# Robust unbounded attractors for differential equations in $\mathbb{R}^3$

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**Abstract**

We construct unbounded strange attractors for vector fields in $\mathbb{R}^3$ that are robust transitive under uniformly small perturbations. Their geometry is reminiscent of geometric Lorenz and other singular hyperbolic attractors, but they contain no equilibria.

**Keywords**: Strange attractor, unbounded attractor

## 1 Introduction

Central topics in dynamical systems are the characterization of robust dynamical properties and the description of attractors. In this paper we consider vector fields in $\mathbb{R}^3$ and provide a geometric construction of robust transitive unbounded attractors.

It is well known that compact hyperbolic attractors, and various singular hyperbolic attractors [11] such as geometric Lorenz attractors, occur robustly with respect to smooth perturbations. In the reverse direction it is known that compact attractors for differential equations in $\mathbb{R}^3$ that are robust transitive in the $C^1$ topology, are hyperbolic or singular hyperbolic [12]. Unbounded invariant sets that are robust transitive have attracted much less interest. Unbounded attractors do seem to appear naturally in applications; numerical experiments on the Rikitake model [8, 10, 14], see also Section 5, indicate the robust occurrence of unbounded attractors.

The topology on the class of vector fields we will consider is the uniform $C^j$ topology. Write $\mathcal{X}^j(\mathbb{R}^3)$ for the set of $C^j$ vector fields on $\mathbb{R}^3$ endowed with the uniform $C^j$ topology. A closed transitive invariant set $\Lambda$ (transitivity means the existence of a dense forward orbit) for a differential equation $\dot{u} = f(u)$ is robust transitive if there is an open neighborhood $U$ of $\Lambda$ for which $\Lambda$ is the maximal invariant set, and if for any differential equation $\dot{u} = g(u)$ with $g$ sufficiently close to $f$, the maximal invariant set for $g$ in $U$ is a transitive invariant set. An attractor for a vector field $f$ is a closed transitive invariant set $A$, which attracts nearby orbits: there is an open neighborhood $U$ of $A$ for which positive orbits for $f$ of points $x \in U$ have $\omega$-limit sets contained in $A$ (if $A$ is unbounded there may exist such positive orbits that escape to infinity, with empty $\omega$-limit set). We speak of a robust unbounded attractor $A$, if it is robust transitive and also the properties of being an attractor and being unbounded are robust.

**Theorem 1.1.** *There is an open set $\mathcal{U} \subset \mathcal{X}^j(\mathbb{R}^3)$, $j \geq 3$, so that each vector field from $\mathcal{U}$ possesses a robust unbounded attractor.*

The construction uses compactifications of differential equations to the closed unit ball, where the vector field restricted to the unit sphere that bounds the unit ball can be seen as the "vector field at infinity". It generalizes to higher dimensions, yielding higher dimensional analogs of Theorem 1.1.

The construction starts with a vector field whose compactification has an attractor intersecting the unit sphere. The original vector field therefore has un unbounded attractor. We construct the vector field such that uniformly small perturbations of it have compactifications with nearby attractors as well. It is a consequence of the construction that the vector field restricted to the attractor is not structurally stable. Where the attractor of the compactified vector fields contains equilibria (in fact, 3 equilibria) on the unit sphere, the attractors of the original vector fields contain no equilibria.

To explain the construction in more detail, we first recall the notions of partially hyperbolic set and singular hyperbolic set.

A compact invariant set $\Lambda$ of $\dot{u} = f(u)$ is a *partially hyperbolic set* if, up to time reversal, there is an invariant dominated splitting $T\Lambda = E^{\mathrm{s}} \oplus E^{\mathrm{c}}$ and positive constants $K, \lambda$ such that

(i) $E^{\mathrm{s}}$ is contracting: $|D\varphi_t/E^{\mathrm{s}}_u| \leq Ke^{-\lambda t}$, for all $u \in \Lambda$ and $t > 0$.

