ROBUST MINIMALITY OF ITERATED FUNCTION SYSTEMS WITH TWO GENERATORS

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Abstract. We prove that every compact manifold without boundary admits a pair of diffeomorphisms that generates C^1 robustly minimal dynamics. We apply the results to the construction of blenders and robustly transitive skew product diffeomorphisms.

Contents

1. Introduction

In this paper we study robust minimality of iterated function systems (IFS, for short). By an iterated function system one means the action of the semigroup generated by a family of diffeomorphisms. The study of iterated function systems, besides its own importance, has a remarkable role for understanding certain (single) dynamical systems. For instance, one can embed an IFS into a skew-product over the full shift with sufficient number of symbols; and therefore in many dynamical systems exhibiting some form of hyperbolicity.

Let $\mathcal{F}, \mathcal{G}$ be two families of diffeomorphisms on a compact manifold M. Denote

$$\mathcal{F} \circ \mathcal{G} = \{f \circ g \mid f \in \mathcal{F}, g \in \mathcal{G}\},$$

and for $k \in \mathbb{N}$,

$$\mathcal{F}^k := \mathcal{F}^{k-1} \circ \mathcal{F},\ \mathcal{F}^0 := \{Id\}.$$

Write $\langle \mathcal{F} \rangle^+$ for the semi-group generated by $\mathcal{F}$, that is, $\langle \mathcal{F} \rangle^+ = \bigcup_{n=0}^{\infty} \mathcal{F}^k$. The action of the semi-group $\langle \mathcal{F} \rangle^+$ is called the *iterated function system (or IFS) associated with* $\mathcal{F}$ and we denote it by IFS$\,(\mathcal{F})$. For $x \in M$, we write the orbits of the

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action of this semi-group as

$$\langle \mathcal{F} \rangle^+(x) = \{f(x) : f \in \langle \mathcal{F} \rangle^+\}.$$

A sequence $\{x_n : n \in \mathbb{N}\}$ is called a branch of an orbit of IFS$(\mathcal{F})$ if for each $n \in \mathbb{N}$ there is $f_n \in \mathcal{F}$ such that $x_{n+1} = f_n(x_n)$. We say that IFS$(\mathcal{F})$ is minimal if every orbit has a branch which is dense in M. We say that a property P holds C^r robustly for IFS$(\mathcal{F})$ if it holds for IFS$(\tilde{\mathcal{F}})$ for every family $\tilde{\mathcal{F}}$ whose elements are C^r perturbations of elements of $\mathcal{F}$.

Here, we prove the following

Theorem A. *Every boundaryless compact manifold admits a pair of diffeomorphisms that generates a C^1 robustly minimal iterated function system.*

The number of generators in Theorem A is optimal. Indeed, a single diffeomorphism can not be C^1 robustly minimal. Recall that a diffeomorphism is minimal if every orbit is dense, but Pugh's closing lemma yields the existence of periodic points for C^1 generic diffeomorphisms. There are several examples of robustly minimal iterated function systems: in dimension one by [8] and [5]; in any dimension but local and with many generators by [14]; on every boundaryless compact manifold and with many generators by [7] and by [6] where at least three generators are required.

The study of minimal IFSs has important consequences to the study and the construction of robustly transitive diffeomorphisms. Recall that a diffeomorphism is transitive if it has a dense orbit, and it is (C^1) robustly transitive if every nearby diffeomorphism (in the C^1 topology) is also transitive. Some parts of the proof of Theorem A are closely related to the construction of blenders. The notion of blender introduced by Bonatti and Díaz [3] is the main tool to construct robustly transitive diffeomorphisms. The results of this paper permit us to give a straightforward and general construction of blenders and robustly transitive dynamics in arbitrary dimension along the lines of [15]. It can be summarized as follows (cf. Theorem 5.6):

> *Let F be a diffeomorphism with invariant compact partially hyperbolic set Γ such that $F|_\Gamma$ is conjugate (appropriately) to an iterated function system* IFS$(\mathcal{F})$*. If* IFS$(\mathcal{F})$ *is* strongly robustly minimal*, then F is* robustly transitive *on* Γ.

The strong robust minimality of an IFS roughly means that one may change the perturbed family of maps at each iteration and still obtain dense orbits for all points (cf. Definition 4.1). The main result in this direction is Theorem 4.2, which implies the following improvement of Theorem A.

Theorem B. *Every boundaryless compact manifold admits a pair of diffeomorphisms that generates a C^1 strongly robustly minimal IFS.*

We provide two constructions that prove Theorem A, in Sections 2 and 3. These constructions yield different types of examples in the applications to robust transitivity. In Section 4 we study the strong robust minimality of IFSs and then we complete the proof of Theorem B. The connection to blenders, and an application to the construction of robustly transitive skew product diffeomorphisms, is presented in Section 5.

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2. Minimal iterated function systems

In this section we prove Theorem A.

2.1. **Local robust minimality.**

Definition 2.1. Let $\mathcal{F}$ be a family of maps on the metric space X, and $Y \subset X$. We say that IFS$(\mathcal{F})$ is *minimal on* Y if for every $x \in Y$, the orbit $\langle \mathcal{F} \rangle^+(x)$ has a branch which is dense in Y.

Recall that a map ϕ on a metric space (X, d) is a contraction if and only if there is a constant $0 < \kappa < 1$ such that $d(\phi(x), \phi(y)) \leq \kappa d(x, y)$, for all $x, y \in X$.

The following proposition is a modification of Propositions 2.3 of [15] (see also [14]). We give a short proof in Section 4.

Proposition 2.2. *Let U and V be two open disks in $\mathbb{R}^n$ containing 0, and $\phi : U \to V$ be a diffeomorphism with $\phi(0) = 0$. If $D\phi_0$ is a contraction, then there exists $k \in \mathbb{N}$ depending only on $D\phi_0$ such that for any small $\varepsilon > 0$ there exist $r > 0$ and vectors $c_1, \ldots, c_k \in B_\varepsilon(0)$ such that*

$$B_r(0) \subset \overline{\mathcal{O}^+_{\mathcal{F}}(0)},$$

where $\mathcal{F} = \{\phi + c_1, \ldots, \phi + c_k\}$ and $\mathcal{O}^+_{\mathcal{F}}(0)$ is a branch of the orbit $\langle \mathcal{F} \rangle^+(0)$. Moreover, IFS$(\mathcal{F})$ *is robustly minimal on $B_r(0)$.*

2.2. **Proof of Theorem A.**

Definition 2.3 (weak hyperbolic point)**.** Let p be a hyperbolic periodic point of g of period k, we say that p is δ-*weak hyperbolic* if

$$1 - \delta < m(D_p g^k|_{E^s_p}) < \| D_p g^k|_{E^s_p} \| < 1 < m(D_p g^k|_{E^u_p}) < \| D_p g^k|_{E^u_p} \| < \frac{1}{1 - \delta}.$$

Here E^s_p and E^u_p are the stable and unstable subspaces at p.

