqslant K_1, 1 \leqslant j \leqslant K_2$. Write

$$\mathcal{I}_x = \bigcup_{\{1\leqslant i\leqslant K_1\}} I_i \times \mathbb{T}, \qquad \mathcal{I}_y = \bigcup_{\{1\leqslant j\leqslant K_2\}} \mathbb{T} \times I_j,$$

and

$$\mathcal{K}_x = \mathbb{T}^2\backslash\mathcal{I}_x, \qquad \mathcal{K}_y = \mathbb{T}^2\backslash\mathcal{I}_y.$$

Take a diffeomorphism h_0, near the identity map, of the form $h_0(x,y) = (x+\phi^x(x,y),y)$ where $\phi^x = 0$ on $\mathcal{I}_y$ and $\phi^x > 0$ on $\mathcal{K}_y$. Let $h_1 = h_0^{-1}$. Take another diffeomorphism h_2, near the

identity map, of the form $h_2(x,y) = (x, y + \phi^y(x,y))$ where $\phi^y = 0$ on $\mathcal{I}_x$ and $\phi^y > 0$ on $\mathcal{K}_x$. Let $h_3 = h_2^{-1}$. For $i = 1,2,3,4$, let $H_i : \Sigma_2 \times \Sigma_2 \times \mathbb{T}^2 \circlearrowleft$ be given by

$$H_i(\omega, \zeta, x, y) = (\omega, \zeta, h_i(x,y)).$$

Define $H : \Sigma_4 \times \Omega \times \Sigma_2 \times \Sigma_2 \times \mathbb{T}^2 \circlearrowleft$ by

$$H(\xi, \eta, \omega, \zeta, x, y) = \left(\sigma\xi, \sigma\eta, H_{\xi_0} \circ F^{\eta_0}(\omega, \zeta, x, y)\right).$$

Endow Σ_4 with $(1/4, 1/4, 1/4, 1/4)$-Bernoulli measure $\nu_{1/4,1/4,1/4,1/4}$ and Ω with $(p, 1-p)$-Bernoulli measure ν_p for $1/2 < p < 1$.

Theorem 4.4. *Consider the skew product system H on $\Sigma_4 \times \Omega \times \Sigma_2 \times \Sigma_2 \times \mathbb{T}^2$. Take $\nu_{1/4,1/4,1/4,1/4} \times \nu_p \times \nu_{1/2} \times \nu_{1/2} \times \lambda$ as reference measure. The sets $\Sigma_4 \times \Omega \times \Lambda_i \times \Xi_j$, $1 \leqslant i \leqslant K_1, 1 \leqslant j \leqslant K_2$ are thick metric attractors for H with mutually intermingled basins.*

Sketch of proof. The proof follows the lines of Theorem 4.3. Properties (4.2) and (4.3) get replaced by the following. For any $(x,y) \in \mathbb{T}^2$, there is a cylinder in $\Sigma_4 \times \Omega \times \Sigma_2 \times \Sigma_2$ so that the torus coordinate of $H^n(\xi, \eta, \omega, \zeta, x, y)$ is in a given square $I_i \times J_j \subset \mathcal{I}_x \cap \mathcal{I}_y$. □

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