(ii) $E^{\mathrm{s}}$ dominates $E^{\mathrm{c}}$: $|D\varphi_t/E^{\mathrm{s}}_u||D\varphi_{-t}/E^{\mathrm{c}}_{\varphi_t(u)}| \leq Ke^{-\lambda t}$, for all $u \in \Lambda$ and $t > 0$.

The central direction $E^{\mathrm{c}}$ of $\Lambda$ is said to be volume expanding if the additional condition

$$|\det(D\varphi_t/E^{\mathrm{c}}_u)| \geq Ke^{\lambda t}$$

holds for all $u \in \Lambda$ and $t > 0$. Let $\Lambda$ be a compact invariant set of $\dot{u} = f(u)$ containing at least one equilibrium. Then $\Lambda$ is called a *singular hyperbolic set* if it is partially hyperbolic with volume expanding central directions and all its equilibria are hyperbolic [11]. A singular hyperbolic set is called a *singular hyperbolic attractor* if it moreover is an attractor. We refer to the book chapter [4, Chapter 9], the article [2] and references therein for general information on singular hyperbolic attractors.

The main arguments to arrive at Theorem 1.1 are as follows. For a vector field $f$ on $\mathbb{R}^3$, a compactification $\Gamma f$ on the unit ball $\mathbb{D}^3$ is defined. This is a vector field of the form $\alpha\phi_* f$, where $\phi$ is a diffeomorphism $\mathbb{R}^3 \to \mathbb{D}^3$ and $\alpha$ is a suitable positive function so that $\phi_* f$ extends continuously to the boundary $\mathbb{S}^2$ of $\mathbb{D}^3$. In Section 3 we review this construction. We construct $\Gamma f$ so that $\Gamma f$ is has a $C^j$ extension to $\mathbb{S}^2$ and possesses a singular hyperbolic attractor $A$ that intersects $\mathbb{S}^2$, see Section 2. For fixed $\alpha$ in the definition of compactification, $\Gamma f$ determines $f$. The existence of a singular hyperbolic attractor for $\Gamma f$ implies that $f$ has un unbounded attractor $\phi^{-1}(A)$. We find $\alpha$ so that uniformly $C^j$, $j \geq 3$, small perturbations $f + r$ of $f$ yield, under compactification $\Gamma(f + r)$, vector fields $C^j$ close to $\Gamma f$. Then $\Gamma(f + r)$ has a singular hyperbolic attractor close to $A$. Hence, $f + r$ has an unbounded attractor close to $\phi^{-1}(A)$. Theorem 1.1 directly follows from combining Proposition 3.1 and Theorem 2.1.

We note that [9] contains constructions of vector fields with "singular horseshoes" on the closed unit ball; invariant sets (not attractors) containing a heteroclinic connection between an equilibrium and a periodic orbit inside the unit sphere.

We are grateful to Hamid Zangeneh for discussions about the Rikitake model.

## 2 Singular hyperbolic attractors on the closed unit ball

We construct a smooth vector field

$$\dot{\bar{u}} = \bar{f}(\bar{u}),$$

on $\mathbb{R}^3$ which admits the unit sphere $\mathbb{S}^2$ as an invariant manifold and which has a transitive invariant set $\Lambda$ containing orbits on $\mathbb{S}^2$. The set $\Lambda$ has a singular hyperbolic structure and is a (singular hyperbolic) attractor for the vector field restricted to the closure of $\mathbb{D}^3$.

**Theorem 2.1.** *There exists a smooth vector field $\dot{\bar{u}} = \bar{f}(\bar{u})$ on the closure of $\mathbb{D}^3$ with a singular hyperbolic attractor intersecting $\mathbb{S}^2$. The attractor is robust in the $C^j$ topology, $j \geq 3$ (under perturbations that keep $\mathbb{S}^2$ invariant).*

*Proof.* The construction of $\bar{f}$ is similar to that of geometric Lorenz attractors [1, 6, 18], the singular hyperbolic attractor will however contain three equilibria instead of one.