Proof of Theorem A. Let M be a compact manifold of dimension $m \geq 2$. We refer to [5] or [8] for iterated functions systems in dimension one. The proof has three steps.

Step 1. It is a standard consequence from Morse theory that one can take a Morse-Smale diffeomorphism $f_\circ$ on M with a unique attracting fixed point o. We assume that the norm of $Df_\circ^{-1}(o)$ is sufficiently close to one (i.e. o is a δ-weak hyperbolic attractor with sufficiently small δ).

We then modify slightly the dynamics of $f_\circ$ in a neighborhood U of o in order to obtain a new Morse-Smale diffeomorphism f such that it has a periodic attracting orbit $\{f^i(p)\}$ in U with sufficiently large period n and with very weak attraction. The period of the attracting orbit and the rate of its weak attraction depend to the dimension of the manifold.

To do the modification we may assume that the open set U is diffeomorphic to the product $\mathbb{D}^2 \times \mathbb{D}^{m-2}$. If $m = 2$, then we replace the dynamics on U by the local diffeomorphism f on $\mathbb{D}^2$ as in Figure 1.

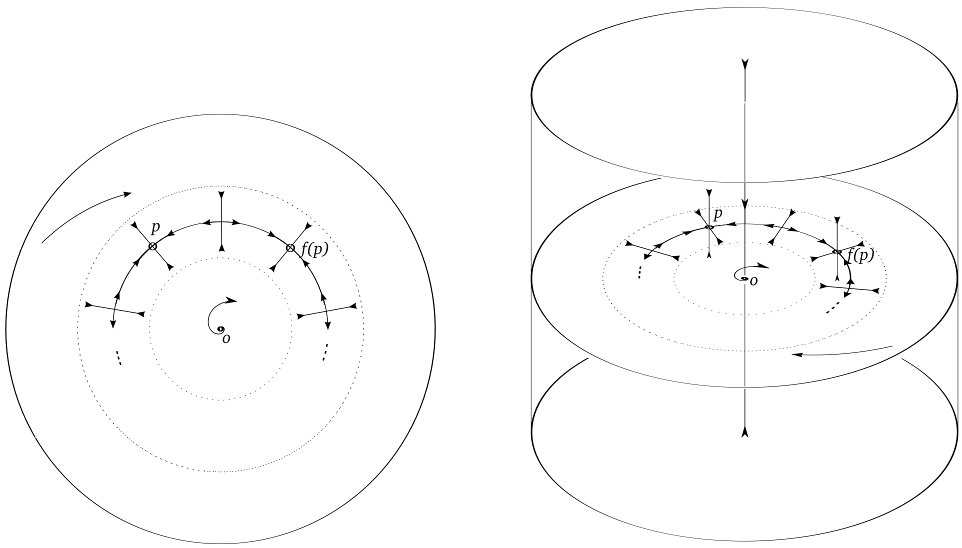

FIGURE 1. Dynamics of f if $\dim(M) = 2$ (left picture) and if $\dim(M) > 2$ (right picture).

If $m > 2$ then we define h_1 as the two dimensional map on $\mathbb{D}^2$ defined for the case $m = 2$, and a contraction h_2 with a unique fixed point o_2 on $\mathbb{D}^{m-2}$ so that o_2 is a δ-weak hyperbolic attractor, with a small $\delta > 0$. Then we modify f on U to be the product $h_1 \times h_2$.

Step 2. (local minimality) Let $\mathcal{V}$ be a small neighborhood of f such that it only contains Morse-Smale diffeomorphisms. We find a diffeomorphism $g \in \mathcal{V}$ arbitrarily close to f in such a way that its dynamics close to the periodic orbit $\mathcal{O}_f(p) = \{f^i(p)\}$ is slightly different to f and is chosen so that the IFS of $\{f, g\}$ is robustly minimal in a neighborhood of periodic orbit $\{f^i(p)\}_i$. Here we use the results of Section 2.1.

Let $V_i := f^i(V_0)$ be sufficiently small neighborhoods of $f^i(p)$, for $i = 0, \dots, n-1$. By the assumptions, $1 - \epsilon < m(Df^n(p)) < ||Df^n(p)|| < 1$ for some small $\epsilon > 0$. So, we may apply Proposition 2.2 for the map f^n restricted to V_0, which gives very small vectors $c_1, \dots, c_k$, (where $k < n$ if δ is small enough) on the local coordinates on V_0 so that the IFS of $\{f^n, T_{c_1} \circ f^n, \dots, T_{c_k} \circ f^n\}$ is robustly minimal on a neighborhood D_0 of p. Here, T_c is defined in local coordinates as the translation by the vector c.

Let $g_\circ$ be a diffeomorphism on M such that on V_{n-i} it is equal to $f^{1-i} \circ T_{c_i} \circ f^i$. Observe that $g_\circ$ is well defined as V_i are disjoint and c_i are sufficiently small.

Moreover, we may assume that $g_\circ$ is arbitrarily close to f by taking c_i small enough. In particular, $g_\circ \in \mathcal{V}$, so it is Morse-Smale. Further, $g_\circ^n$ on V_0 is equal to $T_{c_1} \circ T_{c_n} \circ f^n$ and $f^{i-1} \circ g_\circ \circ f^{n-i}$ on V_0 is equal to $T_{c_i} \circ f^n$. Let g be any diffeomorphism sufficiently close to $g_\circ$ in the C^1 topology. Then the IFS of $\{f, g\}$ is robustly minimal on D_0. We call such a local set with robustly minimal dynamics, a *blending region* of the iterated function system.

Observe that $\hat{g} := g \circ f^{-2}$ is very close to f^{-1} and so it is Morse-Smale, provided that $g_\circ$ is close enough to f. Moreover, the semigroup generated by $\{f, g \circ f^{-2}\}$ contains the semigroup generated by $\{f, g\}$.

From now on, we denote $g \circ f^{-2}$ by $\hat{g}$ for a given g. Thus, for any g close to $g_\circ$, the IFS $(\{f, \hat{g}\})$ is robustly minimal on D_0. We may also assume that for any g close to $g_\circ$, $\hat{g}$ has a periodic repeller $p' \in D_0$ such that $W^u(p')$ contains D_0.

Step 3. (globalization) We show that there exists a diffeomorphism g close to $g_\circ$ such that both positive and negative orbits of the blending region D_0 of IFS $(\{f, \hat{g}\})$ cover the entire manifold M. If so, then for any point $x \in M$ and any open set $O \subset M$, there exist diffeomorphisms $h_1, h_2 \in \langle \{f, \hat{g}\} \rangle^+$ such that, $h_1(x) \in D_0$ and $h_2^{-1}(O) \cap D_0 \neq \emptyset$. Since D_0 is a blending region of IFS $(\{f, \hat{g}\})$, it follows that there exists $h_3 \in \langle \{f, \hat{g}\} \rangle^+$ such that $h_3(h_1(x)) \in h_2^{-1}(O) \cap D_0$, and so for $h := h_2 \circ h_3 \circ h_1 \in \langle \{f, \hat{g}\} \rangle^+$ we have $h(x) \in O$.