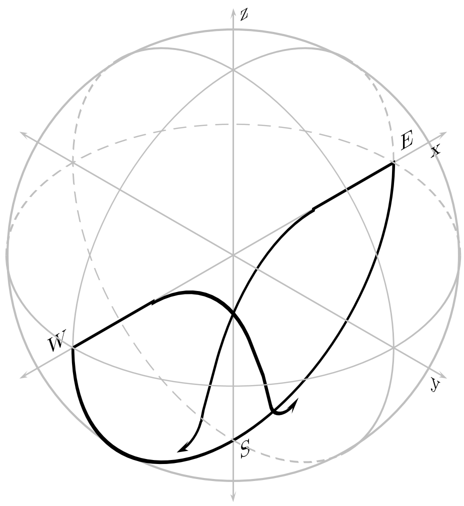


Figure 2.1: *Separatrices of the flow in $W^u(S)$ flow into $W$ and $E$.*

We construct $\bar{f}$ to be $\mathbb{Z}_2$ equivariant under the action $R(x, y, z) = (-x, -y, z)$;

$$\bar{f} \circ R = R\bar{f}. \tag{2.1}$$

The vector field $\bar{f}$ has hyperbolic equilibria on $\mathbb{S}^2$ at $S = (0, 0, -1)$ and $W = (-1, 0, 0)$, $E = (1, 0, 0)$. These equilibria have one dimensional unstable manifolds, with tangent directions parallel to the $x$-axis. Invariance of $\mathbb{S}^2$ forces $W^u(S) \subset \mathbb{S}^2$; for the vector field $\bar{f}$, the closure of $W^u(S)$ is a circle arc connecting $W$ to $E$, see Figure 2.1. The unstable manifolds of $E$ and $W$ are transverse to $\mathbb{S}^2$.

Within the class of vector fields for which $\mathbb{S}^2$ is invariant, these heteroclinic connections are robust as they give saddle-sink connections inside $\mathbb{S}^2$.

We pose the following spectral conditions on the linearized differential equation about the equilibria. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $Df(S)$ satisfy

$$\lambda_1 < \lambda_2 < 0 < \lambda_3, \qquad \lambda_2 + \lambda_3 > 0.$$

The eigendirections corresponding to $\lambda_1, \lambda_2, \lambda_3$ are respectively the $y$-axis, the $z$-axis and the $x$-axis. The eigenvalues $\nu_1, \nu_2, \nu_3$ of $Df(W)$ and $Df(E)$ satisfy

$$\nu_1 < \nu_2 < 0 < \nu_3, \qquad \nu_2 + \nu_3 > 0.$$

The eigendirections corresponding to $\nu_1, \nu_2, \nu_3$ are respectively the $y$-axis, the $z$-axis and the $x$-axis.

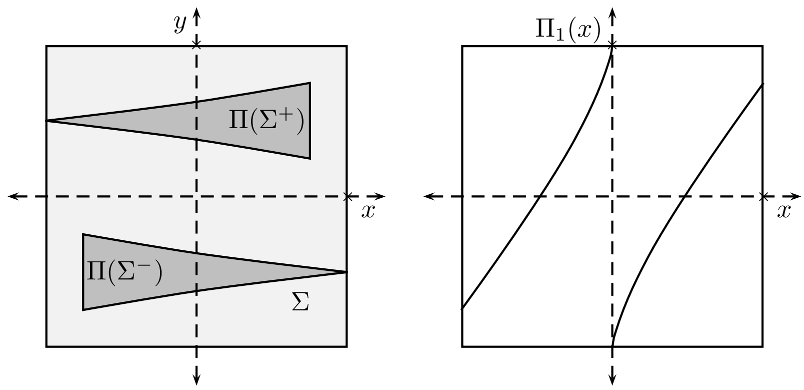


Figure 2.2: *The first return map* $\Pi = (\Pi_1, \Pi_2)$*, with indicated the images of the parts* $\Sigma^+ = \Sigma \cap \{x > 0\}$*,* $\Sigma^- = \Sigma \cap \{x < 0\}$*, and the interval map* $\Pi_1$ *.*