Thus to complete the proof of Theorem A it remains to show the above statement. Recall that for a Morse-Smale diffeomorphism the manifold M is the union of the stable manifolds of finitely many periodic points. Also, the union of basins of attraction of periodic attractors for a Morse-Smale diffeomorphism is an open and dense subset of the manifold: the stable manifold $W^s(p)$ for f and the unstable manifold $W^u(p')$ for $\hat{g}$ are open and dense.

Pick g so that $\hat{g}$ and f have no common periodic points, while $\hat{g}$ maps the periodic points of f into the basin of attraction of $\{f^i(p)\}$. The negative orbits of D_0 then cover M: pick a point $x \in M$, then iterating x by f sufficiently often brings it close to some periodic orbit of f since f is Morse-Smale, a further iterate by $\hat{g}$ maps it into the basin of attraction of $\{f^i(p)\}$ after which an iterate of f brings it into D_0. Similarly for positive orbits of D_0. They cover M if f^{-1} maps the periodic points of $\hat{g}$ into the basin of attraction of the periodic attractor for $\hat{g}^{-1}$.

Since these basins of attraction are open and dense in M, these properties can be achieved for g arbitrarily close to $g_\circ$. This completes the proof of Theorem A. $\square$

3. Alternative construction

In this section we describe an alternative argument leading to Theorem A. The local part of the argument, producing a blending region, is through an alternative construction. The global part, the mechanism to iterate into and out of the blending region, will be the same. We start with the construction of a minimal iterated function systems generated by affine maps on Euclidean spaces.

3.1. **Iterated function systems generated by affine maps.** We will provide an affine contraction S and an affine expansion T (i.e. T^{-1} is a contraction) so that the iterated function system generated by S and T is minimal on all of $\mathbb{R}^m$. Here we assume $m \geq 2$, compare [2] for minimal iterated functions systems on $[0, \infty)$ generated by affine maps.

A set Δ is called invariant for an iterated function system $\mathrm{IFS}\,(\mathcal{F})$ if $\mathcal{F}(\Delta) = \Delta$. Consider an iterated function system $\mathrm{IFS}\,(\mathcal{F})$ generated by contractions on $\mathbb{R}^n$. Such an iterated function system admits a unique compact invariant set that contains all limit points of branches. This follows from the observation that $\mathcal{F}$ is a contraction in the Hausdorff metric on the class of compact sets [10]. We call such a set the attractor of the iterated function system.

The constructed affine maps S, T will be so that $S \circ T$ is a contraction and the iterated function system generated by the two affine contractions S and $S \circ T$ possesses an attractor with nonempty interior. Similar ideas were independently used by Volk [18] in his construction of persistent attractors for endomorphisms.

Consider the rotation R in $\mathbb{R}^m$, $m \geq 2$,

$$R(x_1, \ldots, x_m) = (\pm x_m, x_1, \ldots, x_{m-1}),$$

where the sign is such that $R \in SO(m)$, i.e., a minus sign for even m and a plus sign for odd m. Define further the translation $H(x_1, \ldots, x_m) = (x_1 + s, x_2, \ldots, x_m)$ and the affine map S on $\mathbb{R}^m$ by

$$S(x_1, \ldots, x_m) = H \circ rR(x_1, \ldots, x_m) = (\pm r x_m + s, r x_1, \ldots, r x_{m-1}), \tag{1}$$

for constants $0 < r < 1$, $s > 0$. Likewise, define an affine map T on $\mathbb{R}^m$ by

$$T(x_1, \ldots, x_m) = (-a x_1, a x_2, \ldots, a x_{m-1}, -a x_m - 2\frac{s}{r}) \tag{2}$$

with $a > 1$. Similar to S, T is the composition of a map from $SO(m)$ which is multiplied by a factor, a, and a translation. Note that S is a contraction by $r < 1$, while T is an expansion as $a > 1$. Compute

$$\begin{aligned} S \circ T(x_1, \ldots, x_m) &= H^{-1} \circ arR(x_1, \ldots, x_m) \\ &= (\mp arx_m - s, -arx_1, arx_2, \ldots, arx_{m-1}). \end{aligned}$$

The affine map $S \circ T$ is a contraction for $ar < 1$.

Lemma 3.1. *There are constants $0 < r < 1$, $s > 0$, $a > 1$ with $ar < 1$ so that the iterated function system $\mathrm{IFS}\,(\mathcal{G})$ with $\mathcal{G} = \{S, S \circ T\}$ has an attractor $\triangle$ with nonempty interior. Moreover, this interior contains the fixed point of T.*

Proof. Define the box $B(1, v_2, \ldots, v_m)$ with corners $(\pm 1, \pm v_2, \ldots, \pm v_m)$. We will find r, s, a and $v_2, \ldots, v_m$ so that

$$S(B) \cup S \circ T(B) \supset B \tag{3}$$

This is a consequence of the following conditions,

$$r v_m + s > 1,\ 0 > -r v_m + s,\ r > v_2,\ r v_2 > v_3,\ \ldots,\ r v_{m-1} > v_m. \tag{4}$$

In fact, (4) implies

(5) $\quad arv_m + s > 1,\ 0 > -arv_m + s,\ ar > v_2,\ arv_2 > v_3,\ \ldots,\ arv_{m-1} > v_m,$

and (4) and (5) together give (3).

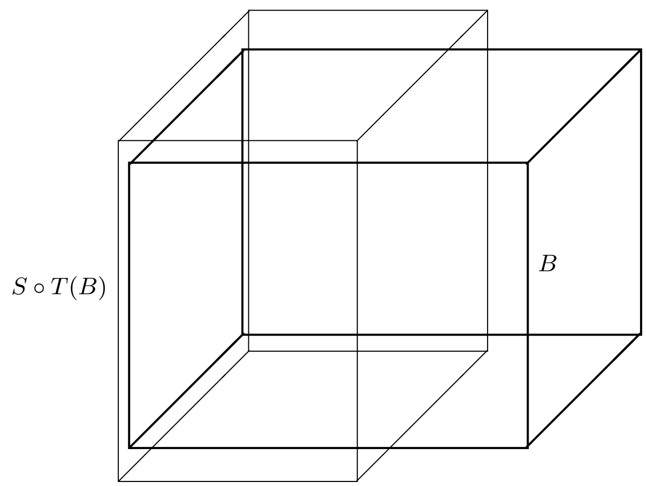

FIGURE 2. The image $S \circ T(B)$ covers over half the box B. The image $S(T)$ covers the remaining part of B, so that $S(B) \cup S \circ T(B)$ contains B.

The repelling fixed point of T is located at $(0, \ldots, 0, \frac{-2s}{r(a+1)})$. It lies in B if

(6) $$2s < v_m r(a+1).$$

Now, (4) and (6) can be satisfied by taking suitable $v_m < \ldots < v_2 < r$ all near 1 and $s = 1 - v_m^2$ near 0 so that (4) holds, and a with $ar < 1$.

Since S and $S \circ T$ are contractions, there is a ball O that is mapped into itself by both S and $S \circ T$, i.e., $\mathcal{G}(O) \subset O$. Thus

$$\triangle = \lim_{p \to \infty} \mathcal{G}^p(O)$$

is a nonempty compact set that is invariant for $\mathrm{IFS}\,(\mathcal{G})$. Since S and $S \circ T$ are contractions, $\triangle$ is the unique compact set that is invariant for $\mathcal{G}$ [10]. Because $\mathcal{G}(B) \supset B$, the set $\triangle$ contains B. $\square$

Corollary 3.2. *The iterated function system* $\mathrm{IFS}\,(\{\mathrm{S}, \mathrm{T}\})$ *is minimal on* $\mathbb{R}^m$.

Proof. Note first that $\mathcal{G}$ is minimal on $\triangle$. Indeed, $\mathcal{G}^n(\triangle)$ consists of 2^n sets with diameter shrinking to zero as $n \to \infty$ and together covering $\triangle$. So given $x \in \triangle$ and open V intersecting $\triangle$, $\langle \mathcal{G} \rangle^+(x)$ intersects V. This implies that $\mathcal{G}$ is minimal on $\triangle$.

The corollary follows, as in the globalization step in the proof of Theorem A in the previous section, since both backward iterates of $\triangle$ under $\mathrm{IFS}\,(\{\mathrm{S}, \mathrm{T}\})$ cover $\mathbb{R}^n$ (true since S is a contraction with attracting fixed point inside $\triangle$) and forward iterates of $\triangle$ under IFS$(\{\mathrm{S},\mathrm{T}\})$ cover $\mathbb{R}^n$ (true since T^{-1} is a contraction with attracting fixed point inside $\triangle$). $\square$

The proof of Lemma 3.1 gives more than its statement as it includes arguments for C^1 robust occurrence of invariant sets with nonempty interior. Denote by $\mathrm{Diff}^1(\mathbb{R}^m)$ the set of diffeomorphisms on $\mathbb{R}^m$, endowed with the C^1 compact-open topology.

Corollary 3.3. *There exists a neighborhood $W \subset \mathrm{Diff}^1(\mathbb{R}^m) \times \mathrm{Diff}^1(\mathbb{R}^m)$ of $(S, S \circ T)$ such that for each (f_1, f_2) in this neighborhood,* IFS$(\mathcal{F})$ *with $\mathcal{F} = \{f_1, f_2\}$ admits an invariant set with non-empty interior.*

In rescaled coordinates $(y_1, \ldots, y_m) = h(x_1, \ldots, x_m) = (\delta x_1, \ldots, \delta x_m)$, $\delta > 0$, the attractor $\triangle$ is multiplied by a factor δ. The affine maps computed in the $(y_1, \ldots, y_m)$ coordinates become

$$h \circ S \circ h^{-1}(y_1, \ldots, y_m) = (\pm r y_m + \delta s, r y_1, \ldots, r y_{m-1})$$

and

$$h \circ T \circ h^{-1}(y_1, \ldots, y_m) = (-a y_1, a y_2, \ldots, a y_{m-1}, -a y_m - 2\frac{\delta s}{r});$$

the maps are unaltered except for the translation vector which is multiplied by δ. In other words, if S and T are affine maps as above so that $\mathcal{G}(\mathbb{R}^m; S, T)$ has an attractor $\triangle$, than replacing s in their expressions by δs yields an iterated function system with attractor $\delta\triangle$.

3.2. **Second proof of Theorem A.** The proof of Theorem A in Section 2.2 consists of a local part, the construction of a blending region in steps 1 and 2, and a global part in step 3 involving a mechanism to move from the blending region to other parts of the manifold, and back, by iterating. The same construction can be followed here, where Lemma 3.1 used in a chart on the manifold provides a blending region. Compare [7].

Specifically we obtain two diffeomorphisms g_1, g_2 on M generating a robustly minimal iterated function system with the following properties:

(1) for $i = 1, 2$, g_i has a unique attracting fixed point p_i and a unique repelling fixed point q_i,
(2) there is a blending region, containing p_1 and q_2, on which g_1 and $g_1 \circ g_2$ are contractions.

The diffeomorphisms g_1, g_2 are not C^1 close to the identity, but the construction can be done so that $m(Dg_1), |Dg_1|, m(Dg_2), |Dg_2|$ are everywhere close to 1.

4. Strong robust minimality

In this section we introduce the notion of strong robust minimality for iterated function systems.

4.1. **Notations and definitions.** Let X be a metric space, and $\mathcal{F}$ be a family of maps on X.

- For any $Y \subset X$ and $x \in X$, we denote $\mathcal{F}(Y) := \bigcup_{\phi\in\mathcal{F}} \phi(Y)$ and $\mathcal{F}(x) := \mathcal{F}(\{x\})$.
- If each element of $\mathcal{F}$ is invertible, then we denote $\mathcal{F}^{-1} = \{f^{-1} \mid f \in \mathcal{F}\}$.
- Let $\mathbf{F} = (\mathcal{F}_{\mathbf{1}}, \mathcal{F}_{\mathbf{2}}, \dots)$, where $\mathcal{F}_n$ is a family of maps on X, for any $n \in \mathbb{N}$. Then we denote $\mathcal{F}^{(n)} = \mathcal{F}_n \circ \cdots \circ \mathcal{F}_1$, $\mathcal{F}^{(1)} = \mathcal{F}_1$, $\mathcal{F}^{(0)} = \{Id\}$,
$$\langle\langle \mathbf{F} \rangle\rangle^+ := \bigcup_{n=0}^{\infty} \mathcal{F}^{(n)},$$
and its orbits by $\langle\langle \mathbf{F} \rangle\rangle^+(x) := \{f(x) : f \in \langle\langle \mathbf{F} \rangle\rangle^+\}$, for $x \in X$.
- We also denote $\mathcal{F}_m^{(n)} := \mathcal{F}_{m+n} \circ \cdots \circ \mathcal{F}_{m+1}$, for $n, m \in \mathbb{N} \cup \{0\}$.
- Let $x \in X$. A sequence $\{x_n : n \in \mathbb{N}\} \subset X$ is called a branch of the orbit $\langle\langle \mathbf{F} \rangle\rangle^+(x)$ if for any $n \in \mathbb{N}$ there is $f_n \in \mathcal{F}^{(n)}$ such that $x_{n+1} = f_n(x_n)$.