On a cross section $\Sigma \subset \{z = -1/2\}$ a first return map $\Pi$ is defined, mapping $\Sigma$ (apart from $\{x = 0\} \cap \Sigma$ which is contained in $W^s(S)$) inside itself. As for the geometric Lorenz model one obtains $\Pi$ as a composition of local transition maps through neighborhoods of the equilibria $S, W, E$ and global transition maps from the flow boxes connecting these neighborhoods. For suitable cross sections $\Sigma_S^{in}, \Sigma_S^{out}$ near $S$, and suitable coordinates $(x, y)$ on them, the local transition map $\Pi_S^{loc} : \Sigma_S^{in} \to \Sigma_S^{out}$ reads

$$\Pi_S^{loc} : (x, y) = (x^{-\lambda_2/\lambda_3}, yx^{-\lambda_1/\lambda_3})$$

(this is a transition map for a linear vector field). Likewise, for suitable cross sections $\Sigma_W^{in}, \Sigma_W^{out}$ near $W$, the local transition map $\Pi_W^{loc} : \Sigma_W^{in} \to \Sigma_W^{out}$ writes

$$\Pi_W^{loc} : (x, y) = (x^{-\nu_2/\nu_3}, yx^{-\nu_1/\nu_3}).$$

With the global transition map $\Pi_{S,W} : \Sigma_S^{out} \to \Sigma_W^{in}$ equal to the identity map, the global transition map $\Pi_{W,S} : \Sigma_W^{out} \to \Sigma_S^{in}$ equal to $(x, y) \mapsto (1 - ax, y/2 + 1/2)$ (with $a < 2$, close to 2) and the other transition maps with $W$ replaced by $E$ obtained by symmetry, one gets an expression $\Pi : [-1, 1]^2 \to [-1, 1]^2$:

$$\Pi(x, y) = (\Pi_1(x), \Pi_2(x, y)) = \begin{cases} (1 - a|x|^{\lambda_2\nu_2/\lambda_3\nu_3}, \Pi_2(x, y)), & x < 0, \\ (-1 + a|x|^{\lambda_2\nu_2/\lambda_3\nu_3}, \Pi_2(x, y)), & x > 0, \end{cases}$$

where $\Pi_2$ is a contraction and $\Pi_1$ is an expansion (see Figure 2.2). This is the same type of expression as for the geometric Lorenz model. Because of this and the assumed spectral conditions, $\bar{f}$ possesses a singular hyperbolic attractor.

To prepare for a proof of robustness of the singular hyperbolic attractor, we consider stable foliations. Near the singular hyperbolic attractor there is a stable foliation $\mathcal{G}^s$ for the flow, compare [6, 15, 18]. Note that $\mathcal{G}^s$ defines a foliation $\mathcal{F}^s$ on the cross sections, invariant under the transition maps, by

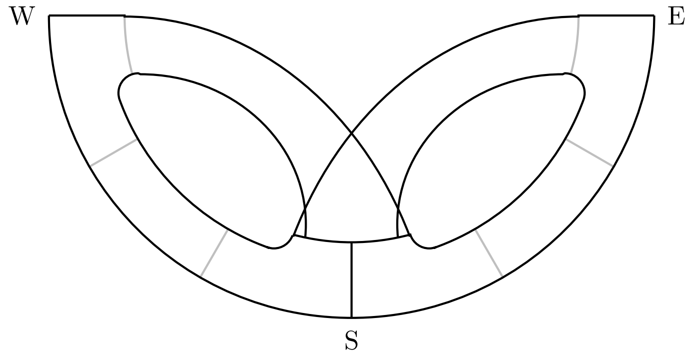


Figure 2.3: *Branched surface $W$.*

projecting along flow lines. On $\Sigma_S^{in}$, $\mathcal{F}^s = \{x = \text{constant}\}$. Take a branched surface $W$ transverse to leaves of the stable foliation, as for the Lorenz model [18], see Figure 2.3. Projecting along leaves of $\mathcal{F}^s$ to the branched surface, yields one dimensional maps. This gives $\Pi_1$ as a composition $\pi_{W,S} \circ \pi_W^{loc} \circ \pi_{S,W} \circ \pi_S^{loc}$ if $x < 0$ and $\pi_{E,S} \circ \pi_E^{loc} \circ \pi_{S,E} \circ \pi_S^{loc}$ if $x > 0$. Here $\pi_{W,S}(x) = -1 + ax$, $\pi_W^{loc}(x) = x^{-\nu_2/\nu_3}$, $\pi_{S,E}(x) = x$, $\pi_S^{loc}(x) = x^{-\lambda_2/\lambda_3}$.