Let $\mathcal{F}$ be a family (with k elements) of C^r diffeomorphisms of a manifold M. A neighborhood $\mathcal{U}$ of $\mathcal{F}$ is the set of all families $\mathcal{F}'$ whose elements are C^r perturbations of elements of $\mathcal{F}$.

Definition 4.1. We say that IFS$(\mathcal{F})$ is C^r *strongly* robustly minimal if there is a neighborhood $\mathcal{U}$ of $\mathcal{F}$ such that for any $x \in M$ and any sequence $\mathbf{F} = (\mathcal{F}_{\mathbf{1}}, \mathcal{F}_{\mathbf{2}}, \dots)$ in $\mathcal{U}$ there is a branch of the orbit of $\langle\langle \mathbf{F} \rangle\rangle^+(x)$ which is dense in M.

It is clear that this is stronger than robust minimality defined in the introduction. We will see how this notion is natural and effective when one applies the robust minimality of IFS to construct examples of robustly transitive diffeomorphisms.

We say that a map ϕ on a metric space (X, d) is bi-Lipschitz with constants $\lambda > 0$ and κ, if $\lambda d(x, y) \le d(\phi(x), \phi(y)) \le \kappa d(x, y)$ for all $x, y \in X$. If $0 < \kappa < 1$ then ϕ is a contraction. Then, we also say λ is the *contraction-lower-bound* of ϕ.

4.2. **Sufficient conditions for strong robustness.**

Theorem 4.2. *Let M be a boundaryless compact manifold and D be an open subset of M. Let $\mathcal{F}$ be a finite family of diffeomorphisms of M such that*

(1) *for any $f \in \mathcal{F}$, $f|_{\overline{D}}$ is a contraction,*
(2) $\overline{D} \subset \mathcal{F}(D)$

Then IFS$(\mathcal{F})$ *is strongly robustly minimal on D.*

Let $\mathcal{G}$ be a family of diffeomorphisms of M such that $\mathcal{F} \subset \mathcal{G}$ and $M = \mathcal{G}(D) = \mathcal{G}^{-1}(D)$. Then IFS$(\mathcal{G})$ *is strongly robustly minimal (on M).*

Before proving the theorem we prove a basic lemma.

Lemma 4.3. *Let X be a connected metric space, and $D \subset X$ be a bounded open set. Let $0 < \lambda < \kappa < 1$. Let $\mathbf{F} = (\mathcal{F}_{\mathbf{1}}, \mathcal{F}_{\mathbf{2}}, \dots)$ be such that for any $n \in \mathbb{N}$, $\mathcal{F}_n$ is a family of homeomorphisms of X such that*

(1) *for any $f \in \mathcal{F}_n$, $f|_D$ is bi-Lipschitz with constants λ and κ,*
(2) $\overline{D} \subset \mathcal{F}_n(D)$,
(3) *Z_n is δ-dense in D,*

where $Z_n = \{x \in D \mid f(x) = x$ for some $f \in \mathcal{F}_n\}$, and $2\delta > 0$ is smaller than the Lebesgue number of a covering $\mathcal{A}$ of $\overline{D}$ with $\mathcal{A} \subseteq \{f(D) \cap \overline{D} \mid f \in \mathcal{F}_n\}$. Then for any $x \in D$, $\langle\langle \mathbf{F} \rangle\rangle^+(x)$ has a branch which is dense in D.

Proof. It is enough to prove the following.

Claim 4.4. *Let $B \subset D$ be an open ball. Then there is $n \in \mathbb{N}$ such that for any $m \in \mathbb{N} \cup \{0\}$ there is $f \in \mathcal{F}_m^{(n)} = \mathcal{F}_{m+n} \circ \cdots \circ \mathcal{F}_{m+1}$ such that $f(D) \subset B$.*

Proof. Assume that $B = B_r(x)$ for some $x \in D$. Let $k \in \mathbb{N}$ be (the smallest integer) larger than $\ln(\delta/\mathrm{diam}(D))/\ln(\kappa)$.

(a) If $r \geq 2\delta$, $B' = B_\delta(x)$ be the ball of radius δ and same center. Then, it follows from (3) that for any $i \in \mathbb{N}$, B' contains some $z \in Z_i$. Thus, there is some $f_i \in \mathcal{F}_i$ with a contacting fixed point in B'. By applying this fact at most k times we see that there is $f \in \mathcal{F}_{m+k} \circ \cdots \circ \mathcal{F}_{n+2} \circ \mathcal{F}_{m+1}$ such that $f(D) \subset B$. In particular, the lemma follows for $n = k$.

(b) If $r < 2\delta$, let $k(r) \in \mathbb{N}$ be the smallest integer larger than $\ln(r/(2\delta))/\ln(\lambda)$. Then, it follows from (2) that for any $i \in \mathbb{N}$, there is some $g_i \in \mathcal{F}_i$ such that $B \subset g_i(D)$. Then by (1), $g_i^{-1}(B) \subset D$ and it contains a ball of radius $r\lambda^{-1}$. By applying this fact at most $k(r)$ times we see that there is $g \in \mathcal{F}_{i+k(r)} \circ \cdots \circ \mathcal{F}_{i+1}$ such that $g^{-1}(B \cap g(D))$ contains a ball of radius 2δ. Thus from (a), for any $m \in N \cup \{0\}$, there is $h \in \mathcal{F}_{m+k} \circ \cdots \circ \mathcal{F}_{m+2} \circ \mathcal{F}_{m+1}$ such that $h(D) \subset g^{-1}(B \cap g(D))$. In particular, $(g \circ h)(D) \subset B$. For $i = m+k$, $n = k + k(r)$ and $f = g \circ h$, it follows that $f(D) \subset B$ and $f \in \mathcal{F}_{m+n} \circ \cdots \circ \mathcal{F}_{m+2} \circ \mathcal{F}_{m+1}$, completing the proof of claim. □

Now, by considering a countable topological base for D and applying the claim repeatedly one obtains a branch of $\langle\langle \mathbf{F} \rangle\rangle^+(x)$ which is dense in D. □

Remark 4.5. The condition (2) and (3) in Lemma 4.3 are equivalent to what are called the covering and the well-distributed properties (respectively) in [15, Definition 2.4].

Proof of Theorem 4.2. Let $\delta > 0$ be a number such that 2δ is smaller that the Lebesgue number of the covering $\mathcal{A} = \{f(D) \cap \overline{D} \mid f \in \mathcal{F}\}$ of $\overline{D}$.

Let Y_n be the set of fixed points of elements of $\mathcal{F}^n$. It follows from the main result of [10] that there is $n_0 \in \mathbb{N}$ large enough such that for any $n \geq n_0$, Y_n is $(\delta/2)$-dense in D.

It is easy to see that the family $\mathcal{F} \cup \mathcal{F}^{n_0}$ satisfies the hypothesis (1)-(3) of Lemma 4.3 for some bi-Lipschitz constants $0 < \lambda^{1/2} < \kappa^2 < 1$. Let $\mathcal{V}$ be a sufficiently small neighborhood of $\mathcal{F}$, and let $\mathbf{F} = (\mathcal{F}_\mathbf{1}, \mathcal{F}_\mathbf{2}, \dots)$ be a sequence of perturbations of $\mathcal{F}$ in $\mathcal{V}$. Then, for any $m \in \mathbb{N}$, $\mathcal{F}_m^{(n_0)}$ is close to $\mathcal{F}^{n_0}$, and the set of its fixed points (denoted by $Z_m^{(n_0)}$) is δ-dense in D. Consequently, for any $n, m \in \mathbb{N}$, the family $\mathcal{F}_n \cup \mathcal{F}_m^{(n_0)}$ satisfies the hypothesis (1)-(3) of Lemma 4.3 for bi-Lipschitz constants λ and κ.

Now, let $\{B_i\}$ be a topological base of D, so that each B_i is a ball. Fix $i \in \mathbb{N}$, and let $B = B_i$ be a ball of radius r. By the proof of Claim 4.4 (to apply part (a) we consider the families $\mathcal{F}_i^{(n_0)}$, and to apply part (b) we consider the families

$\mathcal{F}_i$), there are non-negative integers k and $k(r)$ such that for any $m \in \mathbb{N}$, there is $f \in \mathcal{F}_m^{(n)}$ such that $f(D) \subset B$, where $n = kn_0 + k(r)$.

Using this statement inductively, one find a dense branch in D for any orbit $\langle\langle \mathbf{F} \rangle\rangle^+(x)$, $x \in D$. This completes the proof of the first part.

If $\mathcal{G}$ is a family of diffeomorphisms of M such that $\mathcal{F} \subset \mathcal{G}$ and $M = \mathcal{G}(D) = \mathcal{G}^{-1}(D)$. Then by compactness of M there are finite subsets $\mathcal{G}_\pm \subset \langle \mathcal{G}^{\pm 1} \rangle^+$ such that $\mathcal{G}_\pm(D)$ covers the manifold M. This property is robust under small perturbation of $\mathcal{G}$. Thus any point in M has iterations (either backward or forward) in D. This allows to repeat the above argument to show that any orbit $\langle\langle \mathbf{G} \rangle\rangle^+(x)$, $x \in M$, has a branch which is dense in M. In other words, IFS($\mathcal{G}$) is strongly robustly minimal (on M). □

The following gives a short proof for Proposition 2.2.

Proposition 4.6. *There exist a family of maps $\mathcal{F}$ as in Proposition 2.2 such that* IFS($\mathcal{F}$) *is strongly robustly minimal on some open ball $B_\delta(0)$.*

Proof. Let $\lambda > 0$ such that $B_\lambda(0) \subset D\phi_0(B_1(0))$. Then, we consider a cover of the closed unit ball of $\mathbb{R}^n$ by k balls of radius λ, i.e.,

$$\overline{B_1(0)} \subset \bigcup_{i=1}^{k} B_\lambda(b_i) = \bigcup_{i=1}^{k} (B_\lambda(0) + b_i).$$

It follows that for $\delta > 0$ small enough,

$$\overline{B_\delta(0)} \subset \bigcup_{i=1}^{k} (\phi(B_\delta(0)) + \delta b_i).$$

To complete the proof, it is enough to apply the previous theorem for $D = B_\delta(0)$ and the family $\mathcal{F} = \{\phi + c_1, \ldots, \phi + c_k\}$, where $c_i = \delta b_i$. □

4.3. **End of proof of Theorem B.**

Proof of Theorem B. It is easy to see that the pair of diffeomorphism (in both proofs) of Theorem A satisfies the hypothesis of Theorem 4.2 and so it generates a strongly robustly minimal IFS. The same occurs for the one dimensional case. In fact, on the circle the pair of diffeomorphism (one is a Morse-Smale diffeomorphism close to identity and the other one is an irrational rotation) in [8] also satisfies hypothesis of Theorem 4.2. □

5. Skew products and robust transitivity

Following the ideas developed in [15], we discuss relations between robustly minimal iterated function systems constructed in the previous sections and blenders. We demonstrate how, through these relations, the minimal iterated function systems provide elementary constructions of robustly transitive skew products. More precisely, we show how the constructions in this paper provide an alternative construction for one of the main results from [3] on robustly transitive skew products (see Theorem 5.2 below).

The material in the previous sections is readily translated to skew product systems of diffeomorphisms over shift maps; a blending region will give rise to a symbolic blender. Blenders are certain hyperbolic invariant sets used in the construction of robustly transitive dynamics, as explored in [3] (see also [4, 5]). The original notion of blenders is in a context of partial hyperbolicity with one dimensional central directions. The generalization of blenders to higher dimensional central directions was first considered in [14, 15], where symbolic blenders and their geometric models are introduced and applied to robustly transitive dynamics in the symplectic settings. A further study of their geometric and dynamical properties is in [1].

5.1. **Symbolic blenders.** Let g_1, g_2 be two diffeomorphisms on M generating a robustly minimal iterated function system as in the previous sections. For reasons of definiteness and clarity we assume the properties listed in Section 3.2. Write $h_1 = g_1$, $h_2 = g_1 \circ g_2$ and let $\mathcal{H} = \{h_1, h_2\}$. Recall that IFS$\,(\mathcal{H})$ acts minimally on a set Δ. More specific, there are open sets $E_{in} \subset \Delta \subset E_{out}$ on which

$$E_{in} \subset \mathcal{H}(E_{in}) \subset \Delta \subset \mathcal{H}(E_{out}) \subset E_{out} \tag{7}$$

and h_1 and h_2 are contractions on E_{out}.

With $\Sigma = \{1,2\}^{\mathbb{Z}}$, let H_0 on $\Sigma \times M$ be defined by

$$H_0(\omega, y) = (\sigma(\omega), h_{\omega_0}(y)). \tag{8}$$

where $(\sigma(\omega))_k = \omega_{k+1}$ is the left shift operator. For the one-sided symbol space $\Sigma_+ = \{1,2\}^{\mathbb{N}}$, a map $H_{0,+}$ on $\Sigma_+ \times M$ is defined likewise, using the same formula. The symbol spaces Σ_+ and Σ will be endowed with the product topology, and we write $\Sigma = \Sigma_- \times \Sigma_+$.