Now we prove robustness of the singular hyperbolic attractor. As in [7, Proposition A.2.1], $C^j$, $j \geq 3$, small perturbations of $\bar{f}$ give rise to first return maps admitting $C^1$ invariant stable foliations (stable foliations for the perturbed vector fields are merely continuous). Choosing the branched surface near the equilibria to be a local center manifold, one may assume $\pi_W^{loc}(x) = x^{-\nu_2/\nu_3}$ and $\pi_S^{loc}(x) = x^{-\lambda_2/\lambda_3}$ also for the perturbed vector field (where the eigenvalues may have changed slightly) [7, Proposition A.1.1]. The global transition maps are $C^1$ perturbations of those for $\bar{f}$. Hence, the one dimensional map resulting from projecting along foliation leaves is expanding. Conclusion: each vector field near $\bar{f}$ has a singular hyperbolic attractor. □

The proof simplifies somewhat under additional eigenvalue conditions $\lambda_1 < \lambda_2 - \lambda_3$ and $\nu_1 < \nu_2 - \nu_3$. Then the stable foliation $\mathcal{G}^s$ for vector fields near $\bar{f}$ is $C^1$ and local center unstable manifolds near the equilibria are $C^2$ [16]. See [15, 17] for further information on stable foliations for return maps. We assume $C^j$, $j \geq 3$, vector fields. A similar result may be accomplished with less smoothness, working directly with return maps on a cross section and using Shil'nikov variables to compute their asymptotic expansions.

The arguments in the above proof yield an open class of vector fields with robust singular hyperbolic attractors. Similar attractors containing three equilibria, for differential equations in $\mathbb{R}^4$, are considered in [13].

# 3 Compactifications

We consider compactifications of vector fields on $\mathbb{R}^3$: vector fields equivalent to a push-forward to the open unit ball $\mathbb{D}^3$ with a smooth extension to the unit sphere $\mathbb{S}^2$ that bounds $\mathbb{D}^3$. Such techniques are frequently applied to the study of polynomial vector fields in the plane [5]. We identify open classes of vector fields on $\mathbb{R}^3$, in the uniform $C^j$ topology, with identical extension to $\mathbb{S}^2$ when compactifying.

Consider the smooth coordinate transformation $\phi$ mapping $\mathbb{R}^3$ to $\mathbb{D}^3$;

$$\phi(u) = u/\sqrt{1 + |u|^2}.$$

The inverse coordinate transformation is given by

$$\phi^{-1}(\bar{u}) = \bar{u}/\sqrt{1-|\bar{u}|^2}.$$

Start with a differential equation

$$\dot{u} = f(u) \tag{3.1}$$

on $\mathbb{R}^3$ and the push-forward by $\phi^{-1}$,

$$\dot{\bar{u}} = \bar{f}(\bar{u}), \tag{3.2}$$

on $\mathbb{D}^3$. Then

$$\begin{aligned} \bar{f}(\bar{u}) &= \frac{f(u)(1+|u|^2) - u\langle f(u), u\rangle}{(1+|u|^2)^{3/2}} \\ &= f(\bar{u}/\sqrt{1-|\bar{u}|^2})\sqrt{1-|\bar{u}|^2} - \bar{u}\langle f(\bar{u}/\sqrt{1-|\bar{u}|^2}), \bar{u}\rangle\sqrt{1-|\bar{u}|^2}. \end{aligned} \tag{3.3}$$

Suppose (3.1) is such that it equals the push-forward by $\phi^{-1}$ of some vector field (3.2) extending smoothly to $\mathbb{S}^2$. Uniformly $C^j$ small perturbations of (3.1) may not correspond to $C^j$ small perturbations of (3.2). To remedy this difficulty, we introduce additional multiplications of the vector fields with positive functions, implying changes of the topology of perturbations (when multiplying perturbations with these functions).