A natural situation is where Σ occurs, through a topological conjugacy, as an invariant set of a diffeomorphism. This gives sense to notions of (partial) hyperbolicity and of invariant manifolds such as stable manifolds. It will in fact be the context of our applications on robustly transitive skew product systems. Henceforth, we work in the setup of diffeomorphisms $f : N \to N$ on a compact manifold N possessing a maximal hyperbolic invariant set in an open set $U \subset N$ on which f is topologically conjugate to σ on Σ. We consider skew product systems $(x, y) \mapsto (f(x), g_y(x))$ on $N \times M$; write $\mathcal{S}^1(N, M)$ for the set of such diffeomorphisms endowed with the C^1 topology. Note that C^1 small perturbations of f yield a perturbed hyperbolic set in U on which the dynamics remains topologically conjugate to σ acting on Σ. For convenience, as we restrict diffeomorphisms to their hyperbolic sets, we write $\mathcal{S}^1(\Sigma, M)$ for the C^1 skew product maps $H : \Sigma \times M \to \Sigma \times M$,

$$H(\omega, y) = (\sigma(\omega), h_\omega(y)), \tag{9}$$

endowed with the C^1 topology in the sense just described.

Suppose H is partially hyperbolic on $\Sigma \times M$ in the sense that contraction and expansion rates of σ dominate those of $y \mapsto h_\omega(y)$. That is, with $E^s \oplus E^u$ denoting the splitting in stable and unstable directions for σ,

$$|D\sigma(\omega)|_{E^s_\omega}| < m(Dh_\omega(y)) \le |Dh_\omega(y)| < m(D\sigma(\omega)|_{E^u_\omega})$$

for each $y \in M$ and $\omega \in \Sigma$. The partially hyperbolic skew product systems in $\mathcal{S}^1(\Sigma, M)$ form an open subset of $\mathcal{S}^1(\Sigma, M)$; we may assume that H_0 is partially hyperbolic. We refer to [5] for more on partial hyperbolicity. As a consequence of partial hyperbolicity, there are strong stable and strong unstable manifolds of points in $\Sigma \times M$ that project to the stable and unstable manifolds of σ under the projection to the base space Σ. These strong stable manifolds provide a strong stable lamination $\mathcal{F}^{ss}$ of $\Sigma \times M$. For definiteness, a local strong stable manifold is a compact part of a strong stable manifold that projects to some $\Sigma_- \times \{\omega_+\}$ under the projection to the base space Σ. Likewise, strong unstable manifolds of points in $\Sigma \times M$ provide a strong unstable lamination $\mathcal{F}^{uu}$ of $\Sigma \times M$. Local strong unstable manifolds project to $\{\omega-\} \times \Sigma_+$ under the projection to the base space Σ. A strong stable or unstable lamination of a set is called minimal if each of its leaves lies dense in the set.

The maximal invariant set B of H in $\Sigma \times E_{out}$ in the following proposition is called a symbolic blender [15]. The local unstable set of B is to be interpreted as the union of local strong unstable manifolds through points in B.

Proposition 5.1. *Any skew product map $H \in \mathcal{S}^1(\Sigma, M)$ sufficiently close to $H_0(\omega, m) = (\sigma(\omega), h_{\omega_0}(m))$ possesses a maximal invariant set $B \subset \Sigma \times E_{out}$ with the following properties:*

- *H is topologically mixing on B,*
- *the strong unstable lamination of B is minimal,*
- *any local strong stable manifold inside $\Sigma \times E_{in}$ intersects the local unstable set of B.*

Proof. The image $H^k_{0,+}(\Sigma_+ \times E_{in})$ contains 2^k strips $\Sigma_+ \times U_i \subset \Sigma_+ \times E_{out}$ with diameter of U_i going to 0 as $k \to \infty$ and together covering $\Sigma_+ \times E_{in}$. Take an open set $U \subset \Sigma_+ \times \Delta$. A high iterate $H^n_{0,+}(U)$ contains a strip $\Sigma_+ \times J$ in $\Sigma_+ \times \Delta$. By the above description of iterates $H^k_{0,+}(\Sigma_+ \times E_{in})$, we see that further iterates $H^{n+k}_{0,+}(U)$ contain strips in $\Sigma_+ \times \Delta$, that lie increasingly dense in it as $k \to \infty$.

This reasoning also applies to small perturbations H_+ of $H_{0,+}$, where also the fiber maps may depend on all of ω instead of just ω_0 (for one dimensional central directions this is also pursued in [9]). Suppose $H_+(\omega, x) = (\sigma(\omega), h_\omega(x))$ is such that h_ω depends continuously on ω and is uniformly close to h_{ω_0}. We note the following changes in the reasoning. The inclusions (7) get replaced by

$$\Sigma \times E_{in} \subset H_+(\Sigma_+ \times E_{in}), \qquad H_+(\Sigma_+ \times E_{out}) \subset \Sigma_+ \times E_{out} \tag{10}$$

An iterate H^n_+ maps $\Sigma_+ \times E_{out}$ to 2^n strips R_i. The diameter of a strip is the maximal real number r so that each $R_i \cap \{\omega\} \times E_{out}$ is contained in a ball of radius r. The map H_+ on $\Sigma_+ \times E_{out}$ acts by contractions in the fibers $\{\omega\} \times E_{out}$. Hence $H^n_+(\Sigma_+ \times E_{in})$ are 2^n strips with diameter going to zero as $n \to \infty$ and together covering $\Sigma_+ \times E_{in}$. Iterates under H_+ of $\Sigma_+ \times E_{in}$ or $\Sigma_+ \times E_{out}$ therefore converge to an invariant set that contains $\Sigma_+ \times E_{in}$. As above, for any open set U, $H^{n+k}_+(U)$ contain strips lying increasingly dense in the invariant set of H_+ as k increases.

We proceed with skew products over the shift operator on two sided symbol spaces. For H C^1-close to H_0, there is a strong stable foliation close to the affine foliation, i.e. with leaves close to $\Sigma_- \times \{\omega_+\} \times \{m\}$. Write $\pi^+ : \Sigma \times E_{out} \to \Sigma_+ \times E_{out}$ for the continuous projection along local strong stable manifolds. The existence of a strong stable foliation that is close to the affine foliation means that π^+ is C^0-close to the identity.

We can copy the previous reasoning. Observe that H^n maps a curve $\{\omega_-\} \times \Sigma_+ \times \{m\}$ to 2^n curves that are each a graph of a map $\Sigma_+ \to \Sigma_- \times E_{out}$. Likewise H^n maps $\Sigma \times E_{out}$ to 2^n strips R_i. Since π^+ is close to the identity, (10) gets replaced by

$$\Sigma \times E_{in} \subset \pi^+ H(\Sigma_+ \times E_{in}), \qquad \pi^+ H(\Sigma_+ \times E_{out}) \subset \Sigma_+ \times E_{out}.$$

Invariance of the strong stable foliation gives that these inclusions also hold for iterates of H. Further, by uniform C^1-closeness of h_ω to h_{ω_0}:

$$\lim_{i\to\infty} \operatorname{diam}(R_i) = 0.$$

The diameter of a strip, as before, is the maximal real number r so that each $R_i \cap \{\omega\} \times E_{out}$ is contained in a ball of radius r. This proves the proposition. □

5.2. **Robustly transitive skew product diffeomorphisms.** As an application of symbolic blenders from Proposition 5.1 we show how it gives constructions of robustly transitive diffeomorphisms, obtaining a main result in [3] in a straightforward manner.

Let N, M be compact manifolds and let $f : N \mapsto N$ be a diffeomorphism with a compact hyperbolic locally maximal invariant set Λ_f, on which f is topologically mixing. Below we will use the fact that unstable manifolds for f of points in Λ_f lie dense in Λ_f, see e.g. [11, Section 18.3]. Recall that a C^1 small perturbation $\tilde f$ of f possesses a hyperbolic attractor $\Lambda_{\tilde f}$ near Λ_f. Moreover, $\tilde f$ restricted to $\Lambda_{\tilde f}$ is topologically conjugate to f restricted to Λ_f. Let $R_1, \dots, R_k$ be a Markov partition for Λ_f, through which, $f|_{\Lambda_f}$ is conjugate by φ to the full shift $\sigma : \Sigma \to \Sigma$ with k symbols.

Theorem 5.2 ([3])**.** *There is a diffeomorphism $F : N \times M \to N \times M$, $F(x, y) = (f(x), g(x, y))$ with f as above, that is topologically mixing on $\Lambda_f \times M$. Moreover, F is robustly topologically mixing in $\mathcal{S}^1(N \times M)$; i.e. there is an open neighborhood U of F in $\mathcal{S}^1(N \times M)$ so that each $\tilde F(x, y) = (\tilde f(x), \tilde g(x, y))$ from U is topologically mixing on $\Lambda_{\tilde f} \times M$.*

Proof. Let $\Omega_f \subset \Lambda_f$ be a Smale horseshoe for an iterate f^k of f, i.e. an invariant set for f^k on which f^k is topologically conjugate to the shift σ on the symbol space $\Sigma = \{1, 2\}^{\mathbb{Z}}$ (see e.g. [11, Theorem 6.5.5]). For simplicity we assume $k = 1$ for now. With the Smale horseshoe comes a Markov partition of two sets U_1, U_2 covering Ω_f, for which Ω_f is the maximal invariant set in $U_1 \cup U_2$.

Consider a skew product system $(x, y) \mapsto (f(x), g(x, y))$ with the following properties:

(1) through the conjugation of $f|_{\Omega_f}$ with σ, the skew product system on $\Omega_f \times M$ is given by

$$G(\omega, y) = (\sigma(\omega), g_{\omega_0}(y)), \tag{11}$$

where g_1, g_2 are as in Section 3.2,

(2) for $(x, y) \in U_i \times M$, $i = 1, 2$, $g(x, y)$ does not depend on x.

Such a skew product system exists as g_1 and g_2 are isotopic to the identity.

We now consider the diffeomorphism F restricted to $\Omega_f \times M$. The repelling fixed point q_2 of g_2 in Δ gives a fixed point $(2^\infty, q_2)$ for G that is repelling in $\{2^\infty\} \times M$. This corresponds to a fixed point Q_2 for F that is repelling within its central fiber. Similarly F has a fixed point P_1 coming from the fixed point $(1^\infty, p_1)$ for G, this fixed point is contracting within its central fiber. Note that $W^u(Q_2)$ is dense in $\Lambda_f \times M$, since unstable manifolds for f are dense in Λ_f. We claim that

$$W^u(Q_2) \subset \overline{W^{uu}(P_1)},$$

establishing that $W^{uu}(P_1)$ lies dense in $\Lambda_f \times M$. Namely, take a point $q \in W^u(Q_2)$ and a neighborhood V of it, iterate backwards and note that $H^{-m}(V)$ intersects $W^{uu}(P_1)$ by the existence of a symbolic blender of F on $\Omega_f \times M$ (compare [3, Lemma 1.9]). Density of $W^{uu}(P_1)$ implies that H is topologically mixing on $\Lambda_f \times M$, since iterates of an open set intersect the local stable manifold of P_1 and thus accumulate onto $W^{uu}(P_1)$. This construction is robust under perturbations of F, see also Proposition 5.1, proving that F is robustly topologically mixing. □

5.3. **Sufficient conditions for robust transitivity.** Let $f : N \to N$ be a diffeomorphism with a hyperbolic invariant set Λ_f on which f is topologically conjugate to the shift $\sigma : \Sigma \to \Sigma$ with k symbols. First we state an application of the well known results of [13].

Lemma 5.3 (Normal hyperbolicity). *Let $F : N \times M \to N \times M$ be a skew-product diffeomorphism such that $F(x, y) = (f(x), g(x, y))$ with f as above, and such that $g(x, \cdot)$ is uniformly dominated by $f|_{\Lambda_f}$ Then, for any G in a C^1 neighborhood of F in* Diff$(N \times M)$ *the following holds:*

- *There is a unique continuation Γ_G of $\Gamma_F := \Lambda_f \times M$,*
- *$G|_{\Gamma_G}$ is conjugate by ϕ_G to the C^1 skew product H_G on $\Sigma \times M$:*

$$(\omega, y) \mapsto (\sigma(\omega), h_{G,\omega}(y))$$

- *for any $\omega \in \Sigma$, $\phi_G^{-1}(\{\omega\} \times M)$ is diffeomorphic to M and it is C^1 close to $\phi_F^{-1}(\{\omega\} \times M) = \{\varphi^{-1}(\omega)\} \times M$.*

Definition 5.4. If F satisfies the previous lemma, then we say F is *robustly transitive on* Γ_F if for any G some neighborhood of F, Γ_G is a transitive set for G. Analogously, one may define robustly topologically mixing.

On the other hand, one can prove the following (see [15] for a proof).

Lemma 5.5. *Let $H : \Sigma \times M \to \Sigma \times M$, $H(\omega, y) = (\sigma(\omega), g_{\omega_0}(y))$, where $\mathcal{G} = \{g_i\}$ be a family of diffeomorphisms such that* IFS$(\mathcal{G})$ *is minimal. Then H is transitive; and the strong-unstable lamination is minimal.*

As a corollary one obtains the following criterion for robust transitivity:

Theorem 5.6 (Sufficient condition for robust transitivity)**.** *Let F be as in Lemma 5.3 and suppose that for $i = 1, \dots, k$,*

$$F|_{R_i \times M} = f|_{R_i} \times g_i$$

with $\cup_{i=1}^{k} R_i = \Lambda_f$ and such that $\mathrm{IFS}(\{g_1, \dots, g_k\})$ *is strongly robustly minimal. Then, F is robustly transitive on Γ_F, and the strong-unstable lamination is robustly minimal in Γ_F.*

Remark that Theorem 4.2 provides a sufficient condition to guarantee the main hypothesis of this theorem, i.e. the IFS being strongly robustly minimal. Therefore, Theorems 4.2 and 5.6 together, provide a general framework to prove robust transitivity, reformulating to the role of blenders.

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