For $\sigma > 0$, define the function

$$\alpha_\sigma(u) = (1+|u|^2)^\sigma$$

and consider vector fields $f_\sigma = \alpha_\sigma f$. As $\alpha_\sigma$ is a positive function, this multiplication is equivalent to a time reparameterization. Write $\bar{\alpha}_\sigma(\bar{u}) = \alpha_\sigma(\phi^{-1}(\bar{u})) = 1/(1-|\bar{u}|^2)^\sigma$. Note that $\phi_* f_\sigma = \phi_* \alpha_\sigma f = \bar{\alpha}_\sigma \phi_* f = \bar{\alpha}_\sigma \bar{f}$. Define the compactification $\Gamma_\sigma f_\sigma$ of $f_\sigma$ as the vector field $\phi_* f_\sigma / \bar{\alpha}_\sigma = \bar{f}$.

Let $r$ be a $C^j$ uniformly small vector field on $\mathbb{R}^3$ and perturb $\dot{u} = f_\sigma(u)$ to $\dot{u} = f_\sigma(u) + r(u)$. Define the compactification $\Gamma_\sigma(f_\sigma + r)$ as $\phi_*(f_\sigma + r)/\bar{\alpha}_\sigma$ and note

$$\Gamma_\sigma(f_\sigma + r) = \bar{f} + (\phi_* r)/\bar{\alpha}_\sigma.$$

**Proposition 3.1.** *Let $f_\sigma$ be a smooth differential equation on $\mathbb{R}^3$, so that its compactification $\Gamma_\sigma f_\sigma = \bar{f}$ has a $C^j$ extension to $\mathbb{S}^2$. Then for $2\sigma > 3j - 1$ there is an open neighborhood $\mathcal{U}$ of $f_\sigma$ in the uniform $C^j$ topology so that any $g \in \mathcal{U}$ has compactification $\Gamma_\sigma g$ that has $C^j$ extension to $\mathbb{S}^2$, equal to $\bar{f}$.*

*Proof.* This is a direct computation using the definition of $\Gamma_\sigma$ and (3.3) with $f$ replaced by $f + r/\alpha$. □

If $\bar{f}$ is $\mathbb{Z}_2$-equivariant as in (2.1), then so are $f$ and $f_\sigma$; note also that the Lorenz model and the Rikitake model (see the next section) have this symmetry.

# 4 Vector fields on the positive half space

Alternatively one can consider vector fields on the positive half space $\mathbb{H}^3 = \{z > 0\}$ that have a smooth limit on the boundary $\{z = 0\}$. For computational ease, we look at vector fields on $\mathbb{H}^3$ and pushforwards by $\psi : \mathbb{H}^3 \to \mathbb{H}^3$ given by $\psi(x, y, z) = (x, y, 1/z)$.

Writing $u = (x, y, z)$, consider differential equations

$$\dot{u} = g(u) \tag{4.1}$$

and $\dot{u} = g_\sigma(u) = z^\sigma g(u)$ on $\mathbb{H}^3$. With $\bar{u} = (\bar{x}, \bar{y}, \bar{z}) = \psi(u)$, let $\bar{g} = \psi_* g$ and assume $\bar{u} \mapsto \bar{g}(\bar{u})$ has a smooth extension to $\{\bar{z} = 0\}$. Write $\bar{g}_\sigma = \psi_* g_\sigma$. Consider the compactification $\dot{\bar{u}} = \bar{z}^\sigma \bar{g}_\sigma(\bar{u}) = \bar{g}(\bar{u})$. For $\sigma > 2j - 2$, the compactification $\bar{g}(\bar{u}) + \bar{z}^\sigma \psi_* r(\bar{u})$ of a uniformly $C^j$ small perturbation $g + r$ of $g$ is a vector field with a $C^j$ extension to the boundary $\{\bar{z} = 0\}$ of $\mathbb{H}^3$.

For the construction of a vector field, equivalent to $\bar{f}$ in Theorem 2.1 but on the upper half space $\mathbb{H}^3$ (see the end of Section 3), the singular hyperbolic attractor would intersect the plane that bounds $\mathbb{H}^3$ in a line segment connecting three equilibria. Another possible construction, closer to the idea of taking a Lorenz vector field and pushing the origin to infinity, is as follows. Consider a vector field $\bar{f}$ with a Lorenz like attractor contained in $\mathbb{H}^3$ and with the origin as hyperbolic equilibrium. Suppose near the origin, the vector field is in normal form

$$\begin{aligned} \dot{x} &= \lambda_3 x, \\ \dot{y} &= \lambda_1 y, \\ \dot{z} &= \lambda_2 z + x^2, \end{aligned}$$

which has a local unstable manifold $z = x^2/(2\lambda_3 - \lambda_2)$. Near the origin along $\{z = 0\}$, away from the local strong stable manifold (the $y$-axis), the vector field moves into $\mathbb{H}^3$. As in the previous section, one produces robust strange unbounded attractors by combining a blow-up with a time reparametrization.

Finally we remark that $D\psi^{-1}$ may not map a splitting in contracting and center unstable directions of a partially hyperbolic set for $\bar{g}$ to (uniform) contracting and center unstable directions for $g$. This can be seen from a construction of a vector field with a compact singular hyperbolic attractor on $\mathbb{H}^3$ containing equilibria on its boundary as in Section 2, combined with the following computation: the push-forward of a vector field $\dot{\bar{x}} = \lambda_3 \bar{x}, \dot{\bar{y}} = \lambda_1 \bar{y}, \dot{\bar{z}} = \lambda_2 \bar{z}$ by $\psi^{-1}$ is $\dot{x} = -\lambda_3 x, \dot{y} = \lambda_1 y, \dot{z} = \lambda_2 z$.

# 5 The Rikitake model

Numerical experiments on the Rikitake model indicate the existence of unbounded strange attractors, with a geometry as in the above constructed geometric model. A compactification of the Rikitake model however leads to a conservative vector field on $\mathbb{S}^2$, different from the geometric model and making a rigorous analysis complicated. For reference we deduce the compactification.

The nondimensionalized Rikitake equations [8, 14] are given by

$$\begin{aligned} \dot{x} &= -\mu x + yz, \\ \dot{y} &= -\mu y + (z - A)x, \\ \dot{z} &= 1 - xy. \end{aligned} \tag{5.1}$$

Figure 5.1 shows what appears to be a strange attractor approaching the $z$-axis arbitrarily close, see also [10]. Restricted to the $z$-axis (5.1) equals $\dot{z} = 1$; orbits on it escape to infinity.

To compute the compactification of (5.1) to $\mathbb{D}^3$, it is convenient to work with the coordinate transformation $\hat{u} = u/(1 + |u|)$, which is smooth away from $u = 0$ and maps $\mathbb{R}^3$ to the interior $\mathbb{D}^3$ of the unit sphere $\mathbb{S}^2$. Note that $\bar{u} = \hat{u}/\sqrt{1 - 2|\hat{u}| + 2|\hat{u}|^2}$. The inverse transformation of $\hat{u} = u/(1 + |u|)$ is given by $u = \hat{u}/(1 - |\hat{u}|)$. Compute

$$\dot{\hat{u}} = (1 - |\hat{u}|)\dot{u} - (1 - |\hat{u}|)\langle \hat{u}, \dot{u} \rangle \frac{\hat{u}}{|\hat{u}|}.$$

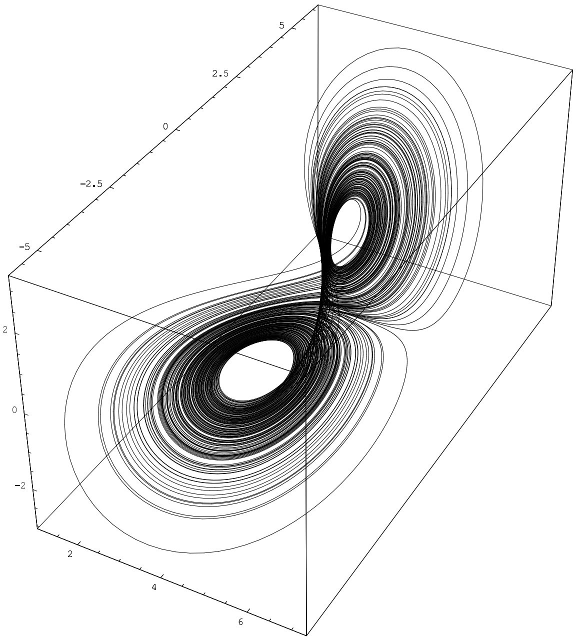


Figure 5.1: *Rikitake attractor for parameter values* $\mu = 1$ *and* $A = 4$.

Plugging in the differential equation yields, for $\hat{u} = (\hat{x}, \hat{y}, \hat{z})$,

$$\begin{aligned}\dot{\hat{x}} &= -\mu\hat{x} + \frac{\hat{y}\hat{z}}{1-|\hat{u}|} - \frac{\hat{x}}{|\hat{u}|}F(\hat{u}),\\ \dot{\hat{y}} &= -\mu\hat{y} + \hat{x}\left(\frac{\hat{z}}{1-|\hat{u}|} - A\right) - \frac{\hat{y}}{|\hat{u}|}F(\hat{u}),\\ \dot{\hat{z}} &= 1-|\hat{u}| - \frac{\hat{x}\hat{y}}{1-|\hat{u}|} - \frac{\hat{z}}{|\hat{u}|}F(\hat{u}),\end{aligned} \tag{5.2}$$

where

$$F(\hat{u}) = -\mu\hat{x}^2 - \mu\hat{y}^2 - A\hat{x}\hat{y} + \frac{\hat{x}\hat{y}\hat{z}}{1-|\hat{u}|} + (1-|\hat{u}|)\hat{z}.$$

Multiplying by a factor $1-|\hat{u}|$ yields a differential equation on $\mathbb{D}^3$ which can be smoothly extended to $\mathbb{S}^2$. In fact, on $\mathbb{S}^2$, one obtains the differential equation

$$\begin{aligned}\dot{\hat{x}} &= \hat{y}\hat{z}(1-\hat{x}^2),\\ \dot{\hat{y}} &= \hat{x}\hat{z}(1-\hat{y}^2),\\ \dot{\hat{z}} &= -\hat{x}\hat{y}(1+\hat{z}^2).\end{aligned} \tag{5.3}$$

It is clear that this statement also applies when using the smooth coordinate transformation $\bar{u} = u/\sqrt{1+|u|^2}$. Note that (5.3) has six equilibria on $\mathbb{S}^2$, at the intersections points of the coordinate axes with $\mathbb{S}^2$. The northpole $N = (0,0,1)$ and southpole $S = (0,0,-1)$ are saddles for (5.3). The linearization of (5.2) about $N$ has an additional eigenvalue 0 for the invariant $z$-axis; along the $z$-axis the flow is thus weakly attracting from inside $\mathbb{D}^3$. Likewise the flow is weakly expanding along the $z$-axis near the southpole. The four other equilibria are foci. Inside the coordinate planes there are heteroclinic connections between $N$ and $S$. The flow consists further of periodic solutions encircling the foci and converging to the heteroclinic cycles. Note that the center stable and center unstable manifolds of $N$ are smooth manifolds yielding, in the original coordinates, stable and unstable manifolds of the $z$-axis [3]. We have

$$\frac{d}{dt}|\hat{u}|^2 = (-\mu\hat{x}^2 - \mu\hat{y}^2)(1-|\hat{u}|)^2 - A\hat{x}\hat{y}(1-|\hat{u}|)^2 + \hat{x}\hat{y}\hat{z}(1-|\hat{u}) + \hat{z}(1-|\hat{u}|)^3,$$

showing that the foci are weakly repelling in directions into $\mathbb{D}^3$.

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